AppellF1

Notations

Traditional name

Appell hypergeometric function F_1

Traditional notation

$F_1(a; b_1, b_2; c; z_1, z_2)$

Mathematica StandardForm notation

AppellF1[a, b_1, b_2, c, z_1, z_2]

Primary definition

07.36.02.0001.01

$$F_1(a; b_1, b_2; c; z_1, z_2) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{(a)_{k+l} (b_1)_k (b_2)_l z_1^k z_2^l}{(c)_{k+l} k! l!} /; |z_1| < 1 \bigwedge |z_2| < 1$$

Specific values

Specialized values

Values at $(z_1, z_2) = (0, 0)$

07.36.03.0001.01

$$F_1(a; b_1, b_2; c; 0, 0) = 1$$

Values at $z_2 = 0$

07.36.03.0002.01

$$F_1(a; b_1, b_2; c; z, 0) = {}_2F_1(a, b_1; c; z)$$

Values at $z_2 = 1$

07.36.03.0003.01

$$F_1(a; b_1, b_2; c; z, 1) = {}_2F_1(a, b_2; c; 1) \, {}_2F_1(a, b_1; c - b_2; z)$$

Values at $z_2 = \pm z_1$

07.36.03.0004.01

$$F_1(a; b_1, b_2; c; z, z) = {}_2F_1(a, b_1 + b_2; c; z)$$

07.36.03.0005.01

$$F_1(a;b_1,b_1;c;z,-z)=={}_3F_2\left(\frac{a+1}{2},\frac{a}{2},b_1;\frac{c+1}{2},\frac{c}{2};z^2\right)$$

Values at fixed points

For fixed a, b_1, b_2, z_1, z_2

07.36.03.0006.01

$$F_1(a;b_1,b_2;b_1+b_2;z_1,z_2)==(1-z_2)^{-a}\,{}_2F_1\left(a,b_1;b_1+b_2;\frac{z_1-z_2}{1-z_2}\right)$$

For fixed z_1, z_2

07.36.03.0011.01

$$F_1(a;b_1,b_2;a-n;z_1,z_2)-(1-z_1)^{-b_1}(1-z_2)^{-b_2}\sum_{k=0}^{n}\sum_{l=0}^{n-k}\frac{(-n)_{k+l}\,(b_1)_k\,(b_2)_l}{(a-n)_{k+l}\,k!\,l!}\left(\frac{z_1}{z_1-1}\right)^k\left(\frac{z_2}{z_2-1}\right)^l\;/;\;n\in\mathbb{N}$$

07.36.03.0012.01

$$F_1\left(n+\frac{1}{2};n-\frac{1}{2},\frac{1}{2};n+\frac{3}{2};z_1,z_2\right)=\frac{(2n+1)\,z_2^{-\frac{1}{2}}}{\left(-\frac{1}{2}\right)_n}\,\frac{\partial^n E\left(\sin^{-1}\left(\sqrt{z_2}\right)\middle|\frac{z_1}{z_2}\right)}{\partial z_1^n}$$

07.36.03.0013.01

$$F_1\left(n+\frac{1}{2};n+\frac{1}{2},\frac{1}{2};n+\frac{3}{2};z_1,z_2\right)=\frac{2n+1}{\left(\frac{1}{2}\right)_n\sqrt{z_2}}\,\frac{\partial^n F\left(\sin^{-1}\left(\sqrt{z_2}\right)\middle|\frac{z_1}{z_2}\right)}{\partial z_1^n}\;/;\;n\in\mathbb{N}$$

07.36.03.0014.01

$$F_1\left(n+\frac{1}{2};n+1,1;n+\frac{3}{2};z_1,z_2\right)==$$

$$\frac{2n+11}{2n!}\sum_{m=0}^{n}\frac{(-1)^{n-m}\,(m)_{2(n-m)}}{(n-m)!\left(2\sqrt{z_1}\right)^{2n-m}}\left((-1)^m\tanh^{-1}\left(\sqrt{z_1}\right)m!\left(\left(\sqrt{z_1}-\sqrt{z_2}\right)^{-m-1}+\left(\sqrt{z_1}+\sqrt{z_2}\right)^{-m-1}\right)+\right.$$

$$\sum_{j=0}^{m}\binom{m}{j}(-1)^{m-j}(m-j)!\left(\frac{1}{\left(\sqrt{z_1}+\sqrt{z_2}\right)^{-j+m+1}}+\frac{1}{\left(\sqrt{z_1}-\sqrt{z_2}\right)^{-j+m+1}}\right)$$

$$\left.\sum_{k=0}^{j-1}\frac{k!\,(-j+2k+2)_{2(j-k-1)}\,(1-z_1)^{-k-1}}{(j-k-1)!\left(2\sqrt{z_1}\right)^{j-2k-1}}\right)-(-1)^n(2n+1)\sqrt{z_2}\,\tanh^{-1}\left(\sqrt{z_2}\right)(z_1-z_2)^{-n-1}\;/;\;n\in\mathbb{N}$$

07.36.03.0007.01

$$F_1\left(\frac{1}{2};\frac{1}{2},-\frac{1}{2};\frac{3}{2};z_1,z_2\right)==\frac{1}{\sqrt{z_1}}\,E\left(\sin^{-1}\left(\sqrt{z_1}\right)\middle|\frac{z_2}{z_1}\right)$$

07.36.03.0008.01

$$F_1\left(\frac{1}{2};\frac{1}{2},\frac{1}{2};\frac{3}{2};z_1,z_2\right)==\frac{1}{\sqrt{z_1}}\,F\left(\sin^{-1}\left(\sqrt{z_1}\right)\middle|\frac{z_2}{z_1}\right)$$

07.36.03.0009.01

$$F_1\left(\frac{1}{2}; \frac{1}{2}, \frac{1}{2}; \frac{3}{2}; z_1, z_2\right) = \frac{1}{\sqrt{z_1}} \operatorname{sn}^{-1}\left(\sqrt{z_1} \,\middle|\, \frac{z_2}{z_1}\right)$$

07.36.03.0015.01

$$F_1\left(\frac{1}{2}; 1, 1; \frac{3}{2}; z_1, z_2\right) = \frac{\tanh^{-1}\left(\sqrt{z_1}\right)\sqrt{z_1} - \tanh^{-1}\left(\sqrt{z_2}\right)\sqrt{z_2}}{z_1 - z_2}$$

07.36.03.0016.01

$$F_1\left(\frac{3}{2}; \frac{1}{2}, 1; \frac{5}{2}; z_1, z_2\right) = \frac{3}{\sqrt{z_1}\, z_2}\left(\frac{1}{\sqrt{\frac{z_2 - z_1}{z_1}}} \tanh^{-1}\left(\frac{\sqrt{z_1}}{\sqrt{1 - z_1}} \sqrt{\frac{z_2 - z_1}{z_1}}\right) - \sin^{-1}\left(\sqrt{z_1}\right)\right)$$

