ChebyshevT

Notations

Traditional name

Chebyshev polynomial of the first kind

Traditional notation

$T_n(z)$

Mathematica StandardForm notation

ChebyshevT[*n*, *z*]

Primary definition

05.04.02.0001.01

$$T_n(z) = \frac{\delta_{n,0}}{2} + \frac{n}{2} \sum_{k=1}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(-1)^k (n-k-1)! (2z)^{n-2k}}{k! (n-2k)!} + 2^{n-1} z^n \; /; \; n \in \mathbb{N}$$

05.04.02.0002.01

$$T_n(z) = T_{-n}(z) \; /; \; n \in \mathbb{Z} \land n < 0$$

Specific values

Specialized values

For fixed *n*

05.04.03.0001.01

$$T_n(0) = \cos\left(\frac{\pi n}{2}\right)$$

05.04.03.0002.01

$$T_n(1) = 1$$

05.04.03.0003.01

$$T_n(-1) = (-1)^n$$

For fixed *z*

05.04.03.0004.01

$$T_0(z) = 1$$

05.04.03.0005.01

$$T_1(z) = z$$

05.04.03.0006.01

$$T_2(z) = 2 z^2 - 1$$

05.04.03.0007.01

$$T_3(z) = 4 z^3 - 3 z$$

05.04.03.0008.01

$$T_4(z) = 8 z^4 - 8 z^2 + 1$$

05.04.03.0009.01

$$T_5(z) = 16 z^5 - 20 z^3 + 5 z$$

05.04.03.0010.01

$$T_6(z) = 32 z^6 - 48 z^4 + 18 z^2 - 1$$

05.04.03.0011.01

$$T_7(z) = 64 z^7 - 112 z^5 + 56 z^3 - 7 z$$

05.04.03.0012.01

$$T_8(z) = 128 z^8 - 256 z^6 + 160 z^4 - 32 z^2 + 1$$

05.04.03.0013.01

$$T_9(z) = 256 z^9 - 576 z^7 + 432 z^5 - 120 z^3 + 9 z$$

05.04.03.0014.01

$$T_{10}(z) = 512 z^{10} - 1280 z^8 + 1120 z^6 - 400 z^4 + 50 z^2 - 1$$

Values at infinities

05.04.03.0015.01

$$T_n(\infty) = \infty \;/;\; n > 0$$

05.04.03.0016.01

$$T_n(-\infty) = (-1)^n \infty \;/;\; n > 0$$

General characteristics

Domain and analyticity

The function $T_n(z)$is defined over $\mathbb{N} \otimes \mathbb{C}$. For fixed n, the function $T_n(z)$ is a polynomial in z of degree n.

05.04.04.0001.01

$$(n * z) \longrightarrow T_n(z) :: (\mathbb{N} \otimes \mathbb{C}) \longrightarrow \mathbb{C}$$

Symmetries and periodicities

Parity

05.04.04.0002.01

$$T_n(-z) = (-1)^n T_n(z)$$

05.04.04.0007.01

$$T_{-n}(z) = T_n(z) \;/;\; n \in \mathbb{N}^+$$

Mirror symmetry

05.04.04.0003.01

$$T_n(\bar{z}) = \overline{T_n(z)}$$

Periodicity

No periodicity

Poles and essential singularities

With respect to z

The function $T_n(z)$ is polynomial and has pole of order n at $z = \tilde{\infty}$.

05.04.04.0004.01

$$\mathcal{S}ing_z(T_n(z)) = \{\{\tilde{\infty}, n\}\}$$

Branch points

With respect to z

The function $T_n(z)$ does not have branch points.

05.04.04.0005.01

$$\mathcal{BP}_z(T_n(z)) = \{\}$$

Branch cuts

With respect to z

The function $T_n(z)$ does not have branch cuts.

05.04.04.0006.01

$$\mathcal{BC}_z(T_n(z)) = \{\}$$

Series representations

Generalized power series

Expansions at generic point $z = z_0$

For the function itself

05.04.06.0018.01

$$T_n(z) \propto T_n(z_0) + n\, U_{n-1}(z_0)\,(z - z_0) + \frac{n}{2\left(z_0^2 - 1\right)}\left(n\, T_n(z_0) - z_0\, U_{n-1}(z_0)\right)(z - z_0)^2 + \ldots /; (z \to z_0)$$

05.04.06.0019.01

$$T_n(z) \propto T_n(z_0) + n\, U_{n-1}(z_0)\,(z - z_0) + \frac{n}{2\left(z_0^2 - 1\right)}\left(n\, T_n(z_0) - z_0\, U_{n-1}(z_0)\right)(z - z_0)^2 + O\left((z - z_0)^3\right)$$

05.04.06.0020.01

$$T_n(z) = T_n(z_0) + n \sum_{k=1}^{n} \frac{2^{k-1}}{k} C_{n-k}^{k}(z_0)\,(z-z_0)^k$$

05.04.06.0021.01

$$T_n(z) = \sqrt{\pi} \sum_{k=0}^{n} \frac{(z_0-1)^{-k}}{k!}\, {}_3\tilde{F}_2\!\left(1, -n, n; \frac{1}{2}, 1-k; \frac{1-z_0}{2}\right)(z-z_0)^k$$

05.04.06.0022.01

$$T_n(z) = T_n(z_0) + n \sum_{k=1}^{n} \frac{2^{k-1}}{k} \left(\sum_{i_1=0}^{n-k} \ldots \sum_{i_k=0}^{n-k} \delta_{\sum_{j=1}^{k} i_j, n=k} \prod_{j=1}^{k} U_{i_j}(z_0) \right) (z-z_0)^k$$

05.04.06.0023.01

$$T_n(z) \propto T_n(z_0)\,(1 + O(z-z_0))$$

Expansions at $z = 0$

For the function itself

05.04.06.0001.02

$$T_n(z) \propto \cos\left(\frac{\pi n}{2}\right) + n \sin\left(\frac{\pi n}{2}\right) z - \frac{n^2}{2} \cos\left(\frac{\pi n}{2}\right) z^2 + \ldots /; (z \to 0)$$

05.04.06.0024.01

$$T_n(z) \propto \cos\left(\frac{\pi n}{2}\right) + n \sin\left(\frac{\pi n}{2}\right) z - \frac{n^2}{2} \cos\left(\frac{\pi n}{2}\right) z^2 + O\left(z^3\right)$$

05.04.06.0002.01

$$T_n(z) = F_{1\times 0\times 0}^{2\times 0\times 0}\left(\begin{matrix} -n, n;;; \\ \frac{1}{2};;; \end{matrix} \frac{1}{2}, -\frac{z}{2} \right)$$

