ChebyshevTGeneral

Notations

Traditional name

Chebyshev function of the first kind

Traditional notation

$T_\nu(z)$

Mathematica StandardForm notation

ChebyshevT[ν, z]

Primary definition

07.04.02.0001.01

$$T_\nu(z) = \cos\left(\nu \cos^{-1}(z)\right)$$

Specific values

Specialized values

For fixed ν

07.04.03.0001.01

$$T_\nu(0) = \cos\left(\frac{\pi \nu}{2}\right)$$

07.04.03.0002.01

$$T_\nu(1) = 1$$

07.04.03.0003.01

$$T_\nu(-1) = \cos(\pi \nu)$$

For fixed z

07.04.03.0004.01

$$T_{-\frac{1}{2}}(z) = \sqrt{\frac{z+1}{2}}$$

07.04.03.0005.01

$$T_{\frac{1}{2}}(z) = \sqrt{\frac{z+1}{2}}$$

07.04.03.0006.01

$$T_0(z) == 1$$

07.04.03.0007.01

$$T_1(z) == z$$

07.04.03.0008.01

$$T_2(z) == 2 z^2 - 1$$

07.04.03.0009.01

$$T_3(z) == 4 z^3 - 3 z$$

07.04.03.0010.01

$$T_4(z) == 8 z^4 - 8 z^2 + 1$$

07.04.03.0011.01

$$T_5(z) == 16 z^5 - 20 z^3 + 5 z$$

07.04.03.0012.01

$$T_6(z) == 32 z^6 - 48 z^4 + 18 z^2 - 1$$

07.04.03.0013.01

$$T_7(z) == 64 z^7 - 112 z^5 + 56 z^3 - 7 z$$

07.04.03.0014.01

$$T_8(z) == 128 z^8 - 256 z^6 + 160 z^4 - 32 z^2 + 1$$

07.04.03.0015.01

$$T_9(z) == 256 z^9 - 576 z^7 + 432 z^5 - 120 z^3 + 9 z$$

07.04.03.0016.01

$$T_{10}(z) == 512 z^{10} - 1280 z^8 + 1120 z^6 - 400 z^4 + 50 z^2 - 1$$

07.04.03.0017.01

$$T_n(z) == 2^{n-1} z^n + \frac{\delta_{n,0}}{2} + \frac{n}{2} \sum_{k=1}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(-1)^k (n-k-1)! (2 z)^{n-2k}}{k! (n-2k)!} /; n \in \mathbb{N}$$

07.04.03.0018.01

$$T_n(z) == (-1)^n \cos\left(2 n \sin^{-1}\left(\frac{\sqrt{z+1}}{\sqrt{2}}\right)\right) /; n \in \mathbb{N}$$

General characteristics

Domain and analyticity

$T_\nu(z)$ is an analytical function of ν and z which is defined over $\mathbb{C}^2$. For fixed z the function $T_\nu(z)$ is an entire function of ν. For integer ν, $T_\nu(z)$ degenerates to a polynomial in z.

07.04.04.0001.01

$$(\nu * z) \longrightarrow T_\nu(z) :: (\mathbb{C} \otimes \mathbb{C}) \longrightarrow \mathbb{C}$$

Symmetries and periodicities

Parity

07.04.04.0002.01

$$T_n(-z) == (-1)^n T_n(z) /; n \in \mathbb{N}$$

$T_\nu(z)$ is an even function with respect to its parameter.

07.04.04.0003.01

$$T_{-\nu}(z) == T_\nu(z)$$

Mirror symmetry

07.04.04.0004.02

$$T_{\bar{\nu}}(\bar{z}) == \overline{T_\nu(z)} /; z \notin (-\infty, -1)$$

Periodicity

No periodicity

Poles and essential singularities

With respect to *z*

For fixed ν /; $\nu \notin \mathbb{Z}$, the function $T_\nu(z)$ does not have poles and essential singularities.

07.04.04.0005.01

$$\mathcal{S}ing_z(T_\nu(z)) == \{\} /; \nu \notin \mathbb{Z}$$

For integer ν, the function $T_\nu(z)$ is polynomial and has pole of order $|\nu|$ at $z = \tilde{\infty}$.

07.04.04.0006.01

$$\mathcal{S}ing_z(T_\nu(z)) == \{\{\tilde{\infty}, |\nu|\}\} /; \nu \in \mathbb{Z}$$

With respect to *ν*

For fixed z, the function $T_\nu(z)$ has only one singular point at $\nu = \tilde{\infty}$. It is an essential singular point. .

07.04.04.0007.01

$$\mathcal{S}ing_\nu(T_\nu(z)) == \{\{\tilde{\infty}, \infty\}\}$$

Branch points

With respect to *z*

For fixed noninteger ν, the function $T_\nu(z)$ has two branch points: $z == -1$, $z == \tilde{\infty}$.

For fixed integer ν, the function $T_\nu(z)$ does not have branch points.

07.04.04.0008.01

$$\mathcal{BP}_z(T_\nu(z)) == \{-1, \tilde{\infty}\} /; \nu \notin \mathbb{Z}$$

07.04.04.0009.01

$$\mathcal{BP}_z(T_\nu(z)) == \{\} /; \nu \in \mathbb{Z}$$

07.04.04.0010.01

$$\mathcal{R}_z(T_\nu(z), -1) == 2 /; \nu \notin \mathbb{Z}$$

07.04.04.0011.01

$$\mathcal{R}_z(T_\nu(z), \tilde{\infty}) == \log /; \nu \notin \mathbb{Q}$$

07.04.04.0012.01

$\mathcal{R}_z(T_\nu(z), \tilde{\infty}) == s \;/;\; \nu == \frac{r}{s} \bigwedge \{r, s\} \in \mathbb{Z} \bigwedge s > 1 \bigwedge \gcd(r, s) == 1$

With respect to ν

For fixed z, the function $T_\nu(z)$ does not have branch points.

07.04.04.0013.01

$\mathcal{BP}_\nu(T_\nu(z)) == \{\}$

Branch cuts

With respect to z

For fixed noninteger ν, the function $T_\nu(z)$ is a single-valued function on the z-plane cut along the interval $(-\infty, -1)$ where it is continuous from above.

07.04.04.0014.01

$\mathcal{BC}_z(T_\nu(z)) == \{\{(-\infty, -1), -i\}\} \;/;\; \nu \notin \mathbb{Z}$

07.04.04.0015.01

$\mathcal{BC}_z(T_n(z)) == \{\} \;/;\; n \in \mathbb{Z}$

07.04.04.0016.01

$\lim_{\epsilon \to +0} T_\nu(x + i\,\epsilon) == T_\nu(x) \;/;\; x < -1$

07.04.04.0017.01

$\lim_{\epsilon \to +0} T_\nu(x - i\,\epsilon) == 2\cos(\nu\,\pi)\, T_\nu(-x) - T_\nu(x) \;/;\; x < -1$

With respect to ν

For fixed z, the function $T_\nu(z)$ does not have branch cuts.

07.04.04.0018.01

$\mathcal{BC}_\nu(T_\nu(z)) == \{\}$

Series representations

Generalized power series

Expansions at generic point $\nu == \nu_0$

For the function itself

07.04.06.0030.01

$T_\nu(z) \propto \cos\left(\cos^{-1}(z)\,\nu_0\right) - \cos^{-1}(z)\sin\left(\cos^{-1}(z)\,\nu_0\right)(\nu - \nu_0) - \frac{1}{2}\cos^{-1}(z)^2\cos\left(\cos^{-1}(z)\,\nu_0\right)(\nu - \nu_0)^2 + \ldots \;/;\; (\nu \to \nu_0)$

07.04.06.0031.01

$T_\nu(z) \propto T_{\nu_0}(z) - \sqrt{1 - z^2}\, \cos^{-1}(z)\, U_{\nu_0 - 1}(z)(\nu - \nu_0) - \frac{1}{2}\cos^{-1}(z)^2\, T_{\nu_0}(z)(\nu - \nu_0)^2 + \ldots \;/;\; (\nu \to \nu_0)$

07.04.06.0032.01

$$T_\nu(z) \propto \cos\left(\cos^{-1}(z)\,\nu_0\right) - \cos^{-1}(z)\sin\left(\cos^{-1}(z)\,\nu_0\right)(\nu-\nu_0) - \frac{1}{2}\cos^{-1}(z)^2\cos\left(\cos^{-1}(z)\,\nu_0\right)(\nu-\nu_0)^2 + O\left((\nu-\nu_0)^3\right)$$

07.04.06.0033.01

$$T_\nu(z) = \sum_{k=0}^{\infty} \frac{\cos^{-1}(z)^k}{k!} \cos\left(\frac{\pi k}{2} + \nu_0 \cos^{-1}(z)\right)(\nu-\nu_0)^k$$

07.04.06.0034.01

$$T_\nu(z) = \sum_{k=0}^{\infty} \frac{i^k \cos^{-1}(z)^k}{k!} \left(T_{\nu_0}(z) - \left(T_{\nu_0}(z) - i\sqrt{1-z^2}\; U_{\nu_0-1}(z)\right)(k \bmod 2)\right)(\nu-\nu_0)^k$$

07.04.06.0035.01

$$T_\nu(z) \propto T_{\nu_0}(z)\,(1 + O(\nu-\nu_0))$$

Expansions at $\nu = 0$

For the function itself

07.04.06.0001.02

$$T_\nu(z) \propto 1 - \frac{\cos^{-1}(z)^2}{2}\nu^2 + \frac{\cos^{-1}(z)^4}{24}\nu^4 - \ldots /; (\nu \to 0)$$

07.04.06.0036.01

$$T_\nu(z) \propto 1 - \frac{\cos^{-1}(z)^2}{2}\nu^2 + \frac{\cos^{-1}(z)^4}{24}\nu^4 - O\left(\nu^6\right)$$

07.04.06.0002.01

$$T_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k \cos^{-1}(z)^{2k}\,\nu^{2k}}{(2k)!}$$

07.04.06.0037.01

$$T_\nu(z) = {}_0F_1\left(; \frac{1}{2}; -\frac{\cos^{-1}(z)^2}{4}\nu^2\right)$$

07.04.06.0003.02

$$T_\nu(z) \propto 1 + O\left(\nu^2\right)$$

07.04.06.0038.01

$$T_\nu(z) = F_\infty(z, \nu) /;$$

$$\left(\left(F_n(z,\nu) = \sum_{k=0}^{n} \frac{(-1)^k \cos^{-1}(z)^{2k}\,\nu^{2k}}{(2k)!} = T_\nu(z) + \frac{(-1)^n \sqrt{\pi}\,\cos^{-1}(z)^{2n+2}\,\nu^{2n+2}}{2^{2n+2}}\, {}_1\tilde{F}_2\left(1; n+\frac{3}{2}, n+2; -\frac{\cos^{-1}(z)^2\,\nu^2}{4}\right)\right) \bigwedge n \in \mathbb{N}\right)$$

Summed form of the truncated series expansion.

Expansions at generic point $z = z_0$

For the function itself

07.04.06.0039.01

$$T_\nu(z) \propto T_\nu(-z_0)\cos(\pi\nu)\left(-2\,i\,i^{\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor}\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor\left\lfloor\frac{\arg(z_0+1)+\pi}{2\pi}\right\rfloor+\left(\frac{1}{z_0+1}\right)^{\frac{1}{2}\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor}(z_0+1)^{\frac{1}{2}\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor}\right)+$$
$$U_{\nu-1}(-z_0)\sin(\pi\nu)\left(\frac{1}{z_0+1}\right)^{\frac{1}{2}\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor}\sqrt{1-z_0^2}\,(z_0+1)^{\frac{1}{2}\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor}+$$
$$\left(\frac{\nu\sin(\pi\nu)}{\sqrt{1-z_0^2}}\left(\frac{1}{z_0+1}\right)^{\frac{1}{2}\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor}(z_0+1)^{\frac{1}{2}\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor}T_\nu(-z_0)-\nu\cos(\pi\nu)\,U_{\nu-1}(-z_0)\right.$$
$$\left.\left(-2\,i\,i^{\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor}\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor\left\lfloor\frac{\arg(z_0+1)+\pi}{2\pi}\right\rfloor+\left(\frac{1}{z_0+1}\right)^{\frac{1}{2}\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor}(z_0+1)^{\frac{1}{2}\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor}\right)\right)(z-z_0)+\ldots\;/;\;(z\to z_0)$$

07.04.06.0040.01

$$T_\nu(z) \propto T_\nu(-z_0)\cos(\pi\nu)\left(-2\,i\,i^{\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor}\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor\left\lfloor\frac{\arg(z_0+1)+\pi}{2\pi}\right\rfloor+\left(\frac{1}{z_0+1}\right)^{\frac{1}{2}\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor}(z_0+1)^{\frac{1}{2}\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor}\right)+$$
$$U_{\nu-1}(-z_0)\sin(\pi\nu)\left(\frac{1}{z_0+1}\right)^{\frac{1}{2}\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor}\sqrt{1-z_0^2}\,(z_0+1)^{\frac{1}{2}\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor}+$$
$$\left(\frac{\nu\sin(\pi\nu)}{\sqrt{1-z_0^2}}\left(\frac{1}{z_0+1}\right)^{\frac{1}{2}\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor}(z_0+1)^{\frac{1}{2}\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor}T_\nu(-z_0)-\nu\cos(\pi\nu)\,U_{\nu-1}(-z_0)\right.$$
$$\left.\left(-2\,i\,i^{\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor}\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor\left\lfloor\frac{\arg(z_0+1)+\pi}{2\pi}\right\rfloor+\left(\frac{1}{z_0+1}\right)^{\frac{1}{2}\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor}(z_0+1)^{\frac{1}{2}\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor}\right)\right)(z-z_0)+O\left((z-z_0)^2\right)$$

07.04.06.0041.01

$$T_\nu(z) = \frac{\nu\sin(2\pi\nu)}{2\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{\pi\sec(\pi\nu)}{\sqrt{2}}(z_0+1)^{\frac{1}{2}-k}\left(\frac{1}{z_0+1}\right)^{\frac{1}{2}\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor}(z_0+1)^{\frac{1}{2}\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor}{}_2\tilde{F}_1\left(\nu+\frac{1}{2},\frac{1}{2}-\nu;\frac{3}{2}-k;\frac{z_0+1}{2}\right)-\right.$$
$$2^{-k}\,\Gamma(k-\nu)\,\Gamma(k+\nu)\left(-2\,i\,i^{\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor}\left\lfloor\frac{\arg(z_0+1)+\pi}{2\pi}\right\rfloor\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor+\left(\frac{1}{z_0+1}\right)^{\frac{1}{2}\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor}(z_0+1)^{\frac{1}{2}\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor}\right)$$
$$\left.{}_2\tilde{F}_1\left(k-\nu,k+\nu;k+\frac{1}{2};\frac{z_0+1}{2}\right)\right)(z-z_0)^k$$

07.04.06.0042.01

$$T_\nu(z) \propto \cos(\pi \nu)\left(-2\, i\, i^{\left\lfloor \frac{\arg(z-z_0)}{2\pi} \right\rfloor} \left\lfloor \frac{\arg(z-z_0)}{2\pi} \right\rfloor \left\lfloor \frac{\arg(z_0+1)+\pi}{2\pi} \right\rfloor + \left(\frac{1}{z_0+1}\right)^{\frac{1}{2}\left\lfloor \frac{\arg(z-z_0)}{2\pi} \right\rfloor} (z_0+1)^{\frac{1}{2}\left\lfloor \frac{\arg(z-z_0)}{2\pi} \right\rfloor}\right) T_\nu(-z_0) + \sin(\pi \nu)\sqrt{1-z_0^2}\, \left(\frac{1}{z_0+1}\right)^{\frac{1}{2}\left\lfloor \frac{\arg(z-z_0)}{2\pi} \right\rfloor} (z_0+1)^{\frac{1}{2}\left\lfloor \frac{\arg(z-z_0)}{2\pi} \right\rfloor} U_{\nu-1}(-z_0) + O(z-z_0)$$

Expansions on branch cuts

For the function itself

07.04.06.0043.01

$$T_\nu(z) \propto \cos(\pi \nu)\left(-2\, i\, i^{\left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor + e^{\pi i \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor}\right) T_\nu(-x) + \sqrt{1-x^2}\, e^{\pi i \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} \sin(\pi \nu)\, U_{\nu-1}(-x) + \left(\frac{\nu \sin(\pi \nu)}{\sqrt{1-x^2}} e^{\pi i \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} T_\nu(-x) - \nu \cos(\pi \nu)\left(-2\, i\, i^{\left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor + e^{\pi i \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor}\right) U_{\nu-1}(-x)\right)(z-x) + \ldots /; (z \to x) \land x \in \mathbb{R} \land x < -1$$

07.04.06.0044.01

$$T_\nu(z) \propto \cos(\pi \nu)\left(-2\, i\, i^{\left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor + e^{\pi i \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor}\right) T_\nu(-x) + \sqrt{1-x^2}\, e^{\pi i \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} \sin(\pi \nu)\, U_{\nu-1}(-x) + \left(\frac{\nu \sin(\pi \nu)}{\sqrt{1-x^2}} e^{\pi i \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} T_\nu(-x) - \nu \cos(\pi \nu)\left(-2\, i\, i^{\left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor + e^{\pi i \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor}\right) U_{\nu-1}(-x)\right)(z-x) + O\left((z-x)^2\right) /; x \in \mathbb{R} \land x < -1$$