07.36.03.0017.01

$$F_1\left(\frac{5}{2}; \frac{1}{2}, 1; \frac{7}{2}; z_1, z_2\right) = \frac{5}{4\sqrt{z_1}\, z_2^2}\left(\frac{4}{\sqrt{\frac{z_2 - z_1}{z_1}}} \tanh^{-1}\left(\frac{\sqrt{z_1}}{\sqrt{1 - z_1}} \sqrt{\frac{z_2 - z_1}{z_1}}\right) - 2\sin^{-1}\left(\sqrt{z_1}\right)\left(\frac{z_2}{z_1} + 2\right) + \frac{2\sqrt{1 - z_1}\, z_2}{\sqrt{z_1}}\right)$$

07.36.03.0018.01

$$F_1\left(n + \frac{1}{2}; \frac{1}{2}, 1; n + \frac{3}{2}; z_1, z_2\right) =$$

$$\frac{2n+1}{\sqrt{z_1}}\left(\frac{z_2^{-n}}{\sqrt{\frac{z_2}{z_1} - 1}} \tanh^{-1}\left(\frac{\sqrt{z_1}\sqrt{\frac{z_2}{z_1} - 1}}{\sqrt{1 - z_1}}\right) - 2^{-n} z_1^{-n} \sum_{j=0}^{n-1} (-1)^j \binom{n}{j+1}\left(\frac{z_2 - 2z_1}{z_2}\right)^{j+1} \sum_{i=0}^{j+1} \binom{j+1}{i} z_1^i (z_2 - 2z_1)^{-i} \left(\sum_{q=0}^{\left\lfloor \frac{i-1}{2} \right\rfloor} \frac{1}{q!}\right.\right.$$

$$\left(\binom{i-1}{2q}\left(\frac{1}{2}\right)_q z_1^{-2q}\left(\frac{2z_1 - z_2}{z_1}\right)^{i-2q-1} z_2^{2q}\left(2\sin^{-1}\left(\sqrt{z_1}\right) + \sqrt{1 - z_1}\sqrt{z_1} \sum_{p=1}^{q} \frac{(p-1)!\,(1 - 2z_1)^{2p-1}}{\left(\frac{1}{2}\right)_p}\right)\right) +$$

$$\left.\left.\sum_{q=0}^{\left\lfloor \frac{i}{2} \right\rfloor - 1} 2^{2q+1}\binom{i-1}{2q+1} q!\,(1 - z_1)^{q+\frac{1}{2}} z_1^{-q-\frac{1}{2}}\left(\frac{2z_1 - z_2}{z_1}\right)^{i-2q-2} z_2^{2q+1} \sum_{p=0}^{q} \frac{2^{-2p}(1 - 2z_1)^{2p}(1 - z_1)^{-p} z_1^{-p}}{p!\left(\frac{3}{2}\right)_{q-p}}\right)\right)$$

07.36.03.0019.01

$$F_1\left(\frac{3}{2}; 2, 1; \frac{5}{2}; z_1, z_2\right) =$$

$$\frac{3}{2(1 - z_1)\sqrt{z_1}\,(z_1 - z_2)^2}\left(\sqrt{z_1}\left(-2\tanh^{-1}\left(\sqrt{z_2}\right)\sqrt{z_2}\,(z_1 - 1) + z_1 - z_2\right) + \tanh^{-1}\left(\sqrt{z_1}\right)(z_1 - 1)(z_1 + z_2)\right)$$

07.36.03.0020.01

$$F_1\left(\frac{5}{2}; 3, 1; \frac{7}{2}; z_1, z_2\right) = \frac{5}{8(z_1 - 1)^2 z_1^{3/2}(z_1 - z_2)^3}\left(\tanh^{-1}\left(\sqrt{z_1}\right)\left(3z_1^2 + 6z_2 z_1 - z_2^2\right)(z_1 - 1)^2 +\right.$$

$$\left.\sqrt{z_1}\left(5z_1^3 - 3(2z_2 + 1)z_1^2 + z_2(z_2 + 2)z_1 - 8\tanh^{-1}\left(\sqrt{z_2}\right)(z_1 - 1)^2\sqrt{z_2}\, z_1 + z_2^2\right)\right)$$

07.36.03.0010.01

$$F_1\left(\frac{1}{2}; \frac{1}{2}, 1; 1; z_1, z_2\right) = \frac{2}{\pi}\,\Pi(z_2 \mid z_1)$$

07.36.03.0021.01

$$F_1\left(1; -\frac{1}{2}, 1; n+\frac{3}{2}; z_1, z_2\right) = (2n+1)\left((-1)^{n-1}\sqrt{\frac{z_2-z_1}{z_2-1}}\,\tanh^{-1}\left(\sqrt{\frac{z_2-z_1}{z_2-1}}\right)(1-z_2)^n z_2^{-n-1} + \frac{(-1)^n (1-z_1)^{n+1}}{n!\,(1-z_2)} \sum_{m=0}^{n-1} \frac{(m+1)\,2^{m-2n} z_1^{\frac{m+1}{2}-n} (2n-m-2)!}{(n-m-1)!} \left(\left(\left(\sqrt{z_1}-\sqrt{\frac{z_2-z_1}{z_2-1}}\right)^{-m-2} + \left(\sqrt{\frac{z_2-z_1}{z_2-1}}+\sqrt{z_1}\right)^{-m-2}\right)\tanh^{-1}\left(\sqrt{z_1}\right) - \sum_{j=0}^{m} \frac{(-1)^j}{j+1}\left(\left(\sqrt{z_1}-\sqrt{\frac{z_2-z_1}{z_2-1}}\right)^{j-m-1} + \left(\sqrt{\frac{z_2-z_1}{z_2-1}}+\sqrt{z_1}\right)^{j-m-1}\right)\sum_{k=0}^{j}\frac{2^{j-2k}(1-z_1)^{-j+k-1} z_1^{\frac{j}{2}-k}(j-k)!}{(j-2k)!\,k!}\right)\right)$$

General characteristics

Some abbreviations

07.36.04.0001.01

$$\mathcal{NT}(\{a_1, a_2\}) = \neg\,(-a_1 \in \mathbb{N} \lor -a_2 \in \mathbb{N})$$

Domain and analyticity

$F_1(a; b_1, b_2; c; z_1, z_2)$ is an analytical function of a, b_1, b_2, c, z_1, z_2, which is defined in $\mathbb{C}^6$. For fixed a, b_1, c, z_1, z_2, it is an entire function of b_2. For fixed a, b_2, c, z_1, z_2, it is an entire function of b_1. For fixed b_1, b_2, c, z_1, z_2, it is an entire function of a. For negative integer a, $F_1(a; b_1, b_2; c; z_1, z_2)$ degenerates to a polynomial in z_1, z_2. For negative integer b_k, $F_1(a; b_1, b_2; c; z_1, z_2)$ degenerates to a polynomial in $z_k$of order $-b_k$, $1 \le k \le 2$.