05.04.06.0025.01

$$T_n(z) = \cos\left(\frac{\pi n}{2}\right) \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{\left(-\frac{n}{2}\right)_k \left(\frac{n}{2}\right)_k}{\left(\frac{1}{2}\right)_k k!} z^{2k} + n \sin\left(\frac{\pi n}{2}\right) z \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \frac{\left(\frac{1-n}{2}\right)_k \left(\frac{n+1}{2}\right)_k}{\left(\frac{3}{2}\right)_k k!} z^{2k}$$

05.04.06.0026.01

$$T_n(z) = \cos\left(\frac{\pi n}{2}\right) {}_2F_1\left(-\frac{n}{2}, \frac{n}{2}; \frac{1}{2}; z^2\right) + n \sin\left(\frac{\pi n}{2}\right) z\; {}_2F_1\left(\frac{1-n}{2}, \frac{n+1}{2}; \frac{3}{2}; z^2\right)$$

05.04.06.0027.01

$$T_n(z) = \cos\left(\frac{\pi n}{2}\right) \cos\left(n \sin^{-1}(z)\right) + \sin\left(\frac{\pi n}{2}\right) \sin\left(n \sin^{-1}(z)\right)$$

05.04.06.0028.01

$$T_n(z) = \cos\left(\frac{1}{2} n \left(\pi - 2 \sin^{-1}(z)\right)\right)$$

05.04.06.0003.02

$$T_n(z) \propto \cos\left(\frac{\pi n}{2}\right)(1 + O(z))$$

05.04.06.0004.01

$$T_n(z) == \frac{\delta_{n,0}}{2} + \frac{n}{2} \sum_{k=1}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(-1)^k (n-k-1)! (2z)^{n-2k}}{k! (n-2k)!} + 2^{n-1} z^n$$

05.04.06.0005.02

$$T_n(z) \propto (-1)^{\left\lfloor \frac{n}{2} \right\rfloor} (n z)^{n-2\left\lfloor \frac{n}{2} \right\rfloor} \left(1 + O\left(z^2\right)\right) /; n > 0$$

Expansions at $z == 1$

For the function itself

05.04.06.0006.02

$$T_n(z) \propto 1 + n^2 (z-1) - \frac{(1-n) n^2 (1+n)}{16} (z-1)^2 + \ldots /; (z \to 1)$$

05.04.06.0029.01

$$T_n(z) \propto 1 + n^2 (z-1) - \frac{(1-n) n^2 (1+n)}{16} (z-1)^2 + O\left((z-1)^3\right)$$

05.04.06.0007.01

$$T_n(z) == \sum_{k=0}^{n} \frac{(-n)_k (n)_k}{\left(\frac{1}{2}\right)_k k!} \left(\frac{1-z}{2}\right)^k$$

05.04.06.0008.01

$$T_n(z) == {}_2F_1\left(-n, n; \frac{1}{2}; \frac{1-z}{2}\right)$$

05.04.06.0030.01

$$T_n(z) == \cos\left(2 n \sin^{-1}\left(\frac{\sqrt{1-z}}{\sqrt{2}}\right)\right)$$

05.04.06.0009.02

$$T_n(z) \propto 1 + O(z-1)$$

Expansions at $z == -1$

For the function itself

05.04.06.0010.02

$$T_n(z) \propto (-1)^n \left(1 - n^2 (z+1) - \frac{(1-n) n^2 (1+n)}{6} (z+1)^2 - \ldots\right) /; (z \to -1)$$

05.04.06.0031.01

$$T_n(z) \propto (-1)^n \left(1 - n^2 (z+1) - \frac{(1-n) n^2 (1+n)}{6} (z+1)^2\right) + O\left((z+1)^3\right)$$

05.04.06.0011.01

$$T_n(z) == (-1)^n \sum_{k=0}^{n} \frac{(-n)_k (n)_k}{\left(\frac{1}{2}\right)_k k!} \left(\frac{z+1}{2}\right)^k$$

05.04.06.0012.01

$$T_n(z) \propto (-1)^n \, {}_2F_1\left(-n, n; \frac{1}{2}; \frac{z+1}{2}\right)$$

05.04.06.0032.01

$$T_n(z) \propto (-1)^n \cos\left(2\, n \sin^{-1}\left(\frac{\sqrt{z+1}}{\sqrt{2}}\right)\right)$$

05.04.06.0013.02

$$T_n(z) \propto (-1)^n \, (1 + O(z+1))$$

Expansions at $z \to \infty$

For the function itself

Expansions in $1/z$

05.04.06.0033.01

$$T_n(z) \propto 2^{n-1} z^n \left(1 - \frac{n}{4 z^2} + \frac{n(n-3)}{32 z^4} + \ldots\right) /; (|z| \to \infty) \land n > 0$$

05.04.06.0034.01

$$T_n(z) \propto 2^{n-1} z^n \left(1 - \frac{n}{4 z^2} + \frac{n(n-3)}{32 z^4} + O\left(\frac{1}{z^6}\right)\right) /; n > 0$$

05.04.06.0035.01

$$T_n(z) = 2^{n-1} z^n \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{\left(\frac{1-n}{2}\right)_k \left(-\frac{n}{2}\right)_k z^{-2k}}{k! \, (1-n)_k} /; n > 0$$

05.04.06.0036.01

$$T_n(z) = 2^{n-1} z^n \, {}_2F_1\left(-\frac{n}{2}, \frac{1-n}{2}; 1-n; \frac{1}{z^2}\right) /; n > 2$$

05.04.06.0037.01

$$T_n(z) = \frac{1}{2} (-z)^{-\left(\frac{n+1}{2}\right)} z^{-\frac{n}{2}} \left(z^2 + \sqrt{z^2} \sqrt{z^2-1}\right)^{-n} \left(\cos\left(\frac{n\pi}{2}\right) \sqrt{-z} \left(z^{2n} + (-1)^n \left(z^2 + \sqrt{z^2} \sqrt{z^2-1}\right)^{2n}\right) + \sin\left(\frac{n\pi}{2}\right) \sqrt{z} \left((-1)^n \left(z^2 + \sqrt{z^2} \sqrt{z^2-1}\right)^{2n} - z^{2n}\right)\right) /; n \in \mathbb{N}$$

Expansions in $1/(1-z)$

05.04.06.0014.02

$$T_n(z) \propto 2^n \, n \, (z-1)^n \left(\frac{1}{2n} + \frac{2}{z-1} + \frac{2(2n-3)}{(z-1)^2} + \ldots\right) /; (|z| \to \infty) \land n > 0$$

