ArithmeticGeometricMean

Notations

Traditional name

Arithmetic-geometric mean

Traditional notation

$\operatorname{agm}(a, b)$

Mathematica StandardForm notation

`ArithmeticGeometricMean[a, b]`

Primary definition

09.54.02.0001.01

$$\operatorname{agm}(a, b) = \frac{\pi (a + b)}{4 K\left(\left(\frac{a-b}{a+b}\right)^2\right)}$$

Specific values

Specialized values

For fixed *a*

09.54.03.0001.01

$$\operatorname{agm}(a, a) = a$$

For fixed *b*

09.54.03.0002.01

$$\operatorname{agm}(0, b) = 0$$

09.54.03.0003.01

$$\operatorname{agm}(1, b) = \frac{\pi}{2 K(1 - b^2)}$$

09.54.03.0007.01

$$\operatorname{agm}\left(a, \sqrt{2}\, a\right) = a \sqrt{\frac{2}{\pi}}\, \Gamma\left(\frac{3}{4}\right)^2$$

09.54.03.0004.01

$$\operatorname{agm}\left(1, \frac{\vartheta_4(0, z)^2}{\vartheta_3(0, z)^2}\right) = \frac{1}{\vartheta_3(0, z)^2} \;/;\; -1 < z < 1$$

Values at fixed points

09.54.03.0005.01

$$\operatorname{agm}(0, 1) = 0$$

Values at infinities

09.54.03.0006.01

$$\operatorname{agm}(1, \infty) = \infty$$

General characteristics

Domain and analyticity

agm(a, b) is an analytical function of a and b which is defined over $\mathbb{C}^2$.

09.54.04.0001.01

$$(a * b) \longrightarrow \operatorname{agm}(a, b) :: (\mathbb{C} \otimes \mathbb{C}) \longrightarrow \mathbb{C}$$

Symmetries and periodicities

Parity

agm(a, b) is an odd function.

09.54.04.0002.01

$$\operatorname{agm}(-a, -b) = -\operatorname{agm}(a, b) \;/;\; a \notin \mathbb{R} \bigwedge b \notin \mathbb{R}$$

Mirror symmetry

09.54.04.0003.02

$$\operatorname{agm}\left(\overline{a}, \overline{b}\right) = \overline{\operatorname{agm}(a, b)} \;/;\; \frac{a}{b} \notin (-\infty, 0)$$

Permutation symmetry

09.54.04.0004.01

$$\operatorname{agm}(b, a) = \operatorname{agm}(a, b)$$

Periodicity

No periodicity

Homogeneity

09.54.04.0005.01

$$\operatorname{agm}(c\, a, c\, b) = c \operatorname{agm}(a, b) \;/;\; c > 0$$

Poles and essential singularities

With respect to *a*

The function agm(a, b) does not have poles and essential singularities with respect to a.

09.54.04.0006.01

$$\mathcal{S}ing_a(\text{agm}(a, b)) = \{\}$$

With respect to *b*

The function agm(a, b) does not have poles and essential singularities with respect to b.

09.54.04.0007.01

$$\mathcal{S}ing_b(\text{agm}(a, b)) = \{\}$$

Branch points

The function agm(a, b) on the $\frac{a}{b}$-plane has two branch points: $\frac{a}{b} = 0$, $\frac{a}{b} = \tilde{\infty}$.

09.54.04.0008.01

$$\mathcal{BP}_{\frac{a}{b}}(\text{agm}(a, b)) = \{0, \tilde{\infty}\}$$

09.54.04.0009.01

$$\mathcal{R}_{\frac{a}{b}}(\text{agm}(a, b), 0) = \log$$

09.54.04.0010.01

$$\mathcal{R}_{\frac{a}{b}}(\text{agm}(a, b), \tilde{\infty}) = \log$$

Branch cuts

The function agm(a, b) is a single-valued function on the $\frac{a}{b}$-plane cut along the interval $(-\infty, 0)$, where it is continuous from above.