07.36.04.0002.01

$$(a * b_1 * b_2 * c * z_1 * z_2) \longrightarrow F_1(a; b_1, b_2; c; z_1, z_2) :: (\mathbb{C}\otimes\mathbb{C}\otimes\mathbb{C}\otimes\mathbb{C}\otimes\mathbb{C}\otimes\mathbb{C}) \longrightarrow \mathbb{C}$$

Symmetries and periodicities

Mirror symmetry

07.36.04.0003.01

$$F_1\left(\overline{a}; \overline{b_1}, \overline{b_2}; \overline{c}; \overline{z_1}, \overline{z_2}\right) = \overline{F_1(a; b_1, b_2; c; z_1, z_2)} \;/;\; z_1 \notin (1, \infty) \land z_2 \notin (1, \infty)$$

Permutation symmetry

07.36.04.0004.01

$$F_1(a; b_1, b_2; c; z_1, z_2) = F_1(a; b_2, b_1; c; z_2, z_1)$$

Periodicity

No periodicity

Poles and essential singularities

With respect to z_k

For fixed a, b_k, c, in nonpolynomial cases (when $\neg(-a \in \mathbb{N} \bigvee -b_k \in \mathbb{N})$), the function $F_1(a; b_1, b_2; c; z_1, z_2)$ does not have poles and essential singularities with respect to z_k, $1 \leq k \leq 2$.

07.36.04.0005.01

$$\mathcal{Sing}_{z_k}(F_1(a; b_1, b_2; c; z_1, z_2)) == \{\} /; \mathcal{NT}(\{a, b_k\}) \bigwedge 1 \leq k \leq 2$$

For negative integer a or $b_k$and fixed c, z_j, $j \neq k$, the function $F_1(a; b_1, b_2; c; z_1, z_2)$ is a polynomial and has pole of order $-a$or $-b_k$at $z_k = \tilde{\infty}$, $1 \leq k \leq 2$.

07.36.04.0006.01

$$\mathcal{Sing}_{z_k}(F_1(a; b_1, b_2; c; z_1, z_2)) == \{\{\tilde{\infty}, -\alpha\}\} /; \\ \left(-a \in \mathbb{N}^+ \bigwedge \alpha == a\right) \bigvee \left(-b_k \in \mathbb{N}^+ \bigwedge \alpha == b_k\right) \bigvee \left(-a \in \mathbb{N}^+ \bigwedge -b_k \in \mathbb{N}^+ \bigwedge \alpha == \min(-a, -b_k)\right) \bigwedge 1 \leq k \leq 2$$

With respect to c

For fixed a, b_1, b_2, z_1, z_2, the function $F_1(a; b_1, b_2; c; z_1, z_2)$ has an infinite set of singular points with respect to c:

a) the points $c == -k /; k \in \mathbb{N}$ are the simple poles with residues $\frac{(-1)^k}{k!}\, {}_2\tilde{F}_1(a, b; -k; z)$;

b) $c == \tilde{\infty}$ is the point of convergence of poles, which is an essential singular point.

07.36.04.0007.01

$$\mathcal{Sing}_c(F_1(a; b_1, b_2; c; z_1, z_2)) == \{\{\{-k, 1\} /; k \in \mathbb{N}\}, \{\tilde{\infty}, \infty\}\}$$

07.36.04.0008.01

$$\text{res}_c(F_1(a; b_1, b_2; c; z_1, z_2))(-k) == \frac{(-1)^k}{k!}\, {}_2\tilde{F}_1(a, b; -k; z) /; k \in \mathbb{N}$$

With respect to b_k

The function $F_1(a; b_1, b_2; c; z_1, z_2)$ does not have poles and essential singularities with respect to b_k, $1 \leq k \leq 2$.

07.36.04.0009.01

$$\mathcal{Sing}_{b_k}(F_1(a; b_1, b_2; c; z_1, z_2)) == \{\} /; k \in \{1, 2\}$$

With respect to a

The function $F_1(a; b_1, b_2; c; z_1, z_2)$ does not have poles and essential singularities with respect to a.

07.36.04.0010.01

$$\mathcal{Sing}_a(F_1(a; b_1, b_2; c; z_1, z_2)) == \{\}$$

Branch points

With respect to z_k

For fixed a, b_k, c, in nonpolynomial cases (when $\neg(-a \in \mathbb{N} \bigvee -b_k \in \mathbb{N})$), the function $F_1(a; b_1, b_2; c; z_1, z_2)$ has for fixed $z_1$or fixed $z_2$two singular branch points with respect to $z_2$or z_1: $z_k == 1$, $z_k == \tilde{\infty}$, $k = 1, 2$.

07.36.04.0011.01

$$\mathcal{BP}_{z_k}(F_1(a; b_1, b_2; c; z_1, z_2)) == \{1, \tilde{\infty}\} /; \mathcal{NT}(\{a, b_k\}) \bigwedge k \in \{1, 2\}$$

07.36.04.0012.01

$$\mathcal{R}_{z_k}(F_1(a; b_1, b_2; c; z_1, z_2), 1) == \log /; c - a - b_k \in \mathbb{Z} \bigvee c - a - b_k \notin \mathbb{Q} \bigwedge \mathcal{NT}(\{a, b_k\}) \bigwedge k \in \{1, 2\}$$

07.36.04.0013.01

$$\mathcal{R}_{z_k}(F_1(a; b_1, b_2; c; z_1, z_2), 1) == s /; c - a - b_k == \frac{r}{s} \bigwedge r \in \mathbb{Z} \bigwedge s - 1 \in \mathbb{N}^+ \bigwedge \gcd(r, s) == 1 \bigwedge \mathcal{NT}(\{a, b_k\}) \bigwedge k \in \{1, 2\}$$

07.36.04.0014.01

$$\mathcal{R}_{z_k}(F_1(a; b_1, b_2; c; z_1, z_2), \tilde{\infty}) == \log /; a - b_k \in \mathbb{Z} \bigvee \neg (a \in \mathbb{Q} \bigwedge b_k \in \mathbb{Q}) \bigwedge k \in \{1, 2\}$$

07.36.04.0015.01

$$\mathcal{R}_{z_k}(F_1(a; b_1, b_2; c; z_1, z_2), \tilde{\infty}) == \operatorname{lcm}(s, u) /; a == \frac{r}{s} \bigwedge b_k == \frac{t}{u} \bigwedge \{r, s, t, u\} \in \mathbb{Z} \bigwedge s > 0 \bigwedge u > 0 \bigwedge \operatorname{lcm}(s, u) \neq 1 \bigwedge \gcd(r, s) == 1 \bigwedge \gcd(t, u) == 1 \bigwedge \mathcal{NT}(\{a, b_k\}) \bigwedge k \in \{1, 2\}$$

With respect to c

The function $F_1(a; b_1, b_2; c; z_1, z_2)$ does not have branch points with respect to c.

07.36.04.0016.01

$$\mathcal{BP}_c(F_1(a; b_1, b_2; c; z_1, z_2)) == \{\}$$

With respect to b_k

The function $F_1(a; b_1, b_2; c; z_1, z_2)$ does not have branch points with respect to b_k, $1 \leq k \leq 2$.

07.36.04.0017.01

$$\mathcal{BP}_{b_k}(F_1(a; b_1, b_2; c; z_1, z_2)) == \{\} /; k \in \{1, 2\}$$

With respect to a

The function $F_1(a; b_1, b_2; c; z_1, z_2)$ does not have branch points with respect to a.

07.36.04.0018.01

$$\mathcal{BP}_a(F_1(a; b_1, b_2; c; z_1, z_2)) == \{\}$$

Branch cuts

With respect to z_1

For fixed a, b_1, c, in nonpolynomial cases (when $\neg(-a \in \mathbb{N} \bigvee -b_1 \in \mathbb{N})$), the function $F_1(a; b_1, b_2; c; z_1, z_2)$ is a single-valued function on the z_1-plane cut along the interval $(1, \infty)$, where it is continuous from below.

