BetaRegularized4

Notations

Traditional name

Generalized regularized incomplete beta function

Traditional notation

$I_{(z_1,z_2)}(a, b)$

Mathematica StandardForm notation

`BetaRegularized[`z_1`, `z_2`, `a`, `b`]`

Primary definition

06.22.02.0001.01

$$I_{(z_1,z_2)}(a, b) = \frac{\mathrm{B}(z_1, z_2, a, b)}{\mathrm{B}(a, b)}$$

Specific values

Specialized values

For fixed z_1, z_2, a

06.22.03.0001.01

$$I_{(z_1,z_2)}(a, -n) = 0 \;/;\; n \in \mathbb{N}$$

06.22.03.0002.01

$$I_{(z_1,z_2)}(a, n) = z_2^k \sum_{k=0}^{n-1} \frac{(a)_k (1-z_2)^k}{k!} - z_1^k \sum_{k=0}^{n-1} \frac{(a)_k (1-z_1)^k}{k!} \;/;\; n \in \mathbb{N}$$

For fixed z_1, z_2, b

06.22.03.0003.01

$$I_{(z_1,z_2)}(-n, b) = 0 \;/;\; n \in \mathbb{N}$$

06.22.03.0004.01

$$I_{(z_1,z_2)}(n, b) = (1-z_1)^b \sum_{k=0}^{n-1} \frac{(b)_k z_1^k}{k!} - (1-z_2)^b \sum_{k=0}^{n-1} \frac{(b)_k z_2^k}{k!} \;/;\; n \in \mathbb{N}$$

For fixed z_1, a, b

06.22.03.0005.02

$I_{(z_1,0)}(a, b) = -I_{z_1}(a, b) /; \text{Re}(a) > 0$

06.22.03.0006.01

$I_{(z_1,0)}(a, b) = \tilde{\infty} /; \text{Re}(a) < 0$

06.22.03.0007.01

$I_{(z_1,1)}(a, b) = 1 - I_{z_1}(a, b) /; \text{Re}(b) > 0$

For fixed z_2, a, b

06.22.03.0008.02

$I_{(0,z_2)}(a, b) = I_{z_2}(a, b) /; \text{Re}(a) > 0$

06.22.03.0009.02

$I_{(0,z_2)}(a, b) = \tilde{\infty} /; \text{Re}(a) < 0$

06.22.03.0010.02

$I_{(1,z_2)}(a, b) = I_{z_2}(a, b) - 1 /; \text{Re}(b) > 0$

General characteristics

Domain and analyticity

$I_{(z_1,z_2)}(a, b)$ is an analytical function of z_1, z_2, a, b which is defined in $\mathbb{C}^4$.

06.22.04.0001.01

$(z_1 * z_2 * a * b) \longrightarrow I_{(z_1,z_2)}(a, b) :: (\mathbb{C}^4) \longrightarrow \mathbb{C}$

Symmetries and periodicities

Mirror symmetry

06.22.04.0002.02

$I_{(\overline{z_1},\overline{z_2})}(\overline{a}, \overline{b}) = \overline{I_{(z_1,z_2)}(a, b)} /; z_1 \notin (-\infty, 0) \bigwedge z_1 \notin (1, \infty) \bigwedge z_2 \notin (-\infty, 0) \bigwedge z_2 \notin (1, \infty)$

Permutation symmetry

06.22.04.0003.01

$I_{(z_1,z_2)}(a, b) = -I_{(z_2,z_1)}(a, b)$

Periodicity

No periodicity

Poles and essential singularities

With respect to b

For fixed z_1, z_2, a, the function $I_{(z_1,z_2)}(a, b)$ has only one singular point at $b = \tilde{\infty}$. It is an essential singular point.

06.22.04.0004.01

$\mathcal{S}ing_b(I_{(z_1,z_2)}(a, b)) = \{\{\tilde{\infty}, \infty\}\}$

With respect to a

For fixed z_1, z_2, b, the function $I_{(z_1,z_2)}(a, b)$ has only one singular point at $a = \tilde{\infty}$. It is an essential singular point.

06.22.04.0005.01

$$\mathcal{S}ing_a\left(I_{(z_1,z_2)}(a, b)\right) == \{\{\tilde{\infty}, \infty\}\}$$

With respect to z_k

For fixed a, b, the function $I_{(z_1,z_2)}(a, b)$ does not have poles and essential singularities.

06.22.04.0006.01

$$\mathcal{S}ing_{z_k}\left(I_{(z_1,z_2)}(a, b)\right) == \{\} /; k \in \{1, 2\}$$

Branch points

With respect to b

For fixed z_1, z_2, a, the function $I_{(z_1,z_2)}(a, b)$ does not have branch points.

06.22.04.0007.01

$$\mathcal{BP}_b\left(I_{(z_1,z_2)}(a, b)\right) == \{\}$$

With respect to a

For fixed z_1, z_2, b, the function $I_{(z_1,z_2)}(a, b)$ does not have branch points.

