ChebyshevUGeneral

Notations

Traditional name

Chebyshev function of the second kind

Traditional notation

$U_\nu(z)$

Mathematica StandardForm notation

`ChebyshevU[ν, z]`

Primary definition

07.05.02.0001.01

$$U_\nu(z) == \frac{\sin\left((\nu+1)\cos^{-1}(z)\right)}{\sqrt{1-z^2}}$$

Specific values

Specialized values

For fixed ν

07.05.03.0001.01

$$U_\nu(0) == \cos\left(\frac{\pi\,\nu}{2}\right)$$

07.05.03.0002.01

$$U_\nu(1) == 1+\nu$$

07.05.03.0003.01

$$U_\nu(-1) == (-1)^\nu\,(1+\nu)$$

For fixed z

07.05.03.0004.01

$$U_{-\frac{1}{2}}(z) == \frac{1}{\sqrt{2}\,\sqrt{z+1}}$$

07.05.03.0005.01

$$U_{\frac{1}{2}}(z) = \frac{2z+1}{\sqrt{2}\sqrt{z+1}}$$

07.05.03.0006.01

$$U_0(z) = 1$$

07.05.03.0007.01

$$U_1(z) = 2z$$

07.05.03.0008.01

$$U_2(z) = 4z^2 - 1$$

07.05.03.0009.01

$$U_3(z) = 8z^3 - 4z$$

07.05.03.0010.01

$$U_4(z) = 16z^4 - 12z^2 + 1$$

07.05.03.0011.01

$$U_5(z) = 32z^5 - 32z^3 + 6z$$

07.05.03.0012.01

$$U_6(z) = 64z^6 - 80z^4 + 24z^2 - 1$$

07.05.03.0013.01

$$U_7(z) = 128z^7 - 192z^5 + 80z^3 - 8z$$

07.05.03.0014.01

$$U_8(z) = 256z^8 - 448z^6 + 240z^4 - 40z^2 + 1$$

07.05.03.0015.01

$$U_9(z) = 512z^9 - 1024z^7 + 672z^5 - 160z^3 + 10z$$

07.05.03.0016.01

$$U_{10}(z) = 1024z^{10} - 2304z^8 + 1792z^6 - 560z^4 + 60z^2 - 1$$

07.05.03.0017.01

$$U_n(z) = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(-1)^k (n-k)! (2z)^{n-2k}}{k! (n-2k)!} /; n \in \mathbb{N}$$

07.05.03.0018.01

$$U_n(z) = (-i)^n F_{n+1}(2iz) /; n \in \mathbb{N}$$

07.05.03.0019.01

$$U_n\left(-\frac{i}{2}\right) = (-i)^n F_{n+1} /; n \in \mathbb{N}$$

07.05.03.0020.01

$$U_n(z) = \frac{(-1)^n}{\sqrt{1-z^2}} \sin\left(2(n+1)\sin^{-1}\left(\frac{\sqrt{z+1}}{\sqrt{2}}\right)\right) /; n \in \mathbb{N}$$

General characteristics

Domain and analyticity

$U_\nu(z)$ is an analytical function of ν and z which is defined over $\mathbb{C}^2$. For fixed z the function $U_\nu(z)$ is an entire function of ν. For integer ν, $U_\nu(z)$ degenerates to a polynomial in z.

07.05.04.0001.01

$$(\nu * z) \longrightarrow U_\nu(z) :: (\mathbb{C} \otimes \mathbb{C}) \longrightarrow \mathbb{C}$$

Symmetries and periodicities

Parity

07.05.04.0002.01

$$U_n(-z) == (-1)^n U_n(z) /; n \in \mathbb{N}$$

07.05.04.0003.01

$$U_{-\nu-2}(z) == -U_\nu(z)$$

Mirror symmetry

07.05.04.0004.02

$$U_{\bar{\nu}}(\bar{z}) == \overline{U_\nu(z)} /; z \notin (-\infty, -1)$$

Periodicity

No periodicity

Poles and essential singularities

With respect to z

For fixed ν /; $\nu \notin \mathbb{Z}$, the function $U_\nu(z)$ does not have poles and essential singularities.

07.05.04.0005.01

$$\mathcal{S}ing_z(U_\nu(z)) == \{\} /; \nu \notin \mathbb{Z}$$

For integer ν, the function $U_\nu(z)$ is polynomial and has pole of order ν (for $\nu > 0$) or $-\nu - 2$(for $\nu < -2$) at $z = \tilde{\infty}$.

07.05.04.0006.01

$$\mathcal{S}ing_z(U_\nu(z)) == \{\{\tilde{\infty}, \nu\}\} /; \nu \in \mathbb{N}^+$$

07.05.04.0007.01

$$\mathcal{S}ing_z(U_\nu(z)) == \{\{\tilde{\infty}, -\nu - 2\}\} /; -\nu - 2 \in \mathbb{N}^+$$

With respect to ν

For fixed z, the function $U_\nu(z)$ has only one singular point at $\nu = \tilde{\infty}$. It is an essential singular point.