05.04.06.0038.01

$$T_n(z) \propto 2^n \, n \, (z-1)^n \left(\frac{1}{2n} + \frac{2}{z-1} + \frac{2(2n-3)}{(z-1)^2} + O\left(\frac{1}{z^3}\right)\right) /; n > 0$$

05.04.06.0015.01

$$T_n(z) == 2^{n-1} (z-1)^n \sum_{k=0}^{2n-1} \frac{(-n)_k \left(\frac{1}{2}-n\right)_k}{k!\,(1-2n)_k} \left(\frac{2}{1-z}\right)^k /;\ n>0$$

05.04.06.0039.01

$$T_n(z) == \frac{2^{-n}\sqrt{\pi}\,(z-1)^n}{(n-1)!} \sum_{k=0}^{2n-1} \frac{(2n-k-1)!\,(-n)_k}{k!\,\Gamma\left(-k+n+\frac{1}{2}\right)} \left(\frac{2}{1-z}\right)^k /;\ n>0$$

05.04.06.0040.01

$$T_n(z) == 2^{n-1} (z-1)^n\, {}_2F_1\left(-n, \frac{1}{2}-n; 1-2n; \frac{2}{1-z}\right) /;\ n>1$$

05.04.06.0041.01

$$T_n(z) == 2^{n-1} (z-1)^{-n} \left(\sqrt{\frac{z+1}{z-1}}+1\right)^{-2n} + 2^{-n-1} (z-1)^n \left(\sqrt{\frac{z+1}{z-1}}+1\right)^{2n}$$

05.04.06.0016.02

$$T_n(z) \propto 2^{n-1} z^n \left(1+O\left(\frac{1}{z}\right)\right) /;\ n>0$$

Expansions at $n == \infty$

05.04.06.0042.01

$$T_n(z) = \cos\left(n \cos^{-1}(z)\right)$$

05.04.06.0043.01

$$T_n(z) \propto \begin{cases} \frac{1}{2} e^{i n \cos^{-1}(z)} & -\pi < \arg\left(\cos^{-1}(z)\right) < 0 \\ \frac{1}{2} e^{-i n \cos^{-1}(z)} & 0 < \arg\left(\cos^{-1}(z)\right) < \pi \\ \cos\left(n \cos^{-1}(z)\right) & \text{True} \end{cases} /;\ (n \to \infty)$$

Other series representations

05.04.06.0017.01

$$T_n(z) == \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n}{2k} z^{n-2k} \left(z^2-1\right)^k$$

Integral representations

Integral representations of negative integer order

Rodrigues-type formula.

05.04.07.0001.01

$$T_n(z) == \frac{(-1)^n \sqrt{\pi}\,\sqrt{1-z^2}}{2^n\,\Gamma\left(n+\frac{1}{2}\right)} \frac{\partial^n \left(1-z^2\right)^{n-\frac{1}{2}}}{\partial z^n}$$

Limit representations

05.04.09.0001.01

$$T_n(z) = \frac{n}{2} \lim_{\lambda \to 0} \frac{1}{\lambda} C_n^{\lambda}(z)$$

Generating functions

05.04.11.0001.01

$$T_n(z) = \left([t^n]\, \frac{1 - t\, z}{t^2 - 2\, z\, t + 1}\right) /;\ -1 < z < 1$$

Differential equations

Ordinary linear differential equations and wronskians

For the direct function itself

05.04.13.0001.01

$$\left(1 - z^2\right) w''(z) - z\, w'(z) + n^2\, w(z) = 0 \,/;\, w(z) = c_1\, T_n(z) + c_2 \sin\left(n \cos^{-1}(z)\right)$$

05.04.13.0002.01

$$W_z\left(T_n(z), \sin\left(n \cos^{-1}(z)\right)\right) = -\frac{n}{\sqrt{1 - z^2}}$$

05.04.13.0003.01

$$\left(1 - z^2\right) w''(z) - z\, w'(z) + n^2\, w(z) = 0 \,/;\, w(z) = c_1\, T_n(z) + c_2 \sinh\left(n \cosh^{-1}(z)\right)$$

05.04.13.0004.01

$$W_z\left(T_n(z), \sinh\left(n \cosh^{-1}(z)\right)\right) = \frac{n \cos\left(n \cos^{-1}(z)\right) \cosh\left(n \cosh^{-1}(z)\right)}{\sqrt{z - 1}\, \sqrt{z + 1}} - \frac{n \sin\left(n \cos^{-1}(z)\right) \sinh\left(n \cosh^{-1}(z)\right)}{\sqrt{1 - z^2}}$$

05.04.13.0005.01

$$\left(1 - z^2\right) w''(z) - z\, w'(z) + n^2\, w(z) = 0 \,/;\, w(z) = c_1\, T_n(z) + c_2 \sqrt{1 - z^2}\, U_{n-1}(z)$$

05.04.13.0006.01

$$W_z\left(T_n(z), \sqrt{1 - z^2}\, U_{n-1}(z)\right) = -\frac{n}{\sqrt{1 - z^2}}$$

05.04.13.0007.01

$$w''(z) - \left(\frac{g(z)\, g'(z)}{1 - g(z)^2} + \frac{g''(z)}{g'(z)}\right) w'(z) + \frac{n^2\, g'(z)^2}{1 - g(z)^2}\, w(z) = 0 \,/;\, w(z) = c_1\, T_n(g(z)) + c_2 \sqrt{1 - g(z)^2}\, U_{n-1}(g(z))$$

05.04.13.0008.01

$$W_z\left(T_n(g(z)), \sqrt{1 - g(z)^2}\, U_{n-1}(g(z))\right) = -\frac{n\, g'(z)}{\sqrt{1 - g(z)^2}}$$

05.04.13.0009.01

$$w''(z)-\left(\frac{g(z)\,g'(z)}{1-g(z)^2}+\frac{2\,h'(z)}{h(z)}+\frac{g''(z)}{g'(z)}\right)w'(z)+\left(\frac{n^2\,g'(z)^2}{1-g(z)^2}+\frac{g(z)\,h'(z)\,g'(z)}{\left(1-g(z)^2\right)h(z)}+\frac{h(z)\,h'(z)\,g''(z)+g'(z)\left(2\,h'(z)^2-h(z)\,h''(z)\right)}{h(z)^2\,g'(z)}\right)w(z)=0\;/;$$
$$w(z)=c_1\,h(z)\,T_n(g(z))+c_2\,h(z)\sqrt{1-g(z)^2}\;U_{n-1}(g(z))$$