07.04.06.0045.01

$$T_\nu(z) = \frac{\nu \sin(2 \pi \nu)}{2\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{\pi \sec(\pi \nu)}{\sqrt{2}} (x+1)^{\frac{1}{2}-k} e^{\pi i \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} {}_2\tilde{F}_1\left(\nu+\frac{1}{2}, \frac{1}{2}-\nu; \frac{3}{2}-k; \frac{x+1}{2}\right) - 2^{-k}\, \Gamma(k-\nu)\, \Gamma(k+\nu) \left(-2\, i\, i^{\left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor + e^{\pi i \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor}\right) {}_2\tilde{F}_1\left(k-\nu, k+\nu; k+\frac{1}{2}; \frac{x+1}{2}\right)\right)(z-x)^k /; x \in \mathbb{R} \land x < -1$$

07.04.06.0046.01

$$T_\nu(z) \propto \cos(\pi \nu)\left(-2\, i\, i^{\left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor + e^{\pi i \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor}\right) T_\nu(-x) + \sqrt{1-x^2}\, e^{\pi i \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} \sin(\pi \nu)\, U_{\nu-1}(-x) + O(z-x) /; x \in \mathbb{R} \land x < -1$$

Expansions at $z = 0$

For the function itself

General case

07.04.06.0004.02

$$T_\nu(z) \propto \cos\left(\frac{\pi\nu}{2}\right) + \nu\sin\left(\frac{\pi\nu}{2}\right) z - \frac{\nu^2}{2}\cos\left(\frac{\pi\nu}{2}\right) z^2 + \ldots /; (z \to 0)$$

07.04.06.0047.01

$$T_\nu(z) \propto \cos\left(\frac{\pi\nu}{2}\right) + \nu\sin\left(\frac{\pi\nu}{2}\right) z - \frac{\nu^2}{2}\cos\left(\frac{\pi\nu}{2}\right) z^2 + O(z^3)$$

07.04.06.0005.01

$$T_\nu(z) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-\nu)_{j+k}\,(\nu)_{j+k}\,(-z)^j}{\left(\frac{1}{2}\right)_{j+k}\, j!\, k!\, 2^{j+k}} /; |z| < 1$$

07.04.06.0006.01

$$T_\nu(z) = F_{1\times0\times0}^{2\times0\times0}\left(\begin{matrix} -\nu, \nu;;; \\ \frac{1}{2};;; \end{matrix} \frac{1}{2}, -\frac{z}{2}\right)$$

07.04.06.0048.01

$$T_\nu(z) = -\frac{\nu\sin(\pi\nu)}{4\pi}\sum_{j=0}^{\infty}\frac{(-2)^j\,\Gamma\left(\frac{j}{2}-\frac{\nu}{2}\right)\Gamma\left(\frac{j}{2}+\frac{\nu}{2}\right)z^j}{j!} /; |z| < 1$$

07.04.06.0049.01

$$T_\nu(z) = \cos\left(\frac{\pi\nu}{2}\right)\sum_{k=0}^{\infty}\frac{\left(-\frac{\nu}{2}\right)_k\left(\frac{\nu}{2}\right)_k}{\left(\frac{1}{2}\right)_k\, k!} z^{2k} + z\,\nu\sin\left(\frac{\pi\nu}{2}\right)\sum_{k=0}^{\infty}\frac{\left(\frac{1-\nu}{2}\right)_k\left(\frac{\nu+1}{2}\right)_k}{\left(\frac{3}{2}\right)_k\, k!} z^{2k} /; |z| < 1$$

07.04.06.0050.01

$$T_\nu(z) = \cos\left(\frac{\pi\nu}{2}\right)\,{}_2F_1\left(-\frac{\nu}{2}, \frac{\nu}{2}; \frac{1}{2}; z^2\right) + z\,\nu\sin\left(\frac{\pi\nu}{2}\right)\,{}_2F_1\left(\frac{1-\nu}{2}, \frac{\nu+1}{2}; \frac{3}{2}; z^2\right)$$

07.04.06.0051.01

$$T_\nu(z) = \cos\left(\frac{\pi\nu}{2}\right)\cos\left(\nu\sin^{-1}(z)\right) + \sin\left(\frac{\pi\nu}{2}\right)\sin\left(\nu\sin^{-1}(z)\right)$$

07.04.06.0052.01

$$T_\nu(z) = \cos\left(\frac{\nu}{2}\left(\pi - 2\sin^{-1}(z)\right)\right)$$

07.04.06.0007.02

$$T_\nu(z) \propto \cos\left(\frac{\pi\nu}{2}\right)(1 + O(z))$$

07.04.06.0053.01

$$T_\nu(z) = F_\infty(z, \nu) /;$$

$$\left(\left(F_m(z, \nu) = -\frac{\nu\sin(\pi\nu)}{4\pi}\sum_{j=0}^{m}\frac{(-2)^j\,\Gamma\left(\frac{j}{2}-\frac{\nu}{2}\right)\Gamma\left(\frac{j}{2}+\frac{\nu}{2}\right)}{j!} z^j = T_\nu(z) - \frac{2^m\,(-1)^{m+1}\,z^{m+2}\,\nu\sin(\pi\nu)\,\Gamma\left(\frac{1}{2}(m-\nu+2)\right)\Gamma\left(\frac{1}{2}(m+\nu+2)\right)}{\pi\,(m+2)!}\right.\right.$$

$${}_3F_2\left(1, \frac{m}{2}-\frac{\nu}{2}+1, \frac{m}{2}+\frac{\nu}{2}+1; \frac{m}{2}+\frac{3}{2}, \frac{m}{2}+2; z^2\right) + \frac{2^{m-1}\,(-z)^{m+1}\,\nu\sin(\pi\nu)\,\Gamma\left(\frac{1}{2}(m-\nu+1)\right)\Gamma\left(\frac{1}{2}(m+\nu+1)\right)}{\pi\,(m+1)!}$$

$$\left.\left.{}_3F_2\left(1, \frac{m}{2}-\frac{\nu}{2}+\frac{1}{2}, \frac{m}{2}+\frac{\nu}{2}+\frac{1}{2}; \frac{m}{2}+1, \frac{m}{2}+\frac{3}{2}; z^2\right)\right) \bigwedge m \in \mathbb{N}\right)$$

Summed form of the truncated series expansion.

Special cases

07.04.06.0008.01

$$T_n(z) = \frac{\delta_{n,0}}{2} + \frac{n}{2} \sum_{k=1}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(-1)^k (n-k-1)! \, (2z)^{n-2k}}{k! \, (n-2k)!} + 2^{n-1} z^n \;/;\; n \in \mathbb{N}$$

07.04.06.0009.01

$$T_n(z) \propto (-1)^{\left\lfloor \frac{n}{2} \right\rfloor} (n z)^{n-2\left\lfloor \frac{n}{2} \right\rfloor} \left(1 + O\left(z^2\right)\right) \;/;\; (z \to 0) \bigwedge n \in \mathbb{N}^+$$

Generic formulas for main term

07.04.06.0054.01

$$T_\nu(z) \propto \begin{cases} 1 & \nu = 0 \\ (-1)^{\left\lfloor \frac{\nu}{2} \right\rfloor} (\nu z)^{\nu-2\left\lfloor \frac{\nu}{2} \right\rfloor} & \nu \in \mathbb{Z} \land \nu \neq 0 \\ \cos\left(\frac{\pi \nu}{2}\right) & \text{True} \end{cases} \;/;\; (z \to 0)$$

Expansions at $z = 1$

For the function itself

General case

07.04.06.0010.02

$$T_\nu(z) \propto 1 + \nu^2 (z-1) - \frac{(1-\nu)\,\nu^2\,(1+\nu)}{16} (z-1)^2 + \ldots \;/;\; (z \to 1)$$

07.04.06.0055.01

$$T_\nu(z) \propto 1 + \nu^2 (z-1) - \frac{(1-\nu)\,\nu^2\,(1+\nu)}{16} (z-1)^2 + O\left((z-1)^3\right)$$

07.04.06.0011.01

$$T_\nu(z) = \sum_{k=0}^{\infty} \frac{(-\nu)_k \, (\nu)_k}{\left(\frac{1}{2}\right)_k k!} \left(\frac{1-z}{2}\right)^k \;/;\; \left|\frac{1-z}{2}\right| < 1$$

07.04.06.0012.01

$$T_\nu(z) = {}_2F_1\left(-\nu, \nu; \frac{1}{2}; \frac{1-z}{2}\right)$$

07.04.06.0056.01

$$T_\nu(z) = \cos\left(2 \nu \sin^{-1}\left(\frac{\sqrt{1-z}}{\sqrt{2}}\right)\right)$$

07.04.06.0014.02

$$T_\nu(z) \propto 1 + O(z-1)$$

07.04.06.0057.01

$$T_\nu(z) == F_\infty(z,\nu)\ /;\ \left(\left(F_m(z,\nu) == \sum_{k=0}^{m} \frac{(-\nu)_k\,(\nu)_k}{\left(\frac{1}{2}\right)_k\,k!}\left(\frac{1-z}{2}\right)^k == T_\nu(z) - \frac{2^{-m-1}\,(-\nu)_{m+1}\,(\nu)_{m+1}\,(1-z)^{m+1}}{(m+1)!\left(\frac{1}{2}\right)_{m+1}}\,{}_3F_2\left(1, m-\nu+1, m+\nu+1; m+\frac{3}{2}, m+2; \frac{1-z}{2}\right)\right) \bigwedge m \in \mathbb{N}\right)$$

Summed form of the truncated series expansion.

Special cases

07.04.06.0013.01

$$T_n(z) == \sum_{k=0}^{n} \frac{(-n)_k\,(n)_k}{\left(\frac{1}{2}\right)_k\,k!}\left(\frac{1-z}{2}\right)^k\ /;\ n \in \mathbb{N}$$

Expansions at $z == -1$

For the function itself

General case

07.04.06.0015.02

$$T_\nu(z) \propto \cos(\pi\nu)\left(1-\nu^2(z+1) - \frac{(1-\nu)\,\nu^2\,(1+\nu)}{6}(z+1)^2 - \ldots\right) + \sqrt{2}\,\sqrt{z+1}\,\nu\sin(\pi\nu)\left(1+\frac{1}{3}\left(\frac{1}{2}-\nu\right)\left(\frac{1}{2}+\nu\right)(z+1) + \frac{1}{30}\left(\frac{1}{2}-\nu\right)\left(\frac{3}{2}-\nu\right)\left(\frac{1}{2}+\nu\right)\left(\frac{3}{2}+\nu\right)(z+1)^2 + \ldots\right)\ /;\ (z \to -1)$$

07.04.06.0058.01

$$T_\nu(z) \propto \cos(\pi\nu)\left(1-\nu^2\,(z+1) - \frac{(1-\nu)\,\nu^2\,(1+\nu)}{6}(z+1)^2 - O\left((z+1)^3\right)\right) + \sqrt{2}\,\sqrt{z+1}\,\nu\sin(\pi\nu)\left(1+\frac{1}{3}\left(\frac{1}{2}-\nu\right)\left(\frac{1}{2}+\nu\right)(z+1) + \frac{1}{30}\left(\frac{1}{2}-\nu\right)\left(\frac{3}{2}-\nu\right)\left(\frac{1}{2}+\nu\right)\left(\frac{3}{2}+\nu\right)(z+1)^2 + O\left((z+1)^3\right)\right)$$

07.04.06.0016.01

$$T_\nu(z) == \cos(\nu\pi)\sum_{k=0}^{\infty} \frac{(-\nu)_k\,(\nu)_k}{\left(\frac{1}{2}\right)_k\,k!}\left(\frac{z+1}{2}\right)^k + \sqrt{2}\,\sqrt{z+1}\,\nu\sin(\nu\pi)\sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}-\nu\right)_k\left(\nu+\frac{1}{2}\right)_k}{\left(\frac{3}{2}\right)_k\,k!}\left(\frac{z+1}{2}\right)^k\ /;\ \left|\frac{z+1}{2}\right| < 1$$

07.04.06.0017.01

$$T_\nu(z) == \cos(\nu\pi)\,{}_2F_1\left(-\nu, \nu; \frac{1}{2}; \frac{z+1}{2}\right) + \sqrt{2}\,\sqrt{z+1}\,\nu\sin(\nu\pi)\,{}_2F_1\left(\nu+\frac{1}{2}, \frac{1}{2}-\nu; \frac{3}{2}; \frac{z+1}{2}\right)$$

07.04.06.0059.01

$$T_\nu(z) == \cos\left(\nu\left(\pi - 2\sin^{-1}\left(\frac{\sqrt{z+1}}{\sqrt{2}}\right)\right)\right)$$

07.04.06.0018.02

$$T_\nu(z) \propto \cos(\pi\,\nu)\,(1 + O(z+1)) + \sqrt{2}\;\nu \sin(\pi\,\nu)\,\sqrt{z+1}\;(1 + O(z+1))$$

07.04.06.0060.01

$$T_\nu(z) = F_\infty(z,\,\nu)\;/;\;\Bigg(\Bigg(F_m(z,\,\nu) = \cos(\nu\,\pi) \sum_{k=0}^{m} \frac{(-\nu)_k\,(\nu)_k}{\left(\frac{1}{2}\right)_k k!}\left(\frac{z+1}{2}\right)^k + \sqrt{2}\,\sqrt{z+1}\;\nu \sin(\nu\,\pi) \sum_{k=0}^{m} \frac{\left(\frac{1}{2}-\nu\right)_k \left(\nu+\frac{1}{2}\right)_k}{\left(\frac{3}{2}\right)_k k!}\left(\frac{z+1}{2}\right)^k =$$

$$T_\nu(z) - \frac{2^{-m-1}\cos(\pi\,\nu)\,(-\nu)_{m+1}\,(\nu)_{m+1}}{(m+1)!\left(\frac{1}{2}\right)_{m+1}}\,(z+1)^{m+1}\,{}_3F_2\!\left(1,\,m-\nu+1,\,m+\nu+1;\,m+\frac{3}{2},\,m+2;\,\frac{z+1}{2}\right) -$$

$$\frac{2^{-m-\frac{1}{2}}\,\nu\sin(\pi\,\nu)\left(\frac{1}{2}-\nu\right)_{m+1}\left(\nu+\frac{1}{2}\right)_{m+1}}{(m+1)!\left(\frac{3}{2}\right)_{m+1}}\,(z+1)^{m+\frac{3}{2}}\,{}_3F_2\!\left(1,\,m-\nu+\frac{3}{2},\,m+\nu+\frac{3}{2};\,m+2,\,m+\frac{5}{2};\,\frac{z+1}{2}\right)\Bigg) \bigwedge m \in \mathbb{N}\Bigg)$$

Summed form of the truncated series expansion.

