Beta3

Notations

Traditional name

Incomplete beta function

Traditional notation

$B_z(a, b)$

Mathematica StandardForm notation

`Beta[z, a, b]`

Primary definition

Basic definition

06.19.02.0001.01

$$B_z(a, b) == \int_0^z t^{a-1} (1-t)^{b-1} \, dt \,/;\, \mathrm{Re}(a) > 0 \bigwedge \mathrm{Re}(b) > 0 \bigwedge |z| < 1$$

The incomplete beta function $B_z(a, b)$ is defined by the above definite integral, that under the stated conditions, is convergent. For other values of the arguments z, a, b, the function $B_z(a, b)$ is defined as the analytic continuation with respect to its arguments of this integral.

Complete definition

06.19.02.0002.01

$$B_z(a, b) == \Gamma(a)\, z^a \, {}_2\tilde{F}_1(a, 1-b; a+1; z) \,/;\, -a \notin \mathbb{N}$$

The function $B_z(a, b)$ can be equivalently defined through the above generalized hypergeometric function.

For negative integers $\alpha == -n$ and positive integers $b == m \,/;\, m \leq n$, the function $B_z(a, b)$ cannot be uniquely defined by a limiting procedure based on the above definition because the two variables a, b can approach integers $-n$, $m \,/;\, m \leq n$ at different speeds. In the case $a == -n$, $b == m \,/;\, m \leq n$ one defines:

06.19.02.0003.01

$$B_z(-n, m) = z^{-n} \sum_{k=0}^{m-1} \frac{(1-m)_k \, z^k}{(k-n)\, k!} \,/;\, n \in \mathbb{N}^+ \bigwedge m \in \mathbb{N}^+ \bigwedge m \leq n$$

For $a == -n$ and arbitrary b (except the case of positive integer $b == m \,/;\, m \leq n$), the function $B_z(a, b)$ is not finite:

06.19.02.0004.01

$$B_z(-n, b) = \tilde{\infty} \;/;\; n = 0 \lor \left(n \in \mathbb{N}^+ \land \neg \left(b \in \mathbb{N}^+ \land b \leq n\right)\right)$$

Specific values

Specialized values

For fixed *z*, *a*

06.19.03.0001.01

$$B_z(a, n) = B(a, n)\, z^a \sum_{k=0}^{n-1} \frac{(a)_k\, (1-z)^k}{k!} \;/;\; n \in \mathbb{N}$$

For fixed *z*, *b*

06.19.03.0002.01

$$B_z(-n, b) = \tilde{\infty} \;/;\; n = 0 \lor \left(n \in \mathbb{N}^+ \land \neg \left(b \in \mathbb{N}^+ \land b \leq n\right)\right)$$

06.19.03.0007.01

$$B_z(-n, b) = z^{-n} \sum_{k=0}^{b-1} \frac{(1-b)_k\, z^k}{(k-n)\, k!} \;/;\; n \in \mathbb{N}^+ \land b \in \mathbb{N}^+ \land b \leq n$$

06.19.03.0003.01

$$B_z(n, b) = B(n, b) \left(1 - (1-z)^b \sum_{k=0}^{n-1} \frac{(b)_k\, z^k}{k!}\right) \;/;\; n \in \mathbb{N}$$

For fixed *a*, *b*

06.19.03.0004.01

$$B_0(a, b) = 0 \;/;\; \mathrm{Re}(a) > 0$$

06.19.03.0005.01

$$B_0(a, b) = \tilde{\infty} \;/;\; \mathrm{Re}(a) < 0$$

06.19.03.0006.01

$$B_1(a, b) = B(a, b) \;/;\; \mathrm{Re}(b) > 0$$

General characteristics

Domain and analyticity

$B_z(a, b)$ is an analytical function of z, a, and b which is defined in $\mathbb{C}^3$.

06.19.04.0001.01

$$(z * a * b) \longrightarrow B_z(a, b) :: (\mathbb{C} \otimes \mathbb{C} \otimes \mathbb{C}) \longrightarrow \mathbb{C}$$

Symmetries and periodicities

Mirror symmetry

06.19.04.0002.02

$$B_{\bar{z}}\left(\bar{a}, \bar{b}\right) = \overline{B_z(a, b)} \;/;\; z \notin (-\infty, 0) \land z \notin (1, \infty)$$

Periodicity

No periodicity

Poles and essential singularities

With respect to *b*

For fixed z, a, the function $B_z(a, b)$ has only one singular point at $b = \tilde{\infty}$. It is an essential singular point.

06.19.04.0003.01

$$\mathcal{S}ing_b(B_z(a, b)) == \{\{\tilde{\infty}, \infty\}\}$$

With respect to *a*

For fixed z, b, the function $B_z(a, b)$ has an infinite set of singular points:

a) $a == -k\ /;\ k \in \mathbb{N}$, are the simple poles with residues $\frac{(1-b)_k}{k!}$;

b) $a == \tilde{\infty}$ is the point of convergence of poles, which is an essential singular point.

06.19.04.0004.01

$$\mathcal{S}ing_a(B_z(a, b)) == \{\{\{-k, 1\}\ /;\ k \in \mathbb{N}\}, \{\tilde{\infty}, \infty\}\}$$

06.19.04.0005.01

$$\text{res}_z(B_z(a, b))\,(-k) == \frac{(1-b)_k}{k!}\ /;\ k \in \mathbb{N}$$

With respect to *z*

For fixed a, b, the function $B_z(a, b)$ does not have poles and essential singularities.

06.19.04.0006.01

$$\mathcal{S}ing_z(B_z(a, b)) == \{\}$$

Branch points

With respect to *b*

For fixed z, a, the function $B_z(a, b)$ does not have branch points.

06.19.04.0007.01

$$\mathcal{BP}_b(B_z(a, b)) == \{\}$$

With respect to *a*

For fixed z, b, the function $B_z(a, b)$ does not have branch points.

06.19.04.0008.01

$$\mathcal{BP}_a(B_z(a, b)) == \{\}$$

With respect to *z*

For fixed a, b, the function $B_z(a, b)$ has three branch points: $z == 0$, $z == 1$ and $z == \tilde{\infty}$.

06.19.04.0009.01

$$\mathcal{BP}_z(B_z(a, b)) == \{0, 1, \tilde{\infty}\}$$

06.19.04.0010.01

$\mathcal{R}_z(B_z(a, b), 0) = \log /; a \notin \mathbb{Z} \land a \notin \mathbb{Q}$

06.19.04.0011.01

$\mathcal{R}_z(B_z(a, b), 0) = q /; a = \frac{p}{q} \bigwedge p \in \mathbb{Z} \bigwedge q - 1 \in \mathbb{N}^+ \bigwedge \gcd(p, q) = 1$

06.19.04.0012.01

$\mathcal{R}_z(B_z(a, b), 1) = \log /; b \notin \mathbb{Z} \land b \notin \mathbb{Q}$

06.19.04.0013.01

$\mathcal{R}_z(B_z(a, b), 1) = q /; b = \frac{p}{q} \bigwedge p \in \mathbb{Z} \bigwedge q - 1 \in \mathbb{N}^+ \bigwedge \gcd(p, q) = 1$

06.19.04.0014.01

$\mathcal{R}_z(B_z(a, b), \tilde{\infty}) = \log /; a + b \in \mathbb{Z} \lor a + b \notin \mathbb{Q}$

06.19.04.0015.01

$\mathcal{R}_z(B_z(a, b), \tilde{\infty}) = s /; a + b = \frac{r}{s} \bigwedge r \in \mathbb{Z} \bigwedge s - 1 \in \mathbb{N}^+ \bigwedge \gcd(r, s) = 1$

Branch cuts

With respect to *b*

For fixed z, a, the function $B_z(a, b)$ does not have branch cuts.

06.19.04.0016.01

$\mathcal{BC}_b(B_z(a, b)) = \{\}$

With respect to *a*

For fixed z, b, the function $B_z(a, b)$ does not have branch cuts.

06.19.04.0017.01

$\mathcal{BC}_a(B_z(a, b)) = \{\}$

With respect to *z*

For fixed a, b, the function $B_z(a, b)$ is a single-valued function on the z-plane cut along the intervals $(-\infty, 0)$ and $(1, \infty)$. The function $B_z(a, b)$ is continuous from above on the interval $(-\infty, 0)$ and from below on the interval $(1, \infty)$.

06.19.04.0018.01

$\mathcal{BC}_z(B_z(a, b)) = \{\{(-\infty, 0), -i\}, \{(1, \infty), i\}\}$

06.19.04.0019.01

$\lim_{\epsilon \to +0} B_{x+i\epsilon}(a, b) = B_x(a, b) /; x < 0$

06.19.04.0020.01

$\lim_{\epsilon \to +0} B_{x-i\epsilon}(a, b) = e^{-2 i a \pi} B_x(a, b) /; x < 0$

06.19.04.0021.01

$\lim_{\epsilon \to +0} B_{x-i\epsilon}(a, b) = B_x(a, b) /; x > 1$

06.19.04.0022.01

$\lim_{\epsilon \to +0} B_{x+i\epsilon}(a, b) = e^{-2 i b \pi} B_x(a, b) + 2 e^{-i b \pi} i B(a, b) \sin(b \pi) /; x > 1$

Series representations

Generalized power series

Expansions at generic point $z == z_0$

For the function itself

06.19.06.0024.01

$$
\begin{aligned}
&B_z(a, b) \propto \frac{1}{\Gamma(1-b)\,\Gamma(a+b)} \left(\frac{1}{z_0}\right)^{a \left\lfloor \frac{\arg(z-z_0)}{2\pi} \right\rfloor} z_0^{a\left(\left\lfloor \frac{\arg(z-z_0)}{2\pi} \right\rfloor + 1\right)} \\
&\Bigg(-2\, i\, e^{i\, b\, \pi \left\lfloor \frac{\arg(z_0-z)}{2\pi} \right\rfloor} \pi \left\lfloor \frac{\arg(1-z_0)+\pi}{2\pi} \right\rfloor \left\lfloor \frac{\arg(z_0-z)}{2\pi} \right\rfloor \Gamma(a)\, z_0^{-a} + G_{2,2}^{2,2}\!\left(1-z_0 \,\middle|\, \begin{matrix} 1-a,\, b \\ 0,\, b \end{matrix}\right) \left(\frac{1}{1-z_0}\right)^{b \left\lfloor \frac{\arg(z_0-z)}{2\pi} \right\rfloor} (1-z_0)^{b \left\lfloor \frac{\arg(z_0-z)}{2\pi} \right\rfloor} + \\
&\Bigg(\left(\frac{1}{1-z_0}\right)^{b \left\lfloor \frac{\arg(z_0-z)}{2\pi} \right\rfloor} (1-z_0)^{b \left\lfloor \frac{\arg(z_0-z)}{2\pi} \right\rfloor} z_0^a \left(a\, G_{2,2}^{2,2}\!\left(1-z_0 \,\middle|\, \begin{matrix} 1-a,\, b \\ 0,\, b \end{matrix}\right) + G_{2,2}^{2,2}\!\left(1-z_0 \,\middle|\, \begin{matrix} -a,\, b-1 \\ 0,\, b-1 \end{matrix}\right) z_0\right) - \\
&2\, i\, e^{i\, b\, \pi \left\lfloor \frac{\arg(z_0-z)}{2\pi} \right\rfloor} \pi \left\lfloor \frac{\arg(1-z_0)+\pi}{2\pi} \right\rfloor \left\lfloor \frac{\arg(z_0-z)}{2\pi} \right\rfloor (a\,\Gamma(a) - \Gamma(a+1))\Bigg) z_0^{-a-1} (z-z_0) + \\
&\frac{1}{2}\Bigg(\left(\frac{1}{1-z_0}\right)^{b \left\lfloor \frac{\arg(z_0-z)}{2\pi} \right\rfloor} (1-z_0)^{b \left\lfloor \frac{\arg(z_0-z)}{2\pi} \right\rfloor} z_0^a \Bigg((a-1)\, a\, G_{2,2}^{2,2}\!\left(1-z_0 \,\middle|\, \begin{matrix} 1-a,\, b \\ 0,\, b \end{matrix}\right) + \\
&z_0 \left(2\, a\, G_{2,2}^{2,2}\!\left(1-z_0 \,\middle|\, \begin{matrix} -a,\, b-1 \\ 0,\, b-1 \end{matrix}\right) + G_{2,2}^{2,2}\!\left(1-z_0 \,\middle|\, \begin{matrix} -a-1,\, b-2 \\ 0,\, b-2 \end{matrix}\right) z_0\right)\Bigg) - 2\, i\, e^{i\, b\, \pi \left\lfloor \frac{\arg(z_0-z)}{2\pi} \right\rfloor} \pi \\
&\left\lfloor \frac{\arg(1-z_0)+\pi}{2\pi} \right\rfloor \left\lfloor \frac{\arg(z_0-z)}{2\pi} \right\rfloor ((a-1)\, a\, \Gamma(a) - 2\, a\, \Gamma(a+1) + \Gamma(a+2))\Bigg) z_0^{-a-2} (z-z_0)^2 + \ldots\Bigg) \;/;\; (z \to z_0)
\end{aligned}
$$