07.05.04.0008.01

$$\mathcal{S}ing_\nu(U_\nu(z)) == \{\{\tilde{\infty}, \infty\}\}$$

Branch points

With respect to z

For fixed noninteger ν, the function $U_\nu(z)$ has two branch points: $z == -1$, $z == \tilde{\infty}$.

For fixed integer ν, the function $U_\nu(z)$ does not have branch points.

07.05.04.0009.01

$\mathcal{BP}_z(U_\nu(z)) == \{-1, \tilde{\infty}\} /; \nu \notin \mathbb{Z}$

07.05.04.0010.01

$\mathcal{BP}_z(U_\nu(z)) == \{\} /; \nu \in \mathbb{Z}$

07.05.04.0011.01

$\mathcal{R}_z(U_\nu(z), -1) == 2 /; \nu \notin \mathbb{Z}$

07.05.04.0012.01

$\mathcal{R}_z(U_\nu(z), \tilde{\infty}) == \log /; \nu \notin \mathbb{Q}$

07.05.04.0013.01

$\mathcal{R}_z(U_\nu(z), \tilde{\infty}) == s /; \nu == \frac{r}{s} \bigwedge \{r, s\} \in \mathbb{Z} \bigwedge s > 1 \bigwedge \gcd(r, s) == 1$

With respect to ν

For fixed z, the function $U_\nu(z)$ does not have branch points.

07.05.04.0014.01

$\mathcal{BP}_\nu(U_\nu(z)) == \{\}$

Branch cuts

With respect to z

For fixed noninteger ν, the function $U_\nu(z)$ is a single-valued function on the z-plane cut along the interval $(-\infty, -1)$ where it is continuous from above.

07.05.04.0015.01

$\mathcal{BC}_z(U_\nu(z)) == \{\{(-\infty, -1), -i\}\} /; \nu \notin \mathbb{Z}$

07.05.04.0016.01

$\mathcal{BC}_z(U_n(z)) == \{\} /; n \in \mathbb{Z}$

07.05.04.0017.01

$\lim_{\epsilon \to +0} U_\nu(x + i\,\epsilon) == U_\nu(x) /; x < -1$

07.05.04.0018.01

$\lim_{\epsilon \to +0} U_\nu(x - i\,\epsilon) == 2\cos(\nu\,\pi)\, U_\nu(-x) - U_\nu(x) /; x < -1$

With respect to ν

For fixed z, the function $U_\nu(z)$ does not have branch cuts.

07.05.04.0019.01

$\mathcal{BC}_\nu(U_\nu(z)) == \{\}$

Series representations

Generalized power series

Expansions at generic point $\nu == \nu_0$

For the function itself

07.05.06.0033.01

$$U_\nu(z) \propto \frac{1}{\sqrt{1-z^2}} \left(\sin\left(\cos^{-1}(z)(\nu_0+1)\right) + \cos^{-1}(z)\cos\left(\cos^{-1}(z)(\nu_0+1)\right)(\nu-\nu_0) - \frac{1}{2}\cos^{-1}(z)^2 \sin\left(\cos^{-1}(z)(\nu_0+1)\right)(\nu-\nu_0)^2 \right) + \ldots /; (\nu \to \nu_0)$$

07.05.06.0034.01

$$U_\nu(z) \propto U_{\nu_0}(z) + \frac{\cos^{-1}(z)}{\sqrt{1-z^2}} T_{\nu_0+1}(z)(\nu-\nu_0) - \frac{1}{2}\cos^{-1}(z)^2 U_{\nu_0}(z)(\nu-\nu_0)^2 + \ldots /; (\nu \to \nu_0)$$

07.05.06.0035.01

$$U_\nu(z) \propto \frac{1}{\sqrt{1-z^2}} \left(\sin\left(\cos^{-1}(z)(\nu_0+1)\right) + \cos^{-1}(z)\cos\left(\cos^{-1}(z)(\nu_0+1)\right)(\nu-\nu_0) - \frac{1}{2}\cos^{-1}(z)^2 \sin\left(\cos^{-1}(z)(\nu_0+1)\right)(\nu-\nu_0)^2 \right) + O\left((\nu-\nu_0)^3\right)$$

07.05.06.0036.01

$$U_\nu(z) == \frac{1}{\sqrt{1-z^2}} \sum_{k=0}^{\infty} \frac{\cos^{-1}(z)^k}{k!} \sin\left(\frac{\pi k}{2} + \cos^{-1}(z)(\nu_0+1)\right)(\nu-\nu_0)^k$$

07.05.06.0037.01

$$U_\nu(z) == \frac{1}{\sqrt{1-z^2}} \sum_{k=0}^{\infty} \frac{i^k \cos^{-1}(z)^k}{k!} \left(\sqrt{1-z^2}\, U_{\nu_0}(z)(1-k \bmod 2) - i\, T_{\nu_0+1}(z)(k \bmod 2)\right)(\nu-\nu_0)^k$$