Special cases

07.04.06.0019.01

$$T_n(z) = (-1)^n \sum_{k=0}^{n} \frac{(-n)_k\,(n)_k}{\left(\frac{1}{2}\right)_k k!}\left(\frac{z+1}{2}\right)^k \;/;\; n \in \mathbb{N}$$

07.04.06.0020.02

$$T_n(z) \propto (-1)^n\,(1 + O(z+1))\;/;\;n \in \mathbb{N}$$

Expansions at $z = \infty$

For the function itself

Expansions in $1/z$

07.04.06.0061.01

$$T_\nu(z) \propto 2^{-\nu-1} z^\nu \left(1 + \frac{\nu}{4 z^2} + \frac{(\nu+3)\,\nu}{32 z^4} + \ldots\right) + 2^{\nu-1} z^{-\nu}\left(1 - \frac{\nu}{4 z^2} + \frac{(\nu-3)\,\nu}{32 z^4} + \ldots\right) /;\; (|z| \to \infty)$$

07.04.06.0062.01

$$T_\nu(z) \propto 2^{-\nu-1} z^\nu \left(1 + \frac{\nu}{4 z^2} + \frac{(\nu+3)\,\nu}{32 z^4} + O\!\left(\frac{1}{z^6}\right)\right) + 2^{\nu-1} z^{-\nu}\left(1 - \frac{\nu}{4 z^2} + \frac{(\nu-3)\,\nu}{32 z^4} + O\!\left(\frac{1}{z^6}\right)\right)$$

07.04.06.0063.01

$$T_\nu(z) = 2^{-\nu-1} z^{-\nu} \sum_{k=0}^{\infty} \frac{\left(\frac{\nu}{2}\right)_k \left(\frac{\nu+1}{2}\right)_k}{(\nu+1)_k\,k!} z^{-2k} + 2^{\nu-1} z^\nu \sum_{k=0}^{\infty} \frac{\left(-\frac{\nu}{2}\right)_k \left(\frac{1-\nu}{2}\right)_k}{(1-\nu)_k\,k!} z^{-2k} \;/;\; |z| > 1 \bigwedge \nu \notin \mathbb{Z}$$

07.04.06.0064.01

$$T_\nu(z) = 2^{-\nu-1} z^{-\nu}\,{}_2F_1\!\left(\frac{\nu}{2},\,\frac{\nu+1}{2};\,\nu+1;\,\frac{1}{z^2}\right) + 2^{\nu-1} z^\nu\,{}_2F_1\!\left(-\frac{\nu}{2},\,\frac{1}{2}-\frac{\nu}{2};\,1-\nu;\,\frac{1}{z^2}\right) /;\; z \notin (-1,\,0) \bigwedge \nu \notin \mathbb{Z}$$

07.04.06.0065.01

$$T_\nu(z) = \frac{1}{2} z^{-\nu} \left(\frac{\sqrt{z^2} \sqrt{z^2-1}}{z^2} + 1 \right)^{-\nu} + \frac{1}{2} z^\nu \left(\frac{\sqrt{z^2} \sqrt{z^2-1}}{z^2} + 1 \right)^{\nu} /; z \notin (-1, 0) \bigwedge \nu \notin \mathbb{Z}$$

07.04.06.0066.01

$$T_n(z) = 2^{n-1} z^n \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{\left(\frac{1-n}{2}\right)_k \left(-\frac{n}{2}\right)_k z^{-2k}}{k! \, (1-n)_k} /; n \in \mathbb{N}^+$$

07.04.06.0067.01

$$T_n(z) = 2^{n-1} z^n \, {}_2F_1\left(-\frac{n}{2}, \frac{1-n}{2}; 1-n; \frac{1}{z^2}\right) /; n-2 \in \mathbb{N}^+$$

07.04.06.0024.02

$$T_\nu(z) \propto 2^{-\nu-1} z^{-\nu} \left(1 + O\left(\frac{1}{z}\right)\right) + 2^{\nu-1} z^\nu \left(1 + O\left(\frac{1}{z}\right)\right) /; \nu \notin \mathbb{Z}$$

07.04.06.0026.02

$$T_n(z) \propto 2^{n-1} z^n \left(1 + O\left(\frac{1}{z}\right)\right) /; n \in \mathbb{N}^+$$

07.04.06.0028.02

$$T_\nu(z) \propto 2^{-|\nu|-1} z^{-|\nu|} \left(1 + O\left(\frac{1}{z}\right)\right) + 2^{|\nu|-1} z^{|\nu|} \left(1 + O\left(\frac{1}{z}\right)\right) /; \nu - \frac{1}{2} \in \mathbb{Z}$$

07.04.06.0068.01

$$T_\nu(z) = F_\infty(z, \nu) /; \Bigg(F_m(z, \nu) = 2^{-\nu-1} z^{-\nu} \sum_{k=0}^{m} \frac{\left(\frac{\nu}{2}\right)_k \left(\frac{\nu+1}{2}\right)_k}{(\nu+1)_k \, k!} z^{-2k} + 2^{\nu-1} z^\nu \sum_{k=0}^{m} \frac{\left(-\frac{\nu}{2}\right)_k \left(\frac{1-\nu}{2}\right)_k}{(1-\nu)_k \, k!} z^{-2k} =$$

$$T_\nu(z) - \frac{2^{-2m-\nu-3} z^{-2(m+1)-\nu} \nu \, \Gamma(2m+\nu+2)}{(m+1)! \, \Gamma(m+\nu+2)} \, {}_3F_2\left(1, m+\frac{\nu}{2}+1, m+\frac{\nu}{2}+\frac{3}{2}; m+2, m+\nu+2; \frac{1}{z^2}\right) +$$

$$\frac{2^{-2m+\nu-3} z^{\nu-2(m+1)} \nu \, \Gamma(2m-\nu+2)}{(m+1)! \, \Gamma(m-\nu+2)} \, {}_3F_2\left(1, m-\frac{\nu}{2}+1, m-\frac{\nu}{2}+\frac{3}{2}; m+2, m-\nu+2; \frac{1}{z^2}\right) \bigwedge m \in \mathbb{N} \Bigg) \bigwedge \nu \notin \mathbb{Z}$$

Summed form of the truncated series expansion.

Expansions in $1/(1-z)$

07.04.06.0021.02

$$T_\nu(z) \propto 2^{-\nu-1} (z-1)^{-\nu} \left(1 + \frac{\nu}{1-z} + \frac{\nu(3+2\nu)}{4(1-z)^2} + \ldots\right) + 2^{\nu-1} (z-1)^\nu \left(1 - \frac{\nu}{1-z} - \frac{\nu(3-2\nu)}{4(1-z)^2} - \ldots\right) /; (|z| \to \infty) \bigwedge 2\nu \notin \mathbb{Z}$$

07.04.06.0069.01

$$T_\nu(z) \propto 2^{-\nu-1} (z-1)^{-\nu} \left(1 + \frac{\nu}{1-z} + \frac{\nu(3+2\nu)}{4(1-z)^2} + O\left(\frac{1}{z^3}\right)\right) + 2^{\nu-1} (z-1)^\nu \left(1 - \frac{\nu}{1-z} - \frac{\nu(3-2\nu)}{4(1-z)^2} - O\left(\frac{1}{z^3}\right)\right) /; \bigwedge 2\nu \notin \mathbb{Z}$$

07.04.06.0022.01

$$T_\nu(z) = 2^{-\nu-1} (z-1)^{-\nu} \sum_{k=0}^{\infty} \frac{(\nu)_k \left(\nu+\frac{1}{2}\right)_k}{(2\nu+1)_k \, k!} \left(\frac{2}{1-z}\right)^k + 2^{\nu-1} (z-1)^\nu \sum_{k=0}^{\infty} \frac{(-\nu)_k \left(\frac{1}{2}-\nu\right)_k}{(1-2\nu)_k \, k!} \left(\frac{2}{1-z}\right)^k /; \left|\frac{1-z}{2}\right| > 1 \bigwedge 2\nu \notin \mathbb{Z}$$

07.04.06.0023.01

$$T_\nu(z) = 2^{-\nu-1} (z-1)^{-\nu} {}_2F_1\left(\nu, \nu+\frac{1}{2}; 2\nu+1; \frac{2}{1-z}\right) + 2^{\nu-1} (z-1)^{\nu} {}_2F_1\left(-\nu, \frac{1}{2}-\nu; 1-2\nu; \frac{2}{1-z}\right) /; z \notin (-\infty, 1) \bigwedge 2\nu \notin \mathbb{Z}$$

07.04.06.0070.01

$$T_\nu(z) = 2^{\nu-1} (z-1)^{-\nu} \left(1+\sqrt{\frac{z+1}{z-1}}\right)^{-2\nu} + 2^{-\nu-1} (z-1)^{\nu} \left(1+\sqrt{\frac{z+1}{z-1}}\right)^{2\nu} /; z \notin (-1, 1)$$

07.04.06.0071.01

$$T_n(z) = 2^{n-1} (z-1)^n \sum_{k=0}^{n} \frac{(-n)_k \left(\frac{1}{2}-n\right)_k}{k!\,(1-2n)_k} \left(\frac{2}{1-z}\right)^k /;\ n \in \mathbb{N}^+$$

07.04.06.0072.01

$$T_n(z) = \frac{2^{-n}\sqrt{\pi}\,(z-1)^n}{(n-1)!} \sum_{k=0}^{n} \frac{(2n-k-1)!\,(-n)_k}{k!\,\Gamma\left(-k+n+\frac{1}{2}\right)} \left(\frac{2}{1-z}\right)^k /; n \in \mathbb{N}^+$$

07.04.06.0073.01

$$T_n(z) = 2^{n-1} (z-1)^n {}_2F_1\left(-n, \frac{1}{2}-n; 1-2n; \frac{2}{1-z}\right) /; n-1 \in \mathbb{N}^+$$

07.04.06.0027.01

$$T_\nu(z) = 2^{-|\nu|} (z-1)^{-|\nu|} {}_2F_1\left(|\nu|, |\nu|+\frac{1}{2}; 2|\nu|+1; \frac{2}{1-z}\right) + 2^{|\nu|-1}\,(z-1)^{|\nu|} \sum_{k=0}^{|\nu|-\frac{1}{2}} \frac{\left(\frac{1}{2}-|\nu|\right)_k (-|\nu|)_k}{k!\,(1-2|\nu|)_k} \left(\frac{2}{1-z}\right)^k /; \nu-\frac{1}{2} \in \mathbb{Z}$$

07.04.06.0074.01

$$T_\nu(z) = F_\infty(z, \nu) /; \left(\left(F_m(z, \nu) = 2^{-\nu-1} (z-1)^{-\nu} \sum_{k=0}^{m} \frac{(\nu)_k \left(\nu+\frac{1}{2}\right)_k}{(2\nu+1)_k\,k!} \left(\frac{2}{1-z}\right)^k + 2^{\nu-1} (z-1)^{\nu} \sum_{k=0}^{m} \frac{(-\nu)_k \left(\frac{1}{2}-\nu\right)_k}{(1-2\nu)_k\,k!} \left(\frac{2}{1-z}\right)^k = \right.\right.$$

$$\frac{2^{m-\nu} (-1)^m (\nu)_{m+1} \left(\nu+\frac{1}{2}\right)_{m+1} (z-1)^{-m-\nu-1}}{(m+1)!\,(2\nu+1)_{m+1}} {}_3F_2\left(1, m+\nu+1, m+\nu+\frac{3}{2}; m+2, m+2\nu+2; \frac{2}{1-z}\right) +$$

$$T_\nu(z) + \frac{2^{m+\nu} (-1)^m \left(\frac{1}{2}-\nu\right)_{m+1} (-\nu)_{m+1} (z-1)^{-m+\nu-1}}{(m+1)!\,(1-2\nu)_{m+1}}$$

$$\left.\left.{}_3F_2\left(1, m-\nu+1, m-\nu+\frac{3}{2}; m+2, m-2\nu+2; \frac{2}{1-z}\right) \bigwedge m \in \mathbb{N}\right) \bigwedge \neg\, 2\nu \in \mathbb{Z}\right)$$

Summed form of the truncated series expansion.

Generic formulas for main term

07.04.06.0075.01

$$T_\nu(z) \propto \begin{cases} 2^{\nu-1} z^{\nu} & \operatorname{Re}(\nu) > 0 \\ 2^{-\nu-1} z^{-\nu} & \operatorname{Re}(\nu) < 0 \\ 2^{-\nu-1} z^{-\nu} + 2^{\nu-1} z^{\nu} & \text{True} \end{cases} /; (|z| \to \infty)$$

Asymptotic series expansions

Expansions at $\nu == \infty$

07.04.06.0076.01

$$T_\nu(z) \propto \begin{cases} \frac{1}{2} e^{i\nu\cos^{-1}(z)} & -\pi < \arg(\nu\cos^{-1}(z)) < 0 \\ \frac{1}{2} e^{-i\nu\cos^{-1}(z)} & 0 < \arg(\nu\cos^{-1}(z)) < \pi \\ \cos(\nu\cos^{-1}(z)) & \text{True} \end{cases} \quad /;\ (|\nu| \to \infty)$$

Other series representations

07.04.06.0029.01

$$T_n(z) == \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n}{2k} z^{n-2k} (z^2-1)^k \ /;\ n \in \mathbb{N}$$

Residue representations

General case

Expansions at $z == 0$

07.04.06.0077.01

$$T_\nu(z) == \frac{z\,\nu\sin(\pi\nu)}{4\sqrt{\pi}} \sum_{j=0}^{\infty} \operatorname{res}_s\left(\frac{\Gamma\left(\frac{1-\nu}{2}-s\right)\Gamma\left(\frac{\nu+1}{2}-s\right)(-z^2)^{-s}}{\Gamma\left(\frac{3}{2}-s\right)}\Gamma(s)\right)(-j) - \frac{\nu\sin(\pi\nu)}{4\sqrt{\pi}} \sum_{j=0}^{\infty} \operatorname{res}_s\left(\frac{\Gamma\left(-s-\frac{\nu}{2}\right)\Gamma\left(\frac{\nu}{2}-s\right)(-z^2)^{-s}}{\Gamma\left(\frac{1}{2}-s\right)}\Gamma(s)\right)(-j) \ /;$$

$|z| < 1$

07.04.06.0078.01

$$T_\nu(z) == \frac{(-z^2)^{\nu/2}\,\nu}{4\sqrt{\pi}}\left(\cos\left(\frac{\pi\nu}{2}\right) + \frac{z}{\sqrt{-z^2}}\sin\left(\frac{\pi\nu}{2}\right)\right)\left(\sum_{j=0}^{\infty}\operatorname{res}_s\left(\frac{\Gamma(s)\,\Gamma\left(-s-\frac{\nu}{2}\right)\left(-\frac{1}{z^2}\right)^{-s}}{\Gamma(-s-\nu+1)}\Gamma\left(\frac{1-\nu}{2}-s\right),\ \left\{s,\ j+\frac{1-\nu}{2}\right\}\right) + \right.$$

$$\left.\sum_{j=0}^{\infty}\operatorname{res}_s\left(\frac{\Gamma(s)\,\Gamma\left(\frac{1-\nu}{2}-s\right)\left(-\frac{1}{z^2}\right)^{-s}}{\Gamma(-s-\nu+1)}\Gamma\left(-s-\frac{\nu}{2}\right),\ \left\{s,\ j-\frac{\nu}{2}\right\}\right)\right) -$$

$$\frac{(-z^2)^{-\frac{\nu}{2}}\,\nu}{4\sqrt{\pi}}\left(\cos\left(\frac{\pi\nu}{2}\right) - \frac{z}{\sqrt{-z^2}}\sin\left(\frac{\pi\nu}{2}\right)\right)\left(\sum_{j=0}^{\infty}\operatorname{res}_s\left(\frac{\Gamma(s)\,\Gamma\left(\frac{\nu+1}{2}-s\right)\left(-\frac{1}{z^2}\right)^{-s}}{\Gamma(-s+\nu+1)}\Gamma\left(\frac{\nu}{2}-s\right),\ \left\{s,\ j+\frac{\nu}{2}\right\}\right) + \right.$$

$$\left.\sum_{j=0}^{\infty}\operatorname{res}_s\left(\frac{\Gamma(s)\,\Gamma\left(\frac{\nu}{2}-s\right)\left(-\frac{1}{z^2}\right)^{-s}}{\Gamma(-s+\nu+1)}\Gamma\left(\frac{\nu+1}{2}-s\right),\ \left\{s,\ j+\frac{\nu+1}{2}\right\}\right)\right) /;\ |z| < 1 \bigwedge 2\nu \notin \mathbb{Z}$$

07.04.06.0079.01

$$T_\nu(z) = -\frac{(-z^2)^{\nu/2}\,\nu}{4\sqrt{\pi}}\left(\cos\left(\frac{\pi\nu}{2}\right)+\frac{z}{\sqrt{-z^2}}\sin\left(\frac{\pi\nu}{2}\right)\right)\left(\cos\left(\frac{\pi\nu}{2}\right)\left(-\frac{1}{z^2}\right)^{\frac{\nu-1}{2}}\sum_{j=0}^{\infty}\operatorname{res}_s\left(\frac{\Gamma\left(\frac{1-\nu}{2}-s\right)\Gamma\left(\frac{\nu+1}{2}-s\right)(-z^2)^{-s}}{\Gamma\left(\frac{3}{2}-s\right)}\Gamma(s)\right)(-j)+\left(-\frac{1}{z^2}\right)^{\nu/2}\sin\left(\frac{\pi\nu}{2}\right)\sum_{j=0}^{\infty}\operatorname{res}_s\left(\frac{\Gamma\left(-s-\frac{\nu}{2}\right)\Gamma\left(\frac{\nu}{2}-s\right)(-z^2)^{-s}}{\Gamma\left(\frac{1}{2}-s\right)}\Gamma(s)\right)(-j)\right) - \frac{(-z^2)^{-\frac{\nu}{2}}\,\nu}{4\sqrt{\pi}}\left(\cos\left(\frac{\pi\nu}{2}\right)-\frac{z}{\sqrt{-z^2}}\sin\left(\frac{\pi\nu}{2}\right)\right)\left(\left(-\frac{1}{z^2}\right)^{-\frac{\nu}{2}}\sin\left(\frac{\pi\nu}{2}\right)\sum_{j=0}^{\infty}\operatorname{res}_s\left(\frac{\Gamma\left(-s-\frac{\nu}{2}\right)\Gamma\left(\frac{\nu}{2}-s\right)(-z^2)^{-s}}{\Gamma\left(\frac{1}{2}-s\right)}\Gamma(s)\right)(-j)-\left(-\frac{1}{z^2}\right)^{-\frac{1}{2}(\nu+1)}\cos\left(\frac{\pi\nu}{2}\right)\sum_{j=0}^{\infty}\operatorname{res}_s\left(\frac{\Gamma\left(\frac{\nu+1}{2}-s\right)\Gamma\left(\frac{1-\nu}{2}-s\right)(-z^2)^{-s}}{\Gamma\left(\frac{3}{2}-s\right)}\Gamma(s)\right)(-j)\right)\;/;\;|z|<1\land 2\nu\notin\mathbb{Z}$$