05.04.13.0010.01

$$W_z\left(h(z)\,T_n(g(z)),\,h(z)\sqrt{1-g(z)^2}\;U_{n-1}(g(z))\right)=-\frac{n\,h(z)^2\,g'(z)}{\sqrt{1-g(z)^2}}$$

05.04.13.0011.01

$$z^2\left(a^2\,z^{2r}-1\right)w''(z)+\left(r-(2\,s-1)\left(a^2\,z^{2r}-1\right)\right)z\,w'(z)+\left(a^2\,z^{2r}\left(s^2-r^2\,n^2\right)-s\,(r+s)\right)w(z)=0\;/;$$
$$w(z)=c_1\,z^s\,T_n(a\,z^r)+c_2\,z^s\sqrt{1-a^2\,z^{2r}}\;U_{n-1}(a\,z^r)$$

05.04.13.0012.01

$$W_z\left(z^s\,T_n(a\,z^r),\,z^s\sqrt{1-a^2\,z^{2r}}\;U_{n-1}(a\,z^r)\right)=-\frac{a\,r\,n\,z^{r+2s-1}}{\sqrt{1-a^2\,z^{2r}}}$$

05.04.13.0013.01

$$\left(a^2\,r^{2z}-1\right)w''(z)+\left(\log(r)-2\left(a^2\,r^{2z}-1\right)\log(s)\right)w'(z)+\left(a^2\,r^{2z}\left(\log^2(s)-n^2\log^2(r)\right)-\log(s)\,(\log(r)+\log(s))\right)w(z)=0\;/;$$
$$w(z)=c_1\,s^z\,T_n(a\,r^z)+c_2\,s^z\sqrt{1-a^2\,r^{2z}}\;U_{n-1}(a\,r^z)$$

05.04.13.0014.01

$$W_z\left(s^z\,T_n(a\,r^z),\,s^z\sqrt{1-a^2\,r^{2z}}\;U_{n-1}(a\,r^z)\right)=-\frac{a\,r^z\,n\,s^{2z}\log(r)}{\sqrt{1-a^2\,r^{2z}}}$$

Transformations

Transformations and argument simplifications

Argument involving basic arithmetic operations

05.04.16.0001.01

$$T_n(-z)=(-1)^n\,T_n(z)$$

Multiple arguments

05.04.16.0005.01

$$T_{2n}\left(\sqrt{\frac{z+1}{2}}\right)=T_n(z)$$

Brychkov Yu.A. (2007)

05.04.16.0006.01

$$T_{2n}(z)=(-1)^n\,T_n\left(1-2\,z^2\right)$$

Products, sums, and powers of the direct function

Products of the direct function

05.04.16.0002.01

$$T_n(z)\, T_m(z) \equiv \frac{1}{2}\left(T_{m+n}(z) + T_{|n-m|}(z)\right) /;\; m > 0 \land n > 0$$

Addition formulas

05.04.16.0004.02

$$T_n(\cos(\theta_0)) \equiv \sum_{k=0}^{n} \sum_{j_1=0}^{\left\lfloor \frac{n-k}{2} \right\rfloor} \sum_{j_2=0}^{\left\lfloor \frac{n-k}{2} \right\rfloor} \sum_{j_3=0}^{\left\lfloor \frac{k}{2} \right\rfloor} \frac{n\, 4^k\, (n-k)!\, \left(\frac{1}{2}-k\right)}{\pi\, \Gamma(k+n)}$$

$$\begin{cases} 1 & k = j_1 = j_2 = 0 \\ 0 & k + j_1 = 0 \lor k + j_2 = 0 \\ \frac{\Gamma(k+j_1)\,\Gamma(k+j_2)}{\Gamma(k)\,\Gamma(k)} & \text{True} \end{cases} \frac{\Gamma(n-j_1)}{j_1!\,\Gamma(-k+n-j_1+1)} \frac{\Gamma(n-j_2)}{j_2!\,\Gamma(-k+n-j_2+1)} \frac{\Gamma\left(j_3-\frac{1}{2}\right)\Gamma\left(k-j_3-\frac{1}{2}\right)}{j_3!\,\Gamma(k-j_3+1)}$$

$$\left(1-\frac{1}{2}\delta_{2 j_1, n-k}\right)\left(1-\frac{1}{2}\delta_{2 j_2, n-k}\right)\left(1-\frac{1}{2}\delta_{2 j_3, k}\right)$$

$$\sin^k(\theta)\sin^k(\vartheta)\, T_{n-k-2 j_1}(\cos(\theta))\, T_{n-k-2 j_2}(\cos(\vartheta))\, T_{k-2 j_3}(\cos(\phi)) /;\; n \in \mathbb{N} \land \cos(\theta_0) \equiv \cos(\theta)\cos(\vartheta) + \cos(\phi)\sin(\theta)\sin(\vartheta)$$

Related transformations

05.04.16.0003.01

$$T_n(T_m(z)) \equiv T_{n m}(z) /;\; m > 0 \land n > 0$$

Identities

Recurrence identities

Consecutive neighbors

05.04.17.0001.01

$$T_n(z) \equiv 2 z\, T_{n+1}(z) - T_{n+2}(z)$$

05.04.17.0002.01

$$T_n(z) \equiv 2 z\, T_{n-1}(z) - T_{n-2}(z)$$

Distant neighbors

05.04.17.0008.01

$$T_n(z) \equiv C_m(n, z)\, T_{n+m}(z) - C_{m-1}(n, z)\, T_{n+m+1}(z) /;\; C_0(n, z) \equiv 1 \land C_1(n, z) \equiv 2 z \land C_m(n, z) \equiv 2 z\, C_{m-1}(n, z) - C_{m-2}(n, z) \land m > 0$$

05.04.17.0009.01

$$T_n(z) \equiv C_m(n, z)\, T_{n-m}(z) - C_{m-1}(n, z)\, T_{n-m-1}(z) /;\; C_0(n, z) \equiv 1 \land C_1(n, z) \equiv 2 z \land C_m(n, z) \equiv 2 z\, C_{m-1}(n, z) - C_{m-2}(n, z) \land m > 0$$