07.05.06.0038.01

$$T_\nu(z) \propto T_{\nu_0}(z)(1+O(\nu-\nu_0))$$

Expansions at $\nu == 0$

For the function itself

07.05.06.0001.02

$$U_\nu(z) \propto 1 + \frac{z\cos^{-1}(z)}{\sqrt{1-z^2}}\nu - \frac{\cos^{-1}(z)^2}{2}\nu^2 - \ldots /; (\nu \to 0)$$

07.05.06.0039.01

$$U_\nu(z) \propto 1 + \frac{z\cos^{-1}(z)}{\sqrt{1-z^2}}\nu - \frac{\cos^{-1}(z)^2}{2}\nu^2 - O\left(\nu^3\right)$$

07.05.06.0002.01

$$U_\nu(z) == \frac{1}{\sqrt{1-z^2}} \sum_{k=0}^{\infty} \frac{(-1)^k \sin\left(\cos^{-1}(z) - \frac{k\pi}{2}\right)\cos^{-1}(z)^k \nu^k}{k!}$$

07.05.06.0040.01

$$U_\nu(z) = {}_0F_1\left(;\frac{1}{2};-\frac{\cos^{-1}(z)^2\,\nu^2}{4}\right)+\frac{\nu\,z\cos^{-1}(z)}{\sqrt{1-z^2}}\,{}_0F_1\left(;\frac{3}{2};-\frac{\cos^{-1}(z)^2\,\nu^2}{4}\right)$$

07.05.06.0003.02

$$U_\nu(z) \propto 1 + O(\nu)$$

07.05.06.0041.01

$$U_\nu(z) = F_\infty(z,\nu) /; \left(\left(F_n(z,\nu) = \frac{1}{\sqrt{1-z^2}}\sum_{k=0}^{n}\frac{(-1)^k \sin\left(\cos^{-1}(z)-\frac{k\pi}{2}\right)\cos^{-1}(z)^k\,\nu^k}{k!} = \frac{e^{-i\nu\cos^{-1}(z)}}{2\,n!\,\sqrt{1-z^2}}\left(\left(i\,z+\sqrt{1-z^2}\right)\Gamma\left(n+1,-i\,\nu\cos^{-1}(z)\right)+e^{2i\nu\cos^{-1}(z)}\left(\sqrt{1-z^2}-i\,z\right)\Gamma\left(n+1,i\,\nu\cos^{-1}(z)\right)\right)\right)\bigwedge n\in\mathbb{N}\right)$$

Summed form of the truncated series expansion.

Expansions at generic point $z = z_0$

For the function itself

07.05.06.0042.01

$$U_\nu(z) \propto -\frac{\sin(\pi\nu)}{\sqrt{1-z_0^2}}\left(\frac{1}{z_0+1}\right)^{-\frac{1}{2}\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor}(z_0+1)^{-\frac{1}{2}\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor}T_{\nu+1}(-z_0)+ \cos(\pi\nu)\left(2\,i\,i^{-\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor}\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor\left\lfloor\frac{\arg(z_0+1)+\pi}{2\pi}\right\rfloor+\left(\frac{1}{z_0+1}\right)^{-\frac{1}{2}\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor}(z_0+1)^{-\frac{1}{2}\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor}\right)U_\nu(-z_0) - \left(\frac{\sin(\pi\nu)}{\left(1-z_0^2\right)^{3/2}}\left(\frac{1}{z_0+1}\right)^{-\frac{1}{2}\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor}(z_0+1)^{-\frac{1}{2}\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor}\left(z_0\,T_{\nu+1}(-z_0)+(\nu+1)\left(z_0^2-1\right)U_\nu(-z_0)\right)+ \frac{1}{z_0^2-1}\cos(\pi\nu)\left(2\,i\,i^{-\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor}\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor\left\lfloor\frac{\arg(z_0+1)+\pi}{2\pi}\right\rfloor+\left(\frac{1}{z_0+1}\right)^{-\frac{1}{2}\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor}(z_0+1)^{-\frac{1}{2}\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor}\right) ((\nu+1)\,T_{\nu+1}(-z_0)+z_0\,U_\nu(-z_0))\right)(z-z_0)+\ldots /; (z\to z_0)$$

07.05.06.0043.01

$$U_\nu(z) \propto -\frac{\sin(\pi\nu)}{\sqrt{1-z_0^2}} \left(\frac{1}{z_0+1}\right)^{-\frac{1}{2}\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor} (z_0+1)^{-\frac{1}{2}\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor} T_{\nu+1}(-z_0) + \cos(\pi\nu)\left(2\,i\,i^{-\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor}\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor\left\lfloor\frac{\arg(z_0+1)+\pi}{2\pi}\right\rfloor + \left(\frac{1}{z_0+1}\right)^{-\frac{1}{2}\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor}(z_0+1)^{-\frac{1}{2}\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor}\right) U_\nu(-z_0) - \left(\frac{\sin(\pi\nu)}{(1-z_0^2)^{3/2}}\left(\frac{1}{z_0+1}\right)^{-\frac{1}{2}\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor}(z_0+1)^{-\frac{1}{2}\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor}\left(z_0\,T_{\nu+1}(-z_0) + (\nu+1)\left(z_0^2-1\right)U_\nu(-z_0)\right) + \frac{1}{z_0^2-1}\cos(\pi\nu)\left(2\,i\,i^{-\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor}\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor\left\lfloor\frac{\arg(z_0+1)+\pi}{2\pi}\right\rfloor + \left(\frac{1}{z_0+1}\right)^{-\frac{1}{2}\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor}(z_0+1)^{-\frac{1}{2}\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor}\right)\left((\nu+1)\,T_{\nu+1}(-z_0) + z_0\,U_\nu(-z_0)\right)\right)(z-z_0) + O\left((z-z_0)^2\right)$$