06.19.06.0025.01

$$B_z(a, b) \propto \frac{1}{\Gamma(1-b)\,\Gamma(a+b)} \left(\frac{1}{z_0}\right)^{a \left\lfloor \frac{\arg(z-z_0)}{2\pi} \right\rfloor} z_0^{a \left(\left\lfloor \frac{\arg(z-z_0)}{2\pi} \right\rfloor + 1\right)} \Bigg(-2\, i\, e^{i\, b\, \pi \left\lfloor \frac{\arg(z_0-z)}{2\pi} \right\rfloor} \pi \left\lfloor \frac{\arg(1-z_0)+\pi}{2\pi} \right\rfloor \left\lfloor \frac{\arg(z_0-z)}{2\pi} \right\rfloor \Gamma(a)\, z_0^{-a} + G_{2,2}^{2,2}\!\left(1-z_0 \,\middle|\, \begin{matrix} 1-a,\, b \\ 0,\, b \end{matrix}\right) \left(\frac{1}{1-z_0}\right)^{b \left\lfloor \frac{\arg(z_0-z)}{2\pi} \right\rfloor} (1-z_0)^{b \left\lfloor \frac{\arg(z_0-z)}{2\pi} \right\rfloor} +$$

$$\Bigg(\left(\frac{1}{1-z_0}\right)^{b \left\lfloor \frac{\arg(z_0-z)}{2\pi} \right\rfloor} (1-z_0)^{b \left\lfloor \frac{\arg(z_0-z)}{2\pi} \right\rfloor} z_0^a \left(a\, G_{2,2}^{2,2}\!\left(1-z_0 \,\middle|\, \begin{matrix} 1-a,\, b \\ 0,\, b \end{matrix}\right) + G_{2,2}^{2,2}\!\left(1-z_0 \,\middle|\, \begin{matrix} -a,\, b-1 \\ 0,\, b-1 \end{matrix}\right) z_0\right) -$$

$$2\, i\, e^{i\, b\, \pi \left\lfloor \frac{\arg(z_0-z)}{2\pi} \right\rfloor} \pi \left\lfloor \frac{\arg(1-z_0)+\pi}{2\pi} \right\rfloor \left\lfloor \frac{\arg(z_0-z)}{2\pi} \right\rfloor (a\,\Gamma(a) - \Gamma(a+1))\Bigg) z_0^{-a-1}(z-z_0) +$$

$$\frac{1}{2}\Bigg(\left(\frac{1}{1-z_0}\right)^{b \left\lfloor \frac{\arg(z_0-z)}{2\pi} \right\rfloor} (1-z_0)^{b \left\lfloor \frac{\arg(z_0-z)}{2\pi} \right\rfloor} z_0^a \left((a-1)\, a\, G_{2,2}^{2,2}\!\left(1-z_0 \,\middle|\, \begin{matrix} 1-a,\, b \\ 0,\, b \end{matrix}\right) + \right.$$

$$\left. z_0 \left(2\, a\, G_{2,2}^{2,2}\!\left(1-z_0 \,\middle|\, \begin{matrix} -a,\, b-1 \\ 0,\, b-1 \end{matrix}\right) + G_{2,2}^{2,2}\!\left(1-z_0 \,\middle|\, \begin{matrix} -a-1,\, b-2 \\ 0,\, b-2 \end{matrix}\right) z_0\right)\right) - 2\, i\, e^{i\, b\, \pi \left\lfloor \frac{\arg(z_0-z)}{2\pi} \right\rfloor} \pi$$

$$\left\lfloor \frac{\arg(1-z_0)+\pi}{2\pi} \right\rfloor \left\lfloor \frac{\arg(z_0-z)}{2\pi} \right\rfloor ((a-1)\, a\, \Gamma(a) - 2\, a\, \Gamma(a+1) + \Gamma(a+2))\Bigg) z_0^{-a-2}(z-z_0)^2 + O\left((z-z_0)^3\right)\Bigg)$$

06.19.06.0026.01

$$B_z(a, b) = \frac{1}{\Gamma(1-b)\,\Gamma(a+b)} \left(\frac{1}{z_0}\right)^{a \left\lfloor \frac{\arg(z-z_0)}{2\pi} \right\rfloor} z_0^{a \left(\left\lfloor \frac{\arg(z-z_0)}{2\pi} \right\rfloor + 1\right)}$$

$$\sum_{k=0}^{\infty} \sum_{j=0}^{k} \frac{(-1)^{k-j}\, (-a)_{k-j}\, z_0^{j-k}}{j!\,(k-j)!} \Bigg(\left(\frac{1}{1-z_0}\right)^{b \left\lfloor \frac{\arg(z_0-z)}{2\pi} \right\rfloor} (1-z_0)^{b \left\lfloor \frac{\arg(z_0-z)}{2\pi} \right\rfloor} G_{2,2}^{2,2}\!\left(1-z_0 \,\middle|\, \begin{matrix} -a-j+1,\, b-j \\ 0,\, b-j \end{matrix}\right) -$$

$$2\,\pi\, i \left\lfloor \frac{\arg(z_0-z)}{2\pi} \right\rfloor e^{i\, b\, \pi \left\lfloor \frac{\arg(z_0-z)}{2\pi} \right\rfloor} \left\lfloor \frac{\arg(1-z_0)+\pi}{2\pi} \right\rfloor (-1)^j\, \Gamma(a+j)\, z_0^{-a-j}\Bigg) (z-z_0)^k$$

06.19.06.0027.01

$$B_z(a,b) = \frac{\pi}{\Gamma(1-b)\,\Gamma(a+b)} \left(\frac{1}{z_0}\right)^{a\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor} z_0^{a\left(\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor+1\right)} \sum_{k=0}^{\infty}\sum_{j=0}^{k} \frac{(-1)^k\,(-a)_{k-j}\,z_0^{j-k}}{j!\,(k-j)!} \left(\Gamma(a+j)\,z_0^{-a-j}\left(\csc(b\pi)\left(\frac{1}{1-z_0}\right)^{b\left\lfloor\frac{\arg(z_0-z)}{2\pi}\right\rfloor}(1-z_0)^{b\left\lfloor\frac{\arg(z_0-z)}{2\pi}\right\rfloor} - 2\,i\,e^{i\,b\,\pi\left\lfloor\frac{\arg(z_0-z)}{2\pi}\right\rfloor}\left\lfloor\frac{\arg(1-z_0)+\pi}{2\pi}\right\rfloor\left\lfloor\frac{\arg(z_0-z)}{2\pi}\right\rfloor\right) - \csc(b\pi)\,\Gamma(a+b)\left(\frac{1}{1-z_0}\right)^{b\left\lfloor\frac{\arg(z_0-z)}{2\pi}\right\rfloor}(1-z_0)^{\left\lfloor\frac{\arg(z_0-z)}{2\pi}\right\rfloor b+b-j}\,{}_2\tilde{F}_1(1,a+b;b-j+1;1-z_0)\right)(z-z_0)^k \;/;\; b\notin\mathbb{Z}$$

06.19.06.0028.01

$$B_z(a,b) \propto \frac{1}{\Gamma(1-b)\,\Gamma(a+b)}\left(\frac{1}{z_0}\right)^{a\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor} z_0^{a\left(\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor+1\right)}\left(-2\,i\,e^{i\,b\,\pi\left\lfloor\frac{\arg(z_0-z)}{2\pi}\right\rfloor}\pi\left\lfloor\frac{\arg(1-z_0)+\pi}{2\pi}\right\rfloor\left\lfloor\frac{\arg(z_0-z)}{2\pi}\right\rfloor\Gamma(a)\,z_0^{-a} + G_{2,2}^{2,2}\!\left(1-z_0\,\middle|\,\begin{matrix}1-a,\,b\\0,\,b\end{matrix}\right)\left(\frac{1}{1-z_0}\right)^{b\left\lfloor\frac{\arg(z_0-z)}{2\pi}\right\rfloor}(1-z_0)^{b\left\lfloor\frac{\arg(z_0-z)}{2\pi}\right\rfloor}\right) + O(z-z_0)$$

Expansions on branch cuts

For the function itself

In the left half-plane

06.19.06.0029.01

$$B_z(a,b) \propto B_z(a,b) \propto \frac{1}{\Gamma(1-b)\,\Gamma(a+b)}\left(\frac{1}{x}\right)^{a\left\lfloor\frac{\arg(z-x)}{2\pi}\right\rfloor} x^{a\left(\left\lfloor\frac{\arg(z-x)}{2\pi}\right\rfloor+1\right)} \left(G_{2,2}^{2,2}\!\left(1-x\,\middle|\,\begin{matrix}1-a,\,b\\0,\,b\end{matrix}\right) + \left(G_{2,2}^{2,2}\!\left(1-x\,\middle|\,\begin{matrix}1-a,\,b\\0,\,b\end{matrix}\right) + x\,G_{2,2}^{2,2}\!\left(1-x\,\middle|\,\begin{matrix}-a,\,b-1\\0,\,b-1\end{matrix}\right)\right)(z-x) + \frac{1}{2x^2}\left(G_{2,2}^{2,2}\!\left(1-x\,\middle|\,\begin{matrix}-a-1,\,b-2\\0,\,b-2\end{matrix}\right)x^2 + a\left((a-1)\,G_{2,2}^{2,2}\!\left(1-x\,\middle|\,\begin{matrix}1-a,\,b\\0,\,b\end{matrix}\right) + 2\,x\,G_{2,2}^{2,2}\!\left(1-x\,\middle|\,\begin{matrix}-a,\,b-1\\0,\,b-1\end{matrix}\right)\right)\right)(z-x)^2 + \ldots\right) /;\; (z\to x)\land x\in\mathbb{R}\land x<0$$

06.19.06.0030.01

$$B_z(a,b)\propto B_z(a,b)\propto \frac{1}{\Gamma(1-b)\,\Gamma(a+b)}\left(\frac{1}{x}\right)^{a\left\lfloor\frac{\arg(z-x)}{2\pi}\right\rfloor} x^{a\left(\left\lfloor\frac{\arg(z-x)}{2\pi}\right\rfloor+1\right)} \left(G_{2,2}^{2,2}\!\left(1-x\,\middle|\,\begin{matrix}1-a,\,b\\0,\,b\end{matrix}\right)+\left(G_{2,2}^{2,2}\!\left(1-x\,\middle|\,\begin{matrix}1-a,\,b\\0,\,b\end{matrix}\right)+x\,G_{2,2}^{2,2}\!\left(1-x\,\middle|\,\begin{matrix}-a,\,b-1\\0,\,b-1\end{matrix}\right)\right)(z-x)+\frac{1}{2x^2}\left(G_{2,2}^{2,2}\!\left(1-x\,\middle|\,\begin{matrix}-a-1,\,b-2\\0,\,b-2\end{matrix}\right)x^2+a\left((a-1)\,G_{2,2}^{2,2}\!\left(1-x\,\middle|\,\begin{matrix}1-a,\,b\\0,\,b\end{matrix}\right)+2\,x\,G_{2,2}^{2,2}\!\left(1-x\,\middle|\,\begin{matrix}-a,\,b-1\\0,\,b-1\end{matrix}\right)\right)\right)(z-x)^2+O\!\left((z-x)^3\right)\right)\,/;\,x\in\mathbb{R}\land x<0$$