06.22.04.0008.01

$$\mathcal{BP}_a\left(I_{(z_1,z_2)}(a, b)\right) == \{\}$$

With respect to z_k

The function $I_{(z_1,z_2)}(a, b)$ has for fixed $z_1$or fixed $z_2$three singular branch points with respect to $z_2$or z_1: $z_k == 0$, $z_k == 1$, $z_k == \tilde{\infty}$, $k = 1, 2$.

06.22.04.0009.01

$$\mathcal{BP}_{z_k}\left(I_{(z_1,z_2)}(a, b)\right) == \{0, 1, \tilde{\infty}\} /; k \in \{1, 2\}$$

06.22.04.0010.01

$$\mathcal{R}_{z_k}\left(I_{(z_1,z_2)}(a, b), 0\right) == \log /; a \notin \mathbb{Z} \bigwedge a \notin \mathbb{Q} \bigwedge k \in \{1, 2\}$$

06.22.04.0011.01

$$\mathcal{R}_{z_k}\left(I_{(z_1,z_2)}(a, b), 0\right) == q /; a == \frac{p}{q} \bigwedge p \in \mathbb{Z} \bigwedge q - 1 \in \mathbb{N}^+ \bigwedge \gcd(p, q) == 1 \bigwedge k \in \{1, 2\}$$

06.22.04.0012.01

$$\mathcal{R}_{z_k}\left(I_{(z_1,z_2)}(a, b), 1\right) == \log /; b \notin \mathbb{Z} \bigwedge b \notin \mathbb{Q} \bigwedge k \in \{1, 2\}$$

06.22.04.0013.01

$$\mathcal{R}_{z_k}\left(I_{(z_1,z_2)}(a, b), 1\right) == q /; b == \frac{p}{q} \bigwedge p \in \mathbb{Z} \bigwedge q - 1 \in \mathbb{N}^+ \bigwedge \gcd(p, q) == 1 \bigwedge k \in \{1, 2\}$$

06.22.04.0014.01

$$\mathcal{R}_{z_k}\left(I_{(z_1,z_2)}(a, b), \tilde{\infty}\right) == \log /; a + b \in \mathbb{Z} \bigvee a + b \notin \mathbb{Q} \bigwedge k \in \{1, 2\}$$

06.22.04.0015.01

$$\mathcal{R}_{z_k}\left(I_{(z_1,z_2)}(a,b),\tilde{\infty}\right) == s\ /;\ a+b == \frac{r}{s} \bigwedge r\in\mathbb{Z} \bigwedge s-1\in\mathbb{N}^+ \bigwedge \gcd(r,s)==1 \bigwedge k\in\{1,2\}$$

Branch cuts

With respect to b

For fixed z_1, z_2, a, the function $I_{(z_1,z_2)}(a,b)$ does not have branch cuts.

06.22.04.0016.01

$$\mathcal{BC}_b\left(I_{(z_1,z_2)}(a,b)\right) == \{\}$$

With respect to a

For fixed z_1, z_2, b, the function $I_{(z_1,z_2)}(a,b)$ does not have branch cuts.

06.22.04.0017.01

$$\mathcal{BC}_a\left(I_{(z_1,z_2)}(a,b)\right) == \{\}$$

With respect to z_1

For fixed a, b, z_2, the function $I_{(z_1,z_2)}(a,b)$ is a single-valued function on the z_1-plane cut along the intervals $(-\infty, 0)$ and $(1, \infty)$.

The function $I_{(z_1,z_2)}(a,b)$ is continuous from above on the interval $(-\infty, 0)$ and from below on the interval $(1, \infty)$.

06.22.04.0018.01

$$\mathcal{BC}_{z_1}\left(I_{(z_1,z_2)}(a,b)\right) == \{\{(-\infty,0),-i\},\{(1,\infty),i\}\}$$

06.22.04.0019.01

$$\lim_{\epsilon\to+0} I_{(x_1+i\epsilon,z_2)}(a,b) == I_{(x_1,z_2)}(a,b)\ /;\ x_1<0$$

06.22.04.0020.01

$$\lim_{\epsilon\to+0} I_{(x_1-i\epsilon,z_2)}(a,b) == I_{(x_1,z_2)}(a,b) + \left(1-e^{-2 i a\pi}\right) I_{x_1}(a,b)\ /;\ x_1<0$$

06.22.04.0021.01

$$\lim_{\epsilon\to+0} I_{(x_1-i\epsilon,z_2)}(a,b) == I_{(x_1,z_2)}(a,b)\ /;\ x_1>1$$

06.22.04.0022.01

$$\lim_{\epsilon\to+0} I_{(x_1+i\epsilon,z_2)}(a,b) == I_{(x_1,z_2)}(a,b) + \left(1-e^{-2 i b\pi}\right) I_{x_1}(a,b) - 2 i e^{-i b\pi}\sin(b\pi)\ /;\ x_1>1$$

With respect to z_2

For fixed a , b, z_1, the function $I_{(z_1,z_2)}(a,b)$ is a single-valued function on the z_2-plane cut along the intervals $(-\infty, 0)$ and $(1, \infty)$.

The function $I_{(z_1,z_2)}(a,b)$ is continuous from above on the interval $(-\infty, 0)$ and from below on the interval $(1, \infty)$.