Expansions at $z = 1$

07.04.06.0080.01

$$T_\nu(z) = -\frac{\nu\sin(\pi\nu)}{\sqrt{\pi}}\sum_{j=0}^{\infty}\operatorname{res}_s\left(\frac{\Gamma(\nu-s)\,\Gamma(-s-\nu)\left(\frac{z-1}{2}\right)^{-s}}{\Gamma\left(\frac{1}{2}-s\right)}\Gamma(s)\right)(-j)\;/;\;|z-1|<2$$

07.04.06.0081.01

$$T_\nu(z) = \frac{2^{-\nu-1}(z-1)^\nu\,\nu}{\sqrt{\pi}}\left(\sum_{j=0}^{\infty}\operatorname{res}_s\left(\frac{\Gamma(s)\,\Gamma\left(-s-\nu+\frac{1}{2}\right)\left(\frac{2}{z-1}\right)^{-s}}{\Gamma(-s-2\nu+1)}\Gamma(-s-\nu)\right)(j-\nu)+\sum_{j=0}^{\infty}\operatorname{res}_s\left(\frac{\Gamma(s)\,\Gamma(-s-\nu)\left(\frac{2}{z-1}\right)^{-s}}{\Gamma(-s-2\nu+1)}\Gamma\left(-s-\nu+\frac{1}{2}\right)\right)\left(j-\nu+\frac{1}{2}\right)\right) - \frac{2^{\nu-1}(z-1)^{-\nu}\,\nu}{\sqrt{\pi}}\left(\sum_{j=0}^{\infty}\operatorname{res}_s\left(\frac{\Gamma(s)\,\Gamma\left(-s+\nu+\frac{1}{2}\right)\left(\frac{2}{z-1}\right)^{-s}}{\Gamma(-s+2\nu+1)}\Gamma(\nu-s)\right)(j+\nu)+\sum_{j=0}^{\infty}\operatorname{res}_s\left(\frac{\Gamma(s)\,\Gamma(\nu-s)\left(\frac{2}{z-1}\right)^{-s}}{\Gamma(-s+2\nu+1)}\Gamma\left(-s+\nu+\frac{1}{2}\right)\right)\left(j+\nu+\frac{1}{2}\right)\right)\;/;\;|z-1|<2\land 2\nu\notin\mathbb{Z}$$

07.04.06.0082.01

$$\begin{aligned} T_{\nu}(z) = {} & \frac{z \sin\left(\frac{\pi \nu}{2}\right) \sin(\pi \nu)}{2 \pi^{3/2}} \left(\sum_{j=0}^{\infty} \mathrm{res}_s\left(\Gamma\left(\frac{1-\nu}{2}-s\right) \Gamma\left(s+\frac{1}{2}\right) \Gamma\left(\frac{\nu+1}{2}-s\right) \left(1-z^2\right)^{-s}\right)(-j) + \right. \\ & \left. \sum_{j=0}^{\infty} \mathrm{res}_s\left(\left(\Gamma(s)\, \Gamma\left(\frac{1-\nu}{2}-s\right) \Gamma\left(\frac{\nu+1}{2}-s\right) \left(1-z^2\right)^{-s}\right) \Gamma\left(s+\frac{1}{2}\right)\right)\left(-j-\frac{1}{2}\right) \right) - \\ & \frac{\nu \cos\left(\frac{\pi \nu}{2}\right) \sin(\pi \nu)}{4 \pi^{3/2}} \left(\sum_{j=0}^{\infty} \mathrm{res}_s\left(\left(\Gamma\left(s+\frac{1}{2}\right) \Gamma\left(-s-\frac{\nu}{2}\right) \Gamma\left(\frac{\nu}{2}-s\right) \left(1-z^2\right)^{-s}\right) \Gamma(s)\right)(-j) + \right. \\ & \left. \sum_{j=0}^{\infty} \mathrm{res}_s\left(\left(\Gamma(s)\, \Gamma\left(-s-\frac{\nu}{2}\right) \Gamma\left(\frac{\nu}{2}-s\right) \left(1-z^2\right)^{-s}\right) \Gamma\left(s+\frac{1}{2}\right)\right)\left(-j-\frac{1}{2}\right) \right) /; \left|1-z^2\right| < 1 \end{aligned}$$

07.04.06.0083.01

$$\begin{aligned} T_{\nu}(z) = {} & \frac{\cos(\pi \nu) \sin^2(\pi \nu)}{\sqrt{2}\, \pi^{3/2}} \sqrt{z+1} \left(\sum_{j=0}^{\infty} \mathrm{res}_s\left(\left(\Gamma\left(s+\frac{1}{2}\right) \Gamma\left(-s-\nu+\frac{1}{2}\right) \Gamma\left(-s+\nu+\frac{1}{2}\right) \left(\frac{1-z}{2}\right)^{-s}\right) \Gamma(s)\right)(-j) + \right. \\ & \left. \sum_{j=0}^{\infty} \mathrm{res}_s\left(\left(\Gamma(s)\, \Gamma\left(-s-\nu+\frac{1}{2}\right) \Gamma\left(-s+\nu+\frac{1}{2}\right) \left(\frac{1-z}{2}\right)^{-s}\right) \Gamma\left(s+\frac{1}{2}\right)\right)\left(-j-\frac{1}{2}\right) \right) - \\ & \frac{\nu \cos^2(\pi \nu) \sin(\pi \nu)}{\pi^{3/2}} \left(\sum_{j=0}^{\infty} \mathrm{res}_s\left(\left(\Gamma\left(s+\frac{1}{2}\right) \Gamma(-s-\nu)\, \Gamma(\nu-s) \left(\frac{1-z}{2}\right)^{-s}\right) \Gamma(s)\right)(-j) + \right. \\ & \left. \sum_{j=0}^{\infty} \mathrm{res}_s\left(\left(\Gamma(s)\, \Gamma(-s-\nu)\, \Gamma(\nu-s) \left(\frac{1-z}{2}\right)^{-s}\right) \Gamma\left(s+\frac{1}{2}\right)\right)\left(-j-\frac{1}{2}\right) \right) /; |1-z| < 2 \end{aligned}$$

07.04.06.0084.01

$$\begin{aligned} T_{\nu}(z) = {} & -\frac{2^{-\nu-1} (z-1)^{\nu}\, \nu}{\sqrt{\pi}} \left(2^{\nu} \sin(\pi \nu) \left(\frac{1}{z-1}\right)^{\nu} \sum_{j=0}^{\infty} \mathrm{res}_s\left(\frac{\Gamma(-s-\nu)\, \Gamma(\nu-s) \left(\frac{z-1}{2}\right)^{-s}}{\Gamma\left(\frac{1}{2}-s\right)} \Gamma(s)\right)(-j) + \right. \\ & \left. 2^{\nu-\frac{1}{2}} \cos(\pi \nu) \left(\frac{1}{z-1}\right)^{\nu-\frac{1}{2}} \sum_{j=0}^{\infty} \mathrm{res}_s\left(\frac{\Gamma\left(-s-\nu+\frac{1}{2}\right) \Gamma\left(-s+\nu+\frac{1}{2}\right) \left(\frac{z-1}{2}\right)^{-s}}{\Gamma\left(\frac{3}{2}-s\right)} \Gamma(s)\right)(-j) \right) - \\ & \frac{2^{\nu-1} (z-1)^{-\nu}\, \nu}{\sqrt{\pi}} \left(2^{-\nu} \left(\frac{1}{z-1}\right)^{-\nu} \sin(\pi \nu) \sum_{j=0}^{\infty} \mathrm{res}_s\left(\frac{\Gamma(-s-\nu)\, \Gamma(\nu-s) \left(\frac{z-1}{2}\right)^{-s}}{\Gamma\left(\frac{1}{2}-s\right)} \Gamma(s)\right)(-j) - \right. \\ & \left. 2^{-\nu-\frac{1}{2}} \left(\frac{1}{z-1}\right)^{-\nu-\frac{1}{2}} \cos(\pi \nu) \sum_{j=0}^{\infty} \mathrm{res}_s\left(\frac{\Gamma\left(-s-\nu+\frac{1}{2}\right) \Gamma\left(-s+\nu+\frac{1}{2}\right) \left(\frac{z-1}{2}\right)^{-s}}{\Gamma\left(\frac{3}{2}-s\right)} \Gamma(s)\right)(-j) \right) /; |z-1| < 2 \end{aligned}$$

Expansions at $z = -1$

07.04.06.0085.01

$$T_\nu(z) = \frac{\sqrt{z+1}\,\nu\sin(2\pi\nu)}{2\sqrt{2\pi}}\sum_{j=0}^{\infty}\operatorname{res}_s\left(\frac{\Gamma\left(-s-\nu+\frac{1}{2}\right)\Gamma\left(-s+\nu+\frac{1}{2}\right)\left(-\frac{z+1}{2}\right)^{-s}}{\Gamma\left(\frac{3}{2}-s\right)}\Gamma(s)\right)(-j) - \frac{\nu\sin(2\pi\nu)}{2\sqrt{\pi}}\sum_{j=0}^{\infty}\operatorname{res}_s\left(\frac{\Gamma(-s-\nu)\Gamma(\nu-s)\left(-\frac{z+1}{2}\right)^{-s}}{\Gamma\left(\frac{1}{2}-s\right)}\Gamma(s)\right)(-j)\;/;\;|z+1|<2$$

07.04.06.0086.01

$$T_\nu(z) = -\frac{\nu\sin(2\pi\nu)}{2\pi^{3/2}}\left(\sum_{j=0}^{\infty}\operatorname{res}_s\left(\left(\Gamma\left(s+\frac{1}{2}\right)\Gamma(-s-\nu)\Gamma(\nu-s)\left(\frac{z+1}{2}\right)^{-s}\right)\Gamma(s)\right)(-j) + \sum_{j=0}^{\infty}\operatorname{res}_s\left(\left(\Gamma(s)\Gamma(-s-\nu)\Gamma(\nu-s)\left(\frac{z+1}{2}\right)^{-s}\right)\Gamma\left(\frac{1}{2}+s\right)\right)\left(-j-\frac{1}{2}\right)\right)/;\;|z+1|<2$$

07.04.06.0087.01

$$T_\nu(z) = -\frac{\nu\sin(2\pi\nu)}{2\pi^{3/2}}\left(\sum_{j=0}^{\infty}\operatorname{res}_s\left(\left(\Gamma\left(s+\frac{1}{2}\right)\Gamma(-s-\nu)\Gamma(\nu-s)\left(\frac{z+1}{2}\right)^{-s}\right)\Gamma(s)\right)(-j) + \sum_{j=0}^{\infty}\operatorname{res}_s\left(\left(\Gamma(s)\Gamma(-s-\nu)\Gamma(\nu-s)\left(\frac{z+1}{2}\right)^{-s}\right)\Gamma\left(s+\frac{1}{2}\right)\right)\left(-j-\frac{1}{2}\right)\right)/;\;|z+1|<2$$

Expansions at $z == \infty$

07.04.06.0088.01

$$T_\nu(z) = \frac{\nu\sin(\pi\nu)}{\sqrt{\pi}}\left(\sum_{j=0}^{\infty}\operatorname{res}_s\left(\frac{\Gamma(-s-\nu)\Gamma(s)\left(\frac{z-1}{2}\right)^{-s}}{\Gamma\left(\frac{1}{2}-s\right)}\Gamma(\nu-s)\right)(\nu+j) + \sum_{j=0}^{\infty}\operatorname{res}_s\left(\frac{\Gamma(\nu-s)\Gamma(s)\left(\frac{z-1}{2}\right)^{-s}}{\Gamma\left(\frac{1}{2}-s\right)}\Gamma(-\nu-s)\right)(-\nu+j)\right)/;\; |z-1|>2\bigwedge 2\nu\notin\mathbb{Z}$$

07.04.06.0089.01

$$T_\nu(z) = \frac{2^{\nu-1}(z-1)^{-\nu}\nu}{\sqrt{\pi}}\sum_{j=0}^{\infty}\operatorname{res}_s\left(\frac{\Gamma(\nu-s)\Gamma\left(-s+\nu+\frac{1}{2}\right)\left(\frac{2}{z-1}\right)^{-s}}{\Gamma(-s+2\nu+1)}\Gamma(s)\right)(-j) - \frac{2^{-\nu-1}(z-1)^{\nu}\nu}{\sqrt{\pi}}\sum_{j=0}^{\infty}\operatorname{res}_s\left(\frac{\Gamma(-s-\nu)\Gamma\left(-s-\nu+\frac{1}{2}\right)\left(\frac{2}{z-1}\right)^{-s}}{\Gamma(-s-2\nu+1)}\Gamma(s)\right)(-j)\;/;\;|z-1|>2\bigwedge 2\nu\notin\mathbb{Z}$$

07.04.06.0090.01

$$T_\nu(z) = \frac{\left(-z^2\right)^{-\frac{\nu}{2}}\nu}{4\sqrt{\pi}}\left(\cos\left(\frac{\pi\nu}{2}\right) - \frac{z}{\sqrt{-z^2}}\sin\left(\frac{\pi\nu}{2}\right)\right)\sum_{j=0}^{\infty}\operatorname{res}_s\left(\frac{\Gamma\left(\frac{\nu}{2}-s\right)\Gamma\left(\frac{\nu+1}{2}-s\right)\left(-\frac{1}{z^2}\right)^{-s}}{\Gamma(-s+\nu+1)}\Gamma(s)\right)(-j) - \frac{\left(-z^2\right)^{\nu/2}\nu}{4\sqrt{\pi}}\left(\cos\left(\frac{\pi\nu}{2}\right) + \frac{z}{\sqrt{-z^2}}\sin\left(\frac{\pi\nu}{2}\right)\right)\sum_{j=0}^{\infty}\operatorname{res}_s\left(\frac{\Gamma\left(\frac{1-\nu}{2}-s\right)\Gamma\left(-s-\frac{\nu}{2}\right)\left(-\frac{1}{z^2}\right)^{-s}}{\Gamma(-s-\nu+1)}\Gamma(s)\right)(-j)\;/;\;|z|>1\bigwedge 2\nu\notin\mathbb{Z}$$

07.04.06.0091.01

$$T_\nu(z) = \frac{\nu\sin(\pi\nu)}{4\sqrt{\pi}}\left(\sum_{j=0}^{\infty}\operatorname{res}_s\left(\frac{\Gamma(s)\,\Gamma\left(-s-\frac{\nu}{2}\right)\left(-z^2\right)^{-s}}{\Gamma\left(\frac{1}{2}-s\right)}\,\Gamma\left(\frac{\nu}{2}-s\right)\right)\left(j+\frac{\nu}{2}\right)+\sum_{j=0}^{\infty}\operatorname{res}_s\left(\frac{\Gamma(s)\,\Gamma\left(\frac{\nu}{2}-s\right)\left(-z^2\right)^{-s}}{\Gamma\left(\frac{1}{2}-s\right)}\,\Gamma\left(-s-\frac{\nu}{2}\right)\right)\left(j+\frac{\nu}{2}\right)\right)-$$
$$\frac{z\,\nu\sin(\pi\nu)}{4\sqrt{\pi}}\left(\sum_{j=0}^{\infty}\operatorname{res}_s\left(\frac{\Gamma(s)\,\Gamma\left(\frac{\nu+1}{2}-s\right)\left(-z^2\right)^{-s}}{\Gamma\left(\frac{3}{2}-s\right)}\,\Gamma\left(\frac{1-\nu}{2}-s\right)\right)\left(j+\frac{1-\nu}{2}\right)+\right.$$
$$\left.\sum_{j=0}^{\infty}\operatorname{res}_s\left(\frac{\Gamma(s)\,\Gamma\left(\frac{1-\nu}{2}-s\right)\left(-z^2\right)^{-s}}{\Gamma\left(\frac{3}{2}-s\right)}\,\Gamma\left(\frac{\nu+1}{2}-s\right)\right)\left(j+\frac{\nu+1}{2}\right)\right)/;\ |z|>1\land\nu\notin\mathbb{Z}$$

07.04.06.0092.01

$$T_\nu(z) = \frac{\nu\sin(\pi\nu)}{\sqrt{\pi}}\left(\sum_{j=0}^{\infty}\operatorname{res}_s\left(\frac{\Gamma(s)\,\Gamma(\nu-s)\left(\frac{z-1}{2}\right)^{-s}}{\Gamma\left(\frac{1}{2}-s\right)}\,\Gamma(-s-\nu)\right)(j-\nu)+\sum_{j=0}^{\infty}\operatorname{res}_s\left(\frac{\Gamma(s)\,\Gamma(-s-\nu)\left(\frac{z-1}{2}\right)^{-s}}{\Gamma\left(\frac{1}{2}-s\right)}\,\Gamma(\nu-s)\right)(j+\nu)\right)/;$$
$$|z-1|>2\land 2\nu\notin\mathbb{Z}$$

