BetaRegularized

Notations

Traditional name

Regularized incomplete beta function

Traditional notation

$I_z(a, b)$

Mathematica StandardForm notation

`BetaRegularized[z, a, b]`

Primary definition

Basic definition

06.21.02.0001.01

$$I_z(a, b) = \frac{B_z(a, b)}{B(a, b)}$$

Above generic formula cannot directly be used for nonpositive integers a, b, which lead to indeterminate expressions like 0/0. In these cases, it is more necessary to use an equivalent complete definition, presented below.

Complete definition

06.21.02.0002.01

$$I_z(a, b) = \frac{\Gamma(a+b)}{\Gamma(b)} z^a \, {}_2\tilde{F}_1(a, 1-b; a+1; z) \,/;\, -a \notin \mathbb{N}$$

The function $I_z(a, b)$ can be equivalently defined through the following generalized hypergeometric function.

For negative integers $\alpha = -n$ and positive integers $b = m \,/;\, m \le n$, the function $I_z(a, b)$ cannot be uniquely defined by a limiting procedure based on the above definition because the two variables a, b can approach integers $-n$, $m \,/;\, m \le n$ at different speeds. In the case $a = -n$, $b = m \,/;\, m \le n$ one defines:

06.21.02.0003.01

$$I_z(-n, m) = \frac{(-1)^m \, n! \, z^{-n}}{(n-m)! \, (m-1)!} \sum_{k=0}^{m-1} \frac{(1-m)_k \, z^k}{(k-n) \, k!} \,/;\, n \in \mathbb{N}^+ \land m \in \mathbb{N}^+ \land m \le n$$

For $a = -n$ and arbitrary b (except case of positive integer $b = m \,/;\, m \le n$), the function $I_z(a, b)$ is defined as having the value 1:

06.21.02.0004.01

$$I_z(-n, b) = 1 /; n \in \mathbb{N} \land \neg \left(b \in \mathbb{N}^+ \land b \leq n\right)$$

Specific values

Specialized values

For fixed *z*, *a*

06.21.03.0001.01

$$I_z(a, n) = z^a \sum_{k=0}^{n-1} \frac{(a)_k (1-z)^k}{k!} /; n \in \mathbb{N}$$

06.21.03.0002.01

$$I_z(a, -n) = 0 /; n \in \mathbb{N}$$

For fixed *z*, *b*

06.21.03.0003.01

$$I_z(-n, b) = 1 /; n \in \mathbb{N} \land \neg \left(b \in \mathbb{N}^+ \land b \leq n\right)$$

06.21.03.0008.01

$$I_z(-n, b) = \frac{(-1)^b n! z^{-n}}{(n-b)! (b-1)!} \sum_{k=0}^{b-1} \frac{(1-b)_k z^k}{(k-n) k!} /; n \in \mathbb{N}^+ \land b \in \mathbb{N}^+ \land b \leq n$$

06.21.03.0004.01

$$I_z(n, b) = 1 - (1-z)^b \sum_{k=0}^{n-1} \frac{(b)_k z^k}{k!} /; n \in \mathbb{N}$$

For fixed *a*, *b*

06.21.03.0005.01

$$I_0(a, b) = 0 /; \mathrm{Re}(a) > 0$$

06.21.03.0006.01

$$I_0(a, b) = \tilde{\infty} /; \mathrm{Re}(a) < 0$$

06.21.03.0007.01

$$I_1(a, b) = 1 /; \mathrm{Re}(b) > 0$$

General characteristics

Domain and analyticity

$I_z(a, b)$ is an analytical function of z, a, and b which is defined in $\mathbb{C}^3$.

06.21.04.0001.01

$$(z * a * b) \longrightarrow I_z(a, b) :: (\mathbb{C} \otimes \mathbb{C} \otimes \mathbb{C}) \longrightarrow \mathbb{C}$$

Symmetries and periodicities

Mirror symmetry

06.21.04.0002.02

$I_{\bar{z}}(\bar{a}, \bar{b}) = \overline{I_z(a, b)} \;/;\; z \notin (-\infty, 0) \land z \notin (1, \infty)$

Periodicity

No periodicity

Poles and essential singularities

With respect to *b*

For fixed z, a, the function $I_z(a, b)$ has only one singular point at $b = \tilde{\infty}$. It is an essential singular point.

06.21.04.0003.01

$\mathcal{Sing}_b(I_z(a, b)) = \{\{\tilde{\infty}, \infty\}\}$

With respect to *a*

For fixed z, b, the function $I_z(a, b)$ has only one singular point at $a = \tilde{\infty}$. It is an essential singular point.

06.21.04.0004.01

$\mathcal{Sing}_a(I_z(a, b)) = \{\{\tilde{\infty}, \infty\}\}$

With respect to *z*

For fixed a, b, the function $I_z(a, b)$ does not have poles and essential singularities.

06.21.04.0005.01

$\mathcal{Sing}_z(I_z(a, b)) = \{\}$

Branch points

With respect to *b*

For fixed z, a, the function $I_z(a, b)$ does not have branch points.

06.21.04.0006.01

$\mathcal{BP}_b(I_z(a, b)) = \{\}$

With respect to *a*

For fixed z, b, the function $I_z(a, b)$ does not have branch points.

06.21.04.0007.01

$\mathcal{BP}_a(I_z(a, b)) = \{\}$

With respect to *z*

The function $I_z(a, b)$ has three branch points: $z = 0$, $z = 1$, and $z = \tilde{\infty}$.

06.21.04.0008.01

$\mathcal{BP}_z(I_z(a, b)) = \{0, 1, \tilde{\infty}\}$

06.21.04.0009.01

$\mathcal{R}_z(I_z(a, b), 0) = \log \;/;\; a \notin \mathbb{Z} \land a \notin \mathbb{Q}$

06.21.04.0010.01

$$\mathcal{R}_z(I_z(a,\,b),\,0) = q\;/;\;a = \frac{p}{q}\bigwedge p \in \mathbb{Z}\bigwedge q-1 \in \mathbb{N}^+ \bigwedge \gcd(p,\,q) = 1$$

06.21.04.0011.01

$$\mathcal{R}_z(I_z(a,\,b),\,1) = \log\;/;\;b \notin \mathbb{Z} \land b \notin \mathbb{Q}$$

06.21.04.0012.01

$$\mathcal{R}_z(I_z(a,\,b),\,1) = q\;/;\;b = \frac{p}{q}\bigwedge p \in \mathbb{Z}\bigwedge q-1 \in \mathbb{N}^+ \bigwedge \gcd(p,\,q) = 1$$

06.21.04.0013.01

$$\mathcal{R}_z(I_z(a,\,b),\,\tilde{\infty}) = \log\;/;\;a+b \in \mathbb{Z} \lor a+b \notin \mathbb{Q}$$

06.21.04.0014.01

$$\mathcal{R}_z(I_z(a,\,b),\,\tilde{\infty}) = s\;/;\;a+b = \frac{r}{s}\bigwedge r \in \mathbb{Z}\bigwedge s-1 \in \mathbb{N}^+ \bigwedge \gcd(r,\,s) = 1$$

Branch cuts

With respect to *b*

For fixed z, a, the function $I_z(a, b)$ does not have branch cuts.

06.21.04.0015.01

$$\mathcal{BC}_b(I_z(a,\,b)) = \{\}$$

With respect to *a*

For fixed z, b, the function $I_z(a, b)$ does not have branch cuts.

06.21.04.0016.01

$$\mathcal{BC}_a(I_z(a,\,b)) = \{\}$$

With respect to *z*

For fixed a, b, the function $I_z(a, b)$ is a single-valued function on the z-plane cut along the intervals $(-\infty,\ 0)$ and $(1,\ \infty)$. The function $I_z(a, b)$ is continuous from above on the interval $(-\infty, 0)$ and from below on the interval $(1,\ \infty)$.

06.21.04.0017.01

$$\mathcal{BC}_z(I_z(a,\,b)) = \{\{(-\infty,\,0),\,-i\},\,\{(1,\,\infty),\,i\}\}$$

06.21.04.0018.01

$$\lim_{\epsilon\to+0} I_{x+i\epsilon}(a,\,b) = I_x(a,\,b)\;/;\;x<0$$

06.21.04.0019.01

$$\lim_{\epsilon\to+0} I_{x-i\epsilon}(a,\,b) = e^{-2ia\pi}\, I_x(a,\,b)\;/;\;x<0$$

06.21.04.0021.01

$$\lim_{\epsilon\to+0} I_{x-i\epsilon}(a,\,b) = I_x(a,\,b)\;/;\;x>1$$

06.21.04.0022.01

$$\lim_{\epsilon\to+0} I_{x+i\epsilon}(a,\,b) = e^{-2ib\pi}\, I_x(a,\,b) + 2\,e^{-ib\pi}\, i\sin(b\,\pi)\;/;\;x>1$$

Series representations

Generalized power series

Expansions at generic point $z == z_0$

For the function itself

06.21.06.0021.01

$$I_z(a, b) \propto \frac{\sin(\pi b)}{\pi \Gamma(a)} \left(\frac{1}{z_0}\right)^{a \left\lfloor \frac{\arg(z-z_0)}{2\pi} \right\rfloor} z_0^{a \left(\left\lfloor \frac{\arg(z-z_0)}{2\pi} \right\rfloor + 1\right)} \left(-2 i e^{i b \pi \left\lfloor \frac{\arg(z_0-z)}{2\pi} \right\rfloor} \pi \left\lfloor \frac{\arg(1-z_0)+\pi}{2\pi} \right\rfloor \left\lfloor \frac{\arg(z_0-z)}{2\pi} \right\rfloor \Gamma(a) z_0^{-a} + G_{2,2}^{2,2}\left(1-z_0 \middle| \begin{matrix} 1-a, b \\ 0, b \end{matrix}\right) \left(\frac{1}{1-z_0}\right)^{b \left\lfloor \frac{\arg(z_0-z)}{2\pi} \right\rfloor} (1-z_0)^{b \left\lfloor \frac{\arg(z_0-z)}{2\pi} \right\rfloor} + \left(\left(\frac{1}{1-z_0}\right)^{b \left\lfloor \frac{\arg(z_0-z)}{2\pi} \right\rfloor} (1-z_0)^{b \left\lfloor \frac{\arg(z_0-z)}{2\pi} \right\rfloor} z_0^a \left(a G_{2,2}^{2,2}\left(1-z_0 \middle| \begin{matrix} 1-a, b \\ 0, b \end{matrix}\right) + G_{2,2}^{2,2}\left(1-z_0 \middle| \begin{matrix} -a, b-1 \\ 0, b-1 \end{matrix}\right) z_0\right) - 2 i e^{i b \pi \left\lfloor \frac{\arg(z_0-z)}{2\pi} \right\rfloor} \pi \left\lfloor \frac{\arg(1-z_0)+\pi}{2\pi} \right\rfloor \left\lfloor \frac{\arg(z_0-z)}{2\pi} \right\rfloor (a \Gamma(a) - \Gamma(a+1))\right) z_0^{-a-1} (z-z_0) + \frac{1}{2} \left(\left(\frac{1}{1-z_0}\right)^{b \left\lfloor \frac{\arg(z_0-z)}{2\pi} \right\rfloor} (1-z_0)^{b \left\lfloor \frac{\arg(z_0-z)}{2\pi} \right\rfloor} z_0^a \left((a-1) a G_{2,2}^{2,2}\left(1-z_0 \middle| \begin{matrix} 1-a, b \\ 0, b \end{matrix}\right) + z_0 \left(2 a G_{2,2}^{2,2}\left(1-z_0 \middle| \begin{matrix} -a, b-1 \\ 0, b-1 \end{matrix}\right) + G_{2,2}^{2,2}\left(1-z_0 \middle| \begin{matrix} -a-1, b-2 \\ 0, b-2 \end{matrix}\right) z_0\right)\right) - 2 i e^{i b \pi \left\lfloor \frac{\arg(z_0-z)}{2\pi} \right\rfloor} \pi \left\lfloor \frac{\arg(1-z_0)+\pi}{2\pi} \right\rfloor \left\lfloor \frac{\arg(z_0-z)}{2\pi} \right\rfloor ((a-1) a \Gamma(a) - 2 a \Gamma(a+1) + \Gamma(a+2))\right) z_0^{-a-2} (z-z_0)^2 + \ldots\right) /; (z \to z_0)$$