09.54.04.0011.01

$$\mathcal{BC}_{\frac{a}{b}}(\text{agm}(a, b)) = \{\{(-\infty, 0), -i\}\}$$

09.54.04.0012.01

$$\lim_{\epsilon \to +0} \text{agm}(a + i\,\epsilon, 1) = \text{agm}(a, 1) \;/;\; a < 0$$

09.54.04.0013.01

$$\lim_{\epsilon \to +0} \text{agm}(a - i\,\epsilon, 1) = a\,\text{agm}\left(1, \frac{1}{a}\right) \;/;\; a < 0$$

Series representations

Generalized power series

Expansions at $b = 0$ for $a = 1$

09.54.06.0001.01

$$\operatorname{agm}(1,b)\propto\frac{\pi}{2(\log(4)-\log(b))}+\frac{\left(\pi\left(\log\left(\frac{b}{4}\right)+1\right)\right)b^2}{8(\log(4)-\log(b))^2}+\left(b^4\pi\left(-40\log^2(2)+\log(1024)+5\log(b)(\log(256)-2\log(b)-1)+8\right)\right)\Big/\left(256(\log(4)-\log(b))^3\right)+\left(b^6\pi\left(4\log(2)(\log(2)(23-132\log(2))+27)+\log(b)\left(792\log^2(2)-92\log(2)+\log(b)(-396\log(2)+66\log(b)+23)-54\right)+24\right)\right)\Big/\left(3072(\log(4)-\log(b))^4\right)+\left(b^8\pi\left(8(\log(2)(\log(2)(\log(2)(1487-11256\log(2))+2230)+744)+96)+\log(b)\left(4\left(\log(2)\left(45024\log^2(2)-4461\log(2)-4460\right)-744\right)+\log(b)(6\log(2)(1487-22512\log(2))+\log(b)(45024\log(2)-5628\log(b)-1487)+4460)\right)\right)\right)\Big/\left(393216(\log(4)-\log(b))^5\right)+\left(b^{10}\pi\left(16(\log(2)(\log(2)(\log(2)(\log(2)(17153-165480\log(2))+31565)+12765)+2640)+240)+\log(b)\left(8\left(\log(2)\left(\log(2)\left(827400\log^2(2)-68612\log(2)-94695\right)-25530\right)-2640\right)+\log(b)\left(12\left(\log(2)\left(-551600\log^2(2)+34306\log(2)+31565\right)+4255\right)+\log(b)(8\log(2)(413700\log(2)-17153)+\log(b)(-827400\log(2)+82740\log(b)+17153)-63130)\right)\right)\right)\right)\Big/\left(7864320(\log(4)-\log(b))^6\right)+O\left(\frac{b^{12}}{\log(b)}\right)\;/;\;(b\to 0)$$

09.54.06.0002.01

$$\operatorname{agm}(1,b)\propto\frac{\pi}{2(\log(4)-\log(b))}+O\left(\frac{b^2}{\log(b)}\right)\;/;\;(b\to 0)$$

Expansions at *b* == 1 for *a* == 1

09.54.06.0003.01

$$\operatorname{agm}(1,b)\propto 1+\frac{b-1}{2}-\frac{(b-1)^2}{16}+\frac{(b-1)^3}{32}-\frac{21(b-1)^4}{1024}+\frac{31(b-1)^5}{2048}-\frac{195(b-1)^6}{16384}+\frac{319(b-1)^7}{32768}-\frac{34325(b-1)^8}{4194304}+\frac{58899(b-1)^9}{8388608}-\frac{410771(b-1)^{10}}{67108864}+O\left((b-1)^{11}\right)\;/;\;(b\to 1)$$