07.04.06.0093.01

$$T_\nu(z) = \frac{\nu\sin(2\pi\nu)}{2\sqrt{\pi}}\left(\sum_{j=0}^{\infty}\operatorname{res}_s\left(\frac{\Gamma(s)\,\Gamma(\nu-s)\left(-\frac{z+1}{2}\right)^{-s}}{\Gamma\left(\frac{1}{2}-s\right)}\,\Gamma(-s-\nu)\right)(j-\nu)+\sum_{j=0}^{\infty}\operatorname{res}_s\left(\frac{\Gamma(s)\,\Gamma(-s-\nu)\left(-\frac{z+1}{2}\right)^{-s}}{\Gamma\left(\frac{1}{2}-s\right)}\,\Gamma(\nu-s)\right)(j+\nu)\right)-$$
$$\frac{\nu\sin(2\pi\nu)\sqrt{z+1}}{2\sqrt{2\pi}}\left(\sum_{j=0}^{\infty}\operatorname{res}_s\left(\frac{\Gamma(s)\,\Gamma\left(-s+\nu+\frac{1}{2}\right)\left(-\frac{z+1}{2}\right)^{-s}}{\Gamma\left(\frac{3}{2}-s\right)}\,\Gamma\left(-s-\nu+\frac{1}{2}\right)\right)\left(j-\nu+\frac{1}{2}\right)+\right.$$
$$\left.\sum_{j=0}^{\infty}\operatorname{res}_s\left(\frac{\Gamma(s)\,\Gamma\left(-s-\nu+\frac{1}{2}\right)\left(-\frac{z+1}{2}\right)^{-s}}{\Gamma\left(\frac{3}{2}-s\right)}\,\Gamma\left(-s+\nu+\frac{1}{2}\right)\right)\left(j+\nu+\frac{1}{2}\right)\right)/;\ |z+1|>2\land 2\nu\notin\mathbb{Z}$$

07.04.06.0094.01

$$T_\nu(z) = \frac{\nu\cos\left(\frac{\pi\nu}{2}\right)\sin(\pi\nu)}{4\pi^{3/2}}\left(\sum_{j=0}^{\infty}\operatorname{res}_s\left(\left(\Gamma(s)\,\Gamma\left(s+\frac{1}{2}\right)\Gamma\left(-s-\frac{\nu}{2}\right)\left(1-z^2\right)^{-s}\right)\Gamma\left(\frac{\nu}{2}-s\right)\right)\left(j+\frac{\nu}{2}\right)+\right.$$
$$\left.\sum_{j=0}^{\infty}\operatorname{res}_s\left(\left(\Gamma(s)\,\Gamma\left(s+\frac{1}{2}\right)\Gamma\left(\frac{\nu}{2}-s\right)\left(1-z^2\right)^{-s}\right)\Gamma\left(-s-\frac{\nu}{2}\right)\right)\left(j-\frac{\nu}{2}\right)\right)-$$
$$\frac{z\sin\left(\frac{\pi\nu}{2}\right)\sin(\pi\nu)}{2\pi^{3/2}}\left(\sum_{j=0}^{\infty}\operatorname{res}_s\left(\left(\Gamma(s)\,\Gamma\left(s+\frac{1}{2}\right)\Gamma\left(\frac{\nu+1}{2}-s\right)\left(1-z^2\right)^{-s}\right)\Gamma\left(\frac{1-\nu}{2}-s\right)\right)\left(j+\frac{1-\nu}{2}\right)+\right.$$
$$\left.\sum_{j=0}^{\infty}\operatorname{res}_s\left(\left(\Gamma(s)\,\Gamma\left(s+\frac{1}{2}\right)\Gamma\left(\frac{1-\nu}{2}-s\right)\left(1-z^2\right)^{-s}\right)\Gamma\left(\frac{\nu+1}{2}-s\right)\right)\left(j+\frac{\nu+1}{2}\right)\right)/;\ \left|1-z^2\right|>1$$

07.04.06.0095.01

$$T_\nu(z) = \frac{\nu\sin(2\pi\nu)}{2\pi^{3/2}}\left(\sum_{j=0}^{\infty}\operatorname{res}_s\left(\left(\Gamma(s)\,\Gamma\left(s+\frac{1}{2}\right)\Gamma(\nu-s)\left(\frac{z+1}{2}\right)^{-s}\right)\Gamma(-s-\nu)\right)(j-\nu)+\right.$$
$$\left.\sum_{j=0}^{\infty}\operatorname{res}_s\left(\left(\Gamma(s)\,\Gamma\left(s+\frac{1}{2}\right)\Gamma(-s-\nu)\left(\frac{z+1}{2}\right)^{-s}\right)\Gamma(\nu-s)\right)(j+\nu)\right)/;\ |z+1|>2$$

07.04.06.0096.01

$$T_\nu(z) = \frac{\nu \cos^2(\pi \nu)\sin(\pi \nu)}{\pi^{3/2}} \left(\sum_{j=0}^{\infty} \operatorname{res}_s\left(\left(\Gamma(s)\,\Gamma\left(s+\frac{1}{2}\right)\Gamma(\nu-s)\left(\frac{1-z}{2}\right)^{-s}\right)\Gamma(-s-\nu)\right)(j-\nu) + \sum_{j=0}^{\infty} \operatorname{res}_s\left(\left(\Gamma(s)\,\Gamma\left(s+\frac{1}{2}\right)\Gamma(-s-\nu)\left(\frac{1-z}{2}\right)^{-s}\right)\Gamma(\nu-s)\right)(j+\nu) \right) - \frac{\sqrt{z+1}\,\cos(\pi \nu)\sin^2(\pi \nu)}{\sqrt{2}\,\pi^{3/2}} \left(\sum_{j=0}^{\infty} \operatorname{res}_s\left(\left(\Gamma(s)\,\Gamma\left(s+\frac{1}{2}\right)\Gamma\left(-s+\nu+\frac{1}{2}\right)\left(\frac{1-z}{2}\right)^{-s}\right)\Gamma\left(-s-\nu+\frac{1}{2}\right)\right)\left(j-\nu+\frac{1}{2}\right) + \sum_{j=0}^{\infty} \operatorname{res}_s\left(\left(\Gamma(s)\,\Gamma\left(s+\frac{1}{2}\right)\Gamma\left(-s-\nu+\frac{1}{2}\right)\left(\frac{1-z}{2}\right)^{-s}\right)\Gamma\left(-s+\nu+\frac{1}{2}\right)\right)\left(j+\nu+\frac{1}{2}\right) \right) /;\ |z-1|>2$$

07.04.06.0097.01

$$T_\nu(z) = -\frac{\nu \cos\left(\frac{\pi \nu}{2}\right)}{4\sqrt{\pi}} \left(\left(-z^2\right)^{-\frac{\nu}{2}} \sum_{j=0}^{\infty} \operatorname{res}_s\left(\frac{\Gamma\left(\frac{\nu}{2}-s\right)\Gamma\left(\frac{\nu+1}{2}-s\right)\left(-\frac{1}{z^2}\right)^{-s}}{\Gamma(-s+\nu+1)}\,\Gamma(s)\right)(-j) + \left(-z^2\right)^{\nu/2} \sum_{j=0}^{\infty} \operatorname{res}_s\left(\frac{\Gamma\left(-s-\frac{\nu}{2}\right)\Gamma\left(\frac{1-\nu}{2}-s\right)\left(-\frac{1}{z^2}\right)^{-s}}{\Gamma(-s-\nu+1)}\,\Gamma(s)\right)(-j) \right) - \frac{z\,\nu}{4\sqrt{\pi}} \sin\left(\frac{\pi \nu}{2}\right) \left(\left(-z^2\right)^{\frac{\nu-1}{2}} \sum_{j=0}^{\infty} \operatorname{res}_s\left(\frac{\Gamma\left(-s-\frac{\nu}{2}\right)\Gamma\left(\frac{1-\nu}{2}-s\right)\left(-\frac{1}{z^2}\right)^{-s}}{\Gamma(-s-\nu+1)}\,\Gamma(s)\right)(-j) - \left(-z^2\right)^{-\frac{\nu+1}{2}} \sum_{j=0}^{\infty} \operatorname{res}_s\left(\frac{\Gamma\left(\frac{\nu}{2}-s\right)\Gamma\left(\frac{\nu+1}{2}-s\right)\left(-\frac{1}{z^2}\right)^{-s}}{\Gamma(-s+\nu+1)}\,\Gamma(s)\right)(-j) \right) /;\ |z|>1 \land \nu \notin \mathbb{Z}$$

07.04.06.0098.01

$$T_\nu(z) = -\frac{\nu}{2\sqrt{\pi}} \left(2^\nu (z-1)^{-\nu} \sum_{j=0}^{\infty} \operatorname{res}_s\left(\frac{\Gamma(\nu-s)\,\Gamma\left(-s+\nu+\frac{1}{2}\right)\left(\frac{2}{z-1}\right)^{-s}}{\Gamma(-s+2\nu+1)}\,\Gamma(s)\right)(-j) + 2^{-\nu}(z-1)^\nu \sum_{j=0}^{\infty} \operatorname{res}_s\left(\frac{\Gamma(-s-\nu)\,\Gamma\left(-s-\nu+\frac{1}{2}\right)\left(\frac{2}{z-1}\right)^{-s}}{\Gamma(-s-2\nu+1)}\,\Gamma(s)\right)(-j) \right) /;\ |z-1|>2$$

07.04.06.0099.01

$$T_\nu(z) = \frac{\sqrt{z+1}\ \nu \sin(\pi\nu)}{2\sqrt{2\pi}} \left(2^{\frac{1}{2}-\nu} (-z-1)^{\nu-\frac{1}{2}} \sum_{j=0}^{\infty} \operatorname{res}_s \left(\frac{\Gamma(-s-\nu)\,\Gamma\left(-s-\nu+\frac{1}{2}\right)\left(-\frac{2}{z+1}\right)^{-s}}{\Gamma(-s-2\nu+1)} \Gamma(s) \right)(-j) - 2^{\nu+\frac{1}{2}} (-z-1)^{-\nu-\frac{1}{2}} \sum_{j=0}^{\infty} \operatorname{res}_s \left(\frac{\Gamma(\nu-s)\,\Gamma\left(-s+\nu+\frac{1}{2}\right)\left(-\frac{2}{z+1}\right)^{-s}}{\Gamma(-s+2\nu+1)} \Gamma(s) \right)(-j) \right) + \frac{\nu\cos(\pi\nu)}{2\sqrt{\pi}} \left(-2^{\nu} (-z-1)^{-\nu} \sum_{j=0}^{\infty} \operatorname{res}_s \left(\frac{\Gamma(\nu-s)\,\Gamma\left(-s+\nu+\frac{1}{2}\right)\left(-\frac{2}{z+1}\right)^{-s}}{\Gamma(-s+2\nu+1)} \Gamma(s) \right)(-j) - 2^{-\nu} (-z-1)^{\nu} \sum_{j=0}^{\infty} \operatorname{res}_s \left(\frac{\Gamma(-s-\nu)\,\Gamma\left(-s-\nu+\frac{1}{2}\right)\left(-\frac{2}{z+1}\right)^{-s}}{\Gamma(-s-2\nu+1)} \Gamma(s) \right)(-j) \right) /;\ |z+1| > 2$$

Other expansions

07.04.06.0100.01

$$T_\nu(z) = \frac{2^{\nu-2}\left(-z^2\right)^{-\frac{\nu}{2}}}{\pi\,\Gamma(\nu)} \left(\cos\left(\frac{\pi\nu}{2}\right) - \frac{z}{\sqrt{-z^2}} \sin\left(\frac{\pi\nu}{2}\right)\right) \left(\sum_{j=0}^{\infty} \operatorname{res}_s\left(\left(\Gamma\left(s+\frac{1}{2}\right) \Gamma\left(\frac{\nu}{2}-s\right) \Gamma\left(\frac{\nu+1}{2}-s\right) \left(\frac{z^2-1}{z^2}\right)^{-s} \right) \Gamma(s) \right)(-j) + \sum_{j=0}^{\infty} \operatorname{res}_s\left(\left(\Gamma(s)\, \Gamma\left(\frac{\nu}{2}-s\right) \Gamma\left(\frac{\nu+1}{2}-s\right) \left(\frac{z^2-1}{z^2}\right)^{-s} \right) \Gamma\left(s+\frac{1}{2}\right) \right)\left(-j-\frac{1}{2}\right) \right) + \frac{2^{-\nu-2}\left(-z^2\right)^{\nu/2}}{\pi\,\Gamma(-\nu)} \left(\cos\left(\frac{\pi\nu}{2}\right) + \frac{z}{\sqrt{-z^2}} \sin\left(\frac{\pi\nu}{2}\right)\right) \left(\sum_{j=0}^{\infty} \operatorname{res}_s\left(\left(\Gamma\left(s+\frac{1}{2}\right) \Gamma\left(\frac{1-\nu}{2}-s\right) \Gamma\left(-s-\frac{\nu}{2}\right) \left(\frac{z^2-1}{z^2}\right)^{-s} \right) \Gamma(s) \right)(-j) + \sum_{j=0}^{\infty} \operatorname{res}_s\left(\left(\Gamma(s)\, \Gamma\left(\frac{1-\nu}{2}-s\right) \Gamma\left(-s-\frac{\nu}{2}\right) \left(\frac{z^2-1}{z^2}\right)^{-s} \right) \Gamma\left(s+\frac{1}{2}\right) \right)\left(-j-\frac{1}{2}\right) \right) /;\ \left|\frac{z^2-1}{z^2}\right| < 1 \bigwedge 2\nu \notin \mathbb{Z}$$

07.04.06.0101.01

$$T_\nu(z) = -\frac{2^{\nu-2}\left(-z^2\right)^{-\frac{\nu}{2}}}{\pi\,\Gamma(\nu)} \left(\cos\left(\frac{\pi\nu}{2}\right) - \frac{z}{\sqrt{-z^2}} \sin\left(\frac{\pi\nu}{2}\right)\right) \left(\sum_{j=0}^{\infty} \operatorname{res}_s\left(\left(\Gamma(s)\, \Gamma\left(s+\frac{1}{2}\right) \Gamma\left(\frac{\nu+1}{2}-s\right) \left(1-\frac{1}{z^2}\right)^{-s} \right) \Gamma\left(\frac{\nu}{2}-s\right) \right)\left(j+\frac{\nu}{2}\right) + \sum_{j=0}^{\infty} \operatorname{res}_s\left(\left(\Gamma(s)\, \Gamma\left(s+\frac{1}{2}\right) \Gamma\left(\frac{\nu}{2}-s\right) \left(1-\frac{1}{z^2}\right)^{-s} \right) \Gamma\left(\frac{\nu+1}{2}-s\right) \right)\left(j+\frac{\nu+1}{2}\right) \right) - \frac{2^{-\nu-2}\left(-z^2\right)^{\nu/2}}{\pi\,\Gamma(-\nu)} \left(\cos\left(\frac{\pi\nu}{2}\right) + \frac{z}{\sqrt{-z^2}} \sin\left(\frac{\pi\nu}{2}\right)\right) \left(\sum_{j=0}^{\infty} \operatorname{res}_s\left(\left(\Gamma(s)\, \Gamma\left(s+\frac{1}{2}\right) \Gamma\left(-s-\frac{\nu}{2}\right) \left(1-\frac{1}{z^2}\right)^{-s} \right) \Gamma\left(\frac{1-\nu}{2}-s\right) \right)\left(j+\frac{1-\nu}{2}\right) + \sum_{j=0}^{\infty} \operatorname{res}_s\left(\left(\Gamma(s)\, \Gamma\left(s+\frac{1}{2}\right) \Gamma\left(\frac{1-\nu}{2}-s\right) \left(1-\frac{1}{z^2}\right)^{-s} \right) \Gamma\left(-s-\frac{\nu}{2}\right) \right)\left(j-\frac{\nu}{2}\right) \right) /;\ \left|1-\frac{1}{z^2}\right| > 1 \bigwedge 2\nu \notin \mathbb{Z}$$

07.04.06.0102.01

$$T_\nu(z) = \frac{2^{-3\nu-2}(z-1)^\nu}{\pi\,\Gamma(-2\nu)}\left(\sum_{j=0}^{\infty}\operatorname{res}_s\left(\left(\Gamma\left(s+\frac{1}{2}\right)\Gamma(-s-\nu)\,\Gamma\left(-s-\nu+\frac{1}{2}\right)\left(\frac{z+1}{z-1}\right)^{-s}\right)\Gamma(s)\right)(-j) + \sum_{j=0}^{\infty}\operatorname{res}_s\left(\left(\Gamma(s)\,\Gamma(-s-\nu)\,\Gamma\left(-s-\nu+\frac{1}{2}\right)\left(\frac{z+1}{z-1}\right)^{-s}\right)\Gamma\left(s+\frac{1}{2}\right)\right)\left(-j-\frac{1}{2}\right)\right) + \frac{2^{3\nu-2}(z-1)^{-\nu}}{\pi\,\Gamma(2\nu)}\left(\sum_{j=0}^{\infty}\operatorname{res}_s\left(\left(\Gamma\left(s+\frac{1}{2}\right)\Gamma(\nu-s)\,\Gamma\left(-s+\nu+\frac{1}{2}\right)\left(\frac{z+1}{z-1}\right)^{-s}\right)\Gamma(s)\right)(-j) + \sum_{j=0}^{\infty}\operatorname{res}_s\left(\left(\Gamma(s)\,\Gamma(\nu-s)\,\Gamma\left(-s+\nu+\frac{1}{2}\right)\left(\frac{z+1}{z-1}\right)^{-s}\right)\Gamma\left(s+\frac{1}{2}\right)\right)\left(-j-\frac{1}{2}\right)\right) /;\ \left|\frac{z+1}{z-1}\right| < 1 \bigwedge 2\nu \notin \mathbb{Z}$$

07.04.06.0103.01

$$T_\nu(z) = -\frac{2^{-3\nu-2}(z-1)^\nu}{\pi\,\Gamma(-2\nu)}\left(\sum_{j=0}^{\infty}\operatorname{res}_s\left(\left(\Gamma(s)\,\Gamma\left(s+\frac{1}{2}\right)\Gamma\left(-s-\nu+\frac{1}{2}\right)\left(\frac{z+1}{z-1}\right)^{-s}\right)\Gamma(-s-\nu)\right)(j-\nu) + \sum_{j=0}^{\infty}\operatorname{res}_s\left(\left(\Gamma(s)\,\Gamma\left(s+\frac{1}{2}\right)\Gamma(-s-\nu)\left(\frac{z+1}{z-1}\right)^{-s}\right)\Gamma\left(-s-\nu+\frac{1}{2}\right)\right)\left(j-\nu+\frac{1}{2}\right)\right) - \frac{2^{3\nu-2}(z-1)^{-\nu}}{\pi\,\Gamma(2\nu)}\left(\sum_{j=0}^{\infty}\operatorname{res}_s\left(\left(\Gamma(s)\,\Gamma\left(s+\frac{1}{2}\right)\Gamma\left(-s+\nu+\frac{1}{2}\right)\left(\frac{z+1}{z-1}\right)^{-s}\right)\Gamma(\nu-s)\right)(j+\nu) + \sum_{j=0}^{\infty}\operatorname{res}_s\left(\left(\Gamma(s)\,\Gamma\left(s+\frac{1}{2}\right)\Gamma(\nu-s)\left(\frac{z+1}{z-1}\right)^{-s}\right)\Gamma\left(-s+\nu+\frac{1}{2}\right)\right)\left(j+\nu+\frac{1}{2}\right)\right) /;\ \left|\frac{z+1}{z-1}\right| > 1 \bigwedge 2\nu \notin \mathbb{Z}$$

Integral representations

Integral representations of negative integer order

Rodrigues-type formula.