06.22.04.0023.01

$$\mathcal{BC}_{z_2}\left(I_{(z_1,z_2)}(a,b)\right) == \{\{(-\infty,0),-i\},\{(1,\infty),i\}\}$$

06.22.04.0024.01

$$\lim_{\epsilon\to+0} I_{(z_1,x_2+i\epsilon)}(a,b) == I_{(z_1,x_2)}(a,b)\ /;\ x_2<0$$

06.22.04.0025.01

$$\lim_{\epsilon\to+0} I_{(z_1,x_2-i\epsilon)}(a,b) = I_{(z_1,x_2)}(a,b) - \left(1-e^{-2ia\pi}\right) I_{x_2}(a,b) \;/;\; x_2<0$$

06.22.04.0026.01

$$\lim_{\epsilon\to+0} I_{(z_1,x_2-i\epsilon)}(a,b) = I_{(z_1,x_2)}(a,b) \;/;\; x_2>1$$

06.22.04.0027.01

$$\lim_{\epsilon\to+0} I_{(z_1,x_2+i\epsilon)}(a,b) = I_{(z_1,x_2)}(a,b) - \left(1-e^{-2ib\pi}\right) I_{x_2}(a,b) + 2\,e^{-ib\pi}\,i\sin(b\pi) \;/;\; x_2>1$$

Series representations

Generalized power series

Expansions at $\{z_1, z_2\} = \{0, 0\}$

06.22.06.0001.02

$$I_{(z_1,z_2)}(a,b) \propto \frac{1}{\mathrm{B}(a,b)} z_2^a \left(\frac{1}{a} + \frac{(1-b)z_2}{1+a} + \frac{(1-b)(2-b)z_2^2}{2(2+a)} + \ldots\right) - \frac{1}{\mathrm{B}(a,b)} z_1^a \left(\frac{1}{a} + \frac{(1-b)z_1}{1+a} + \frac{(1-b)(2-b)z_1^2}{2(2+a)} + \ldots\right) /;\; (z_1\to 0)\land(z_2\to 0)\land -a\notin\mathbb{N}$$

06.22.06.0002.01

$$I_{(z_1,z_2)}(a,b) = \frac{1}{\mathrm{B}(a,b)} \sum_{k=0}^{\infty} \frac{(1-b)_k \left(z_2^{a+k} - z_1^{a+k}\right)}{(a+k)\,k!} \;/;\; |z_1|<1 \land |z_2|<1 \land -a\notin\mathbb{N}$$

06.22.06.0003.01

$$I_{(z_1,z_2)}(a,b) = \frac{1}{a\,\mathrm{B}(a,b)} \left(z_2^a\, {}_2F_1(a, 1-b; a+1; z_2) - z_1^a\, {}_2F_1(a, 1-b; a+1; z_1)\right)$$

06.22.06.0004.01

$$I_{(z_1,z_2)}(a,b) \propto \frac{1}{a\,\mathrm{B}(a,b)} \left(z_2^a(1+O(z_2)) - z_1^a(1+O(z_1))\right) /;\; (z_1\to 0)\land(z_2\to 0)\land -a\notin\mathbb{N}$$

Integral representations

On the real axis

Of the direct function

06.22.07.0001.01

$$I_{(z_1,z_2)}(a,b) = \frac{1}{\mathrm{B}(a,b)} \int_{z_1}^{z_2} t^{a-1}(1-t)^{b-1}\,dt$$

Differential equations

Ordinary linear differential equations and wronskians

For the direct function itself

06.22.13.0001.01

$$(1-z_1)\,z_1\,w''(z_1) + (1-a+(a+b-2)\,z_1)\,w'(z_1) = 0 \;/;\; w(z_1) = c_1\, I_{(z_1,z_2)}(a,b) + c_2$$

06.22.13.0003.01

$$W_{z_1}\left(1, I_{(z_1,z_2)}(a, b)\right) = -\frac{(1-z_1)^{b-1} z_1^{a-1}}{\mathrm{B}(a, b)}$$

06.22.13.0002.01

$$(1-z_2) z_2 w''(z_2) + (1-a+(a+b-2) z_2) w'(z_2) = 0 /; w(z_2) = c_1 I_{(z_1,z_2)}(a, b) + c_2$$

06.22.13.0004.01

$$W_{z_2}\left(1, I_{(z_1,z_2)}(a, b)\right) = \frac{(1-z_2)^{b-1} z_2^{a-1}}{\mathrm{B}(a, b)}$$

Transformations

Transformations and argument simplifications

Argument involving basic arithmetic operations

06.22.16.0001.01

$$I_{(1-z_1,1-z_2)}(a, b) = -I_{(z_1,z_2)}(b, a)$$

06.22.16.0002.01

$$I_{(z_1,z_2)}(a+1, b) = I_{(z_1,z_2)}(a, b) + \frac{\Gamma(a+b)}{\Gamma(a+1)\Gamma(b)}\left((1-z_1)^b z_1^a - (1-z_2)^b z_2^a\right)$$

06.22.16.0003.01

$$I_{(z_1,z_2)}(a-1, b) = I_{(z_1,z_2)}(a, b) + \frac{\Gamma(a+b-1)}{\Gamma(a)\Gamma(b)}\left((1-z_2)^b z_2^{a-1} - (1-z_1)^b z_1^{a-1}\right)$$