07.36.04.0019.01

$$\mathcal{BC}_{z_1}(F_1(a; b_1, b_2; c; z_1, z_2)) == \{\{(1, \infty), i\}\} /; \mathcal{NT}(\{a, b_1\})$$

07.36.04.0020.01

$$\lim_{\epsilon \to +0} F_1(a; b_1, b_2; c; x_1 - i\,\epsilon, z_2) == F_1(a; b_1, b_2; c; x_1, z_2) /; x_1 > 1$$

07.36.04.0021.01

$$\lim_{\epsilon \to +0} F_1(a; b_1, b_2; c; x_1 + i\,\epsilon, z_2) == e^{2 i \pi (a-c+b_1)} F_1(a; b_1, b_2; c; x_1, z_2) + \frac{2\, i\, e^{i \pi (a-c+b_1)} \pi\, \Gamma(c)}{\Gamma(c-a)\, \Gamma(a-c+b_1+1)\, \Gamma(c-b_1)} F_{0 \times 1 \times 1}^{1 \times 1 \times 1}\left(\begin{matrix} a; b_1; b_2; \\ ; a-c+b_1+1; c-b_1; \end{matrix} 1 - x_1, z_2 \right) /; x_1 > 1$$

With respect to z_2

For fixed a, b_2, c, in nonpolynomial cases (when $\neg(-a \in \mathbb{N} \bigvee -b_2 \in \mathbb{N})$), the function $F_1(a; b_1, b_2; c; z_1, z_2)$ is a single-valued function on the z_2-plane cut along the interval $(1, \infty)$, where it is continuous from below.

07.36.04.0022.01

$$\mathcal{BC}_{z_2}(F_1(a; b_1, b_2; c; z_1, z_2)) == \{\{(1, \infty), i\}\} /; \mathcal{NT}(\{a, b_2\})$$

07.36.04.0023.01

$$\lim_{\epsilon \to +0} F_1(a; b_1, b_2; c; z_1, x_2 - i\,\epsilon) == F_1(a; b_1, b_2; c; z_1, x_2) /; x_2 > 1$$

07.36.04.0024.01

$$\lim_{\epsilon \to +0} F_1(a; b_1, b_2; c; z_1, x_2 + i\,\epsilon) ==$$
$$e^{2 i \pi (a-c+b_2)} F_1(a; b_1, b_2; c; z_1, x_2) + \frac{2\, i\, e^{i \pi (a-c+b_2)} \pi\, \Gamma(c)}{\Gamma(c-a)\, \Gamma(a-c+b_2+1)\, \Gamma(c-b_2)} F_{0\times 1\times 1}^{1\times 1\times 1}\left(\begin{matrix} a; b_1; b_2; \\ ; c-b_2; 1+a-c+b_2; \end{matrix} z_1, 1-x_2\right) /; x_2 > 1$$

With respect to c

The function $F_1(a; b_1, b_2; c; z_1, z_2)$ does not have branch cuts with respect to c.

07.36.04.0025.01

$$\mathcal{BC}_c(F_1(a; b_1, b_2; c; z_1, z_2)) == \{\}$$

With respect to b_k

The function $F_1(a; b_1, b_2; c; z_1, z_2)$ does not have branch cuts with respect to b_k, $1 \leq k \leq 2$.

07.36.04.0026.01

$$\mathcal{BC}_{b_k}(F_1(a; b_1, b_2; c; z_1, z_2)) == \{\} /; k \in \{1, 2\}$$

With respect to a

The function $F_1(a; b_1, b_2; c; z_1, z_2)$ does not have branch cuts with respect to a.

07.36.04.0027.01

$$\mathcal{BC}_a(F_1(a; b_1, b_2; c; z_1, z_2)) == \{\}$$

Series representations

Generalized power series

Expansions at $(z_1, z_2) == (0, 0)$

For the function itself

07.36.06.0001.01

$$F_1(a; b_1, b_2; c; z_1, z_2) \propto 1 + \frac{a\, b_1\, z_1}{c} + \frac{a\,(1+a)\, b_1\,(1+b_1)\, z_1^2}{2\, c\,(1+c)} + \frac{a\, b_2\, z_2}{c} + \frac{a\,(1+a)\, b_1\, b_2\, z_1\, z_2}{c\,(1+c)} +$$
$$\frac{a\,(1+a)\,(2+a)\, b_1\,(1+b_1)\, b_2\, z_1^2\, z_2}{2\, c\,(1+c)\,(2+c)} + \frac{a\,(1+a)\, b_2\,(1+b_2)\, z_2^2}{2\, c\,(1+c)} + \frac{a\,(1+a)\,(2+a)\, b_1\, b_2\,(1+b_2)\, z_1\, z_2^2}{2\, c\,(1+c)\,(2+c)} +$$
$$\frac{a\,(1+a)\,(2+a)\,(3+a)\, b_1\,(1+b_1)\, b_2\,(1+b_2)\, z_1^2\, z_2^2}{4\, c\,(1+c)\,(2+c)\,(3+c)} + \ldots /; (z_1 \to 0) \bigwedge (z_2 \to 0)$$

07.36.06.0002.01

$$F_1(a;\, b_1,\, b_2;\, c;\, z_1,\, z_2) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{(a)_{k+l}\, (b_1)_k\, (b_2)_l\, z_1^k\, z_2^l}{(c)_{k+l}\, k!\, l!} \;/;\; |z_1| < 1 \bigwedge |z_2| < 1$$

Expansions at $z_1 == 0$

For the function itself

07.36.06.0003.02

$$F_1(a;\, b_1,\, b_2;\, c;\, z_1,\, z_2) \propto {}_2F_1(a,\, b_2;\, c;\, z_2) + \frac{a\, b_1\, {}_2F_1(a+1,\, b_2;\, c+1;\, z_2)}{c}\, z_1 + \frac{a\,(a+1)\, b_1\,(b_1+1)\, {}_2F_1(a+2,\, b_2;\, c+2;\, z_2)}{2\, c\,(c+1)}\, z_1^2 + \ldots \;/;\; (z_1 \to 0)$$

07.36.06.0022.01

$$F_1(a;\, b_1,\, b_2;\, c;\, z_1,\, z_2) \propto {}_2F_1(a,\, b_2;\, c;\, z_2) + \frac{a\, b_1\, {}_2F_1(a+1,\, b_2;\, c+1;\, z_2)}{c}\, z_1 + \frac{a\,(a+1)\, b_1\,(b_1+1)\, {}_2F_1(a+2,\, b_2;\, c+2;\, z_2)}{2\, c\,(c+1)}\, z_1^2 + O\!\left(z_1^3\right)$$

07.36.06.0004.01

$$F_1(a;\, b_1,\, b_2;\, c;\, z_1,\, z_2) = \sum_{k=0}^{\infty} \frac{(a)_k\, (b_1)_k}{(c)_k\, k!}\, {}_2F_1(a+k,\, b_2;\, c+k;\, z_2)\, z_1^k \;/;\; |z_1| < 1$$

07.36.06.0005.02

$$F_1(a;\, b_1,\, b_2;\, c;\, z_1,\, z_2) \propto {}_2F_1(a,\, b_2;\, c;\, z_2)\,(1 + O(z_1))$$