07.05.06.0044.01

$$U_\nu(z) = \frac{\sin(2\pi\nu)}{4\sqrt{\pi}} \sum_{k=0}^{\infty}\frac{1}{k!}\left(2^{-k}\,\Gamma(k-\nu)\,\Gamma(k+\nu+2)\left(-\left(\frac{1}{z_0+1}\right)^{-\frac{1}{2}\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor}(z_0+1)^{-\frac{1}{2}\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor} - 2\,e^{-\frac{1}{2}i\pi\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor}\,i\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor\left\lfloor\frac{\arg(z_0+1)+\pi}{2\pi}\right\rfloor\right) {}_2\tilde{F}_1\left(k-\nu,\, k+\nu+2;\, k+\frac{3}{2};\, \frac{z_0+1}{2}\right) - \pi\sec(\pi\nu)\sqrt{2}\,(z_0+1)^{-k-\frac{1}{2}}\left(\frac{1}{z_0+1}\right)^{-\frac{1}{2}\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor}(z_0+1)^{-\frac{1}{2}\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor}\,{}_2\tilde{F}_1\left(\nu+\frac{3}{2},\, -\nu-\frac{1}{2};\, \frac{1}{2}-k;\, \frac{z_0+1}{2}\right)\right)(z-z_0)^k$$

07.05.06.0045.01

$$U_\nu(z) \propto -\frac{\sin(\pi\nu)}{\sqrt{1-z_0^2}} \left(\frac{1}{z_0+1}\right)^{-\frac{1}{2}\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor} (z_0+1)^{-\frac{1}{2}\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor} T_{\nu+1}(-z_0) + \cos(\pi\nu)\left(2\,i\,i^{-\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor}\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor\left\lfloor\frac{\arg(z_0+1)+\pi}{2\pi}\right\rfloor + \left(\frac{1}{z_0+1}\right)^{-\frac{1}{2}\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor}(z_0+1)^{-\frac{1}{2}\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor}\right) U_\nu(-z_0) + O(z-z_0)$$

Expansions on branch cuts

For the function itself

07.05.06.0046.01

$$U_\nu(z) \propto \cos(\pi\nu)\left(2\,i\,i^{-\left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor + e^{\pi i \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor}\right) U_\nu(-x) - \frac{\sin(\pi\nu)}{\sqrt{1-x^2}}\, e^{\pi i \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} T_{\nu+1}(-x) - \left(\frac{\cos(\pi\nu)}{x^2-1}\left(2\,i\,i^{-\left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor + e^{\pi i \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor}\right)((\nu+1)\,T_{\nu+1}(-x) + x\,U_\nu(-x)) + \frac{\sin(\pi\nu)}{\left(1-x^2\right)^{3/2}}\, e^{\pi i \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor}\left(x\,T_{\nu+1}(-x) + \left(x^2-1\right)(\nu+1)\,U_\nu(-x)\right)\right)(z-x) + \ldots \;/;\; (z \to x) \land x \in \mathbb{R} \land x < -1$$

07.05.06.0047.01

$$U_\nu(z) \propto \cos(\pi\nu)\left(2\,i\,i^{-\left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor + e^{\pi i \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor}\right) U_\nu(-x) - \frac{\sin(\pi\nu)}{\sqrt{1-x^2}}\, e^{\pi i \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} T_{\nu+1}(-x) - \left(\frac{\cos(\pi\nu)}{x^2-1}\left(2\,i\,i^{-\left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor + e^{\pi i \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor}\right)((\nu+1)\,T_{\nu+1}(-x) + x\,U_\nu(-x)) + \frac{\sin(\pi\nu)}{\left(1-x^2\right)^{3/2}}\, e^{\pi i \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor}\left(x\,T_{\nu+1}(-x) + \left(x^2-1\right)(\nu+1)\,U_\nu(-x)\right)\right)(z-x) + O\left((z-x)^2\right) \;/;\; x \in \mathbb{R} \land x < -1$$