06.22.16.0004.01

$$I_{(z_1,z_2)}(a+n, b) = I_{(z_1,z_2)}(a, b) + \frac{1}{(a+b+n-1)\,\mathrm{B}(a+n, b)} \sum_{k=0}^{n-1} \frac{(1-a-n)_k}{(2-a-b-n)_k}\left((1-z_1)^b z_1^{a+n-k-1} - (1-z_2)^b z_2^{a+n-k-1}\right) /; n \in \mathbb{N}$$

06.22.16.0005.01

$$I_{(z_1,z_2)}(a-n, b) = I_{(z_1,z_2)}(a, b) + \frac{1}{(a+b-1)\,\mathrm{B}(a, b)} \sum_{k=0}^{n-1} \frac{(1-a)_k}{(2-a-b)_k}\left((1-z_2)^b z_2^{a-k-1} - (1-z_1)^b z_1^{a-k-1}\right) /; n \in \mathbb{N}$$

Identities

Recurrence identities

Consecutive neighbors

06.22.17.0001.01

$$I_{(z_1,z_2)}(a, b) = I_{(z_1,z_2)}(a+1, b) + \frac{\Gamma(a+b)}{\Gamma(a+1)\Gamma(b)}\left((1-z_2)^b z_2^a - (1-z_1)^b z_1^a\right)$$

06.22.17.0002.01

$$I_{(z_1,z_2)}(a, b) = I_{(z_1,z_2)}(a-1, b) + \frac{\Gamma(a+b-1)}{\Gamma(a)\Gamma(b)}\left((1-z_1)^b z_1^{a-1} - (1-z_2)^b z_2^{a-1}\right)$$

Distant neighbors

06.22.17.0003.01

$$I_{(z_1,z_2)}(a,\,b) = I_{(z_1,z_2)}(a+n,\,b) + \frac{1}{\mathrm{B}(a,\,b)\,(a+b-1)} \sum_{k=1}^{n} \frac{(a+b-1)_k}{(a)_k} \left((1-z_2)^b\, z_2^{a+k-1} - (1-z_1)^b\, z_1^{a+k-1}\right) /;\, n \in \mathbb{N}$$

06.22.17.0004.01

$$I_{(z_1,z_2)}(a,\,b) = I_{(z_1,z_2)}(a-n,\,b) + \frac{1}{\mathrm{B}(a,\,b)\,(a+b-1)} \sum_{k=0}^{n-1} \frac{(1-a)_k}{(2-a-b)_k} \left((1-z_1)^b\, z_1^{a-k-1} - (1-z_2)^b\, z_2^{a-k-1}\right) /;\, n \in \mathbb{N}$$

Functional identities

Relations between contiguous functions

06.22.17.0005.01

$$I_{(z_1,z_2)}(a,\,b) = \frac{1}{a+b} \left(a\, I_{(z_1,z_2)}(a+1,\,b) + b\, I_{(z_1,z_2)}(a,\,b+1)\right)$$

Major general cases

06.22.17.0006.01

$$I_{(z_1,z_2)}(a,\,b) = -I_{(1-z_1,1-z_2)}(b,\,a)$$

06.22.17.0007.01

$$I_{(1-z_1,z_2)}(a,\,b) = I_{(1-z_2,z_1)}(b,\,a)$$

Differentiation

Low-order differentiation

With respect to z_1

06.22.20.0001.01

$$\frac{\partial I_{(z_1,z_2)}(a,\,b)}{\partial z_1} = -\frac{(1-z_1)^{b-1}\, z_1^{a-1}}{\mathrm{B}(a,\,b)}$$

06.22.20.0002.01

$$\frac{\partial^2 I_{(z_1,z_2)}(a,\,b)}{\partial z_1^2} = \frac{(1-a+(a+b-2)\,z_1)}{\mathrm{B}(a,\,b)}\,(1-z_1)^{b-2}\, z_1^{a-2}$$

With respect to z_2

06.22.20.0003.01

$$\frac{\partial I_{(z_1,z_2)}(a,\,b)}{\partial z_2} = \frac{(1-z_2)^{b-1}\, z_2^{a-1}}{\mathrm{B}(a,\,b)}$$

06.22.20.0004.01

$$\frac{\partial^2 I_{(z_1,z_2)}(a,\,b)}{\partial z_2^2} = \frac{(a-(a+b-2)\,z_2-1)}{\mathrm{B}(a,\,b)}\,(1-z_2)^{b-2}\, z_2^{a-2}$$

With respect to a

06.22.20.0005.01

$$\frac{\partial I_{(z_1,z_2)}(a,b)}{\partial a} = \log(z_2)\, I_{z_2}(a,b) - \log(z_1)\, I_{z_1}(a,b) + (\psi(a+b)-\psi(a))\, I_{(z_1,z_2)}(a,b) + \frac{\Gamma(a)\,\Gamma(a+b)}{\Gamma(b)} \left(z_1^a\, {}_3\tilde{F}_2(a, a, 1-b; a+1, a+1; z_1) - z_2^a\, {}_3\tilde{F}_2(a, a, 1-b; a+1, a+1; z_2)\right)$$