06.19.06.0031.01

$$B_z(a,b)=\frac{1}{\Gamma(1-b)\,\Gamma(a+b)}\left(\frac{1}{x}\right)^{a\left\lfloor\frac{\arg(z-x)}{2\pi}\right\rfloor} x^{a\left(\left\lfloor\frac{\arg(z-x)}{2\pi}\right\rfloor+1\right)}\sum_{k=0}^{\infty}\sum_{j=0}^{k}\frac{(-1)^{k-j}\,(-a)_{k-j}\,x^{j-k}}{j!\,(k-j)!}\,G_{2,2}^{2,2}\!\left(1-x\,\middle|\,\begin{matrix}-a-j+1,\,b-j\\0,\,b-j\end{matrix}\right)(z-x)^k\,/;\,x\in\mathbb{R}\land x<0$$

06.19.06.0032.01

$$B_z(a,b)=\frac{1}{\Gamma(a+b)}\,\Gamma(b)\left(\frac{1}{x}\right)^{a\left\lfloor\frac{\arg(z-x)}{2\pi}\right\rfloor} x^{a\left(\left\lfloor\frac{\arg(z-x)}{2\pi}\right\rfloor+1\right)}\sum_{k=0}^{\infty}\sum_{j=0}^{k}\frac{(-1)^k\,(-a)_{k-j}\,x^{j-k}}{j!\,(k-j)!}\left(\Gamma(a+j)\,x^{-a-j}-\Gamma(a+b)\,(1-x)^{b-j}\,{}_2\tilde{F}_1(1,\,a+b;\,b-j+1;\,1-x)\right)(z-x)^k\,/;\,x\in\mathbb{R}\land x<0$$

06.19.06.0033.01

$$B_z(a,b)\propto\frac{1}{\Gamma(1-b)\,\Gamma(a+b)}\left(\frac{1}{x}\right)^{a\left\lfloor\frac{\arg(z-x)}{2\pi}\right\rfloor} x^{a\left(\left\lfloor\frac{\arg(z-x)}{2\pi}\right\rfloor+1\right)}\,G_{2,2}^{2,2}\!\left(1-x\,\middle|\,\begin{matrix}1-a,\,b\\0,\,b\end{matrix}\right)+O(z-x)\,/;\,x\in\mathbb{R}\land x<0$$

In the right half-plane

06.19.06.0034.01

$$B_z(a,b)\propto\frac{1}{\Gamma(1-b)\,\Gamma(a+b)}\left(-2\,e^{i\,b\,\pi\left\lfloor\frac{\arg(x-z)}{2\pi}\right\rfloor}\,i\,\pi\left\lfloor\frac{\arg(x-z)}{2\pi}\right\rfloor\Gamma(a)+\left(\frac{1}{1-x}\right)^{b\left\lfloor\frac{\arg(x-z)}{2\pi}\right\rfloor}(1-x)^{b\left\lfloor\frac{\arg(x-z)}{2\pi}\right\rfloor}G_{2,2}^{2,2}\!\left(1-x\,\middle|\,\begin{matrix}1-a,\,b\\0,\,b\end{matrix}\right)x^a+\left(\frac{1}{1-x}\right)^{b\left\lfloor\frac{\arg(x-z)}{2\pi}\right\rfloor}(1-x)^{b\left\lfloor\frac{\arg(x-z)}{2\pi}\right\rfloor}x^{a-1}\left(a\,G_{2,2}^{2,2}\!\left(1-x\,\middle|\,\begin{matrix}1-a,\,b\\0,\,b\end{matrix}\right)+x\,G_{2,2}^{2,2}\!\left(1-x\,\middle|\,\begin{matrix}-a,\,b-1\\0,\,b-1\end{matrix}\right)\right)(z-x)+\frac{1}{2}\left(\frac{1}{1-x}\right)^{b\left\lfloor\frac{\arg(x-z)}{2\pi}\right\rfloor}(1-x)^{b\left\lfloor\frac{\arg(x-z)}{2\pi}\right\rfloor}x^{a-2}\left(G_{2,2}^{2,2}\!\left(1-x\,\middle|\,\begin{matrix}-a-1,\,b-2\\0,\,b-2\end{matrix}\right)x^2+a\left((a-1)\,G_{2,2}^{2,2}\!\left(1-x\,\middle|\,\begin{matrix}1-a,\,b\\0,\,b\end{matrix}\right)+2\,x\,G_{2,2}^{2,2}\!\left(1-x\,\middle|\,\begin{matrix}-a,\,b-1\\0,\,b-1\end{matrix}\right)\right)\right)(z-x)^2+\ldots\right)/;\,(z\to x)\land x\in\mathbb{R}\land x>1$$

06.19.06.0035.01

$$B_z(a,b) \propto \frac{1}{\Gamma(1-b)\,\Gamma(a+b)} \left(-2\,e^{i\,b\,\pi \left\lfloor \frac{\arg(x-z)}{2\pi} \right\rfloor} i\,\pi \left\lfloor \frac{\arg(x-z)}{2\pi} \right\rfloor \Gamma(a) + \left(\frac{1}{1-x}\right)^{b \left\lfloor \frac{\arg(x-z)}{2\pi} \right\rfloor} (1-x)^{b \left\lfloor \frac{\arg(x-z)}{2\pi} \right\rfloor} G_{2,2}^{2,2}\!\left(1-x \,\middle|\, \begin{matrix} 1-a,\, b \\ 0,\, b \end{matrix}\right) x^a + \left(\frac{1}{1-x}\right)^{b \left\lfloor \frac{\arg(x-z)}{2\pi} \right\rfloor} (1-x)^{b \left\lfloor \frac{\arg(x-z)}{2\pi} \right\rfloor} x^{a-1} \left(a\,G_{2,2}^{2,2}\!\left(1-x \,\middle|\, \begin{matrix} 1-a,\, b \\ 0,\, b \end{matrix}\right) + x\,G_{2,2}^{2,2}\!\left(1-x \,\middle|\, \begin{matrix} -a,\, b-1 \\ 0,\, b-1 \end{matrix}\right)\right)(z-x) + \frac{1}{2}\left(\frac{1}{1-x}\right)^{b \left\lfloor \frac{\arg(x-z)}{2\pi} \right\rfloor} (1-x)^{b \left\lfloor \frac{\arg(x-z)}{2\pi} \right\rfloor} x^{a-2} \left(G_{2,2}^{2,2}\!\left(1-x \,\middle|\, \begin{matrix} -a-1,\, b-2 \\ 0,\, b-2 \end{matrix}\right) x^2 + a\left((a-1)\,G_{2,2}^{2,2}\!\left(1-x \,\middle|\, \begin{matrix} 1-a,\, b \\ 0,\, b \end{matrix}\right) + 2\,x\,G_{2,2}^{2,2}\!\left(1-x \,\middle|\, \begin{matrix} -a,\, b-1 \\ 0,\, b-1 \end{matrix}\right)\right)\right)(z-x)^2 + O\!\left((z-x)^3\right) \right) /;\ x \in \mathbb{R} \land x > 1$$

06.19.06.0036.01

$$B_z(a,b) = \frac{1}{\Gamma(1-b)\,\Gamma(a+b)} \sum_{k=0}^{\infty} \sum_{j=0}^{k} \frac{(-1)^{k-j}\,(-a)_{k-j}\,x^{j-k}}{j!\,(k-j)!} \left(x^a \left(\frac{1}{1-x}\right)^{b \left\lfloor \frac{\arg(x-z)}{2\pi} \right\rfloor} (1-x)^{b \left\lfloor \frac{\arg(x-z)}{2\pi} \right\rfloor} G_{2,2}^{2,2}\!\left(1-x \,\middle|\, \begin{matrix} -a-j+1,\, b-j \\ 0,\, b-j \end{matrix}\right) - 2\,\pi\,i \left\lfloor \frac{\arg(x-z)}{2\pi} \right\rfloor e^{i\,b\,\pi \left\lfloor \frac{\arg(x-z)}{2\pi} \right\rfloor} (-1)^j\,\Gamma(a+j)\,x^{-j} \right) (z-x)^k /;\ x \in \mathbb{R} \land x > 1$$

06.19.06.0037.01

$$B_z(a,b) = \frac{\pi}{\Gamma(1-b)\,\Gamma(a+b)} \sum_{k=0}^{\infty} \sum_{j=0}^{k} \frac{(-1)^k\,(-a)_{k-j}\,x^{j-k}}{j!\,(k-j)!} \left(\Gamma(a+j)\,x^{-j} \left(\csc(b\,\pi) \left(\frac{1}{1-x}\right)^{b \left\lfloor \frac{\arg(x-z)}{2\pi} \right\rfloor} (1-x)^{b \left\lfloor \frac{\arg(x-z)}{2\pi} \right\rfloor} - 2\,i\,e^{i\,b\,\pi \left\lfloor \frac{\arg(x-z)}{2\pi} \right\rfloor} \left\lfloor \frac{\arg(x-z)}{2\pi} \right\rfloor \right) - \csc(b\,\pi)\,\Gamma(a+b)\,x^a \left(\frac{1}{1-x}\right)^{b \left\lfloor \frac{\arg(x-z)}{2\pi} \right\rfloor} (1-x)^{\left\lfloor \frac{\arg(x-z)}{2\pi} \right\rfloor b+b-j} {}_2\tilde{F}_1(1,\, a+b;\, b-j+1;\, 1-x) \right) (z-x)^k /;\ b \notin \mathbb{Z} \bigwedge x \in \mathbb{R} \bigwedge x > 1$$

06.19.06.0038.01

$$B_z(a,b) \propto \frac{1}{\Gamma(1-b)\,\Gamma(a+b)} \left(-2\,e^{i\,b\,\pi \left\lfloor \frac{\arg(x-z)}{2\pi} \right\rfloor} i\,\pi \left\lfloor \frac{\arg(x-z)}{2\pi} \right\rfloor \Gamma(a) + \left(\frac{1}{1-x}\right)^{b \left\lfloor \frac{\arg(x-z)}{2\pi} \right\rfloor} (1-x)^{b \left\lfloor \frac{\arg(x-z)}{2\pi} \right\rfloor} G_{2,2}^{2,2}\!\left(1-x \,\middle|\, \begin{matrix} 1-a,\, b \\ 0,\, b \end{matrix}\right) x^a \right) + O(z-x) /;\ x \in \mathbb{R} \land x > 1$$

Expansions at $z = 0$

For the function itself

General case

06.19.06.0001.02

$$B_z(a,b) \propto \frac{z^a}{a} \left(1 + \frac{a\,(1-b)\,z}{a+1} + \frac{a\,(1-b)\,(2-b)\,z^2}{2\,(a+2)} + \ldots \right) /;\ (z \to 0) \bigwedge -a \notin \mathbb{N}$$

06.19.06.0039.01

$$B_z(a, b) \propto \frac{z^a}{a}\left(1+\frac{a(1-b)z}{a+1}+\frac{a(1-b)(2-b)z^2}{2(a+2)}+O(z^3)\right) /; -a \notin \mathbb{N}$$

06.19.06.0002.01

$$B_z(a, b) = z^a \sum_{k=0}^{\infty} \frac{(1-b)_k z^k}{(a+k)k!} /; |z| < 1 \bigwedge -a \notin \mathbb{N}$$

06.19.06.0003.01

$$B_z(a, b) = \frac{z^a}{a} {}_2F_1(a, 1-b; a+1; z)$$

06.19.06.0004.02

$$B_z(a, b) \propto \frac{z^a}{a}(1+O(z)) /; -a \notin \mathbb{N}$$

06.19.06.0040.01

$$B_z(a, b) = F_\infty(z, a, b) /; \left(\left(F_n(z) = z^a \sum_{k=0}^{n} \frac{(1-b)_k z^k}{(a+k)k!} = B_z(a, b) - \frac{z^{a+n+1}(1-b)_{n+1}}{(a+n+1)(n+1)!} {}_3F_2(1, a+n+1, -b+n+2; n+2, a+n+2; z)\right) \bigwedge n \in \mathbb{N}\right)$$

Summed form of the truncated series expansion.