07.36.06.0023.01

$$F_1(a;\, b_1,\, b_2;\, c;\, z_1,\, z_2) = F_\infty(z_1,\, a,\, b_1,\, b_2,\, c,\, z_2) \;/;$$
$$\left(\left(F_n(z_1,\, a,\, b_1,\, b_2,\, c,\, z_2) = \sum_{k=0}^{n} \frac{(a)_k\, (b_1)_k\, z_1^k}{(c)_k\, k!}\, {}_2F_1(a+k,\, b_2;\, c+k;\, z_2) = F_1(a;\, b_1,\, b_2;\, c;\, z_1,\, z_2) - \frac{(a)_{n+1}\, (b_1)_{n+1}\, z_1^{n+1}}{(c)_{n+1}\,(n+1)!}\, F_{1\times 2\times 0}^{1\times 2\times 1}\!\left(\begin{matrix} a+n+1;\, 1,\, b_1+n+1;\, b_2; \\ c+n+1;\, n+2;\, ; \end{matrix}\; z_1,\, z_2\right)\right) \bigwedge n \in \mathbb{N}\right)$$

Summed form of the truncated series expansion.

Expansions at $z_2 == 0$

For the function itself

07.36.06.0006.02

$$F_1(a;\, b_1,\, b_2;\, c;\, z_1,\, z_2) \propto {}_2F_1(a,\, b_1;\, c;\, z_2) + \frac{a\, b_2\, {}_2F_1(a+1,\, b_1;\, c+1;\, z_1)}{c}\, z_2 + \frac{a\,(a+1)\, b_2\,(b_2+1)\, {}_2F_1(a+2,\, b_1;\, c+2;\, z_1)}{2\, c\,(c+1)}\, z_2^2 + \ldots \;/;\; (z_2 \to 0)$$

07.36.06.0024.01

$$F_1(a;\, b_1,\, b_2;\, c;\, z_1,\, z_2) \propto {}_2F_1(a,\, b_1;\, c;\, z_2) + \frac{a\, b_2\, {}_2F_1(a+1,\, b_1;\, c+1;\, z_1)}{c}\, z_2 + \frac{a\,(a+1)\, b_2\,(b_2+1)\, {}_2F_1(a+2,\, b_1;\, c+2;\, z_1)}{2\, c\,(c+1)}\, z_2^2 + O\!\left(z_2^3\right)$$

07.36.06.0007.01

$$F_1(a;\, b_1,\, b_2;\, c;\, z_1,\, z_2) = \sum_{k=0}^{\infty} \frac{(a)_k\, (b_2)_k}{(c)_k\, k!}\, {}_2F_1(a+k,\, b_1;\, c+k;\, z_1)\, z_2^k \,/;\, |z_2| < 1$$

07.36.06.0008.02

$$F_1(a;\, b_1,\, b_2;\, c;\, z_1,\, z_2) \propto {}_2F_1(a,\, b_1;\, c;\, z_1)\, (1 + O(z_2))$$

07.36.06.0025.01

$$F_1(a;\, b_1,\, b_2;\, c;\, z_1,\, z_2) = F_\infty(z_2,\, a,\, b_1,\, b_2,\, c,\, z_1) \,/;$$
$$\Bigg(\Bigg(F_n(z_2,\, a,\, b_1,\, b_2,\, c,\, z_1) = \sum_{k=0}^{n} \frac{(a)_k\, (b_2)_k\, z_2^k}{(c)_k\, k!}\, {}_2F_1(a+k,\, b_1;\, c+k;\, z_1) = F_1(a;\, b_1,\, b_2;\, c;\, z_1,\, z_2) -$$
$$\frac{(a)_{n+1}\, (b_2)_{n+1}\, z_2^{n+1}}{(c)_{n+1}\, (n+1)!}\, F_{1\times 2\times 0}^{1\times 2\times 1}\left(\begin{matrix} a+n+1;\, 1,\, b_1;\, b_2+n+1; \\ c+n+1;;\, n+2; \end{matrix}\, z_1,\, z_2\right)\Bigg) \bigwedge n \in \mathbb{N}\Bigg)$$

Summed form of the truncated series expansion.

Expansions at $z_1 = 1$

For the function itself

07.36.06.0009.01

$$F_1(a;\, b_1,\, b_2;\, c;\, z_1,\, z_2) = \frac{\Gamma(c)\, \Gamma(a-c+b_1)}{\Gamma(a)\, \Gamma(b_1)}\, (1-z_1)^{c-a-b_1} \sum_{k=0}^{\infty} \frac{(b_2)_k}{k!}\, {}_2F_1(c-a,\, c+k-b_1;\, c-a-b_1+1;\, 1-z_1)\, z_2^k +$$
$$\frac{\Gamma(c)\, \Gamma(c-a-b_1)}{\Gamma(c-a)} \sum_{k=0}^{\infty} \frac{(a)_k\, (b_2)_k}{k!\, \Gamma(c+k-b_1)}\, {}_2F_1(a+k,\, b_1;\, a-c+b_1+1;\, 1-z_1)\, z_2^k$$

07.36.06.0010.01

$$F_1(a;\, b_1,\, b_2;\, c;\, z_1,\, z_2) = \frac{\Gamma(c)\, \Gamma(a+b_1-c)}{\Gamma(a)\, \Gamma(b_1)}\, (1-z_1)^{c-a-b_1}\, F_{0\times 1\times 1}^{11\times 1}\left(\begin{matrix} c-b_1;\, c-a;\, b_2; \\ ;\, c-a-b_1+1;\, c-b_1; \end{matrix}\, 1-z_1,\, z_2\right) +$$
$$\frac{\Gamma(c)\, \Gamma(c-a-b_1)}{\Gamma(c-a)\, \Gamma(c-b_1)}\, F_{0\times 1\times 1}^{11\times 1}\left(\begin{matrix} a;\, b_1;\, b_2; \\ ;\, a+b_1-c+1;\, c-b_1; \end{matrix}\, 1-z_1,\, z_2\right)$$

07.36.06.0011.02

$$F_1(a;\, b_1,\, b_2;\, c;\, z_1,\, z_2) \propto$$
$$\frac{\Gamma(c)\, \Gamma(a+b_1-c)\, (1-z_2)^{-b_2}}{\Gamma(a)\, \Gamma(b_1)}\, (1-z_1)^{c-a-b_1}\, (1 + O(z_1-1)) + \frac{\Gamma(c)\, \Gamma(c-a-b_1)}{\Gamma(c-a)\, \Gamma(c-b_1)}\, {}_2F_1(a,\, b_2;\, c-b_1;\, z_2)\, (1 + O(z_1-1))$$

Expansions at $z_2 = 1$

For the function itself

07.36.06.0012.01

$$F_1(a;\, b_1,\, b_2;\, c;\, z_1,\, z_2) = \frac{\Gamma(c)\, \Gamma(a-c+b_2)}{\Gamma(a)\, \Gamma(b_2)}\, (1-z_2)^{c-a-b_2} \sum_{k=0}^{\infty} \frac{(b_1)_k}{k!}\, {}_2F_1(c-a,\, c+k-b_2;\, c-a-b_2+1;\, 1-z_2)\, z_1^k +$$
$$\frac{\Gamma(c)\, \Gamma(c-a-b_2)}{\Gamma(c-a)} \sum_{k=0}^{\infty} \frac{(a)_k\, (b_1)_k}{k!\, \Gamma(c+k-b_2)}\, {}_2F_1(a+k,\, b_2;\, a-c+b_2+1;\, 1-z_2)\, z_1^k$$