06.22.20.0006.01

$$\frac{\partial^2 I_{(z_1,z_2)}(a,b)}{\partial a^2} = \frac{2\,\Gamma(a)\,\Gamma(a+b)}{\Gamma(b)} \left(z_1^a (\log(z_1) - \psi(a) + \psi(a+b))\, {}_3\tilde{F}_2(a, a, 1-b; a+1, a+1; z_1) - z_2^a (\log(z_2) - \psi(a) + \psi(a+b))\, {}_3\tilde{F}_2(a, a, 1-b; a+1, a+1; z_2)\right) + \frac{2\,\Gamma(a)^2\,\Gamma(a+b)}{\Gamma(b)} \left(z_2^a\, {}_4\tilde{F}_3(a, a, a, 1-b; a+1, a+1, a+1; z_2) - z_1^a\, {}_4\tilde{F}_3(a, a, a, 1-b; a+1, a+1, a+1; z_1)\right) + I_{z_1}(a,b) \left(-\log^2(z_1) + 2\,\psi(a)\log(z_1) - 2\,\psi(a+b)\log(z_1) - \psi(a)^2 - \psi(a+b)^2 + 2\,\psi(a)\,\psi(a+b) + \psi^{(1)}(a) - \psi^{(1)}(a+b)\right) + I_{z_2}(a,b) \left(\log^2(z_2) + 2\,\psi(a+b)\log(z_2) + \psi(a)^2 + \psi(a+b)^2 - 2\,\psi(a)(\log(z_2) + \psi(a+b)) - \psi^{(1)}(a) + \psi^{(1)}(a+b)\right)$$

With respect to b

06.22.20.0007.01

$$\frac{\partial I_{(z_1,z_2)}(a,b)}{\partial b} = \log(1-z_1)\, I_{1-z_1}(b,a) - \log(1-z_2)\, I_{1-z_2}(b,a) + (\psi(a+b) - \psi(b))\, I_{(z_1,z_2)}(a,b) + \frac{\Gamma(b)\,\Gamma(a+b)}{\Gamma(a)} \left({}_3\tilde{F}_2(b, b, 1-a; b+1, b+1; 1-z_2)\,(1-z_2)^b - {}_3\tilde{F}_2(b, b, 1-a; b+1, b+1; 1-z_1)\,(1-z_1)^b\right)$$

06.22.20.0008.01

$$\frac{\partial^2 I_{(z_1,z_2)}(a,b)}{\partial b^2} = \frac{2\,(1-z_2)^b\,\Gamma(b)\,\Gamma(a+b)}{\Gamma(a)} \left((\log(1-z_2) - \psi(b) + \psi(a+b))\, {}_3\tilde{F}_2(b, b, 1-a; b+1, b+1; 1-z_2) - \Gamma(b)\, {}_4\tilde{F}_3(b, b, b, 1-a; b+1, b+1, b+1; 1-z_2)\right) - I_{1-z_2}(b,a) \left(\log^2(1-z_2) + 2\,\psi(a+b)\log(1-z_2) + \psi(b)^2 + \psi(a+b)^2 - 2\,\psi(b)(\log(1-z_2) + \psi(a+b)) - \psi^{(1)}(b) + \psi^{(1)}(a+b)\right) - \frac{2\,(1-z_1)^b\,\Gamma(b)\,\Gamma(a+b)}{\Gamma(a)} \left((\log(1-z_1) - \psi(b) + \psi(a+b))\, {}_3\tilde{F}_2(b, b, 1-a; b+1, b+1; 1-z_1) - \Gamma(b)\, {}_4\tilde{F}_3(b, b, b, 1-a; b+1, b+1, b+1; 1-z_1)\right) + I_{1-z_1}(b,a) \left(\log^2(1-z_1) + 2\,\psi(a+b)\log(1-z_1) + \psi(b)^2 + \psi(a+b)^2 - 2\,\psi(b)(\log(1-z_1) + \psi(a+b)) - \psi^{(1)}(b) + \psi^{(1)}(a+b)\right)$$

Symbolic differentiation

With respect to z_1

06.22.20.0017.01

$$\frac{\partial^n I_{(z_1,z_2)}(a,b)}{\partial z_1^n} = \delta_n\, I_{(z_1,z_2)}(a,b) + \frac{(1-z_1)^{b-1}\, z_1^{a-n}}{\mathrm{B}(a,b)} \sum_{k=0}^{n-1} (-1)^{n-k} \binom{n-1}{k} (1-b)_k\, (1-a)_{n-k-1} \left(\frac{z_1}{1-z_1}\right)^k /;\ n \in \mathbb{N}$$

06.22.20.0009.02

$$\frac{\partial^n I_{(z_1,z_2)}(a,b)}{\partial z_1^n} = \frac{(-1)^n\, \Gamma(a+b)}{\Gamma(a)} (1-z_1)^{b-n}\, z_1^{a-1}\, {}_2\tilde{F}_1\left(1-a, 1-n; b-n+1; 1-\frac{1}{z_1}\right) /;\ n \in \mathbb{N}$$