09.54.06.0004.01

$$\operatorname{agm}(1,b)\propto 1+O(b-1)\;/;\;(b\to 1)$$

Expansions at *b* == ∞ for *a* == 1

09.54.06.0005.01

$$\operatorname{agm}(1,b)\propto \frac{\pi b}{2\log(4b)}+\frac{\pi(1-\log(4b))}{8b\log^2(4b)}+\frac{\pi}{256b^3\log^3(4b)}\left(-40\log^2(2)+\log(1024)-5\log(b)(\log(256)+2\log(b)-1)+8\right)+\frac{\pi}{3072b^5\log^4(4b)}\left(4\log(2)(\log(2)(23-132\log(2))+27)+\log(b)\left(-792\log^2(2)+92\log(2)+\log(b)(-396\log(2)-66\log(b)+23)+54\right)+24\right)+\frac{\pi}{393216b^7\log^5(4b)}\left(8(\log(2)(\log(2)(\log(2)(1487-11256\log(2))+2230)+744)+96)+\log(b)\left(4\left(\log(2)\left(-45024\log^2(2)+4461\log(2)+4460\right)+744\right)+\log(b)(6\log(2)(1487-22512\log(2))+\log(b)(-45024\log(2)-5628\log(b)+1487)+4460)\right)\right)+\frac{\pi}{7864320b^9\log^6(4b)}\left(16(\log(2)(\log(2)(\log(2)(\log(2)(17153-165480\log(2))+31565)+12765)+2640)+240)+\log(b)\left(8\left(\log(2)\left(\log(2)\left(-827400\log^2(2)+68612\log(2)+94695\right)+25530\right)+2640\right)+\log(b)\left(12\left(\log(2)\left(-551600\log^2(2)+34306\log(2)+31565\right)+4255\right)+\log(b)(8\log(2)(17153-413700\log(2))+\log(b)(-827400\log(2)-82740\log(b)+17153+63130))\right)\right)\right)+O\left(\frac{b^{-11}}{\log(b)}\right)\;/;\;(|b|\to\infty)$$

09.54.06.0006.01

$$\operatorname{agm}(1,b)\propto \frac{\pi b}{2\log(4b)}+O\left(\frac{1}{b\log(b)}\right)\;/;\;(|b|\to\infty)$$

Integral representations

On the real axis

Of the direct function

09.54.07.0001.01

$$\operatorname{agm}(a,b)=\frac{\pi}{2}\Big/\int_0^{\frac{\pi}{2}}\frac{1}{\sqrt{a^2\cos^2(t)+b^2\sin^2(t)}}\,dt\;/;\;a>0\land b>0$$

Product representations

09.54.08.0001.01

$$\operatorname{agm}(1,b)=\prod_{k=0}^{\infty}\frac{1}{2}(q_k+1)\;/;\;q_0=b\land q_{k+1}=\frac{2\sqrt{q_k}}{q_k+1}$$

Limit representations

09.54.09.0001.01

$$\operatorname{agm}(a,b)=\lim_{n\to\infty}a_n=\lim_{n\to\infty}b_n\;/;\;a_0=a>b_0=b>0\land a_{n+1}=\frac{1}{2}(a_n+b_n)=\operatorname{agm}(a_0,b_0)\,\vartheta_3\left(0,z^{2^{n+1}}\right)^2\land b_{n+1}=\sqrt{a_n b_n}=\operatorname{agm}(a_0,b_0)\,\vartheta_4\left(0,z^{2^{n+1}}\right)^2\land z=q\left(1-\left(\frac{b_0}{a_0}\right)^2\right)$$

Differential equations

Ordinary nonlinear differential equations

09.54.13.0001.01

$$2\,a\left(b^2-a^2\right)\left(\frac{\partial w(a)}{\partial a}\right)^2-a\,w(a)^2+\left(\left(3\,a^2-b^2\right)\frac{\partial w(a)}{\partial a}+a\left(a^2-b^2\right)\frac{\partial^2 w(a)}{\partial a^2}\right)w(a)==0\;/;\;w(a)==\mathrm{agm}(a,\,b)$$

Partial differential equations

09.54.13.0002.01

$$\mathrm{agm}(a,\,b)-a\,\frac{\partial\,\mathrm{agm}(a,\,b)}{\partial a}-b\,\frac{\partial\,\mathrm{agm}(a,\,b)}{\partial b}==0$$

Transformations

Transformations and argument simplifications

Argument involving basic arithmetic operations

09.54.16.0001.01

$$\mathrm{agm}(-a,\,-b)==-\mathrm{agm}(a,\,b)\;/;\;a\notin\mathbb{R}\bigwedge b\notin\mathbb{R}$$