07.04.07.0001.01

$$T_n(z) = \frac{(-1)^n\sqrt{\pi}\,\sqrt{1-z^2}}{2^n\,\Gamma\left(n+\frac{1}{2}\right)}\,\frac{\partial^n\left(1-z^2\right)^{n-\frac{1}{2}}}{\partial z^n} /;\ n \in \mathbb{N}$$

Limit representations

07.04.09.0001.01

$$T_\nu(z) = \frac{\nu}{2}\lim_{\lambda\to 0}\frac{1}{\lambda}\,C_\nu^\lambda(z)$$

Generating functions

07.04.11.0001.01

$$T_n(z) = \left([t^n]\, \frac{1 - t\, z}{t^2 - 2\, z\, t + 1}\right) /; n \in \mathbb{N} \land -1 < z < 1$$

Differential equations

Ordinary linear differential equations and wronskians

For the direct function itself

With respect to ν

07.04.13.0005.01

$$w''(\nu) + \cos^{-1}(z)^2\, w(\nu) = 0 /; w(\nu) = T_\nu(z) \bigwedge w(0) = 1 \bigwedge w'(0) = 0$$

07.04.13.0006.01

$$w''(\nu) + \cos^{-1}(z)^2\, w(\nu) = 0 /; w(\nu) = c_1\, T_\nu(z) + c_2\, U_\nu(z)$$

07.04.13.0007.01

$$W_\nu(T_\nu(z),\, U_\nu(z)) = \frac{z \cos^{-1}(z)}{\sqrt{1 - z^2}}$$

07.04.13.0008.01

$$w''(\nu) - \frac{g''(\nu)}{g'(\nu)}\, w'(\nu) + \cos^{-1}(z)^2\, g'(\nu)^2\, w(\nu) = 0 /; w(\nu) = c_1\, T_{g(\nu)}(z) + c_2\, U_{g(\nu)}(z)$$

07.04.13.0009.01

$$W_\nu\left(T_{g(\nu)}(z),\, U_{g(\nu)}(z)\right) = \frac{z \cos^{-1}(z)\, g'(\nu)}{\sqrt{1 - z^2}}$$

07.04.13.0010.01

$$w''(\nu) + \left(-\frac{2\, h'(\nu)}{h(\nu)} - \frac{g''(\nu)}{g'(\nu)}\right) w'(\nu) + \left(\cos^{-1}(z)^2\, g'(\nu)^2 + \frac{h'(\nu)\, g''(\nu)}{h(\nu)\, g'(\nu)} + \frac{2\, h'(\nu)^2 - h(\nu)\, h''(\nu)}{h(\nu)^2}\right) w(\nu) = 0 /;$$
$$w(\nu) = c_1\, h(\nu)\, T_{g(\nu)}(z) + c_2\, h(\nu)\, U_{g(\nu)}(z)$$

07.04.13.0011.01

$$W_\nu\left(h(\nu)\, T_{g(\nu)}(z),\, h(\nu)\, U_{g(\nu)}(z)\right) = \frac{z \cos^{-1}(z)\, h(\nu)^2\, g'(\nu)}{\sqrt{1 - z^2}}$$

07.04.13.0012.01

$$\nu^2\, w''(\nu) - (r + 2\, s - 1)\, \nu\, w'(\nu) + \left(a^2\, r^2 \cos^{-1}(z)^2\, \nu^{2\, r} + s\, (r + s)\right) w(\nu) = 0 /; w(\nu) = c_1\, \nu^s\, T_{a\, \nu^r}(z) + c_2\, \nu^s\, U_{a\, \nu^r}(z)$$

07.04.13.0013.01

$$W_\nu\left(\nu^s\, T_{a\, \nu^r}(z),\, \nu^s\, U_{a\, \nu^r}(z)\right) = \frac{a\, r\, z\, \nu^{r+2\, s-1} \cos^{-1}(z)}{\sqrt{1 - z^2}}$$

07.04.13.0014.01

$$w''(\nu) - (\log(r) + 2\log(s))\, w'(\nu) + \left(a^2 \cos^{-1}(z)^2 \log^2(r)\, r^{2\nu} + \log(s)\,(\log(r) + \log(s))\right) w(\nu) = 0 \;/;\; w(\nu) = c_1\, s^\nu\, T_{a\, r^\nu}(z) + c_2\, s^\nu\, U_{a\, r^\nu}(z)$$

07.04.13.0015.01

$$W_\nu\left(s^\nu\, T_{a\, r^\nu}(z),\, s^\nu\, U_{a\, r^\nu}(z)\right) = \frac{a\, r^\nu\, s^{2\nu}\, z \cos^{-1}(z) \log(r)}{\sqrt{1-z^2}}$$

With respect to *z*

07.04.13.0001.01

$$\left(1-z^2\right) w''(z) - z\, w'(z) + \nu^2\, w(z) = 0 \;/;\; w(z) = c_1\, T_\nu(z) + c_2 \sin\left(\nu \cos^{-1}(z)\right)$$

07.04.13.0002.01

$$W_z\left(T_\nu(z),\, \sin\left(\nu \cos^{-1}(z)\right)\right) = -\frac{\nu}{\sqrt{1-z^2}}$$

07.04.13.0003.01

$$\left(1-z^2\right) w''(z) - z\, w'(z) + \nu^2\, w(z) = 0 \;/;\; w(z) = c_1\, T_\nu(z) + c_2 \sinh\left(\nu \cosh^{-1}(z)\right)$$

07.04.13.0004.01

$$W_z\left(T_\nu(z),\, \sinh\left(\nu \cosh^{-1}(z)\right)\right) = \frac{\nu \cos\left(\nu \cos^{-1}(z)\right) \cosh\left(\nu \cosh^{-1}(z)\right)}{\sqrt{z-1}\,\sqrt{z+1}} - \frac{\nu \sin\left(\nu \cos^{-1}(z)\right) \sinh\left(\nu \cosh^{-1}(z)\right)}{\sqrt{1-z^2}}$$

07.04.13.0016.01

$$\left(1-z^2\right) w''(z) - z\, w'(z) + \nu^2\, w(z) = 0 \;/;\; w(z) = c_1\, T_\nu(z) + c_2 \sqrt{1-z^2}\; U_{\nu-1}(z)$$

07.04.13.0017.01

$$W_z\left(T_\nu(z),\, \sqrt{1-z^2}\; U_{\nu-1}(z)\right) = -\frac{\nu}{\sqrt{1-z^2}}$$

07.04.13.0018.01

$$w''(z) - \left(\frac{g(z)\, g'(z)}{1-g(z)^2} + \frac{g''(z)}{g'(z)}\right) w'(z) + \frac{\nu^2\, g'(z)^2}{1-g(z)^2}\, w(z) = 0 \;/;\; w(z) = c_1\, T_\nu(g(z)) + c_2 \sqrt{1-g(z)^2}\; U_{\nu-1}(g(z))$$

07.04.13.0019.01

$$W_z\left(T_\nu(g(z)),\, \sqrt{1-g(z)^2}\; U_{\nu-1}(g(z))\right) = -\frac{\nu\, g'(z)}{\sqrt{1-g(z)^2}}$$

07.04.13.0020.01

$$w''(z) - \left(\frac{g(z)\, g'(z)}{1-g(z)^2} + \frac{2\, h'(z)}{h(z)} + \frac{g''(z)}{g'(z)}\right) w'(z) + \left(\frac{\nu^2\, g'(z)^2}{1-g(z)^2} + \frac{g(z)\, h'(z)\, g'(z)}{\left(1-g(z)^2\right) h(z)} + \frac{h(z)\, h'(z)\, g''(z) + g'(z)\left(2\, h'(z)^2 - h(z)\, h''(z)\right)}{h(z)^2\, g'(z)}\right) w(z) = 0 \;/;$$
$$w(z) = c_1\, h(z)\, T_\nu(g(z)) + c_2\, h(z) \sqrt{1-g(z)^2}\; U_{\nu-1}(g(z))$$

07.04.13.0021.01

$$W_z\left(h(z)\,T_\nu(g(z)),\,h(z)\,\sqrt{1-g(z)^2}\;U_{\nu-1}(g(z))\right)=-\frac{\nu\,h(z)^2\,g'(z)}{\sqrt{1-g(z)^2}}$$

07.04.13.0022.01

$$z^2\left(a^2\,z^{2r}-1\right)w''(z)+\left(r-(2s-1)\left(a^2\,z^{2r}-1\right)\right)z\,w'(z)+\left(a^2\,z^{2r}\left(s^2-r^2\,\nu^2\right)-s\,(r+s)\right)w(z)=0\;/;\;w(z)=c_1\,z^s\,T_\nu(a\,z^r)+c_2\,z^s\,\sqrt{1-a^2\,z^{2r}}\;U_{\nu-1}(a\,z^r)$$

07.04.13.0023.01

$$W_z\left(z^s\,T_\nu(a\,z^r),\,z^s\,\sqrt{1-a^2\,z^{2r}}\;U_{\nu-1}(a\,z^r)\right)=-\frac{a\,r\,\nu\,z^{r+2s-1}}{\sqrt{1-a^2\,z^{2r}}}$$

07.04.13.0024.01

$$\left(a^2\,r^{2z}-1\right)w''(z)+\left(\log(r)-2\left(a^2\,r^{2z}-1\right)\log(s)\right)w'(z)+\left(a^2\,r^{2z}\left(\log^2(s)-\nu^2\log^2(r)\right)-\log(s)\,(\log(r)+\log(s))\right)w(z)=0\;/;\;w(z)=c_1\,s^z\,T_\nu(a\,r^z)+c_2\,s^z\,\sqrt{1-a^2\,r^{2z}}\;U_{\nu-1}(a\,r^z)$$

07.04.13.0025.01

$$W_z\left(s^z\,T_\nu(a\,r^z),\,s^z\,\sqrt{1-a^2\,r^{2z}}\;U_{\nu-1}(a\,r^z)\right)=-\frac{a\,r^z\,\nu\,s^{2z}\log(r)}{\sqrt{1-a^2\,r^{2z}}}$$

Transformations

Transformations and argument simplifications

Argument involving basic arithmetic operations

07.04.16.0001.01

$$T_{-\nu}(z)=T_\nu(z)$$

07.04.16.0002.01

$$T_n(-z)=(-1)^n\,T_n(z)\;/;\;n\in\mathbb{N}$$

Multiple arguments

07.04.16.0005.01

$$T_{2\nu}\left(\sqrt{\frac{z+1}{2}}\right)=T_\nu(z)$$

Brychkov Yu.A. (2007)

07.04.16.0006.01

$$T_{2n}(z)=(-1)^n\,T_n\left(1-2z^2\right)\;/;\;n\in\mathbb{Z}$$

Products, sums, and powers of the direct function

Products of the direct function

07.04.16.0003.01

$$T_n(z)\,T_m(z)=\frac{1}{2}\left(T_{m+n}(z)+T_{|n-m|}(z)\right)\;/;\;m\in\mathbb{N}^+\land n\in\mathbb{N}^+$$

Related transformations

07.04.16.0004.01

$T_n(T_m(z)) == T_{n\,m}(z) \,/;\, m \in \mathbb{N}^+ \bigwedge n \in \mathbb{N}^+$

Identities

Recurrence identities

Consecutive neighbors

07.04.17.0001.01

$T_\nu(z) == 2\,z\,T_{\nu+1}(z) - T_{\nu+2}(z)$

07.04.17.0002.01

$T_\nu(z) == 2\,z\,T_{\nu-1}(z) - T_{\nu-2}(z)$

Distant neighbors

07.04.17.0008.01

$T_\nu(z) == C_m(\nu, z)\,T_{\nu+m}(z) - C_{m-1}(\nu, z)\,T_{\nu+m+1}(z) \,/;$
$C_0(\nu, z) == 1 \bigwedge C_1(\nu, z) == 2\,z \bigwedge C_m(\nu, z) == 2\,z\,C_{m-1}(\nu, z) - C_{m-2}(\nu, z) \bigwedge m \in \mathbb{N}^+$

07.04.17.0009.01

$T_\nu(z) == C_m(\nu, z)\,T_{\nu-m}(z) - C_{m-1}(\nu, z)\,T_{\nu-m-1}(z) \,/;$
$C_0(\nu, z) == 1 \bigwedge C_1(\nu, z) == 2\,z \bigwedge C_m(\nu, z) == 2\,z\,C_{m-1}(\nu, z) - C_{m-2}(\nu, z) \bigwedge m \in \mathbb{N}^+$

07.04.17.0003.01

$T_\nu(z) == 2\,(-1)^{2\left\lfloor\frac{m}{2}\right\rfloor} z^{m-2\left\lfloor\frac{m}{2}\right\rfloor} \left(z^2\right)^{\frac{1-m}{2}+\left\lfloor\frac{m}{2}\right\rfloor} U_{\frac{m-1}{2}}\left(2\,z^2-1\right) T_{\nu+m}(z) - 2\,z^{1-m+2\left\lfloor\frac{m}{2}\right\rfloor} \left(z^2\right)^{\left\lfloor\frac{m+1}{2}\right\rfloor-\frac{m}{2}} U_{\frac{m}{2}-1}\left(2\,z^2-1\right) T_{\nu+m+1}(z) \,/;\, m \in \mathbb{N}^+$

07.04.17.0004.01

$T_\nu(z) == 2\,(-1)^{2\left\lfloor\frac{m}{2}\right\rfloor} z^{m-2\left\lfloor\frac{m}{2}\right\rfloor} \left(z^2\right)^{\frac{1-m}{2}+\left\lfloor\frac{m}{2}\right\rfloor} U_{\frac{m-1}{2}}\left(2\,z^2-1\right) T_{\nu-m}(z) - 2\,z^{1-m+2\left\lfloor\frac{m}{2}\right\rfloor} \left(z^2\right)^{\left\lfloor\frac{m+1}{2}\right\rfloor-\frac{m}{2}} U_{\frac{m}{2}-1}\left(2\,z^2-1\right) T_{\nu-m-1}(z) \,/;\, m \in \mathbb{N}^+$

Functional identities

Relations between contiguous functions

Recurrence relations

07.04.17.0005.01

$T_{\nu-1}(z) + T_{\nu+1}(z) == 2\,z\,T_\nu(z)$

07.04.17.0006.01

$$T_\nu(z) == \frac{1}{2\,z}\left(T_{\nu-1}(z) + T_{\nu+1}(z)\right)$$

Normalized recurrence relation

07.04.17.0007.01

$$z\,p(\nu, z) == \frac{1}{4}\,p(\nu-1, z) + p(\nu+1, z) \,/;\, p(\nu, z) == 2^{-\nu}\,T_\nu(z)$$

Complex characteristics

Real part

07.04.19.0001.01

$$\operatorname{Re}(T_n(x+i\,y)) = T_n(x) + n\sum_{j=1}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(-1)^j\, 2^{2j-2}\, y^{2j}}{j}\, C_{n-2j}^{2j}(x)\ /;\ x \in \mathbb{R} \bigwedge y \in \mathbb{R} \bigwedge n \in \mathbb{N}$$