With respect to z_2

06.22.20.0018.01

$$\frac{\partial^n I_{(z_1,z_2)}(a,b)}{\partial z_2^n} = \delta_n I_{(z_1,z_2)}(a,b) - \frac{(1-z_2)^{b-1} z_2^{a-n}}{B(a,b)} \sum_{k=0}^{n-1} (-1)^{n-k} \binom{n-1}{k} (1-b)_k (1-a)_{n-k-1} \left(\frac{z_2}{1-z_2}\right)^k /; n \in \mathbb{N}$$

06.22.20.0010.02

$$\frac{\partial^n I_{(z_1,z_2)}(a,b)}{\partial z_2^n} = \frac{(-1)^{n-1}\Gamma(a+b)}{\Gamma(a)} (1-z_2)^{b-n} z_2^{a-1} {}_2\tilde{F}_1\left(1-a, 1-n; b-n+1; 1-\frac{1}{z_2}\right) /; n \in \mathbb{N}$$

With respect to a

06.22.20.0011.02

$$\frac{\partial^n I_{(z_1,z_2)}(a,b)}{\partial a^n} = -\frac{b\sin(\pi b)\Gamma(a)\,n!}{\pi} \left(z_2^a \sum_{j=0}^{n} (-1)^{n-j} \log^j(z_2)\, \Gamma(a+b)^{n-j+1} {}_{n-j+2}\tilde{F}_{n-j+1}(a_1, a_2, \ldots, a_{n-j+1}, 1+b; a_1+1, a_2+1, \ldots, a_{n-j+1}+1; 1) \sum_{i=0}^{j} \frac{1}{(j-i)!} {}_{i+2}\tilde{F}_{i+1}(c_1, c_2, \ldots, c_{i+1}, 1-b; c_1+1, c_2+1, \ldots, c_{i+1}+1; z_2) \left(-\frac{\Gamma(a)}{\log(z_2)}\right)^i - z_1^a \sum_{j=0}^{n} (-1)^{n-j} \log^j(z_1)\, \Gamma(a+b)^{n-j+1} {}_{n-j+2}\tilde{F}_{n-j+1}(a_1, a_2, \ldots, a_{n-j+1}, 1+b; a_1+1, a_2+1, \ldots, a_{n-j+1}+1; 1) \sum_{i=0}^{j} \frac{1}{(j-i)!} {}_{i+2}\tilde{F}_{i+1}(c_1, c_2, \ldots, c_{i+1}, 1-b; c_1+1, c_2+1, \ldots, c_{i+1}+1; z_1) \left(-\frac{\Gamma(a)}{\log(z_1)}\right)^i \right) /; a_1 = a_2 = \ldots = a_{n+1} = a+b \land c_1 = c_2 = \ldots = c_{n+1} = a \land n \in \mathbb{N}$$

With respect to b

06.22.20.0012.02

$$\frac{\partial^n I_{(z_1,z_2)}(a,b)}{\partial b^n} = \frac{a\sin(\pi a)\Gamma(b)\,n!}{\pi} \left((1-z_2)^b \sum_{j=0}^{n} (-1)^{n-j} \log^j(1-z_2)\, \Gamma(a+b)^{n-j+1} {}_{n-j+2}\tilde{F}_{n-j+1}(a_1, a_2, \ldots, a_{n-j+1}, 1+b\,a; a_1+1, a_2+1, \ldots, a_{n-j+1}+1; 1) \sum_{i=0}^{j} \frac{1}{(j-i)!} {}_{i+2}\tilde{F}_{i+1}(c_1, c_2, \ldots, c_{i+1}, 1-a; c_1+1, c_2+1, \ldots, c_{i+1}+1; 1-z_2) \left(-\frac{\Gamma(b)}{\log(1-z_2)}\right)^i - (1-z_1)^b \sum_{j=0}^{n} (-1)^{n-j} \log^j(1-z_1)\, \Gamma(a+b)^{n-j+1} {}_{n-j+2}\tilde{F}_{n-j+1}(a_1, a_2, \ldots, a_{n-j+1}, 1+b\,a; a_1+1, a_2+1, \ldots, a_{n-j+1}+1; 1) \sum_{i=0}^{j} \frac{1}{(j-i)!} {}_{i+2}\tilde{F}_{i+1}(c_1, c_2, \ldots, c_{i+1}, 1-a; c_1+1, c_2+1, \ldots, c_{i+1}+1; 1-z_1) \left(-\frac{\Gamma(b)}{\log(1-z_1)}\right)^i \right) /; a_1 = a_2 = \ldots = a_{n+1} = a+b \land c_1 = c_2 = \ldots = c_{n+1} = b \land n \in \mathbb{N}$$

Fractional integro-differentiation

With respect to z_1

06.22.20.0013.01

$$\frac{\partial^\alpha I_{(z_1,z_2)}(a,b)}{\partial z_1^\alpha} = \frac{z_1^{-\alpha}}{\Gamma(1-\alpha)} I_{z_2}(a,b) - \frac{z_1^{a-\alpha}\,\Gamma(a+b)}{\Gamma(b)}\, {}_2\tilde{F}_1(a,1-b;a-\alpha+1;z_1) \,/;\, -a \notin \mathbb{N}^+$$