05.04.17.0003.01

$$T_n(z) \equiv 2 (-1)^{2\left\lfloor \frac{m}{2} \right\rfloor} z^{m-2\left\lfloor \frac{m}{2} \right\rfloor} \left(z^2\right)^{\frac{1-m}{2}+\left\lfloor \frac{m}{2} \right\rfloor} U_{\frac{m-1}{2}}\left(2 z^2-1\right) T_{n+m}(z) - 2 z^{1-m+2\left\lfloor \frac{m}{2} \right\rfloor} \left(z^2\right)^{\left\lfloor \frac{m+1}{2} \right\rfloor-\frac{m}{2}} U_{\frac{m}{2}-1}\left(2 z^2-1\right) T_{n+m+1}(z) /;\; m > 0$$

05.04.17.0004.01

$$T_n(z) == 2\,(-1)^{2\left\lfloor \frac{m}{2} \right\rfloor} z^{m-2\left\lfloor \frac{m}{2} \right\rfloor} \left(z^2\right)^{\frac{1-m}{2}+\left\lfloor \frac{m}{2} \right\rfloor} U_{\frac{m-1}{2}}\left(2\,z^2-1\right) T_{n-m}(z) - 2\,z^{1-m+2\left\lfloor \frac{m}{2} \right\rfloor} \left(z^2\right)^{\left\lfloor \frac{m+1}{2} \right\rfloor - \frac{m}{2}} U_{\frac{m}{2}-1}\left(2\,z^2-1\right) T_{n-m-1}(z) \;/;\; m > 0$$

Functional identities

Relations between contiguous functions

Recurrence relations

05.04.17.0005.01

$$T_{n-1}(z) + T_{n+1}(z) == 2\,z\,T_n(z)$$

05.04.17.0006.01

$$T_n(z) == \frac{1}{2\,z}\left(T_{n-1}(z) + T_{n+1}(z)\right)$$

Normalized recurrence relation

05.04.17.0007.01

$$z\,p(n,\,z) == \frac{1}{4}\,p(n-1,\,z) + p(n+1,\,z) \;/;\; p(n,\,z) == 2^{-n}\,T_n(z)$$

Complex characteristics

Real part

05.04.19.0001.01

$$\operatorname{Re}(T_n(x+i\,y)) == T_n(x) + n \sum_{j=1}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(-1)^j\,2^{2\,j-2}\,y^{2\,j}}{j}\,C_{n-2\,j}^{2\,j}(x) \;/;\; x \in \mathbb{R} \bigwedge y \in \mathbb{R}$$

Imaginary part

05.04.19.0002.01

$$\operatorname{Im}(T_n(x+i\,y)) == n \sum_{j=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \frac{(-1)^j\,2^{2\,j}\,y^{2\,j+1}}{2\,j+1}\,C_{n-2\,j-1}^{2\,j+)}(x) \;/;\; x \in \mathbb{R} \bigwedge y \in \mathbb{R}$$

Argument

05.04.19.0003.01

$$\arg(T_n(x+i\,y)) == \tan^{-1}\left(T_n(x) + n \sum_{j=1}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(-1)^j\,2^{2\,j-2}}{j}\,C_{n-2\,j}^{2\,j}(x)\,y^{2\,j},\; n \sum_{j=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \frac{(-1)^j\,2^{2\,j}}{2\,j+1}\,C_{-2\,j+n-1}^{2\,j+1}(x)\,y^{2\,j+1}\right) \;/;\; x \in \mathbb{R} \bigwedge y \in \mathbb{R}$$

Conjugate value

05.04.19.0004.01

$$\overline{T_n(x+i\,y)} = T_n(x) - i\,n \sum_{j=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \frac{(-1)^j\, 2^{2j}}{2\,j+1}\, C_{-2\,j+n-1}^{2\,j+1}(x)\, y^{2\,j+1} + n \sum_{j=1}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(-1)^j\, 2^{2\,j-2}}{j}\, C_{n-2\,j}^{2\,j}(x)\, y^{2\,j} \;/;\; x \in \mathbb{R} \bigwedge y \in \mathbb{R}$$

Differentiation

Low-order differentiation

With respect to *z*

Forward shift operator:

05.04.20.0001.01

$$\frac{\partial T_n(z)}{\partial z} = n\, U_{n-1}(z)$$

05.04.20.0002.01

$$\frac{\partial^2 T_n(z)}{\partial z^2} = \frac{n}{z^2-1} \left(n\, T_n(z) - z\, U_{n-1}(z)\right)$$

Symbolic differentiation

With respect to *z*

05.04.20.0003.01

$$\frac{\partial^m T_n(z)}{\partial z^m} = n\, 2^{m-1}\, (m-1)!\, C_{n-m}^m(z) \;/;\; m \in \mathbb{N}^+$$

05.04.20.0004.02

$$\frac{\partial^m T_n(z)}{\partial z^m} = \sqrt{\pi}\, (z-1)^{-m}\, {}_3\tilde{F}_2\!\left(1, -n, n; \frac{1}{2}, 1-m; \frac{1-z}{2}\right) /;\; m \in \mathbb{N}$$

05.04.20.0006.01

$$\frac{\partial^m T_n(z)}{\partial z^m} = 2^{m-1}\, (m-1)!\, n \sum_{i_1=0}^{n-m} \ldots \sum_{i_m=0}^{n-m} \delta_{\sum_{j=1}^m i_j,\, n-m} \prod_{j=1}^m U_{i_j}(z) \;/;\; m \in \mathbb{N}^+ \bigwedge n \in \mathbb{N}$$

Fractional integro-differentiation

With respect to *z*

05.04.20.0005.01

$$\frac{\partial^\alpha T_n(z)}{\partial z^\alpha} = z^{-\alpha}\, \sqrt{\pi}\, \tilde{F}_{1\times 1\times 0}^{2\times 1\times 0}\!\left(\begin{matrix} -n, n; 1; ; \\ \frac{1}{2}; 1-\alpha; ; \end{matrix} \;\; -\frac{z}{2}, \frac{1}{2} \right)$$

Integration

Indefinite integration

Involving only one direct function

05.04.21.0001.01

$$\int T_n(a\,z)\,dz = \frac{1}{2\,a}\left(\frac{T_{n-1}(a\,z)}{1-n}+\frac{T_{n+1}(a\,z)}{n+1}\right)\; /;\; n \neq 1$$

05.04.21.0002.01

$$\int T_n(z)\,dz = \frac{1}{2}\left(\frac{T_{n-1}(z)}{1-n}+\frac{T_{n+1}(z)}{n+1}\right)\; /;\; n \neq 1$$