06.21.06.0022.01

$$I_z(a,b) \propto \frac{\sin(\pi b)}{\pi\,\Gamma(a)} \left(\frac{1}{z_0}\right)^{a \left\lfloor \frac{\arg(z-z_0)}{2\pi} \right\rfloor} z_0^{a\left(\left\lfloor \frac{\arg(z-z_0)}{2\pi} \right\rfloor+1\right)} \left(-2\,i\,e^{i b \pi \left\lfloor \frac{\arg(z_0-z)}{2\pi} \right\rfloor} \pi \left\lfloor \frac{\arg(1-z_0)+\pi}{2\pi} \right\rfloor \left\lfloor \frac{\arg(z_0-z)}{2\pi} \right\rfloor \Gamma(a)\, z_0^{-a} + G_{2,2}^{2,2}\!\left(1-z_0 \,\middle|\, \begin{matrix} 1-a,\, b \\ 0,\, b \end{matrix}\right) \left(\frac{1}{1-z_0}\right)^{b \left\lfloor \frac{\arg(z_0-z)}{2\pi} \right\rfloor} (1-z_0)^{b \left\lfloor \frac{\arg(z_0-z)}{2\pi} \right\rfloor} + \right.$$
$$\left(\left(\frac{1}{1-z_0}\right)^{b \left\lfloor \frac{\arg(z_0-z)}{2\pi} \right\rfloor} (1-z_0)^{b \left\lfloor \frac{\arg(z_0-z)}{2\pi} \right\rfloor} z_0^a \left(a\, G_{2,2}^{2,2}\!\left(1-z_0 \,\middle|\, \begin{matrix} 1-a,\, b \\ 0,\, b \end{matrix}\right) + G_{2,2}^{2,2}\!\left(1-z_0 \,\middle|\, \begin{matrix} -a,\, b-1 \\ 0,\, b-1 \end{matrix}\right) z_0\right) - \right.$$
$$\left. 2\,i\,e^{i b \pi \left\lfloor \frac{\arg(z_0-z)}{2\pi} \right\rfloor} \pi \left\lfloor \frac{\arg(1-z_0)+\pi}{2\pi} \right\rfloor \left\lfloor \frac{\arg(z_0-z)}{2\pi} \right\rfloor (a\,\Gamma(a) - \Gamma(a+1))\right) z_0^{-a-1} (z-z_0) +$$
$$\frac{1}{2}\left(\left(\frac{1}{1-z_0}\right)^{b \left\lfloor \frac{\arg(z_0-z)}{2\pi} \right\rfloor} (1-z_0)^{b \left\lfloor \frac{\arg(z_0-z)}{2\pi} \right\rfloor} z_0^a \left((a-1)\,a\, G_{2,2}^{2,2}\!\left(1-z_0 \,\middle|\, \begin{matrix} 1-a,\, b \\ 0,\, b \end{matrix}\right) + \right.\right.$$
$$\left. z_0 \left(2\,a\, G_{2,2}^{2,2}\!\left(1-z_0 \,\middle|\, \begin{matrix} -a,\, b-1 \\ 0,\, b-1 \end{matrix}\right) + G_{2,2}^{2,2}\!\left(1-z_0 \,\middle|\, \begin{matrix} -a-1,\, b-2 \\ 0,\, b-2 \end{matrix}\right) z_0\right)\right) - 2\,i\,e^{i b \pi \left\lfloor \frac{\arg(z_0-z)}{2\pi} \right\rfloor} \pi$$
$$\left.\left. \left\lfloor \frac{\arg(1-z_0)+\pi}{2\pi} \right\rfloor \left\lfloor \frac{\arg(z_0-z)}{2\pi} \right\rfloor ((a-1)\,a\,\Gamma(a) - 2\,a\,\Gamma(a+1) + \Gamma(a+2))\right) z_0^{-a-2} (z-z_0)^2 + O\left((z-z_0)^3\right)\right)$$

06.21.06.0023.01

$$I_z(a,b) = \frac{\sin(\pi b)}{\pi\,\Gamma(a)} \left(\frac{1}{z_0}\right)^{a \left\lfloor \frac{\arg(z-z_0)}{2\pi} \right\rfloor} z_0^{a\left(\left\lfloor \frac{\arg(z-z_0)}{2\pi} \right\rfloor+1\right)}$$
$$\sum_{k=0}^{\infty} \sum_{j=0}^{k} \frac{(-1)^{k-j}\,(-a)_{k-j}\, z_0^{j-k}}{j!\,(k-j)!} \left(\left(\frac{1}{1-z_0}\right)^{b \left\lfloor \frac{\arg(z_0-z)}{2\pi} \right\rfloor} (1-z_0)^{b \left\lfloor \frac{\arg(z_0-z)}{2\pi} \right\rfloor} G_{2,2}^{2,2}\!\left(1-z_0 \,\middle|\, \begin{matrix} -a-j+1,\, b-j \\ 0,\, b-j \end{matrix}\right) - \right.$$
$$\left. 2\,\pi\,i \left\lfloor \frac{\arg(z_0-z)}{2\pi} \right\rfloor e^{i b \pi \left\lfloor \frac{\arg(z_0-z)}{2\pi} \right\rfloor} \left\lfloor \frac{\arg(1-z_0)+\pi}{2\pi} \right\rfloor (-1)^j\, \Gamma(a+j)\, z_0^{-a-j}\right) (z-z_0)^k$$

06.21.06.0024.01

$$I_z(a,b)=\frac{\sin(\pi b)}{\Gamma(a)}\left(\frac{1}{z_0}\right)^{a\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor} z_0^{a\left(\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor+1\right)}\sum_{k=0}^{\infty}\sum_{j=0}^{k}\frac{(-1)^k(-a)_{k-j}\,z_0^{j-k}}{j!\,(k-j)!}$$

$$\left(\Gamma(a+j)\,z_0^{-a-j}\left(\csc(b\pi)\left(\frac{1}{1-z_0}\right)^{b\left\lfloor\frac{\arg(z_0-z)}{2\pi}\right\rfloor}(1-z_0)^{b\left\lfloor\frac{\arg(z_0-z)}{2\pi}\right\rfloor}-2\,i\,e^{i\,b\,\pi\left\lfloor\frac{\arg(z_0-z)}{2\pi}\right\rfloor}\left\lfloor\frac{\arg(1-z_0)+\pi}{2\pi}\right\rfloor\left\lfloor\frac{\arg(z_0-z)}{2\pi}\right\rfloor\right)-\right.$$

$$\left.\csc(b\pi)\,\Gamma(a+b)\left(\frac{1}{1-z_0}\right)^{b\left\lfloor\frac{\arg(z_0-z)}{2\pi}\right\rfloor}(1-z_0)^{\left\lfloor\frac{\arg(z_0-z)}{2\pi}\right\rfloor b+b-j}\,{}_2\tilde{F}_1(1,a+b;b-j+1;1-z_0)\right)(z-z_0)^k\;/;\;b\notin\mathbb{Z}$$

06.21.06.0025.01

$$I_z(a,b)\propto\frac{\sin(\pi b)}{\pi\,\Gamma(a)}\left(\frac{1}{z_0}\right)^{a\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor} z_0^{a\left(\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor+1\right)}\left(-2\,i\,e^{i\,b\,\pi\left\lfloor\frac{\arg(z_0-z)}{2\pi}\right\rfloor}\pi\left\lfloor\frac{\arg(1-z_0)+\pi}{2\pi}\right\rfloor\left\lfloor\frac{\arg(z_0-z)}{2\pi}\right\rfloor\Gamma(a)\,z_0^{-a}+\right.$$

$$\left.G_{2,2}^{2,2}\!\left(1-z_0\,\middle|\,\begin{matrix}1-a,\,b\\0,\,b\end{matrix}\right)\left(\frac{1}{1-z_0}\right)^{b\left\lfloor\frac{\arg(z_0-z)}{2\pi}\right\rfloor}(1-z_0)^{b\left\lfloor\frac{\arg(z_0-z)}{2\pi}\right\rfloor}\right)+O(z-z_0)$$

Expansions on branch cuts

For the function itself

In the left half-plane

06.21.06.0026.01

$$I_z(a,b)\propto$$

$$\frac{\sin(\pi b)}{\pi\,\Gamma(a)}\left(\frac{1}{x}\right)^{a\left\lfloor\frac{\arg(z-x)}{2\pi}\right\rfloor}x^{a\left(\left\lfloor\frac{\arg(z-x)}{2\pi}\right\rfloor+1\right)}\left(G_{2,2}^{2,2}\!\left(1-x\,\middle|\,\begin{matrix}1-a,\,b\\0,\,b\end{matrix}\right)+\left(G_{2,2}^{2,2}\!\left(1-x\,\middle|\,\begin{matrix}1-a,\,b\\0,\,b\end{matrix}\right)+x\,G_{2,2}^{2,2}\!\left(1-x\,\middle|\,\begin{matrix}-a,\,b-1\\0,\,b-1\end{matrix}\right)\right)(z-x)+\right.$$

$$\frac{1}{2x^2}\left(G_{2,2}^{2,2}\!\left(1-x\,\middle|\,\begin{matrix}-a-1,\,b-2\\0,\,b-2\end{matrix}\right)x^2+a\left((a-1)\,G_{2,2}^{2,2}\!\left(1-x\,\middle|\,\begin{matrix}1-a,\,b\\0,\,b\end{matrix}\right)+2\,x\,G_{2,2}^{2,2}\!\left(1-x\,\middle|\,\begin{matrix}-a,\,b-1\\0,\,b-1\end{matrix}\right)\right)\right)(z-x)^2+$$

$$\left.\ldots\right)/;\;(z\to x)\land x\in\mathbb{R}\land x<0$$

06.21.06.0027.01

$$I_z(a,b)\propto$$

$$\frac{\sin(\pi b)}{\pi\,\Gamma(a)}\left(\frac{1}{x}\right)^{a\left\lfloor\frac{\arg(z-x)}{2\pi}\right\rfloor}x^{a\left(\left\lfloor\frac{\arg(z-x)}{2\pi}\right\rfloor+1\right)}\left(G_{2,2}^{2,2}\!\left(1-x\,\middle|\,\begin{matrix}1-a,\,b\\0,\,b\end{matrix}\right)+\left(G_{2,2}^{2,2}\!\left(1-x\,\middle|\,\begin{matrix}1-a,\,b\\0,\,b\end{matrix}\right)+x\,G_{2,2}^{2,2}\!\left(1-x\,\middle|\,\begin{matrix}-a,\,b-1\\0,\,b-1\end{matrix}\right)\right)(z-x)+\right.$$

$$\frac{1}{2x^2}\left(G_{2,2}^{2,2}\!\left(1-x\,\middle|\,\begin{matrix}-a-1,\,b-2\\0,\,b-2\end{matrix}\right)x^2+a\left((a-1)\,G_{2,2}^{2,2}\!\left(1-x\,\middle|\,\begin{matrix}1-a,\,b\\0,\,b\end{matrix}\right)+2\,x\,G_{2,2}^{2,2}\!\left(1-x\,\middle|\,\begin{matrix}-a,\,b-1\\0,\,b-1\end{matrix}\right)\right)\right)(z-x)^2+$$

$$\left.O\!\left((z-x)^3\right)\right)/;\;x\in\mathbb{R}\land x<0$$

06.21.06.0028.01

$$I_z(a,b) = \frac{\sin(\pi b)}{\pi\,\Gamma(a)} \left(\frac{1}{x}\right)^{a \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} x^{a\left(\left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor+1\right)} \sum_{k=0}^{\infty} \sum_{j=0}^{k} \frac{(-1)^{k-j}\,(-a)_{k-j}\,x^{j-k}}{j!\,(k-j)!}\, G_{2,2}^{2,2}\!\left(1-x \,\middle|\, \begin{matrix} -a-j+1,\, b-j \\ 0,\, b-j \end{matrix}\right) (z-x)^k \,/;\, x \in \mathbb{R} \land x < 0$$