06.22.20.0014.01

$$\frac{\partial^\alpha I_{(z_1,z_2)}(a,b)}{\partial z_1^\alpha} = \frac{z_1^{-\alpha}}{\Gamma(1-\alpha)} I_{z_2}(a,b) - \frac{1}{\mathrm{B}(a,b)} \sum_{k=0}^{\infty} \frac{(1-b)_k\, \mathcal{FC}_{\exp}^{(\alpha)}(z_1, a+k)\, z_1^{a+k-\alpha}}{(a+k)\,k!} \,/;\, |z_1| < 1$$

With respect to z_2

06.22.20.0015.01

$$\frac{\partial^\alpha I_{(z_1,z_2)}(a,b)}{\partial z_2^\alpha} = \frac{\Gamma(a+b)\, z_2^{a-\alpha}}{\Gamma(b)}\, {}_2\tilde{F}_1(a,1-b;a-\alpha+1;z_2) - \frac{z_2^{-\alpha}}{\Gamma(1-\alpha)} I_{z_1}(a,b) \,/;\, -a \notin \mathbb{N}^+$$

06.22.20.0016.01

$$\frac{\partial^\alpha I_{(z_1,z_2)}(a,b)}{\partial z_2^\alpha} = \frac{1}{\mathrm{B}(a,b)} \sum_{k=0}^{\infty} \frac{(1-b)_k\, \mathcal{FC}_{\exp}^{(\alpha)}(z_2, a+k)\, z_2^{a+k-\alpha}}{(a+k)\,k!} - \frac{z_2^{-\alpha}}{\Gamma(1-\alpha)} I_{z_1}(a,b) \,/;\, |z_2| < 1$$

Integration

Indefinite integration

Involving only one direct function with respect to z_1

06.22.21.0001.01

$$\int I_{(a z_1, z_2)}(a,b)\, dz_1 = \frac{a}{a(a+b)} I_{a z_1}(a+1,b) + I_{(a z_1,z_2)}(a,b)\, z_1$$

06.22.21.0002.01

$$\int I_{(z_1, z_2)}(a,b)\, dz_1 = \frac{a}{a+b} I_{z_1}(a+1,b) + z_1\, I_{(z_1,z_2)}(a,b)$$

Involving one direct function and elementary functions with respect to z_1

Involving power function

06.22.21.0003.01

$$\int z_1^{\alpha-1} I_{(a z_1, z_2)}(a,b)\, dz_1 = \frac{z_1^\alpha}{\alpha} \left(\frac{\Gamma(a+b)\,(a z_1)^{-\alpha}}{\Gamma(a)\,\Gamma(b)} \mathrm{B}_{a z_1}(a+\alpha,b) + I_{(a z_1,z_2)}(a,b) \right)$$

06.22.21.0004.01

$$\int z_1^{\alpha-1} I_{(z_1, z_2)}(a,b)\, dz_1 = \frac{z_1^\alpha}{\alpha} I_{(z_1,z_2)}(a,b) + \frac{\Gamma(a+b)\,\Gamma(a+\alpha)}{\alpha\,\Gamma(a)\,\Gamma(a+b+\alpha)} I_{z_1}(a+\alpha,b)$$

Involving only one direct function with respect to z_2

06.22.21.0005.01

$$\int I_{(z_1, a z_2)}(a,b)\, dz_2 = z_2\, I_{(z_1,a z_2)}(a,b) - \frac{a}{a(a+b)} I_{a z_2}(a+1,b)$$

06.22.21.0006.01

$$\int I_{(z_1, z_2)}(a,b)\, dz_2 = z_2\, I_{(z_1,z_2)}(a,b) - \frac{a}{a+b} I_{z_2}(a+1,b)$$

Involving one direct function and elementary functions with respect to z_1

Involving power function

06.22.21.0007.01

$$\int z_2^{\alpha-1} I_{(z_1, a z_2)}(a, b)\, dz_2 = \frac{z_2^{\alpha}}{\alpha}\left(I_{(z_1, a z_2)}(a, b) - \frac{B_{a z_2}(a+\alpha, b)\,(a z_2)^{-\alpha}}{B(a, b)}\right)$$

06.22.21.0008.01

$$\int z_2^{\alpha-1} I_{(z_1, z_2)}(a, b)\, dz_2 = \frac{z_2^{\alpha}}{\alpha} I_{(z_1, z_2)}(a, b) - \frac{\Gamma(a+b)\,\Gamma(a+\alpha)}{\alpha\,\Gamma(a)\,\Gamma(a+b+\alpha)} I_{z_2}(a+\alpha, b)$$

Integral transforms

Laplace transforms

06.22.22.0001.01

$$\mathcal{L}_t\left[I_{(t, z_2)}(a, b)\right](z) = \frac{1}{z}\left(I_{z_2}(a, b) - \frac{(-1)^{-a}\,\Gamma(a+b)}{\Gamma(b)} U(a, a+b, -z)\right) /; \operatorname{Re}(z) > 0 \land \operatorname{Re}(a) > -1$$