Generic formulas for main term

06.19.06.0041.01

$$B_z(a, b) \propto \begin{cases} \tilde{\infty} & a = 0 \bigvee (-a \in \mathbb{N}^+ \bigwedge \neg (b \in \mathbb{N}^+ \bigwedge a+b \le 0)) \\ \frac{z^a}{a} & \text{True} \end{cases} /; (z \to 0)$$

Expansions at $z = 1$

For the function itself

General case

06.19.06.0005.02

$$B_z(a, b) \propto B(a, b) - \frac{(1-z)^b z^a}{b}\left(1-\frac{(a+b)(z-1)}{1+b}+\frac{(a+b)(1+a+b)(z-1)^2}{(1+b)(2+b)}-\ldots\right) /; (z \to 1) \bigwedge -b \notin \mathbb{N}$$

06.19.06.0042.01

$$B_z(a, b) \propto B(a, b) - \frac{(1-z)^b z^a}{b}\left(1-\frac{(a+b)(z-1)}{1+b}+\frac{(a+b)(1+a+b)(z-1)^2}{(1+b)(2+b)}+O\left((z-1)^3\right)\right) /; -b \notin \mathbb{N}$$

06.19.06.0006.01

$$B_z(a, b) = B(a, b) - \frac{(1-z)^b z^a}{b} \sum_{k=0}^{\infty} \frac{(-1)^k (a+b)_k (z-1)^k}{(b+1)_k} /; |z-1| < 1 \bigwedge -b \notin \mathbb{N}$$

06.19.06.0007.01

$$B_z(a, b) = B(a, b) - \frac{(1-z)^b z^a}{b} {}_2F_1(1, a+b; b+1; 1-z)$$

06.19.06.0008.02

$$B_z(a, b) \propto B(a, b) - \frac{(1-z)^b}{b}\left(1 + \frac{(a-1)\,b\,(z-1)}{1+b} + \frac{(a-1)\,(a-2)\,b\,(z-1)^2}{2\,(2+b)} + \ldots\right) /;\ (z \to 1) \bigwedge -b \notin \mathbb{N}$$

06.19.06.0043.01

$$B_z(a, b) \propto B(a, b) - \frac{(1-z)^b}{b}\left(1 + \frac{(a-1)\,b\,(z-1)}{1+b} + \frac{(a-1)\,(a-2)\,b\,(z-1)^2}{2\,(2+b)} + O\left((z-1)^3\right)\right) /;\ -b \notin \mathbb{N}$$

06.19.06.0009.01

$$B_z(a, b) = B(a, b) - \frac{\Gamma(b)\,(1-z)^b}{\Gamma(-a)\,\Gamma(a+b)} \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^{j+k}\,\Gamma(j-a)\,\Gamma(a+b+k)\,(z-1)^{j+k}}{j!\,\Gamma(b+k+1)} /;\ |z-1| < 1 \bigwedge -b \notin \mathbb{N}$$

06.19.06.0010.01

$$B_z(a, b) = B(a, b) - \frac{(1-z)^b}{b}\, F_{0\times 0\times 1}^{0\times 1\times 2}\left(\begin{matrix} ;\ -a;\ 1,\ a+b; \\ ;\ ;\ b+1; \end{matrix}\ 1-z,\ 1-z\right) /;\ -b \notin \mathbb{N}$$

06.19.06.0011.02

$$B_z(a, b) \propto B(a, b) - \frac{(1-z)^b}{b}\,(1 + O(z-1)) /;\ -b \notin \mathbb{N}$$

Logarithmic cases

06.19.06.0044.01

$$B_z(a, 0) \propto -\log(1-z) + \left(-\psi(a) - \gamma + (1-a)\,(z-1) + \frac{1}{4}\,(2-a)\,(a-1)\,(z-1)^2 + \ldots\right) /;\ (z \to 1)$$

06.19.06.0045.01

$$B_z(a, -1) \propto -\frac{1}{z-1}\left(1 + a\,(z-1) + \frac{a\,(a-1)}{2}\,(z-1)^2 + \ldots\right) + (a-1)\left(\psi(a) + \gamma + (a-1)\,(z-1) + \frac{(a-2)\,(a-1)}{4}\,(z-1)^2 + \ldots\right) + (a-1)\log(1-z) /;\ (z \to 1)$$

06.19.06.0046.01

$$B_z(a, -2) \propto \frac{1}{2\,(z-1)^2}\left(1 + 2\,(a-1)\,(z-1) - \frac{(5-3\,a)\,a}{2}\,(z-1)^2 + \ldots\right) + \frac{(a-2)\,(a-1)}{2}\left(-\psi(a) - \gamma + (1-a)\,(z-1) - \frac{(a-2)\,(a-1)}{4}\,(z-1)^2 + \ldots\right) - \frac{(a-2)\,(a-1)}{2}\log(1-z) /;\ (z \to 1)$$

06.19.06.0047.01

$$B_z(a, b) \propto \frac{(-1)^{b-1}\,\Gamma(a)}{(-b)!\,\Gamma(a+b)}\log(1-z) - \frac{(1-z)^b}{b}\left(1 + \frac{(a-1)\,b}{1+b}\,(z-1) + \frac{\left(2-3\,a+a^2\right)b}{2\,(2+b)}\,(z-1)^2 + \ldots\right) + \frac{(1-a)_{-b}}{(-b)!}\left(-\psi(a) - \gamma + (1-a)\,(z-1) - \frac{1}{4}\,((a-2)\,(a-1))\,(z-1)^2 + \ldots\right) /;\ (z \to 1) \bigwedge -b-3 \in \mathbb{N}$$

06.19.06.0048.01

$$B_z(a, b) \propto \frac{(-1)^{b-1}\,\Gamma(a)}{(-b)!\,\Gamma(a+b)}\log(1-z) - \frac{(1-z)^b}{b}\left(1 + \frac{(a-1)\,b}{1+b}\,(z-1) + \frac{\left(2-3\,a+a^2\right)b}{2\,(2+b)}\,(z-1)^2 + O\left((z-1)^3\right)\right) + \frac{(1-a)_{-b}}{(-b)!}\left(-\psi(a) - \gamma + (1-a)\,(z-1) - \frac{1}{4}\,((a-2)\,(a-1))\,(z-1)^2 + O\left((z-1)^3\right)\right) /;\ -b-3 \in \mathbb{N}$$

06.19.06.0012.01

$$B_z(a,b) = -(1-z)^b z^a \sum_{k=0}^{-b-1} \frac{(a+b)_k (1-z)^k}{(b)_{k+1}} + (1-a)_{-b} z^a \sum_{k=0}^{\infty} \frac{(a)_k (1-b)_k (\psi(k+1) - \psi(a+k)) (1-z)^k}{k!\,(k-b)!} + \frac{(-1)^{b-1}\Gamma(a)}{(-b)!\,\Gamma(a+b)} \log(1-z) \;/;\; |z-1| < 1 \land -b \in \mathbb{N}$$

06.19.06.0013.02

$$B_z(a,b) \propto \frac{(-1)^{b-1}\Gamma(a)}{(-b)!\,\Gamma(a+b)} \log(1-z) - \frac{(1-z)^b}{b} (1 + O(z-1)) \;/;\; -b \in \mathbb{N}^+$$

06.19.06.0014.02

$$B_z(a,0) \propto -\log(1-z) - (\psi(a) + \gamma)(1 + O(z-1))$$

Generic formulas for main term

06.19.06.0049.01

$$B_z(a,b) \propto \begin{cases} -\log(1-z) - \psi(a) - \gamma & b = 0 \\ \frac{(-1)^{b-1}\Gamma(a)\log(1-z)}{(-b)!\,\Gamma(a+b)} - \frac{(1-z)^b}{b} & -b \in \mathbb{N}^+ \\ B(a,b) - \frac{(1-z)^b}{b} & \text{True} \end{cases} \;/;\; (z \to 1)$$

06.19.06.0050.01

$$B_z(a,b) \propto \begin{cases} -\log(1-z) & b = 0 \\ B(a,b) & \operatorname{Re}(b) > 0 \\ -\frac{(1-z)^b}{b} & \operatorname{Re}(b) < 0 \\ B(a,b) - \frac{(1-z)^b}{b} & \text{True} \end{cases} \;/;\; (z \to 1)$$

Expansions at $z = \infty$

General case

06.19.06.0015.02

$$B_z(a,b) \propto \frac{\Gamma(a)\Gamma(1-a-b) z^a (-z)^{-a}}{\Gamma(1-b)} + \frac{(-z)^{b-1} z^a}{a+b-1} \left(1 + \frac{(1-b)(1-a-b)}{(2-a-b) z} + \frac{(1-b)(2-b)(1-a-b)}{2(3-a-b) z^2} + \ldots\right) \;/;\; (|z| \to \infty) \land a+b \notin \mathbb{N}^+$$

06.19.06.0051.01

$$B_z(a,b) \propto \frac{\Gamma(a)\Gamma(1-a-b) z^a (-z)^{-a}}{\Gamma(1-b)} + \frac{(-z)^{b-1} z^a}{a+b-1} \left(1 + \frac{(1-b)(1-a-b)}{(2-a-b) z} + \frac{(1-b)(2-b)(1-a-b)}{2(3-a-b) z^2} + O\left(\frac{1}{z^3}\right)\right) \;/;\; (|z| \to \infty) \land a+b \notin \mathbb{N}^+$$

06.19.06.0016.01

$$B_z(a,b) = \frac{\Gamma(a)\Gamma(1-a-b) z^a (-z)^{-a}}{\Gamma(1-b)} + \frac{z^a (-z)^{b-1}}{a+b-1} \sum_{k=0}^{\infty} \frac{(1-b)_k (1-a-b)_k z^{-k}}{(2-a-b)_k\, k!} \;/;\; |z| > 1 \land a+b \notin \mathbb{N}^+$$

06.19.06.0017.01

$$B_z(a,b) = \frac{\Gamma(a)\Gamma(1-a-b) z^a (-z)^{-a}}{\Gamma(1-b)} + \frac{z^a (-z)^{b-1}}{a+b-1} {}_2F_1\left(1-b, 1-a-b; 2-a-b; \frac{1}{z}\right) \;/;\; z \notin (0,1) \land a+b \notin \mathbb{N}^+$$

06.19.06.0018.02

$$B_z(a,\,b) \propto \frac{\Gamma(a)\,\Gamma(1-a-b)}{\Gamma(1-b)}\,z^a(-z)^{-a} + \frac{z^a\,(-z)^{b-1}}{a+b-1}\left(1+O\left(\frac{1}{z}\right)\right) /;\ a+b \notin \mathbb{N}^+$$

06.19.06.0052.01

$$B_z(a,\,b) \propto \begin{cases} -\frac{e^{i\,b\,\pi}\,z^{a+b-1}}{a+b-1} + \frac{e^{-i\,a\,\pi}\,\Gamma(a)\,\Gamma(-a-b+1)}{\Gamma(1-b)} & \arg(z) \le 0 \\ \frac{e^{i\,a\,\pi}\,\Gamma(a)\,\Gamma(-a-b+1)}{\Gamma(1-b)} - \frac{e^{-i\,b\,\pi}\,z^{a+b-1}}{a+b-1} & \text{True} \end{cases} /;\ a+b \notin \mathbb{N}^+ \bigwedge (|z| \to \infty)$$

06.19.06.0053.01

$$B_z(a,\,b) = F_\infty(z,\,a,\,b) /;$$

$$\left(\left(F_n(z,\,a,\,b) = \frac{z^a(-z)^{b-1}}{a+b-1}\sum_{k=0}^{n}\frac{(1-b)_k\,(1-a-b)_k\,z^{-k}}{(2-a-b)_k\,k!} + B(a,\,1-a-b)\,z^a\,(-z)^{-a} = B_z(a,\,b) + \frac{(1-b)_{n+1}\,(-z)^{b-1}\,z^{a-n-1}}{(n+1)!\,(2-a-b+n)}\right.\right.$$
$$\left.\left.{}_3F_2\left(1,\,2-b+n,\,2-a-b+n;\,n+2,\,3-a-b+n;\,\frac{1}{z}\right)\right) \bigwedge n \in \mathbb{N}\right) \bigwedge a+b \notin \mathbb{N}^+$$

Summed form of the truncated series expansion.