09.54.16.0002.01

$$\mathrm{agm}(c\,a,\,c\,b)==c\,\mathrm{agm}(a,\,b)\;/;\;c>0$$

09.54.16.0003.01

$$\mathrm{agm}(1,\,z)==\frac{1}{a}\,\mathrm{agm}(a,\,a\,z)\;/;\;a>0$$

09.54.16.0004.01

$$\mathrm{agm}\left(\frac{a+b}{2},\,\sqrt{a\,b}\right)==\mathrm{agm}(a,\,b)$$

09.54.16.0005.01

$$\mathrm{agm}\left(1,\,\sqrt{1-z^2}\right)==\mathrm{agm}(z+1,\,1-z)$$

09.54.16.0006.01

$$\mathrm{agm}\left(1,\,\frac{2\sqrt{b}}{b+1}\right)==\frac{2}{b+1}\,\mathrm{agm}(1,\,b)$$

Identities

Functional identities

09.54.17.0001.01

$$\mathrm{agm}(c\,a,\,c\,b)==c\,\mathrm{agm}(a,\,b)\;/;\;c>0$$

09.54.17.0002.01

$$\text{agm}(a,\,b) == a\,\text{agm}\left(1,\,\frac{b}{a}\right)\,/;\,a>0$$

09.54.17.0003.01

$$\text{agm}(a,\,b) == \text{agm}\left(\frac{a+b}{2},\,\sqrt{a\,b}\right)$$

09.54.17.0004.01

$$\text{agm}(a,\,2-a) == \text{agm}\left(1,\,\sqrt{a\,(2-a)}\right)$$

09.54.17.0005.01

$$\text{agm}(1,\,b) == \frac{b+1}{2}\,\text{agm}\left(1,\,\frac{2\sqrt{b}}{b+1}\right)$$

Differentiation

Low-order differentiation

With respect to *a*

09.54.20.0001.01

$$\frac{\partial\,\text{agm}(a,\,b)}{\partial a} == \frac{\text{agm}(a,\,b)}{a\,(a-b)\,\pi}\left(a\,\pi-2\,\text{agm}(a,\,b)\,E\left(\frac{(a-b)^2}{(a+b)^2}\right)\right)$$

09.54.20.0002.01

$$\frac{\partial^2\,\text{agm}(a,\,b)}{\partial a^2} == \frac{2\,\text{agm}(a,\,b)^2}{a^2\,(a-b)^2\,(a+b)\,\pi^2}\left(4\,(a+b)\,\text{agm}(a,\,b)\,E\left(\frac{(a-b)^2}{(a+b)^2}\right)^2-\pi\left(\left(a^2+4\,b\,a+b^2\right)E\left(\frac{(a-b)^2}{(a+b)^2}\right)-2\,a\,b\,K\left(\frac{(a-b)^2}{(a+b)^2}\right)\right)\right)$$

With respect to *b*

09.54.20.0003.01

$$\frac{\partial\,\text{agm}(a,\,b)}{\partial b} == \frac{\text{agm}(a,\,b)}{(a-b)\,b\,\pi}\left(2\,\text{agm}(a,\,b)\,E\left(\frac{(a-b)^2}{(a+b)^2}\right)-b\,\pi\right)$$

09.54.20.0004.01

$$\frac{\partial^2\,\text{agm}(a,\,b)}{\partial b^2} == \frac{2\,\text{agm}(a,\,b)^2}{(a-b)^2\,b^2\,(a+b)\,\pi^2}\left(4\,(a+b)\,\text{agm}(a,\,b)\,E\left(\frac{(a-b)^2}{(a+b)^2}\right)^2-\pi\left(\left(a^2+4\,b\,a+b^2\right)E\left(\frac{(a-b)^2}{(a+b)^2}\right)-2\,a\,b\,K\left(\frac{(a-b)^2}{(a+b)^2}\right)\right)\right)$$