Representations through more general functions

Through hypergeometric functions

Involving ${}_2\tilde{F}_1$

06.22.26.0001.01

$$I_{(z_1, z_2)}(a, b) = \frac{\Gamma(a+b)}{\Gamma(b)}\left(z_2^a\, {}_2\tilde{F}_1(a, 1-b; a+1; z_2) - z_1^a\, {}_2\tilde{F}_1(a, 1-b; a+1; z_1)\right) /; -a \notin \mathbb{N}$$

06.22.26.0002.01

$$I_{(z_1, z_2)}(a, b) = \frac{\Gamma(a+b)}{\Gamma(a)}\left((1-z_1)^b z_1^a\, {}_2\tilde{F}_1(1, a+b; b+1; 1-z_1) - (1-z_2)^b z_2^a\, {}_2\tilde{F}_1(1, a+b; b+1; 1-z_2)\right) /; -b \notin \mathbb{N}$$

06.22.26.0003.01

$$I_{(z_1, z_2)}(a, b) = \csc(\pi(a+b))\sin(b\pi)\left((-z_2)^{-a} z_2^a - (-z_1)^{-a} z_1^a\right) - \frac{\pi\csc(\pi(a+b))}{\Gamma(a)\,\Gamma(b)}$$
$$\left((-z_2)^{b-1} z_2^a\, {}_2\tilde{F}_1\left(1-b, 1-a-b; 2-a-b; \frac{1}{z_2}\right) - (-z_1)^{b-1} z_1^a\, {}_2\tilde{F}_1\left(1-b, 1-a-b; 2-a-b; \frac{1}{z_1}\right)\right) /; a+b \notin \mathbb{N}^+$$

Involving ${}_2F_1$

06.22.26.0004.01

$$I_{(z_1, z_2)}(a, b) = \frac{1}{a\,B(a, b)}\left(z_2^a\, {}_2F_1(a, 1-b; a+1; z_2) - z_1^a\, {}_2F_1(a, 1-b; a+1; z_1)\right) /; -a \notin \mathbb{N}$$

06.22.26.0005.01

$$I_{(z_1, z_2)}(a, b) = \frac{1}{b\,B(a, b)}\left((1-z_1)^b z_1^a\, {}_2F_1(1, a+b; b+1; 1-z_1) - (1-z_2)^b z_2^a\, {}_2F_1(1, a+b; b+1; 1-z_2)\right) /; -b \notin \mathbb{N}$$

06.22.26.0006.01

$$I_{(z_1,z_2)}(a,b) = \csc(\pi(a+b))\sin(b\pi)\left((-z_2)^{-a} z_2^a - (-z_1)^{-a} z_1^a\right) + \frac{1}{(a+b-1)\,\mathrm{B}(a,b)} \left((-z_2)^{b-1} z_2^a\, {}_2F_1\!\left(1-b, 1-a-b; 2-a-b; \frac{1}{z_2}\right) - (-z_1)^{b-1} z_1^a\, {}_2F_1\!\left(1-b, 1-a-b; 2-a-b; \frac{1}{z_1}\right)\right) /; a+b \notin \mathbb{N}^+$$

Through Meijer *G*

Classical cases for the direct function itself

06.22.26.0007.01

$$I_{(z_1,z_2)}(a,b) = \frac{\sin(b\pi)\,\Gamma(a+b)}{\pi\,\Gamma(a)} \left(z_2^a\, G_{2,2}^{1,2}\!\left(-z_2 \,\middle|\, \begin{matrix} 1-a, b \\ 0, -a \end{matrix}\right) - z_1^a\, G_{2,2}^{1,2}\!\left(-z_1 \,\middle|\, \begin{matrix} 1-a, b \\ 0, -a \end{matrix}\right)\right)$$

Classical cases involving algebraic functions in the arguments

06.22.26.0008.01

$$I_{\left(\frac{1}{z+1},1\right)}(a,b) = \frac{1}{\Gamma(a)\,\Gamma(b)}\, G_{2,2}^{1,2}\!\left(z \,\middle|\, \begin{matrix} 1, 1-a \\ b, 0 \end{matrix}\right) /; z \notin (-\infty, -1)$$

06.22.26.0009.01

$$(z+1)^{a+b-1}\, I_{\left(\frac{1}{z+1},1\right)}(a,b) = \frac{\sin(\pi a)\,\Gamma(a+b)}{\pi}\, G_{2,2}^{1,2}\!\left(z \,\middle|\, \begin{matrix} b, a+b \\ b, 0 \end{matrix}\right) /; z \notin (-\infty, -1)$$

Representations through equivalent functions

With inverse function

06.22.27.0001.01

$$I_{\left(z_1, I^{-1}_{(z_1,z_2)}(a,b)\right)}(a,b) = z_2$$

With related functions

06.22.27.0002.01

$$I_{(z_1,z_2)}(a,b) = I_{z_2}(a,b) - I_{z_1}(a,b)$$

06.22.27.0003.01

$$I_{(z_1,z_2)}(a,b) = \frac{1}{\mathrm{B}(a,b)}\, \mathrm{B}_{(z_1,z_2)}(a,b)$$

Copyright