07.05.06.0048.01

$$U_\nu(z) = -\frac{\sin(2\pi\nu)}{4\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{1}{k!}\left(\sqrt{2}\,\pi\,\sec(\pi\nu)\, e^{\pi i \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} (x+1)^{-k-\frac{1}{2}}\, {}_2\tilde{F}_1\left(\nu+\frac{3}{2}, -\nu-\frac{1}{2}; \frac{1}{2}-k; \frac{x+1}{2}\right) + 2^{-k}\,\Gamma(k-\nu)\,\Gamma(k+\nu+2) \left(e^{\pi i \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} + 2\,i\,e^{-\frac{1}{2} i \pi \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor\right) {}_2\tilde{F}_1\left(k-\nu, k+\nu+2; k+\frac{3}{2}; \frac{x+1}{2}\right)\right)(z-x)^k \;/;\; x \in \mathbb{R} \land x < -1$$

07.05.06.0049.01

$$U_\nu(z) \propto \cos(\pi\nu)\left(2\,i\,i^{-\left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor + e^{\pi i \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor}\right) U_\nu(-x) - \frac{\sin(\pi\nu)}{\sqrt{1-x^2}}\, e^{\pi i \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} T_{\nu+1}(-x) + O(z-x) \;/;\; x \in \mathbb{R} \land x < -1$$

Expansions at $z = 0$

For the function itself

General case

07.05.06.0004.02

$$U_\nu(z) \propto \cos\left(\frac{\pi\nu}{2}\right) + (\nu+1)\sin\left(\frac{\pi\nu}{2}\right) z - \nu(\nu+2)\cos\left(\frac{\pi\nu}{2}\right) z^2 + \ldots \;/;\; (z \to 0)$$

07.05.06.0050.01

$$U_\nu(z) \propto \cos\left(\frac{\pi\nu}{2}\right) + (\nu+1)\sin\left(\frac{\pi\nu}{2}\right) z - \nu(\nu+2)\cos\left(\frac{\pi\nu}{2}\right) z^2 + O\left(z^3\right)$$

07.05.06.0005.01

$$U_\nu(z) = (\nu+1) \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-\nu)_{j+k}\,(\nu+2)_{j+k}\,(-z)^j}{\left(\frac{3}{2}\right)_{j+k}\, j!\, k!\, 2^{j+k}} \;/;\; |z| < 1$$

07.05.06.0006.01

$$U_\nu(z) == \frac{\sqrt{\pi}}{2}(\nu+1)\sum_{k=0}^{\infty}\sum_{j=0}^{k}\frac{(-\nu)_k(\nu+2)_k(-z)^j}{\Gamma\left(k+\frac{3}{2}\right)j!\,(k-j)!\,2^k} /; |z|<1$$

07.05.06.0007.01

$$U_\nu(z) == (\nu+1)\,F_{1\times0\times0}^{2\times0\times0}\left(\begin{matrix}-\nu,\,\nu+2;;; \\ \frac{3}{2};;;\end{matrix}\ \frac{1}{2},\,-\frac{z}{2}\right)$$

07.05.06.0051.01

$$U_\nu(z) == \cos\left(\frac{\pi\nu}{2}\right)\sum_{k=0}^{\infty}\frac{\left(-\frac{\nu}{2}\right)_k\left(\frac{\nu}{2}+1\right)_k}{\left(\frac{1}{2}\right)_k k!}z^{2k} + z(\nu+1)\sin\left(\frac{\pi\nu}{2}\right)\sum_{k=0}^{\infty}\frac{\left(\frac{1-\nu}{2}\right)_k\left(\frac{\nu+3}{2}\right)_k}{\left(\frac{3}{2}\right)_k k!}z^{2k} /; |z|<1$$

07.05.06.0052.01

$$U_\nu(z) == \cos\left(\frac{\pi\nu}{2}\right){}_2F_1\left(-\frac{\nu}{2},\,\frac{\nu}{2}+1;\,\frac{1}{2};\,z^2\right) + z(\nu+1)\sin\left(\frac{\pi\nu}{2}\right){}_2F_1\left(\frac{1-\nu}{2},\,\frac{\nu+3}{2};\,\frac{3}{2};\,z^2\right)$$

07.05.06.0053.01

$$U_\nu(z) == \frac{\cos\left(\frac{\pi\nu}{2}-(\nu+1)\sin^{-1}(z)\right)}{\sqrt{1-z^2}}$$

07.05.06.0054.01

$$U_\nu(z) == \cos\left(\nu\cos^{-1}(z)\right) + \frac{z}{\sqrt{1-z^2}}\sin\left(\nu\cos^{-1}(z)\right)$$

07.05.06.0055.01

$$U_\nu(z) == \frac{1}{\sqrt{1-z^2}}\left(\cos\left(\frac{\pi\nu}{2}\right)\cos\left((\nu+1)\sin^{-1}(z)\right) + \sin\left(\frac{\pi\nu}{2}\right)\sin\left((\nu+1)\sin^{-1}(z)\right)\right)$$

07.05.06.0056.01

$$U_\nu(z) == \cos\left(\frac{\nu}{2}\left(\pi-2\sin^{-1}(z)\right)\right) + \frac{z}{\sqrt{1-z^2}}\sin\left(\frac{\nu}{2}\left(\pi-2\sin^{-1}(z)\right)\right)$$