07.36.06.0013.01

$$F_1(a;b_1,b_2;c;z_1,z_2) = \frac{\Gamma(c)\,\Gamma(a+b_2-c)}{\Gamma(a)\,\Gamma(b_2)}(1-z_2)^{c-a-b_2}\,F_{0\times1\times1}^{11\times1}\left(\begin{matrix} c-b_2;\, b_1;\, c-a; \\ ;\, c-b_2;\, c-a-b_2+1; \end{matrix}\; z_1,\, 1-z_2\right) + \frac{\Gamma(c)\,\Gamma(c-a-b_2)}{\Gamma(c-a)\,\Gamma(c-b_2)}\,F_{0\times1\times1}^{11\times1}\left(\begin{matrix} a;\, b_1;\, b_2; \\ ;\, c-b_2;\, a+b_2-c+1; \end{matrix}\; z_1,\, 1-z_2\right)$$

07.36.06.0014.02

$$F_1(a;b_1,b_2;c;z_1,z_2) \propto \frac{\Gamma(c)\,\Gamma(a+b_2-c)\,(1-z_1)^{-b_1}}{\Gamma(a)\,\Gamma(b_2)}(1-z_2)^{c-a-b_2}\,(1+O(z_2-1)) + \frac{\Gamma(c)\,\Gamma(c-a-b_2)}{\Gamma(c-a)\,\Gamma(c-b_2)}\,{}_2F_1(a,\, b_1;\, c-b_2;\, z_1)\,(1+O(z_2-1))$$

Expansions at $z_1 = \infty$

For the function itself

07.36.06.0015.02

$$F_1(a;b_1,b_2;c;z_1,z_2) = \frac{\Gamma(c)\,\Gamma(b_1-a)}{\Gamma(c-a)\,\Gamma(b_1)}(-z_1)^{-a}\,F_1\!\left(a;\, a-c+1,\, b_2;\, a-b_1+1;\, \frac{1}{z_1},\, \frac{z_2}{z_1}\right) + \frac{\Gamma(c)}{\Gamma(a)}(-z_1)^{-b_1}\sum_{k=0}^{\infty}\frac{\Gamma(a+k-b_1)\,(b_2)_k}{k!\,\Gamma(c+k-b_1+k)}\,{}_2F_1\!\left(b_1,\, b_1-c-k+1;\, b_1-a-k+1;\, \frac{1}{z_1}\right) z_2^k \;/;\; a-b_1 \notin \mathbb{Z}$$

07.36.06.0016.02

$$F_1(a;b_1,b_2;c;z_1,z_2) = \frac{\Gamma(c)\,\Gamma(b_1-a)}{\Gamma(c-a)\,\Gamma(b_1)}(-z_1)^{-a}\,F_1\!\left(a;\, b_2,\, a-c+1;\, a-b_1+1;\, \frac{z_2}{z_1},\, \frac{1}{z_1}\right) + \frac{\Gamma(c)\,\Gamma(a-b_1)}{\Gamma(a)\,\Gamma(c-b_1)}(-z_1)^{-b_1}\left(F_{1\times1\times1}^{1\times2\times2}\left(\begin{matrix} b_1;\, b_2,\, a-b_1;\, b_1-c+1,\, 1; \\ 1;\, c-b_1;\, b_1-a+1; \end{matrix}\; \frac{z_2}{z_1},\, \frac{1}{z_1}\right) + \frac{(a-b_1)^2\, b_2\, z_2}{(c-b_1)^2}\,F_{2\times0\times1}^{2\times1\times2}\left(\begin{matrix} a-b_1+1,\, b_2+1;\, b_1;\, a-b_1+1,\, 1; \\ c-b_1+1,\, 2;;\, c-b_1+1; \end{matrix}\; \frac{z_2}{z_1},\, z_2\right)\right) /;\; a-b_1 \notin \mathbb{Z}$$

07.36.06.0017.02

$$F_1(a;b_1,b_2;c;z_1,z_2) \propto \frac{\Gamma(c)\,\Gamma(b_1-a)}{\Gamma(c-a)\,\Gamma(b_1)}(-z_1)^{-a}\left(1+O\!\left(\frac{1}{z_1}\right)\right) + \frac{\Gamma(c)\,\Gamma(a-b_1)}{\Gamma(c-b_1)\,\Gamma(a)}(-z_1)^{-b_1}\left(1+O\!\left(\frac{1}{z_1}\right)\right) /;\; a \neq b_1$$

Expansions at $z_2 = \infty$

For the function itself

07.36.06.0018.02

$$F_1(a;b_1,b_2;c;z_1,z_2) = \frac{\Gamma(c)\,\Gamma(b_2-a)}{\Gamma(c-a)\,\Gamma(b_2)}(-z_2)^{-a}\,F_1\!\left(a;\, a-c+1,\, b_1;\, a-b_2+1;\, \frac{1}{z_2},\, \frac{z_1}{z_2}\right) + \frac{\Gamma(c)}{\Gamma(a)}(-z_2)^{-b_2}\sum_{k=0}^{\infty}\frac{\Gamma(a+k-b_2)\,(b_1)_k}{k!\,\Gamma(c+k-b_2+k)}\,{}_2F_1\!\left(b_2,\, b_2-c-k+1;\, b_2-a-k+1;\, \frac{1}{z_2}\right) z_1^k \;/;\; a-b_2 \notin \mathbb{Z}$$

07.36.06.0019.02

$$F_1(a;\,b_1,\,b_2;\,c;\,z_1,\,z_2) = \frac{\Gamma(c)\,\Gamma(b_2-a)}{\Gamma(c-a)\,\Gamma(b_2)}\,(-z_2)^{-a}\,F_1\!\left(a;\,b_1,\,a-c+1;\,a-b_2+1;\,\frac{z_1}{z_2},\,\frac{1}{z_2}\right) + \frac{\Gamma(c)\,\Gamma(a-b_2)}{\Gamma(a)\,\Gamma(c-b_2)}\,(-z_2)^{-b_2}\left(F_{1\times1\times1}^{1\times2\times2}\!\left(\begin{matrix} b_2;\,b_1,\,a-b_2;\,b_2-c+1,\,1; \\ 1;\,c-b_2;\,b_2-a+1; \end{matrix}\;\frac{z_1}{z_2},\,\frac{1}{z_2}\right) + \frac{(a-b_2)^2\,b_1\,z_1}{(c-b_2)^2}\,F_{2\times0\times1}^{2\times1\times2}\!\left(\begin{matrix} a-b_2+1,\,b_1+1;\,b_2;\,a-b_2+1,\,1; \\ c-b_2+1,\,2;;\,c-b_2+1; \end{matrix}\;\frac{z_1}{z_2},\,z_1\right)\right) /;\ a-b_2\notin\mathbb{Z}$$