06.21.06.0029.01

$$I_z(a,b) = \frac{1}{\Gamma(a)} \left(\frac{1}{x}\right)^{a \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} x^{a\left(\left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor+1\right)}$$

$$\sum_{k=0}^{\infty} \sum_{j=0}^{k} \frac{(-1)^{k}\,(-a)_{k-j}\,x^{j-k}}{j!\,(k-j)!} \left(\Gamma(a+j)\,x^{-a-j} - \Gamma(a+b)\,(1-x)^{b-j}\,{}_2\tilde{F}_1(1,\, a+b;\, b-j+1;\, 1-x)\right)(z-x)^k \,/;\, x \in \mathbb{R} \land x < 0$$

06.21.06.0030.01

$$I_z(a,b) \propto \frac{\sin(\pi b)}{\pi\,\Gamma(a)} \left(\frac{1}{x}\right)^{a \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} x^{a\left(\left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor+1\right)} G_{2,2}^{2,2}\!\left(1-x \,\middle|\, \begin{matrix} 1-a,\, b \\ 0,\, b \end{matrix}\right) + O(z-x) \,/;\, x \in \mathbb{R} \land x < 0$$

In the right half-plane

06.21.06.0031.01

$$I_z(a,b) \propto \frac{\sin(\pi b)}{\pi\,\Gamma(a)} \Bigg(-2\, e^{i b \pi \left\lfloor \frac{\arg(x-z)}{2\pi} \right\rfloor} i\,\pi \left\lfloor \frac{\arg(x-z)}{2\pi} \right\rfloor \Gamma(a) + \left(\frac{1}{1-x}\right)^{b \left\lfloor \frac{\arg(x-z)}{2\pi} \right\rfloor} (1-x)^{b \left\lfloor \frac{\arg(x-z)}{2\pi} \right\rfloor} G_{2,2}^{2,2}\!\left(1-x \,\middle|\, \begin{matrix} 1-a,\, b \\ 0,\, b \end{matrix}\right) x^a +$$

$$\left(\frac{1}{1-x}\right)^{b \left\lfloor \frac{\arg(x-z)}{2\pi} \right\rfloor} (1-x)^{b \left\lfloor \frac{\arg(x-z)}{2\pi} \right\rfloor} x^{a-1} \left(a\, G_{2,2}^{2,2}\!\left(1-x \,\middle|\, \begin{matrix} 1-a,\, b \\ 0,\, b \end{matrix}\right) + x\, G_{2,2}^{2,2}\!\left(1-x \,\middle|\, \begin{matrix} -a,\, b-1 \\ 0,\, b-1 \end{matrix}\right)\right)(z-x) +$$

$$\frac{1}{2} \left(\frac{1}{1-x}\right)^{b \left\lfloor \frac{\arg(x-z)}{2\pi} \right\rfloor} (1-x)^{b \left\lfloor \frac{\arg(x-z)}{2\pi} \right\rfloor} x^{a-2} \left(G_{2,2}^{2,2}\!\left(1-x \,\middle|\, \begin{matrix} -a-1,\, b-2 \\ 0,\, b-2 \end{matrix}\right) x^2 +\right.$$

$$\left. a\left((a-1)\, G_{2,2}^{2,2}\!\left(1-x \,\middle|\, \begin{matrix} 1-a,\, b \\ 0,\, b \end{matrix}\right) + 2\,x\, G_{2,2}^{2,2}\!\left(1-x \,\middle|\, \begin{matrix} -a,\, b-1 \\ 0,\, b-1 \end{matrix}\right)\right)\right)(z-x)^2 + \ldots \Bigg) \,/;\, (z \to x) \land x \in \mathbb{R} \land x > 1$$

06.21.06.0032.01

$$I_z(a,b) \propto \frac{\sin(\pi b)}{\pi\,\Gamma(a)} \Bigg(-2\, e^{i b \pi \left\lfloor \frac{\arg(x-z)}{2\pi} \right\rfloor} i\,\pi \left\lfloor \frac{\arg(x-z)}{2\pi} \right\rfloor \Gamma(a) + \left(\frac{1}{1-x}\right)^{b \left\lfloor \frac{\arg(x-z)}{2\pi} \right\rfloor} (1-x)^{b \left\lfloor \frac{\arg(x-z)}{2\pi} \right\rfloor} G_{2,2}^{2,2}\!\left(1-x \,\middle|\, \begin{matrix} 1-a,\, b \\ 0,\, b \end{matrix}\right) x^a +$$

$$\left(\frac{1}{1-x}\right)^{b \left\lfloor \frac{\arg(x-z)}{2\pi} \right\rfloor} (1-x)^{b \left\lfloor \frac{\arg(x-z)}{2\pi} \right\rfloor} x^{a-1} \left(a\, G_{2,2}^{2,2}\!\left(1-x \,\middle|\, \begin{matrix} 1-a,\, b \\ 0,\, b \end{matrix}\right) + x\, G_{2,2}^{2,2}\!\left(1-x \,\middle|\, \begin{matrix} -a,\, b-1 \\ 0,\, b-1 \end{matrix}\right)\right)(z-x) +$$

$$\frac{1}{2} \left(\frac{1}{1-x}\right)^{b \left\lfloor \frac{\arg(x-z)}{2\pi} \right\rfloor} (1-x)^{b \left\lfloor \frac{\arg(x-z)}{2\pi} \right\rfloor} x^{a-2} \left(G_{2,2}^{2,2}\!\left(1-x \,\middle|\, \begin{matrix} -a-1,\, b-2 \\ 0,\, b-2 \end{matrix}\right) x^2 +\right.$$

$$\left. a\left((a-1)\, G_{2,2}^{2,2}\!\left(1-x \,\middle|\, \begin{matrix} 1-a,\, b \\ 0,\, b \end{matrix}\right) + 2\,x\, G_{2,2}^{2,2}\!\left(1-x \,\middle|\, \begin{matrix} -a,\, b-1 \\ 0,\, b-1 \end{matrix}\right)\right)\right)(z-x)^2 + O\!\left((z-x)^3\right) \Bigg) \,/;\, x \in \mathbb{R} \land x > 1$$

06.21.06.0033.01

$$I_z(a,b) = \frac{\sin(\pi b)}{\pi\,\Gamma(a)} \sum_{k=0}^{\infty}\sum_{j=0}^{k} \frac{(-1)^{k-j}\,(-a)_{k-j}\,x^{j-k}}{j!\,(k-j)!} \left(x^a \left(\frac{1}{1-x}\right)^{b\left\lfloor \frac{\arg(x-z)}{2\pi}\right\rfloor} (1-x)^{b\left\lfloor \frac{\arg(x-z)}{2\pi}\right\rfloor} G_{2,2}^{2,2}\!\left(1-x \,\middle|\, \begin{matrix} -a-j+1,\, b-j \\ 0,\, b-j \end{matrix}\right) - 2\pi\,i \left\lfloor \frac{\arg(x-z)}{2\pi} \right\rfloor e^{i b \pi \left\lfloor \frac{\arg(x-z)}{2\pi}\right\rfloor} (-1)^j\,\Gamma(a+j)\,x^{-j} \right) (z-x)^k \;/;\; x \in \mathbb{R} \land x > 1$$

06.21.06.0034.01

$$I_z(a,b) = \frac{\sin(\pi b)}{\Gamma(a)} \sum_{k=0}^{\infty}\sum_{j=0}^{k} \frac{(-1)^k\,(-a)_{k-j}\,x^{j-k}}{j!\,(k-j)!} \left(\Gamma(a+j)\,x^{-j} \left(\csc(b\pi) \left(\frac{1}{1-x}\right)^{b\left\lfloor \frac{\arg(x-z)}{2\pi}\right\rfloor} (1-x)^{b\left\lfloor \frac{\arg(x-z)}{2\pi}\right\rfloor} - 2\,i\,e^{i b \pi \left\lfloor \frac{\arg(x-z)}{2\pi}\right\rfloor} \left\lfloor \frac{\arg(x-z)}{2\pi} \right\rfloor \right) - \csc(b\pi)\,\Gamma(a+b)\,x^a \left(\frac{1}{1-x}\right)^{b\left\lfloor \frac{\arg(x-z)}{2\pi}\right\rfloor} (1-x)^{\left\lfloor \frac{\arg(x-z)}{2\pi}\right\rfloor b+b-j} {}_2\tilde{F}_1(1,\, a+b;\, b-j+1;\, 1-x) \right) (z-x)^k \;/;\; b \notin \mathbb{Z} \bigwedge x \in \mathbb{R} \bigwedge x > 1$$

06.21.06.0035.01

$$I_z(a,b) \propto \frac{\sin(\pi b)}{\pi\,\Gamma(a)} \left(-2\,e^{i b \pi \left\lfloor \frac{\arg(x-z)}{2\pi}\right\rfloor}\, i\,\pi \left\lfloor \frac{\arg(x-z)}{2\pi} \right\rfloor \Gamma(a) + \left(\frac{1}{1-x}\right)^{b\left\lfloor \frac{\arg(x-z)}{2\pi}\right\rfloor} (1-x)^{b\left\lfloor \frac{\arg(x-z)}{2\pi}\right\rfloor} G_{2,2}^{2,2}\!\left(1-x \,\middle|\, \begin{matrix} 1-a,\, b \\ 0,\, b \end{matrix}\right) x^a \right) + O(z-x) \;/;\; x \in \mathbb{R} \land x > 1$$

Expansions at $z = 0$

For the function itself

General case

06.21.06.0001.02

$$I_z(a,b) \propto \frac{z^a}{a\,\mathrm{B}(a,b)} \left(1 + \frac{a\,(1-b)\,z}{a+1} + \frac{a\,(1-b)\,(2-b)\,z^2}{2\,(a+2)} + \ldots\right) /;\; (z \to 0) \bigwedge -a \notin \mathbb{N}$$

06.21.06.0036.01

$$I_z(a,b) \propto \frac{z^a}{a\,\mathrm{B}(a,b)} \left(1 + \frac{a\,(1-b)\,z}{a+1} + \frac{a\,(1-b)\,(2-b)\,z^2}{2\,(a+2)} + O\!\left(z^3\right)\right) /;\; -a \notin \mathbb{N}$$

06.21.06.0002.01

$$I_z(a,b) = \frac{z^a}{\mathrm{B}(a,b)} \sum_{k=0}^{\infty} \frac{(1-b)_k\,z^k}{(a+k)\,k!} \;/;\; |z| < 1 \bigwedge -a \notin \mathbb{N}$$

06.21.06.0003.01

$$I_z(a,b) = \frac{z^a}{a\,\mathrm{B}(a,b)}\, {}_2F_1(a,\, 1-b;\, a+1;\, z)$$

06.21.06.0037.01

$$I_z(a,b) = \frac{z^a\,\Gamma(a+b)}{\Gamma(b)}\, {}_2\tilde{F}_1(a,\, 1-b;\, a+1;\, z)$$

06.21.06.0004.02

$$I_z(a, b) \propto \frac{z^a}{a\,\mathrm{B}(a, b)} (1 + O(z)) \;/;\; -a \notin \mathbb{N}$$

06.21.06.0038.01

$$I_z(a, b) = F_\infty(z, a, b) \;/;\; \left(\left(F_n(z) = \frac{z^a}{\mathrm{B}(a, b)} \sum_{k=0}^{n} \frac{(1-b)_k\, z^k}{(a+k)\, k!} = I_z(a, b) - \frac{z^{a+n+1}\, (1-b)_{n+1}}{\mathrm{B}(a, b)\, (a+n+1)\, (n+1)!}\, {}_3F_2(1, a+n+1, -b+n+2; n+2, a+n+2; z)\right) \land n \in \mathbb{N}\right)$$

Summed form of the truncated series expansion.