Logarithmic cases

06.19.06.0054.01

$$B_z(a,\,1-a) \propto (-z)^{-a}\,z^a\,(\log(-z)-\psi(a)-\gamma) - a\,(-z)^{-a}\,z^{a-1}\left(1+\frac{1+a}{4\,z}+\frac{(1+a)\,(2+a)}{18\,z^2}+\ldots\right) /;\ (|z| \to \infty)$$

06.19.06.0055.01

$$B_z(a,\,2-a) \propto z^a\,(-z)^{1-a} + \frac{a\,(a-1)}{2}\,(-z)^{-a}\,z^{-1+a}\left(1+\frac{1+a}{6\,z}+\frac{(1+a)\,(2+a)}{36\,z^2}+\ldots\right) + (1-a)\,(\log(-z)-\psi(a)-\gamma+1)\,z^a\,(-z)^{-a} /;$$
$$(|z| \to \infty)$$

06.19.06.0056.01

$$B_z(a,\,-a+n+1) \propto \frac{(-a)_{n+1}\,(-z)^{-a}\,z^{a-1}}{(n+1)!}\left(1+\frac{1+a}{2\,(2+n)\,z}+\frac{(1+a)\,(2+a)}{3\,(2+n)\,(3+n)\,z^2}+\ldots\right) +$$
$$\frac{(1-a)_n\,(-z)^{-a}\,z^a}{n!}\,(\log(-z)+\psi(n+1)-\psi(a)) + (-z)^{n-a}\,z^a\sum_{k=0}^{n-1}\frac{(a-n)_k\,z^{-k}}{(n-k)\,k!} /;\ (|z| \to \infty) \land n \in \mathbb{N}$$

06.19.06.0057.01

$$B_z(a,\,1-a) = (-z)^{-a}\,z^a\,(\log(-z)-\psi(a)-\gamma) - a\,(-z)^{-a}\,z^{a-1}\,{}_3F_2\left(1,\,1,\,a+1;\,2,\,2;\,\frac{1}{z}\right) /;\ z \notin (0,\,1)$$

06.19.06.0058.01

$$B_z(a,\,-a+n+1) = \frac{(-a)_{n+1}\,(-z)^{-a}\,z^{a-1}}{(n+1)!}\sum_{k=0}^{\infty}\frac{(a+1)_k\,z^{-k}}{(n+2)_k\,(k+1)} +$$
$$\frac{(1-a)_n\,(-z)^{-a}\,z^a}{n!}\,(\log(-z)+\psi(n+1)-\psi(a)) + (-z)^{n-a}\,z^a\sum_{k=0}^{n-1}\frac{(a-n)_k\,z^{-k}}{(n-k)\,k!} /;\ n \in \mathbb{N} \land |z| > 1$$

06.19.06.0019.02

$$B_z(a, -a+n+1) \propto \frac{(-a)_{n+1} (-z)^{-a} z^{a-1}}{(n+1)!} \, {}_3F_2\left(1, 1, a+1; 2, n+2; \frac{1}{z}\right) + \frac{(1-a)_n (-z)^{-a} z^a}{n!} (\log(-z) + \psi(n+1) - \psi(a)) + (-z)^{n-a} z^a \sum_{k=0}^{n-1} \frac{(a-n)_k z^{-k}}{(n-k)\, k!} \;/;\; n \in \mathbb{N} \bigwedge z \notin (0, 1)$$

06.19.06.0020.02

$$B_z(a, b) \propto \frac{(1-a)_{a+b-1} z^a (-z)^{-a}}{(a+b-1)!} (\log(-z) - \psi(a) + \psi(a+b)) + \frac{z^a (-z)^{b-1}}{a+b-1} \left(1 + O\left(\frac{1}{z}\right)\right) /;\; a+b-1 \in \mathbb{N}^+$$

06.19.06.0021.02

$$B_z(a, b) \propto (-z)^{-a} z^a (\log(-z) - \psi(a) - \gamma) - a (-z)^{-a} z^{a-1} \left(1 + O\left(\frac{1}{z}\right)\right) /;\; a+b = 1$$

06.19.06.0059.01

$$B_z(a, b) \propto \begin{cases} \frac{e^{-i a \pi} \Gamma(b) \log(z)}{\Gamma(1-a)\Gamma(a+b)} - \frac{e^{i b \pi} z^{a+b-1}}{a+b-1} & \arg(z) \leq 0 \\ \frac{e^{i a \pi} \Gamma(b) \log(z)}{\Gamma(1-a)\Gamma(a+b)} - \frac{e^{-i b \pi} z^{a+b-1}}{a+b-1} & \text{True} \end{cases} /;\; a+b \notin \mathbb{N}^+ \bigwedge (|z| \to \infty)$$

06.19.06.0060.01

$$B_z(a, b) \propto \begin{cases} e^{-i a \pi} \log(z) & \arg(z) \leq 0 \\ e^{i a \pi} \log(z) & \text{True} \end{cases} /;\; a+b = 1 \bigwedge (|z| \to \infty)$$

Generic formulas for main term

06.19.06.0061.01

$$B_z(a, b) \propto \begin{cases} \frac{z^a (-z)^{b-1}}{a+b-1} + \frac{z^a (1-a)_{a+b-1} (\log(-z) - \psi(a) + \psi(a+b)) (-z)^{-a}}{(a+b-1)!} & a+b-1 \in \mathbb{N}^+ \\ (\log(-z) - \psi(a) - \gamma)(-z)^{-a} z^a & a+b = 1 \\ \frac{(-1)^{b-1} z^{a+b-1}}{a+b-1} & -a \in \mathbb{N}^+ \bigwedge b \in \mathbb{N}^+ \bigwedge a+b \leq 0 \\ \tilde{\infty} & a = 0 \bigvee (-a \in \mathbb{N}^+ \bigwedge b \in \mathbb{N}^+ \bigwedge a+b > 0) \\ \frac{z^a \Gamma(a) \Gamma(1-a-b) (-z)^{-a}}{\Gamma(1-b)} + \frac{z^a (-z)^{b-1}}{a+b-1} & \text{True} \end{cases} /;\; (|z| \to \infty)$$

06.19.06.0062.01

$$B_z(a, b) \propto \begin{cases} (-z)^{-a} z^a \log(-z) & a+b = 1 \\ \tilde{\infty} & a = 0 \bigvee (-a \in \mathbb{N}^+ \bigwedge b \in \mathbb{N}^+ \bigwedge a+b > 0) \\ \frac{\Gamma(a)\Gamma(1-a-b) z^a (-z)^{-a}}{\Gamma(1-b)} & \mathrm{Re}(a+b) < 1 \\ \frac{z^a (-z)^{b-1}}{a+b-1} & \mathrm{Re}(a+b) > 1 \bigvee (-a \in \mathbb{N}^+ \bigwedge b \in \mathbb{N}^+ \bigwedge a+b \leq 0) \\ \frac{z^a \Gamma(a) \Gamma(1-a-b) (-z)^{-a}}{\Gamma(1-b)} + \frac{z^a (-z)^{b-1}}{a+b-1} & \text{True} \end{cases} /;\; (|z| \to \infty)$$

06.19.06.0063.01

$$B_z(a, b) \propto \begin{cases} \frac{(-1)^{b-1} z^{a+b-1}}{a+b-1} & -a \in \mathbb{N}^+ \land b \in \mathbb{N}^+ \land a+b \leq 0 \\ \tilde{\infty} & a = 0 \lor (-a \in \mathbb{N}^+ \land b \in \mathbb{N}^+ \land a+b > 0) \\ e^{-i a \pi} \left(\frac{e^{i (a+b) \pi} z^{a+b-1}}{1-a-b} - \frac{(1-a)_{a+b-1} (i \pi + \log(z) - \psi(a) + \psi(a+b))}{(a+b-1)!} \right) & \arg(z) \leq 0 \land a+b-1 \in \mathbb{N}^+ \\ e^{-i a \pi} (\log(z) - \psi(a) + i \pi - \gamma) & \arg(z) \leq 0 \land a+b = 1 \\ \frac{e^{i b \pi} z^{a+b-1}}{1-a-b} + \frac{e^{-i a \pi} \Gamma(a) \Gamma(-a-b+1)}{\Gamma(1-b)} & \arg(z) \leq 0 \\ \frac{e^{-i b \pi} z^{a+b-1}}{1-a-b} + \frac{e^{i a \pi} (1-a)_{a+b-1} (\pi i - \log(z) + \psi(a) - \psi(a+b))}{(a+b-1)!} & \arg(z) > 0 \land a+b-\mathbb{N}^+ \\ e^{i a \pi} (\log(z) - \psi(a) - i \pi - \gamma) & \arg(z) > 0 \land a+b = 1 \\ \frac{e^{-i b \pi} z^{a+b-1}}{1-a-b} + \frac{e^{i a \pi} \Gamma(a) \Gamma(-a-b+1)}{\Gamma(1-b)} & \text{True} \end{cases} \quad /; (|z| \to \infty)$$

06.19.06.0064.01

$$B_z(a, b) \propto \begin{cases} \tilde{\infty} & a = 0 \lor (-a \in \mathbb{N}^+ \land b \in \mathbb{N}^+ \land a+b > 0) \\ \frac{\log(z)}{e^{i a \pi}} & \arg(z) \leq 0 \land a+b = 1 \\ \frac{\Gamma(a) \Gamma(1-a-b)}{e^{i a \pi} \Gamma(1-b)} & \arg(z) \leq 0 \land \operatorname{Re}(a+b) < 1 \\ \frac{e^{i b \pi} z^{a+b-1}}{1-a-b} & (\arg(z) \leq 0 \land \operatorname{Re}(a+b) > 1) \lor (-a \in \mathbb{N}^+ \land b \in \mathbb{N}^+ \land a+b \leq 0) \\ \frac{e^{i b \pi} z^{a+b-1}}{1-a-b} + \frac{\Gamma(a) \Gamma(1-a-b)}{e^{i a \pi} \Gamma(1-b)} & \arg(z) \leq 0 \\ e^{i a \pi} \log(z) & \arg(z) > 0 \land a+b = 1 \\ \frac{e^{i a \pi} \Gamma(a) \Gamma(1-a-b)}{\Gamma(1-b)} & \arg(z) > 0 \land \operatorname{Re}(a+b) < 1 \\ \frac{z^{a+b-1}}{e^{i b \pi} (1-a-b)} & (\arg(z) > 0 \land \operatorname{Re}(a+b) > 1) \lor (-a \in \mathbb{N}^+ \land b \in \mathbb{N}^+ \land a+b \leq 0) \\ \frac{z^{a+b-1}}{e^{i b \pi} (1-a-b)} + \frac{e^{i a \pi} \Gamma(a) \Gamma(1-a-b)}{\Gamma(1-b)} & \text{True} \end{cases} \quad /; (|z| \to \infty)$$

Expansions at generic point $a == a_0$

For the function itself

06.19.06.0065.01

$$B_z(a, b) \propto B_z(a_0, b) + \frac{1}{a_0^2} \left(\log(z)\, a_0^2\, B_z(a_0, b) - z^{a_0}\, {}_3F_2(1-b, a_0, a_0; a_0+1, a_0+1; z) \right) (a - a_0) + \frac{1}{2 a_0^3} \left(2 z^{a_0} \left({}_4F_3(1-b, a_0, a_0, a_0; a_0+1, a_0+1, a_0+1; z) - a_0\, {}_3F_2(1-b, a_0, a_0; a_0+1, a_0+1; z) \log(z) \right) + \log^2(z)\, a_0^3\, B_z(a_0, b) \right) (a - a_0)^2 + \ldots /; (a \to a_0)$$

06.19.06.0066.01

$$B_z(a, b) \propto B_z(a_0, b) + \frac{1}{a_0^2} \left(\log(z)\, a_0^2\, B_z(a_0, b) - z^{a_0}\, {}_3F_2(1-b, a_0, a_0; a_0+1, a_0+1; z) \right) (a - a_0) + \frac{1}{2 a_0^3} \left(2 z^{a_0} \left({}_4F_3(1-b, a_0, a_0, a_0; a_0+1, a_0+1, a_0+1; z) - a_0\, {}_3F_2(1-b, a_0, a_0; a_0+1, a_0+1; z) \log(z) \right) + \log^2(z)\, a_0^3\, B_z(a_0, b) \right) (a - a_0)^2 + O\left((a - a_0)^3 \right)$$