05.04.21.0003.01

$$\int T_n(z)\,dz = \frac{2^{n-1}\,z^{n+1}}{n+1}+\frac{1}{2}\,\delta_{n,0}\,z+\frac{1}{4}\,n\sum_{k=1}^{\left\lfloor \frac{n}{2}\right\rfloor}\frac{(-1)^k\,(n-k-1)!\,(2\,z)^{n-2\,k+1}}{k!\,(n-2\,k+1)!}$$

Involving one direct function and elementary functions

Involving power function

05.04.21.0004.01

$$\int z^{\alpha-1}\,T_n(z)\,dz = \frac{\delta_{n,0}\,z^{\alpha+1}}{2\,(\alpha+1)}+\frac{n}{2}\sum_{k=1}^{\left\lfloor \frac{n}{2}\right\rfloor}\frac{(-1)^k\,(n-k-1)!\,2^{n-2\,k}\,z^{n-2\,k+\alpha}}{k!\,(n-2\,k)!\,(n-2\,k+\alpha)}+\frac{2^{n-1}\,z^{n+\alpha+1}}{(n+1)\,(n+\alpha+1)}$$

Involving algebraic functions

05.04.21.0005.01

$$\int \left(1-z^2\right)^{\frac{1}{2}(-n-3)}\,T_n(z)\,dz = \frac{\left(1-z^2\right)^{\frac{1}{2}(-n-1)}}{n+1}\,T_{n+1}(z)$$

05.04.21.0006.01

$$\int \left(1-z^2\right)^{\frac{n-3}{2}}\,T_n(z)\,dz = -\frac{\left(1-z^2\right)^{\frac{n-1}{2}}}{n-1}\,T_{n-1}(z)$$

Definite integration

Involving the direct function

05.04.21.0007.01

$$\mathcal{P}\int_{-1}^{1}\frac{T_n(t)}{\sqrt{1-t^2}\,(t-x)}\,dt = \pi\,U_{n-1}(x)\; /;\; n>1 \land -1<x<1$$

05.04.21.0008.01

$$\int_{-1}^{1} T_n(x)^2\,\log\!\left(T_n(x)^2\right)\frac{1}{\sqrt{1-x^2}}\,dx = -\pi\,(1-\delta_{n,0})\,(\log(2)-1)\; /;\; n\in\mathbb{N}$$

Entropy integral

Orthogonality:

05.04.21.0009.01

$$\int_{-1}^{1} \frac{T_m(t)\, T_n(t)}{\sqrt{1-t^2}}\, dt = \frac{\pi}{2}\, \delta_{m,n} \;/;\; m^2+n^2 \neq 0$$

Summation

Infinite summation

05.04.23.0001.01

$$\sum_{n=0}^{\infty} T_n(z)\, w^n = \frac{1-w\,z}{w^2-2\,z\,w+1} \;/;\; -1<z<1 \land |w|<1$$

05.04.23.0002.01

$$\sum_{n=1}^{\infty} \frac{w^n}{n}\, T_n(z) = -\frac{1}{2} \log\left(w^2-2\,z\,w+1\right) \;/;\; -1<z<1 \land |w|<1$$

05.04.23.0003.01

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n}{n!}\, T_n(z)\, w^n = \frac{\sqrt{1-w\,z+\sqrt{w^2-2\,z\,w+1}}}{\sqrt{2}\,\sqrt{w^2-2\,z\,w+1}} \;/;\; -1<z<1 \land |w|<1$$

05.04.23.0004.01

$$\sum_{n=0}^{\infty} \frac{T_n(z)\, w^n}{\left(\frac{1}{2}\right)_n\, n!} = \cosh\left(\sqrt{2}\,\sqrt{w\,(z-1)}\right) \cosh\left(\sqrt{2}\,\sqrt{w\,(z+1)}\right) \;/;\; -1<z<1 \land |w|<1$$

05.04.23.0005.01

$$\sum_{n=0}^{\infty} \frac{T_n(z)\, w^n}{n!} = e^{w\,z} \cosh\left(\sqrt{w^2\left(z^2-1\right)}\right) \;/;\; -1<z<1 \land |w|<1$$

05.04.23.0006.01

$$\sum_{n=0}^{\infty} \frac{(\gamma)_n\,(-\gamma)_n}{\left(\frac{1}{2}\right)_n\, n!}\, T_n(z)\, w^n = \cos\left(2\,\gamma \sin^{-1}\left(\frac{\sqrt{-w-\sqrt{w^2-2\,z\,w+1}\;+1}}{\sqrt{2}}\right)\right) \cos\left(2\,\gamma \sin^{-1}\left(\frac{\sqrt{w-\sqrt{w^2-2\,z\,w+1}\;+1}}{\sqrt{2}}\right)\right) \;/;$$

$$-1<z<1 \land |w|<1$$

05.04.23.0007.01

$$\sum_{n=0}^{\infty} \frac{(\gamma)_n}{n!}\, T_n(z)\, w^n = (1-w\,z)^{-\gamma}\; {}_2F_1\left(\frac{\gamma}{2}, \frac{\gamma+1}{2}; \frac{1}{2}; \frac{\left(z^2-1\right) w^2}{(1-w\,z)^2}\right) \;/;\; -1<z<1 \land |w|<1$$

05.04.23.0008.01

$$\sum_{n=1}^{\infty} \frac{1}{2^n \prod_{k=1}^{2^{n-1}} T_k\left(\frac{z}{2}\right)} = \frac{1}{2}\left(z-\sqrt{z^2-4}\right) \;/;\; |z|>2$$

05.04.23.0009.01

$$\sum_{n=0}^{\infty} T_n(x)\, T_n(y) = \frac{1}{2}\, \pi \sqrt[4]{1-x^2}\, \sqrt[4]{1-y^2}\; \delta(x-y) \;/;\; -1<x<1 \bigwedge -1<y<1$$

Operations

Orthogonality, completeness, and Fourier expansions

The set of functions $T_n(x)$, $n = 0, 1, \ldots$, forms a complete, orthogonal (with weight $\frac{2}{\pi} \frac{1}{\sqrt{1-x^2}}$) system on the interval $(-1, 1)$.

05.04.25.0001.01

$$\sum_{n=0}^{\infty}\left(\sqrt{\frac{2}{\pi}} \frac{T_n(x)}{\sqrt[4]{1-x^2}}\right)\left(\sqrt{\frac{2}{\pi}} \frac{T_n(y)}{\sqrt[4]{1-y^2}}\right) = \delta(x-y) \;/;\; -1 < x < 1 \land -1 < y < 1$$

05.04.25.0002.01

$$\int_{-1}^{1}\left(\sqrt{\frac{2}{\pi}} \frac{T_m(t)}{\sqrt[4]{1-t^2}}\right)\left(\sqrt{\frac{2}{\pi}} \frac{T_n(t)}{\sqrt[4]{1-t^2}}\right) dt = \delta_{m,n} \;/;\; m^2 + n^2 \neq 0$$

Any sufficiently smooth function $f(x)$ can be expanded in the system $\{T_n(x)\}_{n=0,1,\ldots}$ as a generalized Fourier series, with its sum converging to $f(x)$ almost everywhere.