Generic formulas for main term

06.21.06.0039.01

$$I_z(a, b) \propto \begin{cases} 1 & -a \in \mathbb{N} \land \neg (b \in \mathbb{N}^+ \land a + b \le 0) \\ \frac{z^a}{a\,\mathrm{B}(a,b)} & \text{True} \end{cases} \;/;\; (z \to 0)$$

Expansions at $z = 1$

For the function itself

General case

06.21.06.0005.02

$$I_z(a, b) \propto 1 - \frac{(1-z)^b\, z^a}{b\,\mathrm{B}(a, b)} \left(1 - \frac{(a+b)\,(z-1)}{1+b} + \frac{(a+b)\,(1+a+b)\,(z-1)^2}{(1+b)\,(2+b)} - \ldots\right) /;\; (z \to 1) \land -b \notin \mathbb{N}$$

06.21.06.0040.01

$$I_z(a, b) \propto 1 - \frac{(1-z)^b\, z^a}{b\,\mathrm{B}(a, b)} \left(1 - \frac{(a+b)\,(z-1)}{1+b} + \frac{(a+b)\,(1+a+b)\,(z-1)^2}{(1+b)\,(2+b)} - O\left((z-1)^3\right)\right) /;\; -b \notin \mathbb{N}$$

06.21.06.0006.01

$$I_z(a, b) = 1 - \frac{(1-z)^b\, z^a}{b\,\mathrm{B}(a, b)} \sum_{k=0}^{\infty} \frac{(-1)^k\, (a+b)_k\, (z-1)^k}{(b+1)_k} \;/;\; |z-1| < 1 \land -b \notin \mathbb{N}$$

06.21.06.0007.01

$$I_z(a, b) = 1 - \frac{(1-z)^b\, z^a}{b\,\mathrm{B}(a, b)}\, {}_2F_1(1, a+b; b+1; 1-z)$$

06.21.06.0041.01

$$I_z(a, b) = 1 - \frac{\Gamma(a+b)}{\Gamma(a)} (1-z)^b\, z^a\, {}_2\tilde{F}_1(1, a+b; b+1; 1-z)$$

06.21.06.0008.02

$$I_z(a, b) \propto 1 - \frac{(1-z)^b}{b\,\mathrm{B}(a, b)} \left(1 + \frac{(a-1)\, b\, (z-1)}{1+b} + \frac{(a-1)\,(a-2)\, b\, (z-1)^2}{2\,(2+b)} + \ldots\right) /;\; (z \to 1) \land -b \notin \mathbb{N}$$

06.21.06.0042.01

$$I_z(a,b)\propto 1-\frac{(1-z)^b}{b\,\mathrm{B}(a,b)}\left(1+\frac{(a-1)\,b\,(z-1)}{1+b}+\frac{(a-1)\,(a-2)\,b\,(z-1)^2}{2\,(2+b)}+O\!\left((z-1)^3\right)\right)/;\ -b\notin\mathbb{N}$$

06.21.06.0009.01

$$I_z(a,b)=1+\frac{a\sin(\pi\,a)\,(1-z)^b}{\pi}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{j+k}\,\Gamma(j-a)\,\Gamma(a+b+k)\,(z-1)^{j+k}}{j!\,\Gamma(b+k+1)}\ /;\ |z-1|<1\bigwedge -b\notin\mathbb{N}$$

06.21.06.0010.01

$$I_z(a,b)=1-\frac{(1-z)^b}{b\,\mathrm{B}(a,b)}\,F_{0\times0\times1}^{0\times1\times2}\!\left(\begin{matrix};-a;1,a+b;\\;;b+1;\end{matrix}\ 1-z,\,1-z\right)/;\ -b\notin\mathbb{N}$$

06.21.06.0011.02

$$I_z(a,b)\propto 1-\frac{(1-z)^b}{b\,\mathrm{B}(a,b)}\,(1+O(z-1))\ /;\ -b\notin\mathbb{N}$$

Generic formulas for main term

06.21.06.0043.01

$$I_z(a,b)\propto\begin{cases}0 & -b\in\mathbb{N}\\ 1-\frac{(1-z)^b}{b\,\mathrm{B}(a,b)} & \text{True}\end{cases}\ /;\ (z\to1)$$

06.21.06.0044.01

$$I_z(a,b)\propto\begin{cases}0 & -b\in\mathbb{N}\\ 1 & \mathrm{Re}(b)>0\\ -\frac{(1-z)^b}{b\,\mathrm{B}(a,b)} & \mathrm{Re}(b)<0\\ 1-\frac{(1-z)^b}{b\,\mathrm{B}(a,b)} & \text{True}\end{cases}\ /;\ (z\to1)$$

Expansions at $z=\infty$

General case

06.21.06.0012.02

$$I_z(a,b)\propto\csc(\pi\,(a+b))\sin(b\,\pi)\,z^a(-z)^{-a}+\frac{(-z)^{b-1}\,z^a}{(a+b-1)\,\mathrm{B}(a,b)}\left(1+\frac{(1-b)\,(1-a-b)}{(2-a-b)\,z}+\frac{(1-b)\,(2-b)\,(1-a-b)}{2\,(3-a-b)\,z^2}+\ldots\right)/;$$
$$(|z|\to\infty)\bigwedge a+b\notin\mathbb{N}^+$$

06.21.06.0045.01

$$I_z(a,b)\propto\csc(\pi\,(a+b))\sin(b\,\pi)\,z^a(-z)^{-a}+\frac{(-z)^{b-1}\,z^a}{(a+b-1)\,\mathrm{B}(a,b)}\left(1+\frac{(1-b)\,(1-a-b)}{(2-a-b)\,z}+\frac{(1-b)\,(2-b)\,(1-a-b)}{2\,(3-a-b)\,z^2}+O\!\left(\frac{1}{z^3}\right)\right)/;$$
$$a+b\notin\mathbb{N}^+$$

06.21.06.0013.01

$$I_z(a,b)=\csc(\pi\,(a+b))\sin(b\,\pi)\,z^a(-z)^{-a}+\frac{z^a\,(-z)^{b-1}}{(a+b-1)\,\mathrm{B}(a,b)}\sum_{k=0}^{\infty}\frac{(1-b)_k\,(1-a-b)_k\,z^{-k}}{(2-a-b)_k\,k!}\ /;\ |z|>1\bigwedge a+b\notin\mathbb{N}^+$$

06.21.06.0014.01

$$I_z(a,b)=\csc(\pi\,(a+b))\sin(b\,\pi)\,z^a(-z)^{-a}+\frac{z^a\,(-z)^{b-1}}{(a+b-1)\,\mathrm{B}(a,b)}\,{}_2F_1\!\left(1-b,1-a-b;2-a-b;\frac{1}{z}\right)/;\ z\notin(0,1)\bigwedge a+b\notin\mathbb{N}^+$$

06.21.06.0015.02

$$I_z(a,\,b) \propto \csc(\pi\,(a+b))\sin(b\,\pi)\,z^a(-z)^{-a} + \frac{z^a\,(-z)^{b-1}}{(a+b-1)\,\mathrm{B}(a,\,b)}\left(1+O\!\left(\frac{1}{z}\right)\right) /;\ a+b\notin\mathbb{N}^+$$

06.21.06.0046.01

$$I_z(a,\,b) \propto \begin{cases} e^{-i\,a\,\pi}\csc((a+b)\,\pi)\sin(b\,\pi) - \frac{e^{i\,b\,\pi}\,z^{a+b-1}}{(a+b-1)\,\mathrm{B}(a,b)} & \arg(z)\le 0 \\ e^{i\,a\,\pi}\csc((a+b)\,\pi)\sin(b\,\pi) - \frac{e^{-i\,b\,\pi}\,z^{a+b-1}}{(a+b-1)\,\mathrm{B}(a,b)} & \text{True} \end{cases} /;\ a+b\notin\mathbb{N}^+ \bigwedge (|z|\to\infty)$$

06.21.06.0047.01

$$I_z(a,\,b) = F_\infty(z,\,a,\,b) /;$$
$$\left(\left(F_n(z,\,a,\,b) = \frac{z^a(-z)^{b-1}}{(a+b-1)\,\mathrm{B}(a,\,b)}\sum_{k=0}^{n}\frac{(1-b)_k\,(1-a-b)_k\,z^{-k}}{(2-a-b)_k\,k!} + \csc(\pi\,(a+b))\sin(b\,\pi)\,(-z)^{-a}\,z^a = I_z(a,\,b) + \frac{(1-b)_{n+1}\,(-z)^{b-1}\,z^{a-n-1}}{(n+1)!\,(2-a-b+n)\,\mathrm{B}(a,\,b)}\,{}_3F_2\!\left(1,\,-b+n+2,\,-a-b+n+2;\,n+2,\,-a-b+n+3;\,\frac{1}{z}\right)\right)\bigwedge n\in\mathbb{N}\right)\bigwedge a+b\notin\mathbb{N}^+$$

Summed form of the truncated series expansion.