06.19.06.0067.01

$$B_z(a,b) == \Gamma(a_0)\, z^{a_0} \sum_{k=0}^{\infty} \log^k(z) \sum_{j=0}^{k} \frac{1}{(k-j)!}\, {}_{j+2}\tilde{F}_{j+1}\big(1-b, c_1, c_2, \ldots, c_{j+1}; c_1+1, c_2+1, \ldots, c_{j+1}+1; z\big) \left(-\frac{\Gamma(a_0)}{\log(z)}\right)^j (a-a_0)^k \,/;\, c_1 == c_2 == \ldots == c_{k+1} == a_0 \bigwedge k \in \mathbb{N}$$

06.19.06.0068.01

$$B_z(a,b) \propto B_z(a_0,b)\,(1+O(a-a_0))$$

Expansions at generic point $b == b_0$

For the function itself

06.19.06.0069.01

$$B_z(a,b) \propto B_z(a,b_0) + \big(\Gamma(b_0)^2\, {}_3\tilde{F}_2(1-a, b_0, b_0; b_0+1, b_0+1; 1-z)\,(1-z)^{b_0} - B_{1-z}(b_0,a)\log(1-z) + B(a,b_0)\,(\psi(b_0)-\psi(a+b_0))\big)(b-b_0) + \frac{1}{2}\Bigg(-B_{1-z}(b_0,a)\log^2(1-z) + \big((\psi(b_0)-\psi(a+b_0))^2 + \psi^{(1)}(b_0) - \psi^{(1)}(a+b_0)\big)B(a,b_0) + \frac{2(1-z)^{b_0}}{b_0^3}\big(b_0\log(1-z)\, {}_3F_2(1-a, b_0, b_0; b_0+1, b_0+1; 1-z) - {}_4F_3(1-a, b_0, b_0, b_0; b_0+1, b_0+1, b_0+1; 1-z)\big)\Bigg)(b-b_0)^2 + \ldots \,/;\, (b \to b_0)$$

06.19.06.0070.01

$$B_z(a,b) \propto B_z(a,b_0) + \big(\Gamma(b_0)^2\, {}_3\tilde{F}_2(1-a, b_0, b_0; b_0+1, b_0+1; 1-z)\,(1-z)^{b_0} - B_{1-z}(b_0,a)\log(1-z) + B(a,b_0)\,(\psi(b_0)-\psi(a+b_0))\big)(b-b_0) + \frac{1}{2}\Bigg(-B_{1-z}(b_0,a)\log^2(1-z) + \big((\psi(b_0)-\psi(a+b_0))^2 + \psi^{(1)}(b_0) - \psi^{(1)}(a+b_0)\big)B(a,b_0) + \frac{2(1-z)^{b_0}}{b_0^3}\big(b_0\log(1-z)\, {}_3F_2(1-a, b_0, b_0; b_0+1, b_0+1; 1-z) - {}_4F_3(1-a, b_0, b_0, b_0; b_0+1, b_0+1, b_0+1; 1-z)\big)\Bigg)(b-b_0)^2 + O\big((b-b_0)^3\big)$$

06.19.06.0071.01

$$B_z(a,b) == \sum_{k=0}^{\infty} \Bigg((-1)^k\, \Gamma(b_0)^{k+1}\, {}_{k+2}\tilde{F}_{k+1}(1-a, c_1, c_2, \ldots, c_{k+1}; c_1+1, c_2+1, \ldots, c_{k+1}+1; 1) - \Gamma(b_0)\,(1-z)^{b_0}\log^k(1-z) \sum_{j=0}^{k} \frac{1}{(k-j)!}\, {}_{j+2}\tilde{F}_{j+1}\big(1-a, c_1, c_2, \ldots, c_{j+1}; c_1+1, c_2+1, \ldots, c_{j+1}+1; 1-z\big)\left(-\frac{\Gamma(b_0)}{\log(1-z)}\right)^j\Bigg)(b-b_0)^k \,/;\, c_1 == c_2 == \ldots == c_{k+1} == b_0 \bigwedge k \in \mathbb{N}$$

06.19.06.0072.01

$$B_z(a,b) \propto B_z(a,b_0)\,(1+O(b-b_0))$$

Residue representations

06.19.06.0022.01

$$B_z(a,b)=\frac{z^a}{\Gamma(1-b)}\sum_{j=0}^{\infty}\text{res}_s\left(\frac{\Gamma(1-b-s)(-z)^{-s}}{a-s}\Gamma(s)\right)(-j)\,/;\,|z|<1$$

06.19.06.0023.01

$$B_z(a,b)=-\frac{z^a}{\Gamma(1-b)}\left(\sum_{j=0}^{\infty}\text{res}_s\left(\Gamma(s)(-z)^{-s}\frac{\Gamma(-b-s+1)}{a-s}\right)(a)+\sum_{j=0}^{\infty}\text{res}_s\left(\Gamma(s)(-z)^{-s}\frac{\Gamma(-b-s+1)}{a-s}\right)(j-b+1)\right)/;$$
$$|z|>1\bigwedge a+b\notin\mathbb{N}^+$$

06.19.06.0073.01

$$B_z(a,n+1-a)=-\frac{z^a}{\Gamma(a-n)}\left(\text{res}_s\left(\Gamma(s)(-z)^{-s}\frac{\Gamma(a-n-s)}{a-s}\right)(a)+\right.$$
$$\left.\sum_{j=0}^{n-1}\text{res}_s\left(\frac{\Gamma(s)(-z)^{-s}}{a-s}\Gamma(a-n-s)\right)(a+j-n)+\sum_{j=n+1}^{\infty}\text{res}_s\left(\frac{\Gamma(s)(-z)^{-s}}{a-s}\Gamma(a-n-s)\right)(a+j-n)\right)/;\,|z|>1\bigwedge n\notin\mathbb{N}$$

Integral representations

On the real axis

Of the direct function

06.19.07.0001.01

$$B_z(a,b)=\int_0^z t^{a-1}(1-t)^{b-1}\,dt\,/;\,\text{Re}(a)>0$$

06.19.07.0002.01

$$B_z(a,b)=\int_0^z t^{a-1}\left((1-t)^{b-1}-\sum_{k=0}^{\lfloor-\text{Re}(a)\rfloor}\frac{(1-b)_k z^k}{k!}\right)dt+\sum_{k=0}^{\lfloor-\text{Re}(a)\rfloor}\frac{(1-b)_k z^{a+k}}{(a+k)\,k!}$$

Contour integral representations

06.19.07.0003.01

$$B_z(a,b)=\frac{z^a}{2\pi i\,\Gamma(1-b)}\int_{\mathcal{L}}\frac{\Gamma(s)\Gamma(a-s)\Gamma(1-b-s)}{\Gamma(a+1-s)}(-z)^{-s}\,ds\,/;\,|\arg(-z)|<\pi$$

06.19.07.0004.01

$$B_z(a,b)=\frac{z^a}{2\pi i\,\Gamma(1-b)\Gamma(a+b)}\int_{\mathcal{L}}\Gamma(s)\Gamma(b+s)\Gamma(a-s)\Gamma(1-b-s)(1-z)^{-s}\,ds\,/;\,|\arg(1-z)|<\pi$$

06.19.07.0005.01

$$B_z(a,b)=\frac{z^a}{2\pi i\,\Gamma(1-b)}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\Gamma(s)\Gamma(a-s)\Gamma(1-b-s)}{\Gamma(a+1-s)}(-z)^{-s}\,ds\,/;\,0<\gamma<\min(\text{Re}(a),1-\text{Re}(b))\bigwedge|\arg(-z)|<\pi$$

06.19.07.0006.01

$$B_z(a,b)=\frac{z^a}{2\pi i\,\Gamma(1-b)\Gamma(a+b)}\int_{\gamma-i\infty}^{\gamma+i\infty}\Gamma(s)\Gamma(b+s)\Gamma(a-s)\Gamma(1-b-s)(1-z)^{-s}\,ds\,/;$$
$$\max(0,-\text{Re}(b))<\gamma<\min(\text{Re}(a),1-\text{Re}(b))\bigwedge|\arg(1-z)|<\pi$$

Continued fraction representations

06.19.10.0001.01

$$B_z(a, b) = \frac{z^a (1-z)^b}{a} \frac{1}{1+\frac{r(1)}{1+\frac{r(2)}{1+\frac{r(3)}{1+\frac{r(4)}{1+\frac{r(5)}{1+\ldots}}}}}} /; r(2k+1) = -\frac{(a+k)(a+b+k)z}{(a+2k)(a+2k+1)} \bigwedge r(2k) = \frac{k(b-k)z}{(a+2k-1)(a+2k)}$$

06.19.10.0002.01

$$B_z(a, b) = \frac{z^a (1-z)^b}{a\left(1+K_k(r(k), 1)_1^\infty\right)} /; r(2k+1) = -\frac{(a+k)(a+b+k)z}{(a+2k)(a+2k+1)} \bigwedge r(2k) = \frac{k(b-k)z}{(a+2k-1)(a+2k)}$$

06.19.10.0003.01

$$B_z(a, b) = \frac{z^a (1-z)^b}{a} \frac{1}{1+\frac{r(1)}{1+\frac{r(2)}{1+\frac{r(3)}{1+\frac{r(4)}{1+\frac{r(5)}{1+\ldots}}}}}} /; r(2k+1) = -\frac{(a+k)(a+b+k)z}{(a+2k)(a+2k+1)} \bigwedge r(2k) = \frac{k(b-k)z}{(a+2k-1)(a+2k)}$$

06.19.10.0004.01

$$B_z(a, b) = \frac{z^a (1-z)^b}{a\left(K_k(r(k), 1)_1^\infty + 1\right)} /; r(2k+1) = -\frac{(a+k)(a+b+k)z}{(a+2k)(a+2k+1)} \bigwedge r(2k) = \frac{k(b-k)z}{(a+2k-1)(a+2k)}$$

Differential equations

Ordinary linear differential equations and wronskians

For the direct function itself

06.19.13.0001.01

$$(1-z)\, z\, w''(z) + (1-a+(a+b-2)z)\, w'(z) = 0 /; w(z) = c_1 B_z(a, b) + c_2$$

06.19.13.0002.01

$$W_z(1, B_z(a, b)) = (1-z)^{b-1} z^{a-1}$$

06.19.13.0003.01

$$w''(z) - \left(\frac{(-a+(a+b-2)g(z)+1)\, g'(z)}{(g(z)-1)\, g(z)} + \frac{g''(z)}{g'(z)}\right) w'(z) = 0 /; w(z) = c_1 + c_2 B_{g(z)}(a, b)$$

06.19.13.0004.01

$$W_z\left(1, B_{g(z)}(a, b)\right) = (1-g(z))^{b-1} g(z)^{a-1} g'(z)$$

06.19.13.0005.01

$$w''(z) - h(z)\left(\frac{(-a+(a+b-2)\,g(z)+1)\,g'(z)}{(g(z)-1)\,g(z)\,h(z)} + \frac{2\,h'(z)}{h(z)^2} + \frac{g''(z)}{h(z)\,g'(z)}\right)w'(z) + \left(\frac{2\,h'(z)^2}{h(z)^2} + \frac{(-a+(a+b-2)\,g(z)+1)\,g'(z)\,h'(z)}{(g(z)-1)\,g(z)\,h(z)} + \frac{g''(z)\,h'(z)}{h(z)\,g'(z)} - \frac{h''(z)}{h(z)}\right)w(z) = 0\;/;\; w(z) = c_1\,h(z) + c_2\,h(z)\,\mathrm{B}_{g(z)}(a,\,b)$$

06.19.13.0006.01

$$W_z\big(h(z),\,h(z)\,\mathrm{B}_{g(z)}(a,\,b)\big) = (1-g(z))^{b-1}\,g(z)^{a-1}\,h(z)^2\,g'(z)$$

06.19.13.0007.01

$$(d\,z^r-1)\,z^2\,w''(z) + (-d\,((a+b-1)\,r+2\,s-1)\,z^r + a\,r+2\,s-1)\,z\;w'(z) + s\,((b-1)\,d\,r\,z^r + (a\,r+s)\,(d\,z^r-1))\,w(z) = 0\;/;\; w(z) = c_1\,z^s + c_2\,z^s\,\mathrm{B}_{d\,z^r}(a,\,b)$$