Imaginary part

07.04.19.0002.01

$$\operatorname{Im}(T_n(x+i\,y)) = n\sum_{j=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \frac{(-1)^j\, 2^{2j}\, y^{2j+1}}{2j+1}\, C_{n-2j-1}^{2j+)}(x)\ /;\ x \in \mathbb{R} \bigwedge y \in \mathbb{R} \bigwedge n \in \mathbb{N}$$

Differentiation

Low-order differentiation

With respect to ν

07.04.20.0001.01

$$\frac{\partial T_\nu(z)}{\partial \nu} = -\sqrt{1-z^2}\, \cos^{-1}(z)\, U_{\nu-1}(z)$$

07.04.20.0002.01

$$\frac{\partial^2 T_\nu(z)}{\partial \nu^2} = -\cos^{-1}(z)^2\, T_\nu(z)$$

With respect to z

Forward shift operator:

07.04.20.0003.01

$$\frac{\partial T_\nu(z)}{\partial z} = \nu\, U_{\nu-1}(z)$$

07.04.20.0004.01

$$\frac{\partial^2 T_\nu(z)}{\partial z^2} = \frac{\nu}{z^2-1}\,(\nu\, T_\nu(z) - z\, U_{\nu-1}(z))$$

Symbolic differentiation

With respect to ν

07.04.20.0005.02

$$\frac{\partial^m T_\nu(z)}{\partial \nu^m} = \cos^{-1}(z)^m \cos\left(\frac{\pi\, m}{2} + \nu \cos^{-1}(z)\right) /;\ m \in \mathbb{N}$$

07.04.20.0006.02

$$\frac{\partial^m T_\nu(z)}{\partial \nu^m} = i^m \cos^{-1}(z)^m \left(T_\nu(z) - \left(T_\nu(z) - i \sqrt{1-z^2}\, U_{\nu-1}(z)\right)(m \bmod 2)\right) /; m \in \mathbb{N}$$

With respect to z

07.04.20.0007.01

$$\frac{\partial^m T_\nu(z)}{\partial z^m} = \nu\, 2^{m-1} (m-1)!\, C_{\nu-m}^{m}(z) /; m \in \mathbb{N}^+$$

07.04.20.0008.02

$$\frac{\partial^m T_\nu(z)}{\partial z^m} = \sqrt{\pi}\, (z-1)^{-m} {}_3\tilde{F}_2\left(1, -\nu, \nu; \frac{1}{2}, 1-m; \frac{1-z}{2}\right) /; m \in \mathbb{N}$$

07.04.20.0011.01

$$\frac{\partial^m T_n(z)}{\partial z^m} = 2^{m-1} (m-1)!\, n \sum_{i_1=0}^{n-m} \ldots \sum_{i_m=0}^{n-m} \delta_{\sum_{j=1}^{m} i_j, n-m} \prod_{j=1}^{m} U_{i_j}(z) /; m \in \mathbb{N}^+ \bigwedge n \in \mathbb{N}$$

Fractional integro-differentiation

With respect to ν

07.04.20.0009.01

$$\frac{\partial^\alpha T_\nu(z)}{\partial \nu^\alpha} = 2^\alpha \nu^{-\alpha} \sqrt{\pi}\, {}_1\tilde{F}_2\left(1; \frac{1-\alpha}{2}, 1-\frac{\alpha}{2}; -\frac{1}{4} \nu^2 \cos^{-1}(z)^2\right)$$

With respect to z

07.04.20.0010.01

$$\frac{\partial^\alpha T_\nu(z)}{\partial z^\alpha} = z^{-\alpha} \sqrt{\pi}\, \tilde{F}_{1\times1\times0}^{2\times1\times0}\left(\begin{matrix} -\nu, \nu; 1; ; \\ \frac{1}{2}; 1-\alpha; ; \end{matrix} \; -\frac{z}{2}, \frac{1}{2}\right)$$

Integration

Indefinite integration

Involving only one direct function

07.04.21.0001.01

$$\int T_\nu(a z)\, dz = \frac{1}{2a}\left(\frac{T_{\nu-1}(a z)}{1-\nu} + \frac{T_{\nu+1}(a z)}{\nu+1}\right)$$

07.04.21.0002.01

$$\int T_\nu(z)\, dz = \frac{1}{2}\left(\frac{T_{\nu-1}(z)}{1-\nu} + \frac{T_{\nu+1}(z)}{\nu+1}\right)$$

Involving one direct function and elementary functions

Involving power function

07.04.21.0003.01

$$\int z^{\alpha-1} T_\nu(z)\, dz = 2^{-\alpha-1} z^{\alpha-1} \left(T_{\alpha-1}(z) + i\sqrt{1-z^2}\, U_{\alpha-2}(z)\right)\left(z\left(z + i\sqrt{1-z^2}\right)\right)^{1-\alpha}$$
$$\left(\frac{1}{\alpha-\nu}\left(T_{\nu-\alpha}(z) + i\sqrt{1-z^2}\, U_{-\alpha+\nu-1}(z)\right) {}_2F_1\left(\frac{\nu-\alpha}{2}, 1-\alpha; \frac{\nu-\alpha}{2}+1; -2z^2 - 2i\sqrt{1-z^2}\, z + 1\right) + \right.$$
$$\frac{1}{\alpha+\nu}\left(T_{\alpha+\nu}(z) - i\sqrt{1-z^2}\, U_{\alpha+\nu-1}(z)\right) {}_2F_1\left(-\frac{\alpha+\nu}{2}, 1-\alpha; 1-\frac{\alpha+\nu}{2}; -2z^2 - 2i\sqrt{1-z^2}\, z + 1\right) +$$
$$\frac{1}{\nu-\alpha+2}\left(T_{2-\alpha+\nu}(z) + i\sqrt{1-z^2}\, U_{1-\alpha+\nu}(z)\right) {}_2F_1\left(\frac{\nu-\alpha}{2}+1, 1-\alpha; \frac{\nu-\alpha}{2}+2; -2z^2 - 2i\sqrt{1-z^2}\, z + 1\right) -$$
$$\left.\frac{1}{\alpha+\nu-2}\left(T_{2-\alpha-\nu}(z) + i\sqrt{1-z^2}\, U_{1-\alpha-\nu}(z)\right) {}_2F_1\left(1-\frac{\alpha+\nu}{2}, 1-\alpha; 2-\frac{\alpha+\nu}{2}; -2z^2 - 2i\sqrt{1-z^2}\, z + 1\right)\right)$$

Involving algebraic functions

07.04.21.0004.01

$$\int \left(1-z^2\right)^{\frac{1}{2}(-\nu-3)} T_\nu(z)\, dz = \frac{\left(1-z^2\right)^{\frac{1}{2}(-\nu-1)}}{\nu+1} T_{\nu+1}(z)$$

07.04.21.0005.01

$$\int \left(1-z^2\right)^{\frac{\nu-3}{2}} T_\nu(z)\, dz = -\frac{\left(1-z^2\right)^{\frac{\nu-1}{2}}}{\nu-1} T_{\nu-1}(z)$$

Involving only one direct function with respect to ν

07.04.21.0006.01

$$\int T_\nu(z)\, d\nu = \frac{\sqrt{1-z^2}}{\cos^{-1}(z)} U_{\nu-1}(z)$$

Involving one direct function and elementary functions with respect to ν

Involving power function

07.04.21.0007.01

$$\int \nu^{\alpha-1} T_\nu(z)\, d\nu = \frac{1}{2}\nu^\alpha \left(-\left(-i\nu\cos^{-1}(z)\right)^{-\alpha} \Gamma\left(\alpha, -i\nu\cos^{-1}(z)\right) - \left(i\nu\cos^{-1}(z)\right)^{-\alpha}\Gamma\left(\alpha, i\nu\cos^{-1}(z)\right)\right)$$

Definite integration

Involving the direct function

07.04.21.0008.01

$$\mathcal{P}\int_{-1}^{1} \frac{T_n(t)}{\sqrt{1-t^2}\,(t-x)}\, dt = \pi\, U_{n-1}(x) \,/;\, n-1 \in \mathbb{N}^+ \land -1 < x < 1$$

Orthogonality:

07.04.21.0009.01

$$\int_{-1}^{1} \frac{T_m(t)\, T_n(t)}{\sqrt{1-t^2}}\, dt = \frac{\pi}{2}\, \delta_{m,n} \,/;\, m \in \mathbb{N} \bigwedge n \in \mathbb{N} \bigwedge m^2 + n^2 \neq 0$$

Summation

Infinite summation

07.04.23.0001.01

$$\sum_{n=0}^{\infty} T_n(z)\, w^n = \frac{1 - w\, z}{w^2 - 2\, z\, w + 1} \,/;\, -1 < z < 1 \land |w| < 1$$

07.04.23.0002.01

$$\sum_{n=1}^{\infty} \frac{w^n}{n}\, T_n(z) = -\frac{1}{2} \log\left(w^2 - 2\, z\, w + 1\right) /;\, -1 < z < 1 \land |w| < 1$$

07.04.23.0003.01

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n}{n!}\, T_n(z)\, w^n = \frac{\sqrt{1 - w\, z + \sqrt{w^2 - 2\, z\, w + 1}}}{\sqrt{2}\, \sqrt{w^2 - 2\, z\, w + 1}} \,/;\, -1 < z < 1 \land |w| < 1$$

07.04.23.0004.01

$$\sum_{n=0}^{\infty} \frac{T_n(z)\, w^n}{\left(\frac{1}{2}\right)_n\, n!} = \cosh\left(\sqrt{2}\, \sqrt{w\,(z-1)}\right) \cosh\left(\sqrt{2}\, \sqrt{w\,(z+1)}\right) /;\, -1 < z < 1 \land |w| < 1$$

07.04.23.0005.01

$$\sum_{n=0}^{\infty} \frac{T_n(z)\, w^n}{n!} = e^{w\, z} \cosh\left(\sqrt{w^2\left(z^2 - 1\right)}\right) /;\, -1 < z < 1 \land |w| < 1$$

07.04.23.0006.01

$$\sum_{n=0}^{\infty} \frac{(\gamma)_n\, (2\,\lambda - \gamma)_n}{(2\,\lambda)_n \left(\lambda + \frac{1}{2}\right)_n}\, C_n^{\lambda}(z)\, w^n = {}_2F_1\left(\gamma, 2\,\lambda - \gamma; \lambda + \frac{1}{2}; \frac{1}{2}\left(1 - \sqrt{w^2 - 2\, z\, w + 1} - w\right)\right)$$

$${}_2F_1\left(\gamma, 2\,\lambda - \gamma; \lambda + \frac{1}{2}; \frac{1}{2}\left(1 - \sqrt{w^2 - 2\, z\, w + 1} + w\right)\right) /;\, -1 < z < 1 \land |w| < 1$$

07.04.23.0007.01

$$\sum_{n=0}^{\infty} \frac{(\gamma)_n}{n!}\, T_n(z)\, w^n = (1 - w\, z)^{-\gamma}\, {}_2F_1\left(\frac{\gamma}{2}, \frac{\gamma+1}{2}; \frac{1}{2}; \frac{\left(z^2 - 1\right) w^2}{(1 - w\, z)^2}\right) /;\, -1 < z < 1 \land |w| < 1$$

07.04.23.0008.01

$$\sum_{n=1}^{\infty} \frac{1}{2^n \prod_{k=1}^{2^{n-1}} T_k\left(\frac{z}{2}\right)} = \frac{1}{2}\left(z - \sqrt{z^2 - 4}\right) /;\, |z| > 2$$

07.04.23.0009.01

$$\sum_{n=0}^{\infty} T_n(x)\, T_n(y) \left(\frac{1}{2} + \frac{1}{2}\, \delta_{n,0}\right) = \frac{1}{2}\, \pi\, \sqrt[4]{1 - x^2}\, \sqrt[4]{1 - y^2}\, \delta(x - y) \,/;\, -1 < x < 1 \bigwedge -1 < y < 1$$

Operations

Orthogonality, completeness, and Fourier expansions

The set of functions $T_n(x)$, $n = 0, 1, \ldots$, forms a complete, orthogonal (with weight $\frac{2}{\pi}\frac{1}{\sqrt{1-x^2}}$) system on the interval $(-1, 1)$.

07.04.25.0001.01

$$\sum_{n=0}^{\infty}\left(\sqrt{\frac{2}{\pi}}\frac{T_n(x)}{\sqrt[4]{1-x^2}}\right)\left(\sqrt{\frac{2}{\pi}}\frac{T_n(y)}{\sqrt[4]{1-y^2}}\right) = \delta(x-y) /; -1 < x < 1 \land -1 < y < 1$$

07.04.25.0002.01

$$\int_{-1}^{1}\left(\sqrt{\frac{2}{\pi}}\frac{T_m(t)}{\sqrt[4]{1-t^2}}\right)\left(\sqrt{\frac{2}{\pi}}\frac{T_n(t)}{\sqrt[4]{1-t^2}}\right)dt = \delta_{m,n} /; m^2 + n^2 \neq 0$$

Any sufficiently smooth function $f(x)$ can be expanded in the system $\{T_n(x)\}_{n=0,1,\ldots}$ as a generalized Fourier series, with its sum converging to $f(x)$ almost everywhere.

07.04.25.0003.01

$$f(x) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n\,\psi_n(x) /; c_n = \int_{-1}^{1}\psi_n(t)\,f(t)\,dt \bigwedge \psi_n(x) = \sqrt{\frac{2}{\pi}}\frac{T_n(x)}{\sqrt[4]{1-x^2}} \bigwedge -1 < x < 1$$

Representations through more general functions

Through hypergeometric functions

Involving $_2F_1$

07.04.26.0027.01

$$T_\nu(z) = \cos\left(\frac{\pi\nu}{2}\right) {}_2F_1\left(-\frac{\nu}{2}, \frac{\nu}{2}; \frac{1}{2}; z^2\right) + z\,\nu\sin\left(\frac{\pi\nu}{2}\right) {}_2F_1\left(\frac{1-\nu}{2}, \frac{\nu+1}{2}; \frac{3}{2}; z^2\right)$$

07.04.26.0001.01

$$T_\nu(z) = {}_2F_1\left(-\nu, \nu; \frac{1}{2}; \frac{1-z}{2}\right)$$

07.04.26.0002.01

$$T_\nu(z) = \cos(\nu\pi)\,{}_2F_1\left(-\nu, \nu; \frac{1}{2}; \frac{z+1}{2}\right) + \sqrt{2}\,\nu\sin(\nu\pi)\sqrt{z+1}\;{}_2F_1\left(\nu+\frac{1}{2}, \frac{1}{2}-\nu; \frac{3}{2}; \frac{z+1}{2}\right)$$

07.04.26.0028.01

$$T_\nu(z) = 2^{-\nu-1}\left(-z^2\right)^{-\frac{\nu}{2}}\left(\cos\left(\frac{\pi\nu}{2}\right) - \frac{z}{\sqrt{-z^2}}\sin\left(\frac{\pi\nu}{2}\right)\right){}_2F_1\left(\frac{\nu}{2}, \frac{\nu+1}{2}; \nu+1; \frac{1}{z^2}\right) +$$
$$2^{\nu-1}\left(-z^2\right)^{\nu/2}\left(\cos\left(\frac{\pi\nu}{2}\right) + \frac{z}{\sqrt{-z^2}}\sin\left(\frac{\pi\nu}{2}\right)\right){}_2F_1\left(-\frac{\nu}{2}, \frac{1}{2}-\frac{\nu}{2}; 1-\nu; \frac{1}{z^2}\right) /; z \notin (-1, 0) \land 2\nu \notin \mathbb{Z}$$

07.04.26.0003.01

$$T_\nu(z) == 2^{-\nu-1}(z-1)^{-\nu}\,{}_2F_1\left(\nu, \nu+\frac{1}{2}; 2\nu+1; \frac{2}{1-z}\right) + 2^{\nu-1}(z-1)^{\nu}\,{}_2F_1\left(-\nu, \frac{1}{2}-\nu; 1-2\nu; \frac{2}{1-z}\right) /; z \notin (-\infty, 1) \bigwedge 2\nu \notin \mathbb{Z}$$

Through hypergeometric functions of two variables

07.04.26.0004.01

$$T_\nu(z) == F_{1\times0\times0}^{2\times0\times0}\left(\begin{matrix} -\nu, \nu;;; \\ \frac{1}{2};;; \end{matrix} \; \frac{1}{2}, -\frac{z}{2}\right)$$

Through Meijer *G*

Classical cases for the direct function itself

07.04.26.0029.01

$$T_\nu(z) == -\frac{\nu\sin(\pi\nu)}{4\sqrt{\pi}}\left(G_{2,2}^{1,2}\left(-z^2 \left| \begin{matrix} \frac{\nu}{2}+1, 1-\frac{\nu}{2} \\ 0, \frac{1}{2} \end{matrix}\right.\right) - z\, G_{2,2}^{1,2}\left(-z^2 \left| \begin{matrix} \frac{\nu+1}{2}, \frac{1-\nu}{2} \\ 0, -\frac{1}{2} \end{matrix}\right.\right)\right) /; \nu \notin \mathbb{Z}$$