Logarithmic cases

06.21.06.0048.01

$$I_z(a,\,1-a) \propto \frac{\sin(a\,\pi)}{\pi}\,(-z)^{-a}\,z^a\,(\log(-z)-\psi(a)-\gamma) - a\,\frac{\sin(a\,\pi)}{\pi}\,(-z)^{-a}\,z^{a-1}\left(1+\frac{1+a}{4\,z}+\frac{(1+a)\,(2+a)}{18\,z^2}+\ldots\right) /;\ (|z|\to\infty)$$

06.21.06.0049.01

$$I_z(a,\,2-a) \propto \frac{\sin(a\,\pi)\,z^a\,(-z)^{1-a}}{(1-a)\,\pi} - \frac{a\sin(a\,\pi)}{2\,\pi}\,(-z)^{-a}\,z^{a-1}\left(1+\frac{1+a}{6\,z}+\frac{(1+a)\,(2+a)}{36\,z^2}+\ldots\right) + \frac{\sin(a\,\pi)}{\pi}\,(-z)^{-a}\,z^a(\log(-z)-\psi(a)-\gamma+1) /;\ (|z|\to\infty)$$

06.21.06.0050.01

$$I_z(a,\,1-a+n) = -\frac{a\sin(a\,\pi)\,(-z)^{-a}\,z^{a-1}}{\pi\,(n+1)}\left(1+\frac{1+a}{2\,(2+n)\,z}+\frac{(1+a)\,(2+a)}{3\,(2+n)\,(3+n)\,z^2}+\ldots\right) + \frac{\sin(a\,\pi)\,(-z)^{-a}\,z^a}{\pi}\,(\log(-z)+\psi(n+1)-\psi(a)) + \frac{z^a\,(-z)^{n-a}}{\mathrm{B}(a,\,1-a+n)}\sum_{k=0}^{n-1}\frac{(a-n)_k\,z^{-k}}{(n-k)\,k!} /;\ (|z|\to\infty)\land n\in\mathbb{N}$$

06.21.06.0051.01

$$I_z(a,\,1-a) = \frac{\sin(a\,\pi)\,(-z)^{-a}\,z^a}{\pi}\,(\log(-z)-\psi(a)-\gamma) - \frac{a\sin(a\,\pi)\,(-z)^{-a}\,z^{a-1}}{\pi}\,{}_3F_2\!\left(1,\,1,\,a+1;\,2,\,2;\,\frac{1}{z}\right) /;\ z\notin(0,\,1)$$

06.21.06.0052.01

$$I_z(a,\,1-a+n) = -\frac{a\sin(a\,\pi)\,(-z)^{-a}\,z^{a-1}}{\pi\,(n+1)}\sum_{k=0}^{\infty}\frac{(a+1)_k\,z^{-k}}{(n+2)_k\,(k+1)} + \frac{\sin(a\,\pi)}{\pi}\,z^a\,(-z)^{-a}\,(\log(-z)+\psi(n+1)-\psi(a)) + \frac{z^a\,(-z)^{n-a}}{\mathrm{B}(a,\,1-a+n)}\sum_{k=0}^{n-1}\frac{(a-n)_k\,z^{-k}}{(n-k)\,k!} /;\ n\in\mathbb{N}\land|z|>1$$

06.21.06.0016.02

$$I_z(a,\,1-a+n) \propto -\frac{a\sin(a\pi)}{\pi\,(n+1)}\,(-z)^{-a}\,z^{a-1}\,{}_3F_2\!\left(1,\,1,\,a+1;\,2,\,n+2;\,\frac{1}{z}\right)+\frac{\sin(a\pi)}{\pi}\,(-z)^{-a}\,z^{a}\,(\log(-z)+\psi(n+1)-\psi(a))+\frac{z^a\,(-z)^{n-a}}{B(a,\,1-a+n)}\sum_{k=0}^{n-1}\frac{(a-n)_k\,z^{-k}}{(n-k)\,k!}\;/;\;n\in\mathbb{N}\bigwedge z\notin(0,\,1)$$

06.21.06.0017.02

$$I_z(a,\,b)\propto -\frac{\sin(a\pi)}{\pi}\,(-z)^{-a}\,z^{a}(\log(-z)-\psi(a)+\psi(a+b))+\frac{z^a\,(-z)^{b-1}}{(a+b-1)\,B(a,\,b)}\left(1+O\!\left(\frac{1}{z}\right)\right)\;/;\;a+b-1\in\mathbb{N}^+$$

06.21.06.0018.02

$$I_z(a,\,b)\propto\frac{\sin(\pi a)}{\pi}\,(-z)^{-a}\,z^{a}\,(\log(-z)-\psi(a)-\gamma)-\frac{a\sin(\pi a)}{\pi}\,(-z)^{-a}\,z^{a-1}\left(1+O\!\left(\frac{1}{z}\right)\right)\;/;\;a+b=1$$

06.21.06.0053.01

$$I_z(a,\,b)\propto\begin{cases}-\frac{e^{i b\pi}\,z^{a+b-1}}{(a+b-1)\,B(a,b)}-\frac{e^{-i a\pi}\log(z)\sin(a\pi)}{\pi} & \arg(z)\le 0\\ -\frac{e^{-i b\pi}\,z^{a+b-1}}{(a+b-1)\,B(a,b)}-\frac{e^{i a\pi}\log(z)\sin(a\pi)}{\pi} & \text{True}\end{cases}\;/;\;a+b-1\in\mathbb{N}^+\bigwedge(|z|\to\infty)$$

06.21.06.0054.01

$$I_z(a,\,b)\propto\begin{cases}\frac{e^{-i a\pi}\log(z)\sin(a\pi)}{\pi} & \arg(z)\le 0\\ \frac{e^{i a\pi}\log(z)\sin(a\pi)}{\pi} & \text{True}\end{cases}\;/;\;a+b=1\bigwedge(|z|\to\infty)$$

Generic formulas for main term

06.21.06.0055.01

$$I_z(a,\,b)\propto\begin{cases}\frac{z^a\,(-z)^{b-1}}{(a+b-1)\,B(a,b)}-\frac{\sin(a\pi)\,z^a\,(-z)^{-a}\,(\log(-z)-\psi(a)+\psi(a+b))}{\pi} & a+b-1\in\mathbb{N}^+\\ \frac{\sin(\pi a)\,z^a\,(-z)^{-a}\,(\log(-z)-\psi(a)-\gamma)}{\pi}-\frac{a\sin(\pi a)\,z^{a-1}\,(-z)^{-a}}{\pi} & a+b=1\\ \frac{z^a\,(-z)^{b-1}}{(a+b-1)\,B(a,b)}+\csc(\pi\,(a+b))\sin(b\pi)\,z^a\,(-z)^{-a} & \text{True}\end{cases}\;/;\;(|z|\to\infty)$$

06.21.06.0056.01

$$I_z(a,\,b)\propto\begin{cases}e^{-i a\pi}\left(\frac{e^{i(a+b)\pi}\,z^{a+b-1}}{(1-a-b)\,B(a,b)}-\frac{\sin(a\pi)\,(i\pi+\log(z)-\psi(a)+\psi(a+b))}{\pi}\right) & \arg(z)\le 0\bigwedge a+b-1\in\mathbb{N}^+\\ -\frac{e^{-i a\pi}\sin(a\pi)\,(a-i\pi z+z\,(-\log(z)+\psi(a)+\gamma))}{\pi z} & \arg(z)\le 0\bigwedge a+b=1\\ e^{-i a\pi}\left(\frac{e^{i(a+b)\pi}\,z^{a+b-1}}{(1-a-b)\,B(a,b)}+\csc((a+b)\pi)\sin(b\pi)\right) & \arg(z)\le 0\\ \frac{e^{-i b\pi}\,z^{a+b-1}}{(1-a-b)\,B(a,b)}+\frac{\left(\sin(a\pi)\,e^{i a\pi}\right)(\pi i-\log(z)+\psi(a)-\psi(a+b))}{\pi} & \arg(z)>0\bigwedge a+b-1\in\mathbb{N}^+\\ -\frac{e^{i a\pi}\sin(a\pi)\,(a+i\pi z+z\,(-\log(z)+\psi(a)+\gamma))}{\pi z} & \arg(z)>0\bigwedge a+b=1\\ \frac{e^{-i b\pi}\,z^{a+b-1}}{(1-a-b)\,B(a,b)}+e^{i a\pi}\csc((a+b)\pi)\sin(b\pi) & \text{True}\end{cases}\;/;\;(|z|\to\infty)$$

Expansions at generic point $a = a_0$

For the function itself

06.21.06.0057.01

$$I_z(a,b) \propto I_z(a_0,b) + \frac{1}{a_0^2\,\mathrm{B}(a_0,b)}\left(a_0^2\,\mathrm{B}_z(a_0,b)\,(\log(z)+\psi(b+a_0)-\psi(a_0)) - z^{a_0}\,{}_3F_2(a_0,a_0,1-b;a_0+1,a_0+1;z)\right)(a-a_0) + \frac{1}{2\,a_0^3\,\mathrm{B}(a_0,b)}\Big(2\,({}_4F_3(1-b,a_0,a_0,a_0;a_0+1,a_0+1,a_0+1;z) - {}_3F_2(1-b,a_0,a_0;a_0+1,a_0+1;z)\,(\log(z)-\psi(a_0)+\psi(b+a_0))\,a_0)\,z^{a_0} + \mathrm{B}_z(a_0,b)\left(\log^2(z)+(\psi(a_0)-\psi(b+a_0))\,(-2\log(z)+\psi(a_0)-\psi(b+a_0))-\psi^{(1)}(a_0)+\psi^{(1)}(b+a_0)\right)a_0^3\Big)\,(a-a_0)^2 + \ldots\;/;\;(a\to a_0)$$

06.21.06.0058.01

$$I_z(a,b) \propto I_z(a_0,b) + \frac{1}{a_0^2\,\mathrm{B}(a_0,b)}\left(a_0^2\,\mathrm{B}_z(a_0,b)\,(\log(z)+\psi(b+a_0)-\psi(a_0)) - z^{a_0}\,{}_3F_2(a_0,a_0,1-b;a_0+1,a_0+1;z)\right)(a-a_0) + \frac{1}{2\,a_0^3\,\mathrm{B}(a_0,b)}\Big(2\,({}_4F_3(1-b,a_0,a_0,a_0;a_0+1,a_0+1,a_0+1;z) - {}_3F_2(1-b,a_0,a_0;a_0+1,a_0+1;z)\,(\log(z)-\psi(a_0)+\psi(b+a_0))\,a_0)\,z^{a_0} + \mathrm{B}_z(a_0,b)\left(\log^2(z)+(\psi(a_0)-\psi(b+a_0))\,(-2\log(z)+\psi(a_0)-\psi(b+a_0))-\psi^{(1)}(a_0)+\psi^{(1)}(b+a_0)\right)a_0^3\Big)\,(a-a_0)^2 + O\left((a-a_0)^3\right)$$

06.21.06.0059.01

$$I_z(a,b) = -\frac{b\sin(\pi b)\,\Gamma(a_0)\,z^{a_0}}{\pi}\sum_{k=0}^{\infty}\sum_{j=0}^{k}(-1)^{k-j}\log^j(z)\,\Gamma(b+a_0)^{-j+k+1}\,{}_{k-j+2}\tilde{F}_{k-j+1}\left(1+b,d_1,d_2,\ldots,d_{k-j+1};d_1+1,d_2+1,\ldots,d_{k-j+1}+1;1\right)\sum_{i=0}^{j}\frac{1}{(j-i)!}\,{}_{i+2}\tilde{F}_{i+1}(1-b,c_1,c_2,\ldots,c_{i+1};c_1+1,c_2+1,\ldots,c_{i+1}+1;z)\left(-\frac{\Gamma(a_0)}{\log(z)}\right)^i(a-a_0)^k\;/;$$
$$d_1 = d_2 = \ldots = d_{k+1} = b+a_0 \land c_1 = c_2 = \ldots = c_{k+1} = a_0 \land k\in\mathbb{N}$$