06.19.13.0008.01

$$W_z\big(z^s,\,z^s\,\mathrm{B}_{d\,z^r}(a,\,b)\big) = r\,z^{2\,s-1}\,(d\,z^r)^a\,(1-d\,z^r)^{b-1}$$

06.19.13.0009.01

$$(d\,r^z-1)\,w''(z) + (-d\,((a+b-1)\log(r)+2\log(s))\,r^z + a\log(r)+2\log(s))\,w'(z) + \log(s)\,(d\,((a+b-1)\log(r)+\log(s))\,r^z - a\log(r)-\log(s))\,w(z) = 0\;/;\; w(z) = c_1\,s^z + c_2\,s^z\,\mathrm{B}_{d\,r^z}(a,\,b)$$

06.19.13.0010.01

$$W_z\big(s^z,\,s^z\,\mathrm{B}_{d\,r^z}(a,\,b)\big) = d\,r^z\,(d\,r^z)^{a-1}\,(1-d\,r^z)^{b-1}\,s^{2\,z}\log(r)$$

Transformations

Transformations and argument simplifications

Argument involving basic arithmetic operations

06.19.16.0001.01

$$\mathrm{B}_{1-z}(a,\,b) = \mathrm{B}(a,\,b) - \mathrm{B}_z(b,\,a)$$

06.19.16.0002.01

$$\mathrm{B}_z(a+1,\,b) = \frac{1}{a+b}\left(a\,\mathrm{B}_z(a,\,b) - (1-z)^b\,z^a\right)$$

06.19.16.0003.01

$$\mathrm{B}_z(a-1,\,b) = \frac{1}{a-1}\left((1-z)^b\,z^{a-1} + (a+b-1)\,\mathrm{B}_z(a,\,b)\right)$$

06.19.16.0004.01

$$\mathrm{B}_z(a+n,\,b) = \frac{(a)_n}{(a+b)_n}\,\mathrm{B}_z(a,\,b) - \frac{1}{a+b+n-1}\sum_{k=0}^{n-1}\frac{(1-a-n)_k\,(1-z)^b\,z^{a-k+n-1}}{(2-a-b-n)_k}\;/;\; n\in\mathbb{N}$$

06.19.16.0005.01

$$\mathrm{B}_z(a-n,\,b) = \frac{(1-a-b)_n}{(1-a)_n}\,\mathrm{B}_z(a,\,b) - \frac{(2-a-b)_{n-1}}{(1-a)_n}\sum_{k=0}^{n-1}\frac{(1-a)_k\,(1-z)^b\,z^{a-k-1}}{(2-a-b)_k}\;/;\; n\in\mathbb{N}$$

Identities

Recurrence identities

Consecutive neighbors

06.19.17.0001.01

$$B_z(a, b) = \frac{1}{a}\left((1-z)^b z^a + (a+b) B_z(a+1, b)\right)$$

06.19.17.0002.01

$$B_z(a, b) = \frac{1}{a+b-1}\left((a-1) B_z(a-1, b) - (1-z)^b z^{a-1}\right)$$

Distant neighbors

06.19.17.0003.01

$$B_z(a, b) = \frac{(a+b)_n}{(a)_n} B_z(a+n, b) + \frac{z^{a-1}(1-z)^b}{a+b-1} \sum_{k=1}^{n} \frac{(a+b-1)_k z^k}{(a)_k} /; n \in \mathbb{N}$$

06.19.17.0004.01

$$B_z(a, b) = \frac{(1-a)_n}{(1-a-b)_n} B_z(a-n, b) - \frac{z^{a-1}(1-z)^b}{a+b-1} \sum_{k=0}^{n-1} \frac{(1-a)_k z^{-k}}{(2-a-b)_k} /; n \in \mathbb{N}$$

Functional identities

Relations between contiguous functions

06.19.17.0005.01

$$B_z(a, b) = B_z(a+1, b) + B_z(a, b+1)$$

06.19.17.0006.01

$$B_z(a, b) = \frac{1}{a+b-1}\left((a-1) z B_z(a-1, b) + (b-1)(1-z) B_z(a, b-1)\right)$$

Additional relations between contiguous functions

06.19.17.0007.01

$$B_z(a, b) = \frac{1}{a+b-a z}\left((b-1)(1-z) B_z(a+1, b-1) + (a+b) B_z(a, b+1)\right)$$

Major general cases

06.19.17.0008.01

$$B_z(a, b) = B(a, b) - B_{1-z}(b, a)$$

06.19.17.0009.01

$$B_z(a, b) = (-z)^{-a} z^a \left(\left(\frac{1}{z}\right)^{a+b} (-z)^{a+b} B_{\frac{1}{z}}(1-a-b, b) + B(a, 1-a-b)\right) /; a+b \notin \mathbb{N}^+ \bigwedge z \notin (0, 1)$$

Differentiation

Low-order differentiation

With respect to *z*

06.19.20.0001.01

$$\frac{\partial B_z(a, b)}{\partial z} == (1-z)^{b-1} z^{a-1}$$

06.19.20.0002.01

$$\frac{\partial^2 B_z(a, b)}{\partial z^2} == -(1-z)^{b-2} z^{a-2} ((z-1) a + (b-2) z + 1)$$

With respect to *a*

06.19.20.0003.01

$$\frac{\partial B_z(a, b)}{\partial a} == B_z(a, b) \log(z) - z^a \Gamma(a)^2 \, {}_3\tilde{F}_2(a, a, 1-b; a+1, a+1; z)$$

06.19.20.0004.01

$$\frac{\partial^2 B_z(a, b)}{\partial a^2} == 2 \Gamma(a)^2 \left(\Gamma(a) \, {}_4\tilde{F}_3(a, a, a, 1-b; a+1, a+1, a+1; z) - {}_3\tilde{F}_2(a, a, 1-b; a+1, a+1; z) \log(z)\right) z^a + B_z(a, b) \log^2(z)$$

With respect to *b*

06.19.20.0005.01

$$\frac{\partial B_z(a, b)}{\partial b} == \Gamma(b)^2 (1-z)^b \, {}_3\tilde{F}_2(b, b, 1-a; b+1, b+1; 1-z) - \log(1-z) B_{1-z}(b, a) + (\psi(b) - \psi(a+b)) B(a, b)$$

06.19.20.0006.01

$$\frac{\partial^2 B_z(a, b)}{\partial b^2} == B(a, b) \left((\psi(b) - \psi(a+b))^2 + \psi^{(1)}(b) - \psi^{(1)}(a+b)\right) - \left(2 \Gamma(b)^2 \left(\Gamma(b) \, {}_4\tilde{F}_3(b, b, b, 1-a; b+1, b+1, b+1; 1-z) - {}_3\tilde{F}_2(b, b, 1-a; b+1, b+1; 1-z) \log(1-z)\right) (1-z)^b + B_{1-z}(b, a) \log^2(1-z)\right)$$

Symbolic differentiation

With respect to *z*

06.19.20.0017.01

$$\frac{\partial^n B_z(a, b)}{\partial z^n} == \delta_n B_z(a, b) - (1-z)^{b-1} z^{a-n} \sum_{k=0}^{n-1} (-1)^{n-k} \binom{n-1}{k} (1-b)_k (1-a)_{n-k-1} \left(\frac{z}{1-z}\right)^k /; n \in \mathbb{N}$$

06.19.20.0007.02

$$\frac{\partial^n B_z(a, b)}{\partial z^n} == (-1)^{n-1} \Gamma(b) (1-z)^{b-n} z^{a-1} \, {}_2\tilde{F}_1\left(1-a, 1-n; b-n+1; 1-\frac{1}{z}\right) /; n \in \mathbb{N}$$

With respect to *a*

06.19.20.0008.02

$$\frac{\partial^n B_z(a, b)}{\partial a^n} == \Gamma(a) z^a \log^n(z) \sum_{j=0}^{n} \binom{n}{j} j! \, {}_{j+2}\tilde{F}_{j+1}(a_1, a_2, \ldots, a_{j+1}, 1-b; a_1+1, a_2+1, \ldots, a_{j+1}+1; z) \left(-\frac{\Gamma(a)}{\log(z)}\right)^j /; a_1 == a_2 == \ldots == a_{n+1} == a \bigwedge n \in \mathbb{N}$$

06.19.20.0009.02

$$\frac{\partial^n B_z(a, b)}{\partial a^n} == (-1)^n \sum_{k=0}^{\infty} \frac{(1-b)_k \Gamma(n+1, -(a+k) \log(z))}{(a+k)^{n+1} k!} /; n \in \mathbb{N}$$

With respect to *b*

06.19.20.0010.01

$$\frac{\partial^n \mathrm{B}_z(a,b)}{\partial b^n} = (-1)^n\, n!\, \Gamma(b)^{n+1}\, {}_{n+2}\tilde{F}_{n+1}(a_1, a_2, \ldots, a_{n+1}, 1-a; a_1+1, a_2+1, \ldots, a_{n+1}+1; 1) - \Gamma(b)\,(1-z)^b \log^n(1-z) \sum_{j=0}^{n} \binom{n}{j} j!\, {}_{j+2}\tilde{F}_{j+1}\big(a_1, a_2, \ldots, a_{j+1}, 1-a; a_1+1, a_2+1, \ldots, a_{j+1}+1; 1-z\big) \left(-\frac{\Gamma(b)}{\log(1-z)}\right)^j /; a_1 = a_2 = \ldots = a_{n+1} = b \land n \in \mathbb{N}$$

Fractional integro-differentiation

With respect to *z*

06.19.20.0011.01

$$\frac{\partial^\alpha \mathrm{B}_z(a,b)}{\partial z^\alpha} = z^{a-\alpha}\, \Gamma(a)\, {}_2\tilde{F}_1(a, 1-b; a-\alpha+1; z) /; -a \notin \mathbb{N}^+$$

06.19.20.0012.01

$$\frac{\partial^\alpha \mathrm{B}_z(a,b)}{\partial z^\alpha} = \sum_{k=0}^{\infty} \frac{(1-b)_k\, \mathcal{FC}_{\exp}^{(\alpha)}(z, a+k)\, z^{a+k-\alpha}}{(a+k)\, k!} /; |z| < 1$$

With respect to *a*

06.19.20.0013.01

$$\frac{\partial^\alpha \mathrm{B}_z(a,b)}{\partial a^\alpha} = a^{-\alpha}\,(1-b)\, z^{a+1} \sum_{k=0}^{\infty} \frac{(2-b)_k\, z^k}{(k+1)^2\, k!}\, {}_2\tilde{F}_1\left(1, 1; 1-\alpha; -\frac{a}{k+1}\right) + \mathcal{FC}_{\exp}^{(\alpha)}(z, -1)\, a^{-\alpha-1} + a^{-\alpha} \log(z)\, {}_2\tilde{F}_2(1, 1; 2, 1-\alpha; a\log(z)) /; |z| < 1 \land -a \notin \mathbb{N}$$

06.19.20.0014.01

$$\frac{\partial^\alpha \mathrm{B}_z(a,b)}{\partial a^\alpha} = a^{-\alpha} \int_0^z t^{a-1}\,(1-t)^{b-1}\,(a\log(t))^\alpha\, Q(-\alpha, 0, a\log(t))\, dt /; \mathrm{Re}(a) > 0$$

With respect to *b*

06.19.20.0015.01

$$\frac{\partial^\alpha \mathrm{B}_z(a,b)}{\partial b^\alpha} = \sum_{j=0}^{\infty} \sum_{k=0}^{j} \frac{(-1)^k\, k!\, z^{a+j}\, b^{k-\alpha}}{(a+j)\,\Gamma(k-\alpha+1)\, j!}\, S_j^{(k)}\, {}_2F_1(j+1, a+j; a+j+1; z) /; |z| < 1 \land -a \notin \mathbb{N}$$

06.19.20.0016.01

$$\frac{\partial^\alpha \mathrm{B}_z(a,b)}{\partial b^\alpha} = b^{-\alpha} \int_0^z t^{a-1}\,(1-t)^{b-1}\,(b\log(1-t))^\alpha\, Q(-\alpha, 0, b\log(1-t))\, dt /; \mathrm{Re}(a) > 0$$