Symbolic differentiation

With respect to *a*

09.54.20.0005.01

$$\frac{\partial^n \operatorname{agm}(a,b)}{\partial a^n} = \operatorname{agm}(a,b)\,\delta_n + \frac{\pi}{4\,b^n}\left(\frac{b\,\delta_{n-1}}{K\left(\left(\frac{a-b}{a+b}\right)^2\right)} + b\,n\,n!\sum_{q=1}^{n-1}\frac{(-1)^q}{(q+1)!\,(n-q-1)!}\,K\left(\left(\frac{a-b}{a+b}\right)^2\right)^{-q-1}\right.$$
$$\sum_{k_1=0}^{n-\sum_{j=1}^{p}k_j-1}\sum_{k_2=0}^{n-\sum_{j=1}^{p}k_j-1}\cdots\sum_{k_{q-1}=0}^{n-\sum_{j=1}^{p}k_j-1}\left(\prod_{p=1}^{q-1}\binom{n-\sum_{j=1}^{p-1}k_j-1}{k_p}\right)\left(\prod_{i=1}^{q-1}A(k_i,a,b)\right)A\left(n-\sum_{j=1}^{q-1}k_j-1,a,b\right)+$$
$$(a+b)\,(n+1)!\sum_{q=1}^{n}\frac{(-1)^q}{(q+1)!\,(n-q)!}\,K\left(\left(\frac{a-b}{a+b}\right)^2\right)^{-q-1}\sum_{k_1=0}^{n-\sum_{j=1}^{p}k_j}\sum_{k_2=0}^{n-\sum_{j=1}^{p}k_j}\cdots$$
$$\left.\sum_{k_{q-1}=0}^{n-\sum_{j=1}^{p}k_j}\left(\prod_{p=1}^{q-1}\binom{n-\sum_{j=1}^{p-1}k_j}{k_p}\right)\left(\prod_{i=1}^{q-1}A(k_i,a,b)\right)A\left(n-\sum_{j=1}^{q-1}k_j,a,b\right)\right)\Big/;\ A(r,a,b) = K\left(\left(\frac{a-b}{a+b}\right)^2\right)\delta_r+$$
$$\frac{\pi}{2}\sum_{m=1}^{r}\frac{1}{m!}\sum_{s=0}^{m}\frac{1}{(m-s)!\,2^{m-2s}}\left((2s-m+1)_{2(m-s)}\left(\frac{a+b}{a-b}\right)^m {}_2\tilde{F}_1\left(\frac{1}{2},\frac{1}{2};1-s;\left(\frac{a-b}{a+b}\right)^2\right)\right)\sum_{q=0}^{m}(-1)^q\binom{m}{q}\left(\frac{a-b}{a+b}\right)^q$$
$$\sum_{u_1=0}^{r}\sum_{u_2=0}^{r}\cdots\sum_{u_{m-q}=0}^{r}\delta_{r,\sum_{i=1}^{m-q}u_i}\,(u_1+u_2+\ldots+u_{m-q};u_1,u_2,\ldots,u_{m-q})\prod_{i=1}^{m-q}\left(\delta_{u_i}-\frac{2\,(-1)^{u_i}\,b^{u_i+1}\,u_i!}{(a+b)^{u_i+1}}\right)\Big/;\ n\in\mathbb{N}$$