06.21.06.0060.01

$$I_z(a,b) \propto I_z(a_0,b)\,(1+O(a-a_0))$$

Expansions at generic point $b = b_0$

For the function itself

06.21.06.0061.01

$$I_z(a,b) \propto I_z(a,b_0) + \frac{(1-z)^{b_0}\,\Gamma(a+b_0)}{\Gamma(a)}\Bigg(\left(\Gamma(b_0)\,{}_3\tilde{F}_2(1-a,b_0,b_0;b_0+1,b_0+1;1-z) - (\log(1-z)-\psi(b_0)+\psi(a+b_0))\,{}_2\tilde{F}_1(1-a,b_0;b_0+1;1-z)\right)(b-b_0) - \Bigg(\Gamma(b_0)^2\,{}_4\tilde{F}_3(1-a,b_0,b_0,b_0;b_0+1,b_0+1,b_0+1;1-z) - \Gamma(b_0)\,(\log(1-z)-\psi(b_0)+\psi(a+b_0))\,{}_3\tilde{F}_2(1-a,b_0,b_0;b_0+1,b_0+1;1-z) + \frac{1}{2}\left(\log^2(1-z)+(\psi(b_0)-\psi(a+b_0))\,(-2\log(1-z)+\psi(b_0)-\psi(a+b_0))-\psi^{(1)}(b_0)+\psi^{(1)}(a+b_0)\right){}_2\tilde{F}_1(1-a,b_0;b_0+1;1-z)\Bigg)(b-b_0)^2 + \ldots\Bigg)\;/;\;(b\to b_0)$$

06.21.06.0062.01

$$I_z(a,\,b) \propto I_z(a,\,b_0) + \frac{(1-z)^{b_0}\,\Gamma(a+b_0)}{\Gamma(a)} \left(\left(\Gamma(b_0)\,{}_3\tilde{F}_2(1-a,\,b_0,\,b_0;\,b_0+1,\,b_0+1;\,1-z) - (\log(1-z)-\psi(b_0)+\psi(a+b_0))\,{}_2\tilde{F}_1(1-a,\,b_0;\,b_0+1;\,1-z)\right)(b-b_0) - \left(\Gamma(b_0)^2\,{}_4\tilde{F}_3(1-a,\,b_0,\,b_0,\,b_0;\,b_0+1,\,b_0+1,\,b_0+1;\,1-z) - \Gamma(b_0)\,(\log(1-z)-\psi(b_0)+\psi(a+b_0))\,{}_3\tilde{F}_2(1-a,\,b_0,\,b_0;\,b_0+1,\,b_0+1;\,1-z) + \frac{1}{2}\left(\log^2(1-z) + (\psi(b_0)-\psi(a+b_0))\,(-2\log(1-z)+\psi(b_0)-\psi(a+b_0)) - \psi^{(1)}(b_0) + \psi^{(1)}(a+b_0)\right) {}_2\tilde{F}_1(1-a,\,b_0;\,b_0+1;\,1-z)\right)(b-b_0)^2 + O\left((b-b_0)^3\right)\right)$$

06.21.06.0063.01

$$I_z(a,\,b) = I_z(a,\,b_0) + \frac{a\sin(\pi\,a)\,\Gamma(b_0)\,(1-z)^{b_0}}{\pi} \sum_{k=1}^{\infty}\sum_{j=0}^{k}(-1)^{k-j}\log^j(1-z)\,\Gamma(a+b_0)^{k-j+1}\;{}_{k-j+2}\tilde{F}_{k-j+1}\left(a+1,\,d_1,\,d_2,\,\ldots,\,d_{k-j+1};\,d_1+1,\,d_2+1,\,\ldots,\,d_{k-j+1}+1;\,1\right) \sum_{i=0}^{j}\frac{1}{(j-i)!}\;{}_{i+2}\tilde{F}_{i+1}(1-a,\,c_1,\,c_2,\,\ldots,\,c_{i+1};\,c_1+1,\,c_2+1,\,\ldots,\,c_{i+1}+1;\,1-z)\left(-\frac{\Gamma(b_0)}{\log(1-z)}\right)^i (b-b_0)^k \,/;\; d_1 = d_2 = \ldots = d_{k+1} = a+b_0 \bigwedge c_1 = c_2 = \ldots = c_{k+1} = b_0 \bigwedge k \in \mathbb{N}$$

06.21.06.0064.01

$$I_z(a,\,b) \propto I_z(a,\,b_0)\,(1+O(b-b_0))$$

Residue representations

06.21.06.0019.01

$$I_z(a,\,b) = \frac{\Gamma(a+b)\sin(\pi\,b)\,z^a}{\pi\,\Gamma(a)} \sum_{j=0}^{\infty}\mathrm{res}_s\left(\frac{\Gamma(1-b-s)\,(-z)^{-s}}{a-s}\,\Gamma(s)\right)(-j) \,/;\; |z| < 1$$

06.21.06.0020.02

$$I_z(a,\,b) = -\frac{\Gamma(a+b)\sin(\pi\,b)\,z^a}{\pi\,\Gamma(a)}\left(\sum_{j=0}^{\infty}\mathrm{res}_s\left(\Gamma(s)\,(-z)^{-s}\,\frac{\Gamma(1-b-s)}{a-s}\right)(a) + \sum_{j=0}^{\infty}\mathrm{res}_s\left(\Gamma(s)\,(-z)^{-s}\,\frac{\Gamma(1-b-s)}{a-s}\right)(j-b+1)\right) /;\; |z| > 1 \bigwedge a+b \notin \mathbb{N}^+$$

06.21.06.0065.01

$$I_z(a,\,1-a+n) = \frac{(-1)^{n-1}\,n!\,\sin(a\,\pi)}{\pi\,\Gamma(a)}\,z^a\left(\mathrm{res}_s\left(\Gamma(s)\,(-z)^{-s}\,\frac{\Gamma(a-n-s)}{a-s}\right)(a) + \sum_{j=0}^{n-1}\mathrm{res}_s\left(\frac{\Gamma(s)\,(-z)^{-s}}{a-s}\,\Gamma(a-n-s)\right)(a+j-n) + \sum_{j=n+1}^{\infty}\mathrm{res}_s\left(\frac{\Gamma(s)\,(-z)^{-s}}{a-s}\,\Gamma(a-n-s)\right)(a+j-n)\right) /;\; |z| > 1 \bigwedge n \notin \mathbb{N}$$

Integral representations

On the real axis

Of the direct function

06.21.07.0001.01

$$I_z(a, b) = \frac{1}{B(a, b)} \int_0^z t^{a-1} (1-t)^{b-1} \, dt \,/;\, \mathrm{Re}(a) > 0$$

06.21.07.0002.01

$$I_z(a, b) = \frac{1}{B(a, b)} \left(\int_0^z t^{a-1} \left((1-t)^{b-1} - \sum_{k=0}^{\lfloor -\mathrm{Re}(a) \rfloor} \frac{(1-b)_k z^k}{k!} \right) dt + \sum_{k=0}^{\lfloor -\mathrm{Re}(a) \rfloor} \frac{(1-b)_k z^{a+k}}{(a+k) k!} \right)$$

Contour integral representations

06.21.07.0003.01

$$I_z(a, b) = \frac{\Gamma(a+b) \sin(\pi b) z^a}{2 \pi^2 i \, \Gamma(a)} \int_{\mathcal{L}} \frac{\Gamma(s) \Gamma(a-s) \Gamma(1-b-s)}{\Gamma(a+1-s)} (-z)^{-s} \, ds \,/;\, |\arg(-z)| < \pi$$

06.21.07.0004.01

$$I_z(a, b) = \frac{\sin(\pi b) z^a}{2 \pi^2 i \, \Gamma(a)} \int_{\mathcal{L}} \Gamma(s) \Gamma(b+s) \Gamma(a-s) \Gamma(1-b-s) (1-z)^{-s} \, ds \,/;\, |\arg(1-z)| < \pi$$

06.21.07.0005.01

$$I_z(a, b) = \frac{\Gamma(a+b) \sin(\pi b) z^a}{2 \pi^2 i \, \Gamma(a)} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{\Gamma(s) \Gamma(a-s) \Gamma(1-b-s)}{\Gamma(a+1-s)} (-z)^{-s} \, ds \,/;\, 0 < \gamma < \min(\mathrm{Re}(a), 1 - \mathrm{Re}(b)) \land |\arg(-z)| < \pi$$

06.21.07.0006.01

$$I_z(a, b) = \frac{\sin(\pi b) z^a}{2 \pi^2 i \, \Gamma(a)} \int_{\gamma - i\infty}^{\gamma + i\infty} \Gamma(s) \Gamma(b+s) \Gamma(a-s) \Gamma(1-b-s) (1-z)^{-s} \, ds \,/;$$

$$\max(0, -\mathrm{Re}(b)) < \gamma < \min(\mathrm{Re}(a), 1 - \mathrm{Re}(b)) \land |\arg(1-z)| < \pi$$

Continued fraction representations

06.21.10.0001.01

$$I_z(a, b) = \frac{z^a (1-z)^b}{a \, B(a, b)} \cfrac{1}{1 + \cfrac{r(1)}{1 + \cfrac{r(2)}{1 + \cfrac{r(3)}{1 + \cfrac{r(4)}{1 + \cfrac{r(5)}{1 + \ldots}}}}}} \,/;\, r(2k+1) = -\frac{(a+k)(a+b+k) z}{(a+2k)(a+2k+1)} \land r(2k) = \frac{k(b-k) z}{(a+2k-1)(a+2k)}$$

06.21.10.0002.01

$$I_z(a, b) = \frac{z^a (1-z)^b}{a \, B(a, b) \left(\mathrm{K}_k(r(k), 1)_1^\infty + 1 \right)} \,/;\, r(2k+1) = -\frac{(a+k)(a+b+k) z}{(a+2k)(a+2k+1)} \land r(2k) = \frac{k(b-k) z}{(a+2k-1)(a+2k)}$$

Differential equations

Ordinary linear differential equations and wronskians

For the direct function itself

06.21.13.0001.01

$$(1-z)\,z\,w''(z)+(1-a+(a+b-2)\,z)\,w'(z)=0\;/;\;w(z)=c_1\,I_z(a,b)+c_2$$

06.21.13.0002.01

$$W_z(1,\,I_z(a,b))=\frac{(1-z)^{b-1}\,z^{a-1}}{\mathrm{B}(a,b)}$$

06.21.13.0003.01

$$w''(z)-\left(\frac{(-a+(a+b-2)\,g(z)+1)\,g'(z)}{(g(z)-1)\,g(z)}+\frac{g''(z)}{g'(z)}\right)w'(z)=0\;/;\;w(z)=c_1+c_2\,I_{g(z)}(a,b)$$

06.21.13.0004.01

$$W_z\big(1,\,I_{g(z)}(a,b)\big)=\frac{(1-g(z))^{b-1}\,g(z)^{a-1}\,g'(z)}{\mathrm{B}(a,b)}$$

06.21.13.0005.01

$$w''(z)-h(z)\left(\frac{(-a+(a+b-2)\,g(z)+1)\,g'(z)}{(g(z)-1)\,g(z)\,h(z)}+\frac{2\,h'(z)}{h(z)^2}+\frac{g''(z)}{h(z)\,g'(z)}\right)w'(z)+\left(\frac{2\,h'(z)^2}{h(z)^2}+\frac{(-a+(a+b-2)\,g(z)+1)\,g'(z)\,h'(z)}{(g(z)-1)\,g(z)\,h(z)}+\frac{g''(z)\,h'(z)}{h(z)\,g'(z)}-\frac{h''(z)}{h(z)}\right)w(z)=0\;/;\;w(z)=c_1\,h(z)+c_2\,h(z)\,I_{g(z)}(a,b)$$

06.21.13.0006.01

$$W_z\big(h(z),\,h(z)\,I_{g(z)}(a,b)\big)=\frac{(1-g(z))^{b-1}\,g(z)^{a-1}\,h(z)^2\,g'(z)}{\mathrm{B}(a,b)}$$