07.05.06.0008.02

$$U_\nu(z) \propto \cos\left(\frac{\pi\nu}{2}\right)(1+O(z))$$

07.05.06.0057.01

$$U_\nu(z) == F_\infty(z,\nu) /; \left(\left(F_m(z,\nu) == \cos\left(\frac{\pi\nu}{2}\right)\sum_{k=0}^{m}\frac{\left(-\frac{\nu}{2}\right)_k\left(\frac{\nu}{2}+1\right)_k}{\left(\frac{1}{2}\right)_k k!}z^{2k} + z(\nu+1)\sin\left(\frac{\pi\nu}{2}\right)\sum_{k=0}^{m}\frac{\left(\frac{1-\nu}{2}\right)_k\left(\frac{\nu+3}{2}\right)_k}{\left(\frac{3}{2}\right)_k k!}z^{2k} == \right.\right.$$

$$U_\nu(z) - \frac{z^{2m+2}\cos\left(\frac{\pi\nu}{2}\right)\left(\frac{\nu}{2}+1\right)_{m+1}\left(-\frac{\nu}{2}\right)_{m+1}}{(m+1)!\left(\frac{1}{2}\right)_{m+1}}\,{}_3F_2\left(1,\,m-\frac{\nu}{2}+1,\,m+\frac{\nu}{2}+2;\,m+\frac{3}{2},\,m+2;\,z^2\right) -$$

$$\left.\left.\frac{z^{2m+3}(\nu+1)\sin\left(\frac{\pi\nu}{2}\right)\left(\frac{1-\nu}{2}\right)_{m+1}\left(\frac{\nu+3}{2}\right)_{m+1}}{(m+1)!\left(\frac{3}{2}\right)_{m+1}}\,{}_3F_2\left(1,\,m-\frac{\nu}{2}+\frac{3}{2},\,m+\frac{\nu}{2}+\frac{5}{2};\,m+2,\,m+\frac{5}{2};\,z^2\right)\right)\bigwedge m\in\mathbb{N}\right)$$

Summed form of the truncated series expansion.

Special cases

07.05.06.0009.01

$$U_n(z) = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(-1)^k (n-k)! (2z)^{n-2k}}{k! (n-2k)!} /; n \in \mathbb{N}$$

07.05.06.0010.01

$$U_n(z) \propto (-1)^{\left\lfloor \frac{n}{2} \right\rfloor} ((n+1)z)^{n-2\left\lfloor \frac{n}{2} \right\rfloor} \left(1 + O\left(z^2\right)\right) /; (z \to 0) \bigwedge n \in \mathbb{N}^+$$

Generic formulas for main term

07.05.06.0058.01

$$U_\nu(z) \propto \begin{cases} (-1)^{\left\lfloor \frac{\nu}{2} \right\rfloor} ((\nu+1)z)^{\nu-2\left\lfloor \frac{\nu}{2} \right\rfloor} & \nu \in \mathbb{Z} \\ \cos\left(\frac{\pi \nu}{2}\right) & \text{True} \end{cases} /; (z \to 0)$$

Expansions at $z = 1$

For the function itself

General case

07.05.06.0011.02

$$U_\nu(z) \propto (\nu+1)\left(1 + \frac{\nu(2+\nu)}{3}(z-1) + \frac{-\nu(1-\nu)(2+\nu)(3+\nu)}{30}(z-1)^2 + \ldots\right) /; (z \to 1)$$

07.05.06.0059.01

$$U_\nu(z) \propto (\nu+1)\left(1 + \frac{\nu(2+\nu)}{3}(z-1) + \frac{-\nu(1-\nu)(2+\nu)(3+\nu)}{30}(z-1)^2 + O\left((z-1)^3\right)\right)$$

07.05.06.0012.01

$$U_\nu(z) = (\nu+1)\sum_{k=0}^{\infty} \frac{(-\nu)_k (\nu+2)_k}{\left(\frac{3}{2}\right)_k k!} \left(\frac{1-z}{2}\right)^k /; \left|\frac{1-z}{2}\right| < 1$$

07.05.06.0013.01

$$U_\nu(z) = (\nu+1)\, {}_2F_1\left(-\nu, \nu+2; \frac{3}{2}; \frac{1-z}{2}\right)$$

07.05.06.0060.01

$$U_\nu(z) = \frac{1}{2(z-1)}\left({}_2F_1\left(-\nu-1, \nu+2; \frac{1}{2}; \frac{1-z}{2}\right) - {}_2F_1\left(-\nu, \nu+1; \frac{1}{2}; \frac{1-z}{2}\right)\right)$$

07.05.06.0061.01

$$U_\nu(z) = \frac{\sqrt{\pi}}{2(1-z)^{3/4}\sqrt[4]{z+1}}\left(P_\nu^{\frac{1}{2}}(z) - P_{\nu+1}^{\frac{1}{2}}(z)\right)$$