With respect to b

09.54.20.0006.01

$$\frac{\partial^n \operatorname{agm}(a,b)}{\partial b^n} = \operatorname{agm}(a,b)\,\delta_n + \frac{\pi}{4\,a^n}\left(\frac{a\,\delta_{n-1}}{K\left(\left(\frac{a-b}{a+b}\right)^2\right)} + a\,n\,n!\sum_{q=1}^{n-1}\frac{(-1)^q}{(q+1)!\,(n-q-1)!}\,K\left(\left(\frac{a-b}{a+b}\right)^2\right)^{-q-1}\right.$$
$$\sum_{k_1=0}^{n-\sum_{j=1}^{p}k_j-1}\sum_{k_2=0}^{n-\sum_{j=1}^{p}k_j-1}\cdots\sum_{k_{q-1}=0}^{n-\sum_{j=1}^{p}k_j-1}\left(\prod_{p=1}^{q-1}\binom{n-\sum_{j=1}^{p-1}k_j-1}{k_p}\right)\left(\prod_{i=1}^{q-1}A(k_i,b,a)\right)A\left(n-\sum_{j=1}^{q-1}k_j-1,b,a\right)+$$
$$(a+b)\,(n+1)!\sum_{q=1}^{n}\frac{(-1)^q}{(q+1)!\,(n-q)!}\,K\left(\left(\frac{a-b}{a+b}\right)^2\right)^{-q-1}\sum_{k_1=0}^{n-\sum_{j=1}^{p}k_j}\sum_{k_2=0}^{n-\sum_{j=1}^{p}k_j}\cdots$$
$$\left.\sum_{k_{q-1}=0}^{n-\sum_{j=1}^{p}k_j}\left(\prod_{p=1}^{q-1}\binom{n-\sum_{j=1}^{p-1}k_j}{k_p}\right)\left(\prod_{i=1}^{q-1}A(k_i,b,a)\right)A\left(n-\sum_{j=1}^{q-1}k_j,b,a\right)\right)\Big/;\ A(r,a,b) = K\left(\left(\frac{a-b}{a+b}\right)^2\right)\delta_r+$$
$$\frac{\pi}{2}\sum_{m=1}^{r}\frac{1}{m!}\sum_{s=0}^{m}\frac{1}{(m-s)!\,2^{m-2s}}\left((2s-m+1)_{2(m-s)}\left(\frac{a+b}{a-b}\right)^m {}_2\tilde{F}_1\left(\frac{1}{2},\frac{1}{2};1-s;\left(\frac{a-b}{a+b}\right)^2\right)\right)\sum_{q=0}^{m}(-1)^q\binom{m}{q}\left(\frac{a-b}{a+b}\right)^q$$
$$\sum_{u_1=0}^{r}\sum_{u_2=0}^{r}\cdots\sum_{u_{m-q}=0}^{r}\delta_{r,\sum_{i=1}^{m-q}u_i}\,(u_1+u_2+\ldots+u_{m-q};u_1,u_2,\ldots,u_{m-q})\prod_{i=1}^{m-q}\left(\delta_{u_i}-\frac{2\,(-1)^{u_i}\,b^{u_i+1}\,u_i!}{(a+b)^{u_i+1}}\right)\Big/;\ n\in\mathbb{N}$$

Representations through more general functions

Through hypergeometric functions

Involving ${}_2F_1$

09.54.26.0001.01

$$\operatorname{agm}(a, b) = \frac{a+b}{2\, {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \left(\frac{a-b}{a+b}\right)^2\right)}$$

Through Meijer *G*

Classical cases for the direct function itself

09.54.26.0002.01

$$\operatorname{agm}(a, b) = \frac{\pi (a+b)}{2} \Big/ G_{2,2}^{1,2}\left(-\left(\frac{a-b}{a+b}\right)^2 \,\middle|\, \begin{matrix} \frac{1}{2}, \frac{1}{2} \\ 0, 0 \end{matrix}\right)$$

Through other functions

Involving some hypergeometric-type functions

09.54.26.0003.01

$$\operatorname{agm}(a, b) = \frac{\pi (a+b)}{4\, K\left(\left(\frac{a-b}{a+b}\right)^2\right)}$$

Inequalities

09.54.29.0001.01

$$\sqrt{a\, b} \leq \operatorname{agm}(a, b) \leq \frac{a+b}{2}$$

Theorems

Representation of π

$$\pi = \lim_{n\to\infty} \frac{2\, a_{n+1}^2}{1 - \sum_{k=0}^{n} 2^k\, c_k^2} \,/;\, c_{n+1} = \frac{a_n - b_n}{2} \bigwedge c_0 = \sqrt{a_0^2 - b_0^2}$$

History

– J. Landen (1771,1775)
– J.-L. Lagrange (1784-85)
– C .F. Gauss (1791–1799, 1800, 1876); Gauss (1800) derived the relation to ${}_2F_1(a, b; c; z)$

Applications include fast high-precision computation of π, $\log(z)$, e^z, $\sin(z)$, $\cos(z)$, etc.

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