06.21.13.0007.01

$$(d\,z^r-1)\,z^2\,w''(z)+(-d\,((a+b-1)\,r+2\,s-1)\,z^r+a\,r+2\,s-1)\,z\,w'(z)+s\,((b-1)\,d\,r\,z^r+(a\,r+s)\,(d\,z^r-1))\,w(z)=0\;/;\;w(z)=c_1\,z^s+c_2\,z^s\,I_{d\,z^r}(a,b)$$

06.21.13.0008.01

$$W_z\big(z^s,\,z^s\,I_{d\,z^r}(a,b)\big)=\frac{r\,z^{2s-1}\,(d\,z^r)^a\,(1-d\,z^r)^{b-1}}{\mathrm{B}(a,b)}$$

06.21.13.0009.01

$$(d\,r^z-1)\,w''(z)+(-d\,((a+b-1)\log(r)+2\log(s))\,r^z+a\log(r)+2\log(s))\,w'(z)+\log(s)\,(d\,((a+b-1)\log(r)+\log(s))\,r^z-a\log(r)-\log(s))\,w(z)=0\;/;\;w(z)=c_1\,s^z+c_2\,s^z\,I_{d\,r^z}(a,b)$$

06.21.13.0010.01

$$W_z\big(s^z,\,s^z\,I_{d\,r^z}(a,b)\big)=\frac{d\,r^z\,(d\,r^z)^{a-1}\,(1-d\,r^z)^{b-1}\,s^{2z}\log(r)}{\mathrm{B}(a,b)}$$

Transformations

Transformations and argument simplifications

Argument involving basic arithmetic operations

06.21.16.0001.01

$$I_{1-z}(a,b)=1-I_z(b,a)$$

06.21.16.0002.01

$$I_z(a+1,\, b) = I_z(a,\, b) - \frac{\Gamma(a+b)}{\Gamma(a+1)\,\Gamma(b)} (1-z)^b\, z^a$$

06.21.16.0003.01

$$I_z(a-1,\, b) = I_z(a,\, b) + \frac{\Gamma(a+b-1)}{\Gamma(a)\,\Gamma(b)} (1-z)^b\, z^{a-1}$$

06.21.16.0004.01

$$I_z(a+n,\, b) = I_z(a,\, b) - \frac{1}{(a+b+n-1)\,\mathrm{B}(a+n,\, b)} \sum_{k=0}^{n-1} \frac{(1-a-n)_k\, (1-z)^b\, z^{a-k+n-1}}{(2-a-b-n)_k} \;/;\; n \in \mathbb{N}$$

06.21.16.0005.01

$$I_z(a-n,\, b) = I_z(a,\, b) + \frac{1}{(a+b-1)\,\mathrm{B}(a,\, b)} \sum_{k=0}^{n-1} \frac{(1-a)_k\, (1-z)^b\, z^{a-k-1}}{(2-a-b)_k} \;/;\; n \in \mathbb{N}$$

Identities

Recurrence identities

Consecutive neighbors

06.21.17.0001.01

$$I_z(a,\, b) = I_z(a+1,\, b) + \frac{\Gamma(a+b)}{\Gamma(a+1)\,\Gamma(b)} (1-z)^b\, z^a$$

06.21.17.0002.01

$$I_z(a,\, b) = I_z(a-1,\, b) - \frac{\Gamma(a+b-1)}{\Gamma(a)\,\Gamma(b)} (1-z)^b\, z^{a-1}$$

Distant neighbors

06.21.17.0003.01

$$I_z(a,\, b) = I_z(a+n,\, b) + \frac{z^{a-1}\, (1-z)^b}{\mathrm{B}(a,\, b)\, (a+b-1)} \sum_{k=1}^{n} \frac{(a+b-1)_k\, z^k}{(a)_k} \;/;\; n \in \mathbb{N}$$

06.21.17.0004.01

$$I_z(a,\, b) = I_z(a-n,\, b) - \frac{z^{a-1}\, (1-z)^b}{\mathrm{B}(a,\, b)\, (a+b-1)} \sum_{k=0}^{n-1} \frac{(1-a)_k\, z^{-k}}{(2-a-b)_k} \;/;\; n \in \mathbb{N}$$

Functional identities

Relations between contiguous functions

06.21.17.0005.01

$$I_z(a,\, b) = \frac{1}{a+b} (a\, I_z(a+1,\, b) + b\, I_z(a,\, b+1))$$

06.21.17.0006.01

$$I_z(a,\, b) = z\, I_z(a-1,\, b) + (1-z)\, I_z(a,\, b-1)$$

Additional relations between contiguous functions

06.21.17.0007.01

$$I_z(a,\,b) = \frac{1}{a+b-a\,z}\,(a\,(1-z)\,I_z(a+1,\,b-1) + b\,I_z(a,\,b+1))$$

Major general cases

06.21.17.0008.01

$$I_z(a,\,b) = 1 - I_{1-z}(b,\,a)$$

06.21.17.0009.01

$$I_z(a,\,b) = (-z)^{-a}\,z^a\left(\sin(a\,\pi)\,(-z)^{a+b}\left(\frac{1}{z}\right)^{a+b} I_{\frac{1}{z}}(1-a-b,\,b) + \sin(b\,\pi)\right)\csc((a+b)\,\pi)\;/;\;a+b\notin\mathbb{N}^+\bigwedge z\notin(0,\,1)$$

Differentiation

Low-order differentiation

With respect to *z*

06.21.20.0001.01

$$\frac{\partial I_z(a,\,b)}{\partial z} = \frac{(1-z)^{b-1}\,z^{a-1}}{\mathrm{B}(a,\,b)}$$

06.21.20.0002.01

$$\frac{\partial^2 I_z(a,\,b)}{\partial z^2} = -\frac{(1-z)^{b-2}\,z^{a-2}\,((z-1)\,a + (b-2)\,z + 1)}{\mathrm{B}(a,\,b)}$$

With respect to *a*

06.21.20.0003.01

$$\frac{\partial I_z(a,\,b)}{\partial a} = (\log(z) - \psi(a) + \psi(a+b))\,I_z(a,\,b) - \frac{\Gamma(a)\;\Gamma(a+b)}{\Gamma(b)}\,z^a\,{}_3\tilde{F}_2(a,\,a,\,1-b;\,a+1,\,a+1;\,z)$$

06.21.20.0004.01

$$\frac{\partial^2 I_z(a,\,b)}{\partial a^2} = \frac{2\,z^a\,\Gamma(a)\,\Gamma(a+b)}{\Gamma(b)}\left(\left(\Gamma(a)\,{}_4\tilde{F}_3(a,\,a,\,a,\,1-b;\,a+1,\,a+1,\,a+1;\,z) - (\log(z) - \psi(a) + \psi(a+b))\,{}_3\tilde{F}_2(a,\,a,\,1-b;\,a+1,\,a+1;\,z)\right)\right) + I_z(a,\,b)\left(\log^2(z) + 2\,\psi(a+b)\log(z) + \psi(a)^2 + \psi(a+b)^2 - 2\,\psi(a)\,(\log(z) + \psi(a+b)) - \psi^{(1)}(a) + \psi^{(1)}(a+b)\right)$$

With respect to *b*

06.21.20.0005.01

$$\frac{\partial I_z(a,\,b)}{\partial b} = \frac{\Gamma(b)\,\Gamma(a+b)}{\Gamma(a)}\,(1-z)^b\,{}_3\tilde{F}_2(b,\,b,\,1-a;\,b+1,\,b+1;\,1-z) + I_{1-z}(b,\,a)\,(\psi(b) - \psi(a+b) - \log(1-z))$$

06.21.20.0006.01

$$\frac{\partial^2 I_z(a,\,b)}{\partial a^2} = \frac{2\,(1-z)^b\,\Gamma(b)\,\Gamma(a+b)}{\Gamma(a)}\left((\log(1-z) - \psi(b) + \psi(a+b))\,{}_3\tilde{F}_2(b,\,b,\,1-a;\,b+1,\,b+1;\,1-z) - \Gamma(b)\,{}_4\tilde{F}_3(b,\,b,\,b,\,1-a;\,b+1,\,b+1,\,b+1;\,1-z)\right) - I_{1-z}(b,\,a)\left(\log^2(1-z) + 2\,\psi(a+b)\log(1-z) + \psi(b)^2 + \psi(a+b)^2 - 2\,\psi(b)\,(\log(1-z) + \psi(a+b)) - \psi^{(1)}(b) + \psi^{(1)}(a+b)\right)$$

Symbolic differentiation

With respect to *z*

06.21.20.0012.01

$$\frac{\partial^n I_z(a, b)}{\partial z^n} == \delta_n I_z(a, b) - \frac{(1-z)^{b-1} z^{a-n}}{\mathrm{B}(a, b)} \sum_{k=0}^{n-1} (-1)^{n-k} \binom{n-1}{k} (1-b)_k (1-a)_{n-k-1} \left(\frac{z}{1-z}\right)^k /; n \in \mathbb{N}$$

06.21.20.0007.02

$$\frac{\partial^n I_z(a, b)}{\partial z^n} == \frac{(-1)^{n-1} \Gamma(a+b)}{\Gamma(a)} (1-z)^{b-n} z^{a-1} {}_2\tilde{F}_1\left(1-a, 1-n; b-n+1; 1-\frac{1}{z}\right) /; n \in \mathbb{N}$$

With respect to *a*

06.21.20.0008.02

$$\frac{\partial^n I_z(a, b)}{\partial a^n} ==$$

$$-\frac{b \sin(\pi b) \Gamma(a) n! z^a}{\pi} \sum_{j=0}^{n} (-1)^{n-j} \log^j(z) \Gamma(a+b)^{n-j+1} {}_{n-j+2}\tilde{F}_{n-j+1}\left(a_1, a_2, \ldots, a_{n-j+1}, 1+b; a_1+1, a_2+1, \ldots, a_{n-j+1}+1; 1\right)$$

$$\sum_{i=0}^{j} \frac{1}{(j-i)!} {}_{i+2}\tilde{F}_{i+1}(c_1, c_2, \ldots, c_{i+1}, 1-b; c_1+1, c_2+1, \ldots, c_{i+1}+1; z) \left(-\frac{\Gamma(a)}{\log(z)}\right)^i /;$$

$$a_1 == a_2 == \ldots == a_{n+1} == a+b \bigwedge c_1 == c_2 == \ldots == c_{n+1} == a \bigwedge n \in \mathbb{N}$$

With respect to *b*

06.21.20.0009.02

$$\frac{\partial^n I_z(a, b)}{\partial b^n} == \delta_n + \frac{a \sin(\pi a) \Gamma(b) n! (1-z)^b}{\pi}$$

$$\sum_{j=0}^{n} (-1)^{n-j} \log^j(1-z) \Gamma(a+b)^{n-j+1} {}_{n-j+2}\tilde{F}_{n-j+1}\left(a_1, a_2, \ldots, a_{n-j+1}, 1+b\,a; a_1+1, a_2+1, \ldots, a_{n-j+1}+1; 1\right)$$

$$\sum_{i=0}^{j} \frac{1}{(j-i)!} {}_{i+2}\tilde{F}_{i+1}(c_1, c_2, \ldots, c_{i+1}, 1-a; c_1+1, c_2+1, \ldots, c_{i+1}+1; 1-z) \left(-\frac{\Gamma(b)}{\log(1-z)}\right)^i /;$$

$$a_1 == a_2 == \ldots == a_{n+1} == a+b \bigwedge c_1 == c_2 == \ldots == c_{n+1} == b \bigwedge n \in \mathbb{N}$$