07.05.06.0062.01

$$U_\nu(z) = \frac{1}{\sqrt{2}\,(1-z)\sqrt{z+1}}\left({}_2F_1\left(\frac{1+2\nu}{4}, -\frac{1+2\nu}{4}; \frac{1}{2}; 1-z^2\right) - {}_2F_1\left(\frac{3+2\nu}{4}, -\frac{3+2\nu}{4}; \frac{1}{2}; 1-z^2\right)\right)$$

07.05.06.0063.01

$$U_\nu(z) = \frac{1}{\sqrt{2}\,(1-z)\sqrt{z+1}}\left(\cos\left(\frac{1}{2}(2\nu+1)\sin^{-1}\left(\sqrt{1-z^2}\right)\right) - \cos\left(\frac{1}{2}(2\nu+3)\sin^{-1}\left(\sqrt{1-z^2}\right)\right)\right)$$

07.05.06.0015.02

$$U_\nu(z) \propto (\nu+1)(1+O(z-1))$$

07.05.06.0064.01

$$U_\nu(z) = F_\infty(z,\nu) \,/;\, \left(\left(F_m(z,\nu) = (\nu+1)\sum_{k=0}^{m}\frac{(-\nu)_k(\nu+2)_k}{\left(\frac{3}{2}\right)_k k!}\left(\frac{1-z}{2}\right)^k = U_\nu(z) - \frac{2^{-m-1}(1-z)^{m+1}(\nu+1)(-\nu)_{m+1}(\nu+2)_{m+1}}{(m+1)!\left(\frac{3}{2}\right)_{m+1}}\,{}_3F_2\left(1, m-\nu+1, m+\nu+3; m+2, m+\frac{5}{2}; \frac{1-z}{2}\right)\right) \land m \in \mathbb{N}\right)$$

Summed form of the truncated series expansion.

Special cases

07.05.06.0014.01

$$U_n(z) = (n+1)\sum_{k=0}^{n}\frac{(-n)_k(n+2)_k}{\left(\frac{3}{2}\right)_k k!}\left(\frac{1-z}{2}\right)^k \,/;\, n \in \mathbb{N}$$

Expansions at $z = -1$

For the function itself

General case

07.05.06.0016.02

$$U_\nu(z) \propto (\nu+1)\cos(\pi\nu)\left(1 - \frac{\nu(2+\nu)}{3}(z+1) - \frac{\nu(1-\nu)(2+\nu)(3+\nu)}{30}(z+1)^2 - \ldots\right) - \frac{\sin(\nu\pi)}{\sqrt{2}\sqrt{z+1}}\left(1 + \left(-\frac{1}{2}-\nu\right)\left(\frac{3}{2}+\nu\right)(z+1) + \frac{1}{6}\left(-\frac{1}{2}-\nu\right)\left(\frac{1}{2}-\nu\right)\left(\frac{3}{2}+\nu\right)\left(\frac{5}{2}+\nu\right)(z+1)^2 + \ldots\right) \,/;\, (z \to -1)$$

07.05.06.0065.01

$$U_\nu(z) \propto (\nu+1)\cos(\pi\nu)\left(1 - \frac{\nu(2+\nu)}{3}(z+1) - \frac{\nu(1-\nu)(2+\nu)(3+\nu)}{30}(z+1)^2 - O\left((z+1)^3\right)\right) - \frac{\sin(\nu\pi)}{\sqrt{2}\sqrt{z+1}}\left(1 + \left(-\frac{1}{2}-\nu\right)\left(\frac{3}{2}+\nu\right)(z+1) + \frac{1}{6}\left(-\frac{1}{2}-\nu\right)\left(\frac{1}{2}-\nu\right)\left(\frac{3}{2}+\nu\right)\left(\frac{5}{2}+\nu\right)(z+1)^2 + O\left((z+1)^3\right)\right)$$

07.05.06.0017.01

$$U_\nu(z) = (\nu+1)\cos(\nu\pi)\sum_{k=0}^{\infty}\frac{(-\nu)_k(\nu+2)_k}{\left(\frac{3}{2}\right)_k k!}\left(\frac{z+1}{2}\right)^k - \frac{\sin(\nu\pi)}{\sqrt{2}\sqrt{z+1}}\sum_{k=0}^{\infty}\frac{\left(\nu+\frac{3}{2}\right)_k\left(-\nu-\frac{1}{2}\right)_k}{\left(\frac{1}{2}\right)_k k!}\left(\frac{z+1}{2}\right)^k \,/;\, \left|\frac{z+1}{2}\right| < 1$$

07.05.06.0018.01

$$U_\nu(z) = (\nu+1)\cos(\nu\pi)\,{}_2F_1\left(-\nu,\,\nu+2;\,\frac{3}{2};\,\frac{z+1}{2}\right) - \frac{\sin(\nu\pi)}{\sqrt{2}\,\sqrt{z+1}}\,{}_2F_1\left(\nu+\frac{3}{2},\,-\nu-\frac{1}{2};\,\frac{1}{2};\,\frac{z+1}{2}\right)$$