05.04.25.0003.01

$$f(x) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n\, \psi_n(x) \;/;\; c_n = \int_{-1}^{1} \psi_n(t)\, f(t)\, dt \land \psi_n(x) = \sqrt{\frac{2}{\pi}} \frac{T_n(x)}{\sqrt[4]{1-x^2}} \land -1 < x < 1$$

Representations through more general functions

Through hypergeometric functions

Involving $_2F_1$

05.04.26.0001.01

$$T_n(z) = {}_2F_1\left(-n, n; \frac{1}{2}; \frac{1-z}{2}\right)$$

05.04.26.0002.01

$$T_n(z) = (-1)^n\, {}_2F_1\left(-n, n; \frac{1}{2}; \frac{z+1}{2}\right)$$

Through hypergeometric functions of two variables

05.04.26.0003.01

$$T_n(z) = F_{1\times 0\times 0}^{2\times 0\times 0}\left(\begin{matrix} -n, n;;; \\ \frac{1}{2};;; \end{matrix}\; \frac{1}{2}, -\frac{z}{2}\right)$$

Through Meijer *G*

Classical cases for the direct function itself

05.04.26.0004.01

$$T_n(z) = -\frac{1}{\sqrt{\pi}} \lim_{m\to n} m \sin(\pi m)\, G_{2,2}^{1,2}\left(\frac{z-1}{2} \,\middle|\, \begin{matrix} m+1, 1-m \\ 0, \frac{1}{2} \end{matrix}\right)$$

Classical cases involving algebraic functions

05.04.26.0005.01

$$(z+1)^{-n}\, T_n\left(\frac{1-z}{1+z}\right) = \frac{2^{2n-1}}{\Gamma(2n)}\, G_{2,2}^{1,2}\left(z \,\middle|\, \begin{matrix} \frac{1}{2}-n, 1-n \\ 0, \frac{1}{2} \end{matrix}\right) /; z \notin (-\infty, -1)$$

05.04.26.0006.01

$$(z+1)^{-n}\, T_n\left(\frac{z-1}{z+1}\right) = \frac{2^{2n-1}}{\Gamma(2n)}\, G_{2,2}^{2,1}\left(z \,\middle|\, \begin{matrix} 1-n, \frac{1}{2}-n \\ 0, \frac{1}{2} \end{matrix}\right) /; z \notin (-1, 0)$$

05.04.26.0007.01

$$(z+1)^{-\frac{n}{2}}\, T_n\left(\frac{1}{\sqrt{z+1}}\right) = \frac{2^{n-1}}{\Gamma(n)}\, G_{2,2}^{1,2}\left(z \,\middle|\, \begin{matrix} 1-\frac{n}{2}, \frac{1-n}{2} \\ 0, \frac{1}{2} \end{matrix}\right)$$

05.04.26.0008.01

$$(z+1)^{-\frac{n}{2}}\, T_n\left(\sqrt{\frac{z}{z+1}}\right) = \frac{2^{n-1}}{\Gamma(n)}\, G_{2,2}^{2,1}\left(z \,\middle|\, \begin{matrix} 1-\frac{n}{2}, \frac{1-n}{2} \\ 0, \frac{1}{2} \end{matrix}\right) /; z \notin (-1, 0)$$

Classical cases involving unit step θ

05.04.26.0009.01

$$\frac{\theta(1-|z|)}{\sqrt{1-z}}\, T_n(2z-1) = \sqrt{\pi}\, G_{2,2}^{2,0}\left(z \,\middle|\, \begin{matrix} \frac{1}{2}-n, n+\frac{1}{2} \\ 0, \frac{1}{2} \end{matrix}\right) /; z \notin (-1, 0)$$

05.04.26.0010.01

$$\frac{\theta(|z|-1)}{\sqrt{z-1}}\, T_n(2z-1) = \sqrt{\pi}\, G_{2,2}^{0,2}\left(z \,\middle|\, \begin{matrix} \frac{1}{2}-n, n+\frac{1}{2} \\ 0, \frac{1}{2} \end{matrix}\right)$$

05.04.26.0011.01

$$\frac{\theta(1-|z|)}{\sqrt{1-z}}\, T_n\left(\frac{2}{z}-1\right) = \sqrt{\pi}\, G_{2,2}^{2,0}\left(z \,\middle|\, \begin{matrix} 0, \frac{1}{2} \\ -n, n \end{matrix}\right)$$

05.04.26.0012.01

$$\frac{\theta(|z|-1)}{\sqrt{z-1}}\, T_n\left(\frac{2}{z}-1\right) = \sqrt{\pi}\, G_{2,2}^{0,2}\left(z \,\middle|\, \begin{matrix} 0, \frac{1}{2} \\ -n, n \end{matrix}\right) /; z \notin (-\infty, -1)$$

05.04.26.0013.01

$$\frac{\theta(|z|-1)}{\sqrt{z-1}}\, T_n\left(8z^2-8z+1\right) = \sqrt{\pi}\, G_{2,2}^{0,2}\left(z \,\middle|\, \begin{matrix} \frac{1}{2}-2n, 2n+\frac{1}{2} \\ 0, \frac{1}{2} \end{matrix}\right) /; \mathrm{Re}(z) > 0$$

05.04.26.0014.01

$$\frac{\theta(1-|z|)}{\sqrt{1-z}}\, T_n\left(\frac{8}{z^2}-\frac{8}{z}+1\right) = \sqrt{\pi}\, G_{2,2}^{2,0}\left(z \,\middle|\, \begin{matrix} 0, \frac{1}{2} \\ -2n, 2n \end{matrix}\right) /; \mathrm{Re}(z) > 0$$

Generalized cases involving algebraic functions

05.04.26.0015.01

$$\left(z^2+1\right)^{-\frac{n}{2}} T_n\left(\frac{z}{\sqrt{z^2+1}}\right) = \frac{2^{n-1}}{\Gamma(n)} G_{2,2}^{2,1}\left(z, \frac{1}{2} \middle| \begin{matrix} 1-\frac{n}{2}, \frac{1-n}{2} \\ 0, \frac{1}{2} \end{matrix}\right) /; \operatorname{Re}(z) > 0$$