Fractional integro-differentiation

With respect to *z*

06.21.20.0010.01

$$\frac{\partial^\alpha I_z(a, b)}{\partial z^\alpha} == \frac{\Gamma(a+b)}{\Gamma(b)} z^{a-\alpha} {}_2\tilde{F}_1(a, 1-b; a-\alpha+1; z) /; -a \notin \mathbb{N}^+$$

06.21.20.0011.01

$$\frac{\partial^\alpha I_z(a, b)}{\partial z^\alpha} == \frac{1}{\mathrm{B}(a, b)} \sum_{k=0}^{\infty} \frac{(1-b)_k \mathcal{FC}_{\exp}^{(\alpha)}(z, a+k) z^{a+k-\alpha}}{(a+k) k!} /; |z| < 1$$

Integration

Indefinite integration

Involving only one direct function

06.21.21.0001.01

$$\int I_{a\,z}(a,\,b)\,dz = z\,I_{a\,z}(a,\,b) - \frac{a}{a\,(a+b)}\,I_{a\,z}(a+1,\,b)$$

06.21.21.0002.01

$$\int I_z(a,\,b)\,dz = z\,I_z(a,\,b) - \frac{a}{a+b}\,I_z(a+1,\,b)$$

Involving one direct function and elementary functions

Involving power function

06.21.21.0003.01

$$\int z^{\alpha-1}\,I_z(a,\,b)\,dz = \frac{z^\alpha}{\alpha}\,I_z(a,\,b) - \frac{\Gamma(a+b)\,\Gamma(a+\alpha)}{\alpha\,\Gamma(a)\,\Gamma(a+b+\alpha)}\,I_z(a+\alpha,\,b)$$

06.21.21.0004.01

$$\int z^{\alpha-1}\,I_z(a,\,b)\,dz = \frac{z^\alpha}{\alpha}\,I_z(a,\,b) - \frac{\Gamma(a+b)\,\Gamma(a+\alpha)}{\alpha\,\Gamma(a)\,\Gamma(a+b+\alpha)}\,I_z(a+\alpha,\,b)$$

06.21.21.0005.01

$$\int \frac{I_z(a,\,b)}{z}\,dz = \frac{z^a}{a^2\,\mathrm{B}(a,\,b)}\,{}_3F_2(a,\,a,\,1-b;\,a+1,\,a+1;\,z)$$

06.21.21.0006.01

$$\int (1-z)^{b-1}\,z^{a-1}\,I_z(a,\,b)\,dz = \frac{1}{2}\,\mathrm{B}_z(a,\,b)\,I_z(a,\,b)$$

06.21.21.0007.01

$$\int (1-z)^n\,I_z(a,\,b)\,dz = \frac{\mathrm{B}_z(a,\,b+n+1) - (1-z)^{n+1}\,\mathrm{B}_z(a,\,b)}{(n+1)\,\mathrm{B}(a,\,b)}$$

Involving rational function

06.21.21.0008.01

$$\int \frac{I_z(a,\,b)}{1-z}\,dz = \frac{(1-z)^b\,{}_3F_2(b,\,b,\,1-a;\,b+1,\,b+1;\,1-z)}{b^2\,\mathrm{B}(a,\,b)} - (I_{1-z}(b,\,a) + I_z(a,\,b))\log(1-z)$$

Integral transforms

Fourier cos transforms

06.21.22.0001.01

$$\mathcal{F}c_t[I_t(a,\,b)]\,(x) = \frac{2^{-b-\frac{3}{2}}\,e^{-i\,a\,\pi}\,\sin(\pi\,b)\,\Gamma(a+b)}{\pi^2\,\Gamma(a)}\,G_{2,4}^{3,2}\!\left(\frac{x^2}{4}\,\middle|\,\begin{matrix}\frac{1-a}{2},\,-\frac{a}{2}\\ 0,\,-\frac{a+b}{2},\,\frac{1-a-b}{2},\,-\frac{1}{2}\end{matrix}\right) /;\, x\in\mathbb{R}\land\mathrm{Re}(a)>-1$$

Fourier sin transforms

06.21.22.0002.01

$$\mathcal{F}s_t[I_t(a,b)](x) = -\frac{2^{-b-\frac{3}{2}} e^{-i a \pi} \sin(\pi b)\,\Gamma(a+b)}{\pi^2\,\Gamma(a)} \operatorname{sgn}(x)\, G_{2,4}^{3,2}\!\left(\frac{x^2}{4} \,\middle|\, \begin{array}{c} \frac{1-a}{2}, -\frac{a}{2} \\ -\frac{1}{2}, -\frac{a+b}{2}, \frac{1-a-b}{2}, 0 \end{array}\right) /;\, x \in \mathbb{R} \land \operatorname{Re}(a) > -2$$

Laplace transforms

06.21.22.0003.01

$$\mathcal{L}_t[I_t(a,b)](z) = \frac{(-1)^{-a}\,\Gamma(a+b)}{z\,\Gamma(b)}\, U(a, a+b, -z) /;\, \operatorname{Re}(z) > 0 \land \operatorname{Re}(a) > -1$$

Mellin transforms

06.21.22.0004.01

$$\mathcal{M}_t[I_t(a,b)](z) = -\frac{(i + \cot(\pi(a+z)))\sin(\pi b)\,\Gamma(1-a-b-z)\,\Gamma(a+b)}{z\,\Gamma(1-a-z)\,\Gamma(a)} /;\, \operatorname{Re}(a+z) > 0 \bigwedge \operatorname{Re}(z) < 0 \bigwedge \operatorname{Re}(a+b+z) < 1$$

Representations through more general functions

Through hypergeometric functions

Involving ${}_2\tilde{F}_1$

06.21.26.0001.01

$$I_z(a,b) = \frac{\Gamma(a+b)}{\Gamma(b)}\, z^a\, {}_2\tilde{F}_1(a, 1-b; a+1; z) /;\, -a \notin \mathbb{N}$$

06.21.26.0002.01

$$I_z(a,b) = \frac{\Gamma(a+b)}{\Gamma(b)}\, (1-z)^b\, z^a\, {}_2\tilde{F}_1(1, a+b; a+1; z) /;\, -a \notin \mathbb{N}$$

06.21.26.0003.01

$$I_z(a,b) = 1 - \frac{\Gamma(a+b)}{\Gamma(a)}\, (1-z)^b\, z^a\, {}_2\tilde{F}_1(1, a+b; b+1; 1-z) /;\, -b \notin \mathbb{N}$$

06.21.26.0004.01

$$I_z(a,b) = \csc(\pi(a+b))\sin(b\pi)\,(-z)^{-a} z^a - \frac{\pi\csc(\pi(a+b))}{\Gamma(a)\,\Gamma(b)}\,(-z)^{b-1} z^a\, {}_2\tilde{F}_1\!\left(1-b, -a-b+1; -a-b+2; \frac{1}{z}\right) /;\, a+b \notin \mathbb{N}^+$$

Involving ${}_2F_1$

06.21.26.0005.01

$$I_z(a,b) = \frac{z^a}{a\,\mathrm{B}(a,b)}\, {}_2F_1(a, 1-b; a+1; z) /;\, -a \notin \mathbb{N}$$

06.21.26.0006.01

$$I_z(a,b) = \frac{(1-z)^b}{a\,\mathrm{B}(a,b)}\, z^a\, {}_2F_1(1, a+b; a+1; z) /;\, -a \notin \mathbb{N}$$

06.21.26.0007.01

$$I_z(a,b) = 1 - \frac{(1-z)^b z^a}{b\,\mathrm{B}(a,b)}\, {}_2F_1(1, a+b; b+1; 1-z) /;\, -b \notin \mathbb{N}$$

06.21.26.0008.01

$$I_z(a,b) = \frac{z^a (-z)^{b-1}}{(a+b-1)\,\mathrm{B}(a,b)}\, {}_2F_1\!\left(1-b, 1-a-b; 2-a-b; \frac{1}{z}\right) + \csc(\pi(a+b))\sin(b\pi)\, z^a (-z)^{-a} \,/;\, a+b \notin \mathbb{N}^+$$

Through hypergeometric functions of two variables

06.21.26.0009.01

$$I_z(a,b) = 1 - \frac{(1-z)^b}{b\,\mathrm{B}(a,b)}\, F_{0\times 0\times 1}^{0\times 1\times 2}\!\left(\begin{matrix} ;-a; 1, a+b; \\ ;; b+1; \end{matrix}\; 1-z, 1-z\right) /;\, -b \notin \mathbb{N}$$

Through Meijer *G*

Classical cases for the direct function itself

06.21.26.0010.01

$$I_z(a,b) = \frac{\sin(\pi b)\,\Gamma(a+b)\, z^a}{\pi\,\Gamma(a)}\, G_{2,2}^{1,2}\!\left(-z \,\middle|\, \begin{matrix} 1-a, b \\ 0, -a \end{matrix}\right)$$

Classical cases involving algebraic functions

06.21.26.0011.01

$$(1-z)^{-b}\, I_z(a,b) = \frac{z^a}{\Gamma(b)}\, G_{2,2}^{1,2}\!\left(-z \,\middle|\, \begin{matrix} 0, -a-b+1 \\ 0, -a \end{matrix}\right)$$

06.21.26.0016.01

$$(z-1)^{-b}\, I_{\frac{1}{z}}(a,b) = \frac{z^{-a-b}}{\Gamma(b)}\, G_{2,2}^{2,1}\!\left(-z \,\middle|\, \begin{matrix} 1, a+1 \\ 1, a+b \end{matrix}\right) /;\, z \notin (-\infty, 0)$$

Classical cases involving algebraic functions in the arguments

06.21.26.0012.01

$$I_{\frac{z}{z+1}}(a,b) = \frac{1}{\Gamma(a)\,\Gamma(b)}\, G_{2,2}^{1,2}\!\left(z \,\middle|\, \begin{matrix} 1, 1-b \\ a, 0 \end{matrix}\right) /;\, z \notin (-\infty, -1)$$

06.21.26.0013.01

$$(z+1)^{a+b-1}\, I_{\frac{z}{z+1}}(a,b) = \frac{\sin(\pi b)\,\Gamma(a+b)}{\pi}\, G_{2,2}^{1,2}\!\left(z \,\middle|\, \begin{matrix} a, a+b \\ a, 0 \end{matrix}\right) /;\, z \notin (-\infty, -1)$$

Through other functions

Involving some hypergeometric-type functions

06.21.26.0014.01

$$I_z(a,b) = I_{(1,z)}(a,b) + 1 \,/;\, \mathrm{Re}(b) > 0$$

06.21.26.0015.01

$$I_z(a,b) = \frac{1}{\mathrm{B}(a,b)}\, \mathrm{B}_{(1,z)}(a,b) + 1 \,/;\, \mathrm{Re}(b) > 0$$

Representations through equivalent functions

With inverse function

06.21.27.0001.01

$$I_{I_z^{-1}(a,b)}(a,\, b) == z$$

06.21.27.0002.01

$$I_{I_{(1,z_2)}^{-1}(a,b)}(a,\, b) == z_2 + 1 \;/;\; -1 < z_2 < 0$$

With related functions

06.21.27.0003.01

$$I_z(a,\, b) == \frac{\Gamma(a+b)}{\Gamma(a)\,\Gamma(b)}\, \mathrm{B}_z(a,\, b)$$

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