07.04.26.0005.01

$$T_\nu(z) == -\frac{\nu\sin(\pi\nu)}{\sqrt{\pi}}\, G_{2,2}^{1,2}\left(\frac{z-1}{2} \left| \begin{matrix} \nu+1, 1-\nu \\ 0, \frac{1}{2} \end{matrix}\right.\right) /; \nu \notin \mathbb{Z}$$

07.04.26.0006.01

$$T_n(z) == -\frac{1}{\sqrt{\pi}} \lim_{m\to n} m\sin(\pi m)\, G_{2,2}^{1,2}\left(\frac{z-1}{2} \left| \begin{matrix} m+1, 1-m \\ 0, \frac{1}{2} \end{matrix}\right.\right) /; n \in \mathbb{Z}$$

07.04.26.0007.01

$$T_\nu(2z+1) == -\frac{\nu\sin(\pi\nu)}{\sqrt{\pi}}\, G_{2,2}^{1,2}\left(z \left| \begin{matrix} \nu+1, 1-\nu \\ 0, \frac{1}{2} \end{matrix}\right.\right) /; \nu \notin \mathbb{Z}$$

07.04.26.0030.01

$$T_\nu(z) == \frac{\nu\sin(2\pi\nu)}{2\sqrt{\pi}}\left(\frac{\sqrt{z+1}}{\sqrt{2}}\, G_{2,2}^{1,2}\left(-\frac{z+1}{2} \left| \begin{matrix} \frac{1}{2}-\nu, \nu+\frac{1}{2} \\ 0, -\frac{1}{2} \end{matrix}\right.\right) - G_{2,2}^{1,2}\left(-\frac{z+1}{2} \left| \begin{matrix} \nu+1, 1-\nu \\ 0, \frac{1}{2} \end{matrix}\right.\right)\right)$$

07.04.26.0031.01

$$T_\nu(z) == -\frac{\nu\left(-z^2\right)^{-\frac{\nu+1}{2}}}{4\sqrt{\pi}}\left(\left(-z^2\right)^{\nu}\left(z\sin\left(\frac{\pi\nu}{2}\right) + \sqrt{-z^2}\cos\left(\frac{\pi\nu}{2}\right)\right) G_{2,2}^{1,2}\left(-\frac{1}{z^2} \left| \begin{matrix} \frac{\nu+2}{2}, \frac{\nu+1}{2} \\ 0, \nu \end{matrix}\right.\right) + \right.$$
$$\left.\left(z\sin\left(\frac{\pi\nu}{2}\right) - \sqrt{-z^2}\cos\left(\frac{\pi\nu}{2}\right)\right) G_{2,2}^{1,2}\left(-\frac{1}{z^2} \left| \begin{matrix} 1-\frac{\nu}{2}, \frac{1-\nu}{2} \\ 0, -\nu \end{matrix}\right.\right)\right) /; z \notin (-1, 0) \bigwedge 2\nu \notin \mathbb{Z}$$

07.04.26.0032.01

$$T_\nu(z) == \frac{2^{-\nu-1}\nu(z-1)^{-\nu}}{\sqrt{\pi}}\left(4^{\nu}\, G_{2,2}^{1,2}\left(\frac{2}{z-1} \left| \begin{matrix} 1-\nu, \frac{1}{2}-\nu \\ 0, -2\nu \end{matrix}\right.\right) - (z-1)^{2\nu}\, G_{2,2}^{1,2}\left(\frac{2}{z-1} \left| \begin{matrix} \nu+1, \nu+\frac{1}{2} \\ 0, 2\nu \end{matrix}\right.\right)\right) /; z \notin (-1, 1) \bigwedge 2\nu \notin \mathbb{Z}$$

Classical cases involving algebraic functions

07.04.26.0008.01

$$(z+1)^{-\nu}\, T_\nu\left(\frac{1-z}{1+z}\right) == \frac{2^{2\nu-1}}{\Gamma(2\nu)}\, G_{2,2}^{1,2}\left(z \left| \begin{matrix} \frac{1}{2}-\nu, 1-\nu \\ 0, \frac{1}{2} \end{matrix}\right.\right) /; z \notin (-\infty, -1)$$

07.04.26.0009.01

$$(z+1)^{-\nu}\, T_\nu\left(\frac{z-1}{z+1}\right) = \frac{2^{2\nu-1}}{\Gamma(2\nu)}\, G_{2,2}^{2,1}\left(z \,\middle|\, \begin{array}{c} 1-\nu,\ \frac{1}{2}-\nu \\ 0,\ \frac{1}{2} \end{array}\right) /;\ z \notin (-1, 0)$$

07.04.26.0010.01

$$(z+1)^{-\frac{\nu}{2}}\, T_\nu\left(\frac{1}{\sqrt{z+1}}\right) = \frac{2^{\nu-1}}{\Gamma(\nu)}\, G_{2,2}^{1,2}\left(z \,\middle|\, \begin{array}{c} 1-\frac{\nu}{2},\ \frac{1-\nu}{2} \\ 0,\ \frac{1}{2} \end{array}\right)$$

07.04.26.0011.01

$$(z+1)^{-\frac{\nu}{2}}\, T_\nu\left(\sqrt{\frac{z}{z+1}}\right) = \frac{2^{\nu-1}}{\Gamma(\nu)}\, G_{2,2}^{2,1}\left(z \,\middle|\, \begin{array}{c} 1-\frac{\nu}{2},\ \frac{1-\nu}{2} \\ 0,\ \frac{1}{2} \end{array}\right) /;\ z \notin (-1, 0)$$

Classical cases involving unit step θ

07.04.26.0012.01

$$\frac{\theta(1-|z|)}{\sqrt{1-z}}\, T_\nu(2z-1) = \sqrt{\pi}\, G_{2,2}^{2,0}\left(z \,\middle|\, \begin{array}{c} \frac{1}{2}-\nu,\ \nu+\frac{1}{2} \\ 0,\ \frac{1}{2} \end{array}\right) /;\ z \notin (-1, 0)$$

07.04.26.0013.01

$$\frac{\theta(|z|-1)}{\sqrt{z-1}}\, T_\nu(2z-1) = \sqrt{\pi}\, G_{2,2}^{0,2}\left(z \,\middle|\, \begin{array}{c} \frac{1}{2}-\nu,\ \nu+\frac{1}{2} \\ 0,\ \frac{1}{2} \end{array}\right)$$

07.04.26.0014.01

$$\frac{\theta(1-|z|)}{\sqrt{1-z}}\, T_\nu\left(\frac{2}{z}-1\right) = \sqrt{\pi}\, G_{2,2}^{2,0}\left(z \,\middle|\, \begin{array}{c} 0,\ \frac{1}{2} \\ -\nu,\ \nu \end{array}\right)$$

07.04.26.0015.01

$$\frac{\theta(|z|-1)}{\sqrt{z-1}}\, T_\nu\left(\frac{2}{z}-1\right) = \sqrt{\pi}\, G_{2,2}^{0,2}\left(z \,\middle|\, \begin{array}{c} 0,\ \frac{1}{2} \\ -\nu,\ \nu \end{array}\right) /;\ z \notin (-\infty, -1)$$

07.04.26.0016.01

$$\frac{\theta(|z|-1)}{\sqrt{z-1}}\, T_\nu\left(8z^2-8z+1\right) = \sqrt{\pi}\, G_{2,2}^{0,2}\left(z \,\middle|\, \begin{array}{c} \frac{1}{2}-2\nu,\ 2\nu+\frac{1}{2} \\ 0,\ \frac{1}{2} \end{array}\right) /;\ \mathrm{Re}(z) > 0$$

07.04.26.0017.01

$$\frac{\theta(1-|z|)}{\sqrt{1-z}}\, T_\nu\left(\frac{8}{z^2}-\frac{8}{z}+1\right) = \sqrt{\pi}\, G_{2,2}^{2,0}\left(z \,\middle|\, \begin{array}{c} 0,\ \frac{1}{2} \\ -2\nu,\ 2\nu \end{array}\right) /;\ \mathrm{Re}(z) > 0$$

Generalized cases involving algebraic functions

07.04.26.0018.01

$$\left(z^2+1\right)^{-\frac{\nu}{2}}\, T_\nu\left(\frac{z}{\sqrt{z^2+1}}\right) = \frac{2^{\nu-1}}{\Gamma(\nu)}\, G_{2,2}^{2,1}\left(z, \frac{1}{2} \,\middle|\, \begin{array}{c} 1-\frac{\nu}{2},\ \frac{1-\nu}{2} \\ 0,\ \frac{1}{2} \end{array}\right) /;\ \mathrm{Re}(z) > 0$$

Generalized cases involving unit step θ

07.04.26.0019.01

$$\frac{\theta(1-|z|)}{\sqrt{1-z^2}}\,T_\nu(z) \equiv \sqrt{\pi}\,G_{2,2}^{2,0}\left(z,\frac{1}{2}\,\middle|\,\begin{matrix}\frac{\nu+1}{2},\frac{1-\nu}{2}\\ 0,\frac{1}{2}\end{matrix}\right)\,/;\,\nu\notin\mathbb{Z}$$

07.04.26.0020.01

$$\frac{\theta(|z|-1)}{\sqrt{z^2-1}}\,T_\nu(z) \equiv \sqrt{\pi}\,\sqrt{z^2}\,G_{2,2}^{0,2}\left(z,\frac{1}{2}\,\middle|\,\begin{matrix}-\frac{\nu}{2},\frac{\nu}{2}\\ -\frac{1}{2},0\end{matrix}\right)$$

07.04.26.0021.01

$$\frac{\theta(1-|z|)}{\sqrt{1-z^2}}\,T_\nu\left(\frac{1}{z}\right) \equiv \sqrt{\pi}\,G_{2,2}^{2,0}\left(z,\frac{1}{2}\,\middle|\,\begin{matrix}0,\frac{1}{2}\\ -\frac{\nu}{2},\frac{\nu}{2}\end{matrix}\right)$$

07.04.26.0022.01

$$\frac{\theta(|z|-1)}{\sqrt{z^2-1}}\,T_\nu\left(\frac{1}{z}\right) \equiv \sqrt{\pi}\,\sqrt{z^2}\,G_{2,2}^{0,2}\left(z,\frac{1}{2}\,\middle|\,\begin{matrix}-\frac{1}{2},0\\ -\frac{\nu+1}{2},\frac{\nu-1}{2}\end{matrix}\right)$$

Through other functions

Involving some hypergeometric-type functions

07.04.26.0023.01

$$T_\nu(z) \equiv \sqrt{\frac{\pi}{2}}\,\sqrt[4]{1-z^2}\,P_{\nu-\frac{1}{2}}^{\frac{1}{2}}(z)$$

07.04.26.0024.01

$$T_\nu(z) \equiv \sqrt{\frac{\pi}{2}}\,\sqrt[4]{z-1}\,\sqrt[4]{z+1}\,\mathfrak{P}_{\nu-\frac{1}{2}}^{\frac{1}{2}}(z)$$

07.04.26.0025.01

$$T_\nu(z) \equiv \frac{\nu!}{\left(\frac{1}{2}\right)_\nu}\,P_\nu^{\left(-\frac{1}{2},-\frac{1}{2}\right)}(z)$$

07.04.26.0026.01

$$T_\nu(z) \equiv \frac{P_\nu^{\left(-\frac{1}{2},-\frac{1}{2}\right)}(z)}{P_\nu^{\left(-\frac{1}{2},-\frac{1}{2}\right)}(1)}$$

Involving spheroidal functions

07.04.26.0033.01

$$T_\nu(z) = \frac{\sqrt{2}\,\sqrt[4]{1-z^2}}{\sqrt{\pi}}\,PS_{\nu-\frac{1}{2},\frac{1}{2}}(0,z)$$

Representations through equivalent functions

With related functions

07.04.27.0001.01

$$T_\nu(z) == \frac{\nu}{2} C_\nu^{(0)}(z)$$

07.04.27.0002.01

$$T_\nu(z) == U_\nu(z) - z\, U_{\nu-1}(z)$$

07.04.27.0003.01

$$T_\nu(z) == \frac{1}{2} (U_\nu(z) - U_{\nu-2}(z))$$

07.04.27.0004.01

$$T_\nu(z) == \frac{1}{\nu} \left(\frac{\partial\left(\left(z^2-1\right) U_{\nu-1}(z)\right)}{\partial z} - z\, U_{\nu-1}(z) \right)$$

07.04.27.0005.01

$$T_n(x) == -\frac{1}{\pi} \mathcal{P} \int_{-1}^{1} \frac{\sqrt{1-t^2}\; U_{n-1}(t)}{t-x} \, dt \;/;\; n \in \mathbb{N}^+ \bigwedge -1 < x < 1$$

With elementary functions

07.04.27.0007.01

$$T_\nu(z) == \cos\left(\nu \cos^{-1}(z)\right)$$

07.04.27.0010.01

$$T_\nu(z) == \cos\left(\frac{\pi \nu}{2}\right) \cos\left(\nu \sin^{-1}(z)\right) + \sin\left(\frac{\pi \nu}{2}\right) \sin\left(\nu \sin^{-1}(z)\right)$$

07.04.27.0011.01

$$T_\nu(z) == \cos\left(2 \nu \sin^{-1}\left(\frac{\sqrt{1-z}}{\sqrt{2}}\right)\right)$$

07.04.27.0012.01

$$T_\nu(z) == \cos\left(\nu \left(\pi - 2 \sin^{-1}\left(\frac{\sqrt{z+1}}{\sqrt{2}}\right)\right)\right)$$

07.04.27.0006.01

$$T_\nu(z) == \frac{1}{2} \left(e^{\frac{i \pi \nu}{2}} \left(i z + \sqrt{1-z^2}\right)^{-\nu} + e^{-\frac{i \pi \nu}{2}} \left(i z + \sqrt{1-z^2}\right)^{\nu} \right)$$

07.04.27.0013.01

$$T_\nu(z) == \frac{1}{2} (-z)^{-\frac{\nu+1}{2}} z^{-\frac{\nu}{2}} \left(\frac{\sqrt{z^2}\, \sqrt{z^2-1}}{z^2} + 1 \right)^{-\nu} \left(\cos\left(\frac{\pi \nu}{2}\right) \left(z^\nu (-z)^{\nu+\frac{1}{2}} \left(\frac{\sqrt{z^2}\, \sqrt{z^2-1}}{z^2} + 1 \right)^{2\nu} + \sqrt{-z} \right) + \right.$$

$$\left. \sin\left(\frac{\pi \nu}{2}\right) \left((-z)^\nu z^{\nu+\frac{1}{2}} \left(\frac{\sqrt{z^2}\, \sqrt{z^2-1}}{z^2} + 1 \right)^{2\nu} - \sqrt{z} \right) \right) /;\; z \notin (-1,\, 0)$$

07.04.27.0014.01

$$T_\nu(z) == 2^{\nu-1} (z-1)^{-\nu} \left(1 + \sqrt{\frac{z+1}{z-1}}\right)^{-2\nu} + 2^{-\nu-1} (z-1)^\nu \left(1 + \sqrt{\frac{z+1}{z-1}}\right)^{2\nu} /;\; z \notin (-1,\, 1)$$

07.04.27.0008.01

$$T_n(z) == 2^{-n-1} (-1)^n \left(\left(\sqrt{1-z} - \sqrt{-z-1} \right)^{2n} + \left(\sqrt{1-z} + \sqrt{-z-1} \right)^{2n} \right) /; n \in \mathbb{N}$$

07.04.27.0009.01

$$T_n(z) == \frac{1}{2} z^n \left(\left(1 - \sqrt{1 - \frac{1}{z^2}} \right)^n + \left(1 + \sqrt{1 - \frac{1}{z^2}} \right)^n \right) /; n \in \mathbb{N}$$

Theorems

Expansions in generalized Fourier series

$$f(x) == \sum_{k=0}^{\infty} c_k \, \psi_k(x) /; \quad c_k == \int_{-1}^{1} f(t) \, \psi_k(t) \, dt, \ \psi_k(x) == \sqrt{\frac{2}{\pi}} \left(1 - x^2\right)^{-\frac{1}{4}} T_k(x), \ k \in \mathbb{N}.$$

Minimizing property of Chebyshev polynomials

Chebyshev polynomials $T_n(x)$ have the smallest absolute values among all polynomials of degree n with leading coefficient 1:

$$\max_{-1 \le x \le 1} \left| \sum_{k=0}^{n} a_i \, x^i \right| \ge \max_{-1 \le x \le 1} \left| \frac{T_n(x)}{[x^n](T_n(x))} \right|.$$

Another minimizing property of Chebyshev polynomials

$$\int_{-1}^{1} \frac{P(x)^2}{\sqrt{1-x^2}} \, dx \ge \int_{-1}^{1} \frac{T_n(x)^2}{\sqrt{1-x^2}} \, dx == \frac{\pi}{2} /; \qquad P(x) == 2^{n-1} x^n + \sum_{k=0}^{n-1} c_i \, x^i \bigwedge c_i \in \mathbb{R}$$

Distribution of the zeros of Chebyshev polynomials of high order

In the limit $n \to \infty$, the zeros of $T_n(x)$ fulfill the arcsin distribution; in other words, the relative number of zeros $m_n(a, b)/n$ in the interval (a, b) is $(\arcsin(b) - \arcsin(a))/\pi$.

History

– P. L. Chebyshev (1855, 1859)

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