Generalized cases involving unit step θ

05.04.26.0016.01

$$\frac{\theta(|z|-1)}{\sqrt{z^2-1}} T_n(z) = \sqrt{\pi} \sqrt{z^2} G_{2,2}^{0,2}\left(z, \frac{1}{2} \middle| \begin{matrix} -\frac{n}{2}, \frac{n}{2} \\ -\frac{1}{2}, 0 \end{matrix}\right)$$

05.04.26.0017.01

$$\frac{\theta(1-|z|)}{\sqrt{1-z^2}} T_n\left(\frac{1}{z}\right) = \sqrt{\pi} G_{2,2}^{2,0}\left(z, \frac{1}{2} \middle| \begin{matrix} 0, \frac{1}{2} \\ -\frac{n}{2}, \frac{n}{2} \end{matrix}\right)$$

05.04.26.0018.01

$$\frac{\theta(|z|-1)}{\sqrt{z^2-1}} T_n\left(\frac{1}{z}\right) = \sqrt{\pi} \sqrt{z^2} G_{2,2}^{0,2}\left(z, \frac{1}{2} \middle| \begin{matrix} -\frac{1}{2}, 0 \\ -\frac{n+1}{2}, \frac{n-1}{2} \end{matrix}\right)$$

Through other functions

Involving some hypergeometric-type functions

05.04.26.0019.01

$$T_n(z) = \sqrt{\frac{\pi}{2}} \sqrt[4]{1-z^2}\, P_{n-\frac{1}{2}}^{\frac{1}{2}}(z)$$

05.04.26.0020.01

$$T_n(z) = \sqrt{\frac{\pi}{2}} \sqrt[4]{z-1} \sqrt[4]{z+1}\, \mathfrak{P}_{n-\frac{1}{2}}^{\frac{1}{2}}(z)$$

05.04.26.0021.01

$$T_n(z) = \frac{n!}{\left(\frac{1}{2}\right)_n} P_n^{\left(-\frac{1}{2},-\frac{1}{2}\right)}(z)$$

05.04.26.0022.01

$$T_n(z) = \frac{P_n^{\left(-\frac{1}{2},-\frac{1}{2}\right)}(z)}{P_n^{\left(-\frac{1}{2},-\frac{1}{2}\right)}(1)}$$

Involving spheroidal functions

05.04.26.0023.01

$$T_n(z) = \frac{\sqrt{2} \sqrt[4]{1-z^2}}{\sqrt{\pi}} PS_{n-\frac{1}{2},\frac{1}{2}}(0, z)$$

Representations through equivalent functions

With related functions

05.04.27.0001.01

$$T_n(z) = \frac{n}{2} C_n^{(0)}(z)$$

05.04.27.0002.01

$$T_n(z) = U_n(z) - z\, U_{n-1}(z)$$

05.04.27.0003.01

$$T_n(z) = \frac{1}{2}(U_n(z) - U_{n-2}(z))$$

05.04.27.0004.01

$$T_n(z) = \frac{1}{n}\left(\frac{\partial\left(\left(z^2-1\right) U_{n-1}(z)\right)}{\partial z} - z\, U_{n-1}(z)\right)$$

05.04.27.0005.01

$$T_n(x) = -\frac{1}{\pi} \mathcal{P} \int_{-1}^{1} \frac{\sqrt{1-t^2}\; U_{n-1}(t)}{t-x}\, dt \;/;\; n > 0 \land -1 < x < 1$$

05.04.27.0010.01

$$T_{2n+1}(z) = (-1)^n z\, U_{2n}\left(\sqrt{1-z^2}\right)$$

With elementary functions

05.04.27.0006.01

$$T_n(z) = \frac{1}{2}\left(e^{\frac{i\pi n}{2}}\left(i z + \sqrt{1-z^2}\right)^{-n} + e^{-\frac{i\pi n}{2}}\left(i z + \sqrt{1-z^2}\right)^{n}\right)$$

05.04.27.0007.01

$$T_n(z) = \cos\left(n \cos^{-1}(z)\right)$$

05.04.27.0011.01

$$T_n(z) = (-1)^n \cos\left(2 n \sin^{-1}\left(\frac{\sqrt{z+1}}{\sqrt{2}}\right)\right)$$

05.04.27.0008.01

$$T_n(z) = 2^{-n-1} (-1)^n \left(\left(\sqrt{1-z} - \sqrt{-z-1}\right)^{2n} + \left(\sqrt{1-z} + \sqrt{-z-1}\right)^{2n}\right)$$

05.04.27.0009.01

$$T_n(z) = \frac{1}{2} z^n \left(\left(1 - \sqrt{1 - \frac{1}{z^2}}\right)^n + \left(1 + \sqrt{1 - \frac{1}{z^2}}\right)^n\right)$$

Theorems

Expansions in generalized Fourier series

$$f(x) = \sum_{k=0}^{\infty} c_k\, \psi_k(x) \;/;\; c_k = \int_{-1}^{1} f(t)\, \psi_k(t)\, dt,\; \psi_k(x) = \sqrt{\frac{2}{\pi}} \left(1-x^2\right)^{-\frac{1}{4}} T_k(x),\; k \in \mathbb{N}.$$

Minimizing property of Chebyshev polynomials

Chebyshev polynomials $T_n(x)$ have the smallest absolute values among all polynomials of degree n with leading coefficient 1:

$$\max_{-1\leq x\leq 1}\left|\sum_{k=0}^{n} a_i\, x^i\right| \geq \max_{-1\leq x\leq 1}\left|\frac{T_n(x)}{[x^n](T_n(x))}\right|.$$

Another minimizing property of Chebyshev polynomials

$$\int_{-1}^{1}\frac{P(x)^2}{\sqrt{1-x^2}}\,dx \geq \int_{-1}^{1}\frac{T_n(x)^2}{\sqrt{1-x^2}}\,dx == \frac{\pi}{2}\ /;\qquad P(x) == 2^{n-1}\,x^n + \sum_{k=0}^{n-1} c_i\, x^i \bigwedge c_i \in \mathbb{R}$$

Distribution of the zeros of Chebyshev polynomials of high order

In the limit $n \to \infty$, the zeros of $T_n(x)$ fulfill the arcsin distribution; in other words, the relative number of zeros $m_n(a,\ b)/n$ in the interval (a, b) is $(\arcsin(b) - \arcsin(a))/\pi$.

History

– P. L. Chebyshev (1854,1855,1859)

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