Integration

Indefinite integration

Involving only one direct function

06.19.21.0001.01

$$\int \mathrm{B}_{a\,z}(a,\,b)\,dz = z\,\mathrm{B}_{a\,z}(a,\,b) - \frac{\mathrm{B}_{a\,z}(a+1,\,b)}{a}$$

06.19.21.0002.01

$$\int \mathrm{B}_z(a,\,b)\,dz = z\,\mathrm{B}_z(a,\,b) - \mathrm{B}_z(a+1,\,b)$$

Involving one direct function and elementary functions

Involving power function

06.19.21.0003.01

$$\int z^{\alpha-1}\,\mathrm{B}_{a\,z}(a,\,b)\,dz = \frac{z^{\alpha}}{\alpha}\left(\mathrm{B}_{a\,z}(a,\,b) - (a\,z)^{-\alpha}\,\mathrm{B}_{a\,z}(a+\alpha,\,b)\right)$$

06.19.21.0004.01

$$\int z^{\alpha-1}\,\mathrm{B}_z(a,\,b)\,dz = \frac{1}{\alpha}\left(z^{\alpha}\,\mathrm{B}_z(a,\,b) - \mathrm{B}_z(a+\alpha,\,b)\right)$$

06.19.21.0005.01

$$\int \frac{\mathrm{B}_z(a,\,b)}{z}\,dz = z^a\,\Gamma(a)^2\,{}_3\tilde{F}_2(a,\,a,\,1-b;\,a+1,\,a+1;\,z)$$

06.19.21.0006.01

$$\int (1-z)^{b-1}\,z^{a-1}\,\mathrm{B}_z(a,\,b)\,dz = \frac{1}{2}\,\mathrm{B}_z(a,\,b)^2$$

06.19.21.0007.01

$$\int (1-z)^n\,\mathrm{B}_z(a,\,b)\,dz = \frac{\mathrm{B}_z(a,\,b+n+1) - (1-z)^{n+1}\,\mathrm{B}_z(a,\,b)}{n+1}$$

Involving rational function

06.19.21.0008.01

$$\int \frac{\mathrm{B}_z(a,\,b)}{1-z}\,dz = (1-z)^b\,\Gamma(b)^2\,{}_3\tilde{F}_2(b,\,b,\,1-a;\,b+1,\,b+1;\,1-z) - \left(\mathrm{B}_{1-z}(b,\,a) + \mathrm{B}_z(a,\,b)\right)\log(1-z)$$

Involving only one direct function with respect to *a*

06.19.21.0009.01

$$\int \mathrm{B}_z(a,\,b)\,da = (1-b)\,z^{a+1}\sum_{k=0}^{\infty}\frac{(2-b)_k\,z^k}{(k+1)!}\log\left(1+\frac{a}{k+1}\right) - \Gamma(0,\,-a\log(z)) + \log(a) - \log(-a\log(z)) - \gamma\;/;\;|z|<1\bigwedge -a\notin\mathbb{N}$$

Involving one direct function and elementary functions with respect to *a*

Involving power function

06.19.21.0010.01

$$\int a^{\alpha-1} \mathrm{B}_z(a, b)\, da = \frac{a^\alpha (1-b) z^{a+1}}{\alpha} \sum_{k=0}^{\infty} \frac{(2-b)_k z^k}{(k+1)^2 k!} {}_2F_1\left(\alpha, 1; \alpha+1; -\frac{a}{k+1}\right) + \frac{a^{\alpha-1} z^a}{\alpha-1} - \frac{a^\alpha (-a \log(z))^{-\alpha} \log(z)}{\alpha-1} \Gamma(\alpha, 0, -a \log(z)) \;/;\; |z| < 1 \land -a \notin \mathbb{N}$$

Involving only one direct function with respect to *b*

06.19.21.0011.01

$$\int \mathrm{B}_z(a, b)\, db = \sum_{j=0}^{\infty} \sum_{k=0}^{j} \frac{(-1)^k k!\, z^{a+j} b^{k+1}}{(a+j)(k+1)!\, j!} S_j^{(k)} {}_2F_1(j+1, a+j; a+j+1; z) \;/;\; |z| < 1 \land -a \notin \mathbb{N}$$

Involving one direct function and elementary functions with respect to *b*

Involving power function

06.19.21.0012.01

$$\int \mathrm{B}_z(a, b)\, db = \sum_{j=0}^{\infty} \sum_{k=0}^{j} \frac{(-1)^k k!\, z^{a+j} b^{k+1}}{(a+j)(k+1)!\, j!} S_j^{(k)} {}_2F_1(j+1, a+j; a+j+1; z) \;/;\; |z| < 1 \land -a \notin \mathbb{N}$$

Integral transforms

Fourier cos transforms

06.19.22.0001.01

$$\mathcal{F}c_t[\mathrm{B}_t(a, b)](x) = \frac{2^{-b-\frac{3}{2}} e^{-i a \pi}}{\pi \Gamma(1-b)} G_{2,4}^{3,2}\left(\frac{x^2}{4} \middle| \begin{matrix} \frac{1-a}{2}, -\frac{a}{2} \\ 0, -\frac{a+b}{2}, \frac{1-a-b}{2}, -\frac{1}{2} \end{matrix}\right) /;\; x \in \mathbb{R} \land \mathrm{Re}(a) > -1$$

Fourier sin transforms

06.19.22.0002.01

$$\mathcal{F}s_t[\mathrm{B}_t(a, b)](x) = -\frac{2^{-b-\frac{3}{2}} e^{-i a \pi}}{\pi \Gamma(1-b)} \mathrm{sgn}(x)\, G_{2,4}^{3,2}\left(\frac{x^2}{4} \middle| \begin{matrix} \frac{1-a}{2}, -\frac{a}{2} \\ -\frac{1}{2}, -\frac{a+b}{2}, \frac{1-a-b}{2}, 0 \end{matrix}\right) /;\; x \in \mathbb{R} \land \mathrm{Re}(a) > -2$$

Laplace transforms

06.19.22.0003.01

$$\mathcal{L}_t[\mathrm{B}_t(a, b)](z) = \frac{(-1)^{-a} \Gamma(a)}{z} U(a, a+b, -z) \;/;\; \mathrm{Re}(z) > 0 \land \mathrm{Re}(a) > -1$$

Mellin transforms

06.19.22.0004.01

$$\mathcal{M}_t[\mathrm{B}_t(a, b)](z) = -\frac{\pi (\cot(\pi (a+z)) + i) \Gamma(1-a-b-z)}{z \Gamma(1-b) \Gamma(1-a-z)} \;/;\; \mathrm{Re}(a+z) > 0 \land \mathrm{Re}(z) < 0 \land \mathrm{Re}(a+b+z) < 1$$

Operations

Limit operation

06.19.25.0001.01

$$\lim_{a\to -n} (a+n)\,B_z(a, b) = \frac{(1-b)_n}{n!} \;/;\; n \in \mathbb{N}$$

Representations through more general functions

Through hypergeometric functions

Involving ${}_2\tilde{F}_1$

06.19.26.0001.01

$$B_z(a, b) = z^a\,\Gamma(a)\,{}_2\tilde{F}_1(a, 1-b; a+1; z) \;/;\; -a \notin \mathbb{N}$$

06.19.26.0002.01

$$B_z(a, b) = \Gamma(a)\,(1-z)^b\,z^a\,{}_2\tilde{F}_1(1, a+b; a+1; z) \;/;\; -a \notin \mathbb{N}$$

06.19.26.0003.01

$$B_z(a, b) = B(a, b) - \Gamma(b)\,(1-z)^b\,z^a\,{}_2\tilde{F}_1(1, a+b; b+1; 1-z) \;/;\; -b \notin \mathbb{N}$$

06.19.26.0004.01

$$B_z(a, b) = B(1-a-b, a)\,(-z)^{-a}\,z^a - \Gamma(1-a-b)\,(-z)^{b-1}\,z^a\,{}_2\tilde{F}_1\left(1-b, -a-b+1; -a-b+2; \frac{1}{z}\right) \;/;\; a+b \notin \mathbb{N}^+$$

Involving ${}_2F_1$

06.19.26.0005.01

$$B_z(a, b) = \frac{z^a}{a}\,{}_2F_1(a, 1-b; a+1; z) \;/;\; -a \notin \mathbb{N}$$

06.19.26.0006.01

$$B_z(a, b) = \frac{(1-z)^b\,z^a}{a}\,{}_2F_1(1, a+b; a+1; z) \;/;\; -a \notin \mathbb{N}$$

06.19.26.0007.01

$$B_z(a, b) = B(a, b) - \frac{(1-z)^b\,z^a}{b}\,{}_2F_1(1, a+b; b+1; 1-z) \;/;\; -b \notin \mathbb{N}$$

06.19.26.0008.01

$$B_z(a, b) = \frac{z^a\,(-z)^{b-1}}{a+b-1}\,{}_2F_1\left(1-b, -a-b+1; -a-b+2; \frac{1}{z}\right) + z^a\,B(1-a-b, a)\,(-z)^{-a} \;/;\; a+b \notin \mathbb{N}^+$$

Through hypergeometric functions of two variables

06.19.26.0009.01

$$B_z(a, b) = B(a, b) - \frac{(1-z)^b}{b}\,F_{0\times 0\times 1}^{0\times 1\times 2}\left(\begin{matrix} ;\, -a;\, 1,\, a+b; \\ ;\, ;\, b+1; \end{matrix}\; 1-z,\, 1-z\right) \;/;\; -b \notin \mathbb{N}$$

Through Meijer *G*

Classical cases for the direct function itself

06.19.26.0010.01

$$B_z(a, b) = \frac{z^a}{\Gamma(1-b)} G_{2,2}^{1,2}\left(-z \,\middle|\, \begin{matrix} 1-a, b \\ 0, -a \end{matrix}\right)$$

Classical cases involving algebraic functions

06.19.26.0011.01

$$(1-z)^{-b} B_z(a, b) = \frac{\Gamma(a)\, z^a}{\Gamma(a+b)} G_{2,2}^{1,2}\left(-z \,\middle|\, \begin{matrix} 0, -a-b+1 \\ 0, -a \end{matrix}\right)$$

06.19.26.0016.01

$$(z-1)^{-b} B_{\frac{1}{z}}(a, b) = \frac{z^{-a-b}\, \Gamma(a)}{\Gamma(a+b)} G_{2,2}^{2,1}\left(-z \,\middle|\, \begin{matrix} 1, a+1 \\ 1, a+b \end{matrix}\right) /; z \notin (-\infty, 0)$$

Classical cases involving algebraic functions in the arguments

06.19.26.0012.01

$$B_{\frac{z}{z+1}}(a, b) = \frac{1}{\Gamma(a+b)} G_{2,2}^{1,2}\left(z \,\middle|\, \begin{matrix} 1, 1-b \\ a, 0 \end{matrix}\right) /; z \notin (-\infty, -1)$$

06.19.26.0013.01

$$(z+1)^{a+b-1} B_{\frac{z}{z+1}}(a, b) = \frac{\Gamma(a)}{\Gamma(1-b)} G_{2,2}^{1,2}\left(z \,\middle|\, \begin{matrix} a, a+b \\ a, 0 \end{matrix}\right) /; z \notin (-\infty, -1)$$

Through other functions

Involving some hypergeometric-type functions

06.19.26.0014.01

$$B_z(a, b) = B_{(1,z)}(a, b) + B(a, b) /; \operatorname{Re}(b) > 0$$

06.19.26.0015.01

$$B_z(a, b) = B(a, b)\, (I_{(1,z)}(a, b) + 1) /; \operatorname{Re}(b) > 0$$

Representations through equivalent functions

With inverse function

06.19.27.0001.01

$$B_{I_z^{-1}(a,b)}(a, b) = B(a, b)\, z$$

06.19.27.0002.01

$$B_{I_{(1,z_2)}^{-1}(a,b)}(a, b) = B(a, b)\, (1+z_2) /; -1 < z_2 < 0$$

With related functions

06.19.27.0003.01

$$B_z(a, b) = \frac{\Gamma(a)\, \Gamma(b)}{\Gamma(a+b)} I_z(a, b)$$

History

– I. Newton (1676)
– J. Stirling (1730)
– T. Bayes (1763)
– P.-S. Laplace (1778)

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