07.36.06.0020.02

$$F_1(a;\,b_1,\,b_2;\,c;\,z_1,\,z_2) \propto \frac{\Gamma(c)\,\Gamma(b_2-a)}{\Gamma(c-a)\,\Gamma(b_2)}\,(-z_2)^{-a}\left(1+O\!\left(\frac{1}{z_2}\right)\right) + \frac{\Gamma(c)\,\Gamma(a-b_2)}{\Gamma(c-b_2)\,\Gamma(a)}\,(-z_2)^{-b_2}\left(1+O\!\left(\frac{1}{z_2}\right)\right) /;\ a\neq b_2$$

Residue representations

07.36.06.0021.01

$$F_1(a;\,b_1,\,b_2;\,c;\,z_1,\,z_2) = \frac{\Gamma(c)}{\Gamma(a)\,\Gamma(b_1)\,\Gamma(b_2)}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\operatorname{res}_{s,t}\!\left(\frac{\Gamma(a-s-t)\,\Gamma(b_1-s)\,\Gamma(b_2-t)\,(-z_1)^{-s}\,(-z_2)^{-t}}{\Gamma(c-s-t)}\,\Gamma(s)\,\Gamma(t)\right)(-j,\,-k)$$

Integral representations

On the real axis

Of the direct function

07.36.07.0001.01

$$F_1(a;\,b_1,\,b_2;\,c;\,z_1,\,z_2) = \frac{\Gamma(c)}{\Gamma(a)\,\Gamma(c-a)}\int_0^1 t^{a-1}\,(1-t)^{c-a-1}\,(1-t\,z_1)^{-b_1}\,(1-t\,z_2)^{-b_2}\,dt\ /;\ \operatorname{Re}(a)>0\land\operatorname{Re}(c-a)>0$$

Contour integral representations

07.36.07.0002.01

$$F_1(a;\,b_1,\,b_2;\,c;\,z_1,\,z_2) = \frac{\Gamma(c)}{(2\pi i)^2\,\Gamma(a)\,\Gamma(b_1)\,\Gamma(b_2)}\int_{\mathcal{L}^*}\int_{\mathcal{L}}\frac{\Gamma(a-s-t)\,\Gamma(s)\,\Gamma(b_1-s)\,\Gamma(t)\,\Gamma(b_2-t)}{\Gamma(c-s-t)}\,(-z_1)^{-s}\,(-z_2)^{-t}\,ds\,dt;\ |\arg(-z_1)|<\pi\bigwedge|\arg(-z_2)|<\pi$$

Multiple integral representations

07.36.07.0003.01

$$F_1(a;\,b_1,\,b_2;\,c;\,z_1,\,z_2) = \frac{\Gamma(c)}{\Gamma(c-b_1-b_2)\,\Gamma(b_1)\,\Gamma(b_2)}\int_0^1\int_0^{1-x} z_1^{b_1-1}\,z_2^{b_2-1}\,(-x-y+1)^{c-b_1-b_2-1}\,(1-x\,z_1-y\,z_2)^{-a}\,dy\,dx\ /;\ \operatorname{Re}(b_1)>0\bigwedge\operatorname{Re}(b_2)>0\bigwedge\operatorname{Re}(c-b_1-b_2)>0$$

Differential equations

Ordinary linear differential equations and wronskians

For the direct function itself

07.36.13.0001.01

$$(1-z_1)\, z_1 \frac{\partial^2 w(z_1, z_2)}{\partial z_1^2} + (c-(a+b_1+1)\, z_1) \frac{\partial w(z_1, z_2)}{\partial z_1} - b_1\, z_2 \frac{\partial w(z_1, z_2)}{\partial z_2} + (1-z_1)\, z_2 \frac{\partial^2 w(z_1, z_2)}{\partial z_1\, \partial z_2} - a\, b_1\, w(z_1, z_2) = 0 \bigwedge$$
$$(1-z_2)\, z_2 \frac{\partial^2 w(z_1, z_2)}{\partial z_2^2} + (c-(a+b_2+1)\, z_2) \frac{\partial w(z_1, z_2)}{\partial z_2} - b_2\, z_1 \frac{\partial w(z_1, z_2)}{\partial z_1} + (1-z_2)\, z_1 \frac{\partial^2 w(z_1, z_2)}{\partial z_1\, \partial z_2} - a\, b_2\, w(z_1, z_2) = 0 \;/;$$
$$w(z_1, z_2) = F_1(a; b_1, b_2; c; z_1, z_2)$$

07.36.13.0002.01

$$(z_1-1)\, z_1\, (z_1-z_2)\, w^{(3,0)}(z_1, z_2) + ((b_1+b_2+1)\, (z_1-1)\, z_1 + (a-c+b_1+2)\, (z_1-z_2)\, z_1 + (c-b_2+1)\, (z_1-1)\, (z_1-z_2))\, w^{(2,0)}(z_1, z_2) + (b_1+1)\, ((2\, a+b_1+2)\, z_1 - (a-b_2+1)\, z_2 - c)\, w^{(1,0)}(z_1, z_2) + a\, b_1\, (b_1+1)\, w(z_1, z_2) = 0 \;/;\; w(z_1, z_2) = F_1(a; b_1, b_2; c; z_1, z_2)$$

07.36.13.0003.01

$$(z_2-z_1)\, (z_2-1)\, z_2\, w^{(0,3)}(z_1, z_2) + ((c-b_1+1)\, (z_2-z_1)\, (z_2-1) + (b_1+b_2+1)\, z_2\, (z_2-1) + (a-c+b_2+2)\, (z_2-z_1)\, z_2)\, w^{(0,2)}(z_1, z_2) + (b_2+1)\, (-c-(a-b_1+1)\, z_1 + (2\, a+b_2+2)\, z_2)\, w^{(0,1)}(z_1, z_2) + a\, b_2\, (b_2+1)\, w(z_1, z_2) = 0 \;/;\; w(z_1, z_2) = F_1(a; b_1, b_2; c; z_1, z_2)$$

Identities

Functional identities

Major general cases

07.36.17.0001.01

$$F_1(a; b_1, b_2; c; z_1, z_2) = (1-z_1)^{-b_1}\, (1-z_2)^{-b_2}\, F_1\!\left(c-a; b_1, b_2; c; \frac{z_1}{z_1-1}, \frac{z_2}{z_2-1}\right) /;\; z_1 \notin (1, \infty) \bigwedge z_2 \notin (1, \infty)$$

07.36.17.0002.01

$$F_1(a; b_1, b_2; c; z_1, z_2) = (1-z_1)^{-a}\, F_1\!\left(a; c-b_1-b_2, b_2; c; \frac{z_1}{z_1-1}, \frac{z_1-z_2}{z_1-1}\right) /;\; z_1 \notin (1, \infty) \bigwedge z_2 \notin (1, \infty)$$

07.36.17.0003.01

$$F_1(a; b_1, b_2; c; z_1, z_2) = (1-z_2)^{-a}\, F_1\!\left(a; b_1, c-b_1-b_2; c; \frac{z_2-z_1}{z_2-1}, \frac{z_2}{z_2-1}\right) /;\; z_1 \notin (1, \infty) \bigwedge z_2 \notin (1, \infty)$$

07.36.17.0004.01

$$F_1(a; b_1, b_2; c; z_1, z_2) = (1-z_1)^{c-a-b_1}\, (1-z_2)^{-b_2}\, F_1\!\left(c-a; c-b_1-b_2, b_2; c; z_1, \frac{z_2-z_1}{z_2-1}\right)$$

07.36.17.0005.01

$$F_1(a; b_1, b_2; c; z_1, z_2) = (1-z_1)^{-b_1}\, (1-z_2)^{c-a-b_2}\, F_1\!\left(c-a; b_1, c-b_1-b_2; c; \frac{z_1-z_2}{z_1-1}, z_2\right)$$

Differentiation

Low-order differentiation

With respect to z_1

07.36.20.0001.01

$$\frac{\partial F_1(a; b_1, b_2; c; z_1, z_2)}{\partial z_1} = \frac{a\, b_1}{c}\, F_1(a+1; b_1+1, b_2; c+1; z_1, z_2)$$

07.36.20.0002.01

$$\frac{\partial^2 F_1(a;\, b_1,\, b_2;\, c;\, z_1,\, z_2)}{\partial z_1^2} = \frac{a\,(a+1)\,b_1\,(b_1+1)}{c\,(c+1)}\, F_1(a+2;\, b_1+2,\, b_2;\, c+2;\, z_1,\, z_2)$$

With respect to z_2

07.36.20.0003.01

$$\frac{\partial F_1(a;\, b_1,\, b_2;\, c;\, z_1,\, z_2)}{\partial z_2} = \frac{a\, b_2}{c}\, F_1(a+1;\, b_1,\, b_2+1;\, c+1;\, z_1,\, z_2)$$

07.36.20.0004.01

$$\frac{\partial^2 F_1(a;\, b_1,\, b_2;\, c;\, z_1,\, z_2)}{\partial z_2^2} = \frac{a\,(a+1)\,b_2\,(b_2+1)}{c\,(c+1)}\, F_1(a+2;\, b_1,\, b_2+2;\, c+2;\, z_1,\, z_2)$$

Symbolic differentiation

With respect to z_1

07.36.20.0005.02

$$\frac{\partial^n F_1(a;\, b_1,\, b_2;\, c;\, z_1,\, z_2)}{\partial z_1^n} = \frac{(a)_n\,(b_1)_n}{(c)_n}\, F_1(a+n;\, n+b_1,\, b_2;\, c+n;\, z_1,\, z_2)\, /;\, n \in \mathbb{N}$$

07.36.20.0009.01

$$\frac{\partial^n \left(z_1^{n+b_1-1}\, F_1(a;\, b_1,\, b_2;\, c;\, z_1,\, z_2)\right)}{\partial z_1^n} = (b_1)_n\, z_1^{b_1-1}\, F_1(a;\, n+b_1,\, b_2;\, c;\, z_1,\, z_2)\, /;\, n \in \mathbb{N}$$

With respect to z_2

07.36.20.0006.02

$$\frac{\partial^n F_1(a;\, b_1,\, b_2;\, c;\, z_1,\, z_2)}{\partial z_2^n} = \frac{(a)_n\,(b_2)_n}{(c)_n}\, F_1(a+n;\, b_1,\, b_2+n;\, c+n;\, z_1,\, z_2)\, /;\, n \in \mathbb{N}$$

07.36.20.0010.01

$$\frac{\partial^n \left(z_2^{n+b_2-1}\, F_1(a;\, b_1,\, b_2;\, c;\, z_1,\, z_2)\right)}{\partial z_2^n} = (b_2)_n\, z_2^{b_2-1}\, F_1(a;\, b_1,\, n+b_2;\, c;\, z_1,\, z_2)\, /;\, n \in \mathbb{N}$$

Fractional integro-differentiation

With respect to z_1

07.36.20.0007.01

$$\frac{\partial^\alpha F_1(a;\, b_1,\, b_2;\, c;\, z_1,\, z_2)}{\partial z_1^\alpha} = z_1^{-\alpha}\, \Gamma(c)\, \tilde{F}_{1\times 1\times 0}^{1\times 2\times 1}\left(\begin{matrix} a;\, b_1,\, 1;\, b_2; \\ c;\, 1-\alpha;\, ; \end{matrix}\; z_1,\, z_2\right)$$

With respect to z_2

07.36.20.0008.01

$$\frac{\partial^\alpha F_1(a;\, b_1,\, b_2;\, c;\, z_1,\, z_2)}{\partial z_2^\alpha} = z_2^{-\alpha}\, \Gamma(c)\, \tilde{F}_{1\times 0\times 1}^{1\times 1\times 2}\left(\begin{matrix} a;\, b_1;\, b_2,\, 1; \\ c;;\, 1-\alpha; \end{matrix}\; z_1,\, z_2\right)$$

Integration

Indefinite integration

Involving only one direct function with respect to z_1

07.36.21.0001.01

$$\int F_1(a; b_1, b_2; c; a z_1, z_2)\, dz_1 = \frac{c-1}{a(a-1)(b_1-1)} F_1(a-1; b_1-1, b_2; c-1; a z_1, z_2)$$

07.36.21.0002.01

$$\int F_1(a; b_1, b_2; c; z_1, z_2)\, dz_1 = \frac{c-1}{(a-1)(b_1-1)} F_1(a-1; b_1-1, b_2; c-1; z_1, z_2)$$

Involving only one direct function with respect to z_2

07.36.21.0003.01

$$\int F_1(a; b_1, b_2; c; z_1, a z_2)\, dz_1 = \frac{c-1}{a(a-1)(b_1-1)} F_1(a-1; b_1-1, b_2; c-1; z_1, a z_2)$$

07.36.21.0004.01

$$\int F_1(a; b_1, b_2; c; z_1, z_2)\, dz_2 = \frac{c-1}{(a-1)(b_2-1)} F_1(a-1; b_1, b_2-1; c-1; z_1, z_2)$$

Representations through more general functions

Through hypergeometric functions of two variables

07.36.26.0001.01

$$F_1(a; b_1, b_2; c; z_1, z_2) = F_{1\times 0\times 0}^{1\times 1\times 1}\left(\begin{matrix} a; b_1; b_2; \\ c;;; \end{matrix} z_1, z_2\right)$$

07.36.26.0002.01

$$F_1(a; b_1, b_2; c; z_1, z_2) = \Gamma(c)\, \tilde{F}_{1\times 0\times 0}^{1\times 1\times 1}\left(\begin{matrix} a; b_1; b_2; \\ c;;; \end{matrix} z_1, z_2\right)$$

Through Meijer *G*

Classical cases for the direct function itself

07.36.26.0003.01

$$F_1(a; b_1, b_2; c; z_1, z_2) = \frac{\Gamma(c)}{\Gamma(a)\,\Gamma(b_1)\,\Gamma(b_2)} G_{1,1:1,1:1,1}^{0,1:1,1:1,1}\left(\begin{matrix} 1-a \\ 1-c \end{matrix} \middle| \begin{matrix} 1-b_1 \\ 0 \end{matrix} \middle| \begin{matrix} 1-b_2 \\ 0 \end{matrix} \middle| -z_1, -z_2\right)$$

History

– L. Pochhammer (1870)
– P. E. Appell (1880)
– J. Horn (1889)
– P.E. Appell and J. Kampé de Fériet (1926)

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