07.05.06.0066.01

$$U_\nu(z) = -\frac{1}{\sqrt{1-z^2}}\sin\left(\pi\nu - 2(\nu+1)\sin^{-1}\left(\frac{\sqrt{z+1}}{\sqrt{2}}\right)\right)$$

07.05.06.0067.01

$$U_\nu(z) = \cos\left(\nu\left(\pi - 2\sin^{-1}\left(\frac{\sqrt{z+1}}{\sqrt{2}}\right)\right)\right) + \frac{z}{\sqrt{1-z^2}}\sin\left(\nu\left(\pi - 2\sin^{-1}\left(\frac{\sqrt{z+1}}{\sqrt{2}}\right)\right)\right)$$

07.05.06.0068.01

$$U_\nu(z) = \cos(\pi\nu)\left(\cos\left(2\nu\sin^{-1}\left(\frac{\sqrt{z+1}}{\sqrt{2}}\right)\right) - \frac{z}{\sqrt{1-z^2}}\sin\left(2\nu\sin^{-1}\left(\frac{\sqrt{z+1}}{\sqrt{2}}\right)\right)\right) + \sin(\pi\nu)\left(\sin\left(2\nu\sin^{-1}\left(\frac{\sqrt{z+1}}{\sqrt{2}}\right)\right) + \frac{z}{\sqrt{1-z^2}}\cos\left(2\nu\sin^{-1}\left(\frac{\sqrt{z+1}}{\sqrt{2}}\right)\right)\right)$$

07.05.06.0019.02

$$U_\nu(z) \propto (\nu+1)\cos(\pi\nu)\,(1+O(z+1)) - \frac{\sin(\nu\pi)}{\sqrt{2}\,\sqrt{z+1}}\,(1+O(z+1))$$

07.05.06.0069.01

$$U_\nu(z) = F_\infty(z,\nu)\ /;\ \left(\left(\left(F_m(z,\nu) = (\nu+1)\cos(\nu\pi)\sum_{k=0}^{m}\frac{(-\nu)_k\,(\nu+2)_k}{\left(\frac{3}{2}\right)_k\,k!}\left(\frac{z+1}{2}\right)^k - \frac{\sin(\nu\pi)}{\sqrt{2}\,\sqrt{z+1}}\sum_{k=0}^{m}\frac{\left(\nu+\frac{3}{2}\right)_k\left(-\nu-\frac{1}{2}\right)_k}{\left(\frac{1}{2}\right)_k\,k!}\left(\frac{z+1}{2}\right)^k = U_\nu(z) - \frac{2^{-m-1}(z+1)^{m+1}(\nu+1)\cos(\pi\nu)\,(-\nu)_{m+1}\,(\nu+2)_{m+1}}{(m+1)!\left(\frac{3}{2}\right)_{m+1}}\,{}_3F_2\left(1,\,m-\nu+1,\,m+\nu+3;\,m+2,\,m+\frac{5}{2};\,\frac{z+1}{2}\right) + \frac{2^{-m-\frac{3}{2}}(z+1)^{m+\frac{1}{2}}\left(-\nu-\frac{1}{2}\right)_{m+1}\left(\nu+\frac{3}{2}\right)_{m+1}\sin(\pi\nu)}{(m+1)!\left(\frac{1}{2}\right)_{m+1}}\,{}_3F_2\left(1,\,m-\nu+\frac{1}{2},\,m+\nu+\frac{5}{2};\,m+\frac{3}{2},\,m+2;\,\frac{z+1}{2}\right)\right)\bigwedge m\in\mathbb{N}\right)$$

Summed form of the truncated series expansion.

Special cases

07.05.06.0020.01

$$U_n(z) = (n+1)(-1)^n\sum_{k=0}^{n}\frac{(-n)_k\,(n+2)_k}{\left(\frac{3}{2}\right)_k\,k!}\left(\frac{z+1}{2}\right)^k\ /;\ n\in\mathbb{N}$$

07.05.06.0021.02

$$U_n(z) \propto (1+n)(-1)^n\,(1+O(z+1))\ /;\ n\in\mathbb{N}$$

Expansions at $z = \infty$

For the function itself

Expansions in $1/z$

07.05.06.0070.01

$$U_\nu(z) \propto 2^\nu z^\nu \left(1 + \frac{1-\nu}{4 z^2} + \frac{(2-\nu)(3-\nu)}{32 z^4} + \ldots\right) - 2^{-\nu-2} z^{-\nu-2} \left(1 + \frac{3+\nu}{4 z^2} + \frac{(4+\nu)(5+\nu)}{32 z^4} + \ldots\right) /;\ (|z| \to \infty)$$

07.05.06.0071.01

$$U_\nu(z) \propto 2^\nu z^\nu \left(1 + \frac{1-\nu}{4 z^2} + \frac{(2-\nu)(3-\nu)}{32 z^4} + O\left(\frac{1}{z^6}\right)\right) + 2^{-\nu-2} \left(-z^2\right)^{-\frac{\nu}{2}-1} \left(\cos\left(\frac{\pi \nu}{2}\right) - \frac{z}{\sqrt{-z^2}} \sin\left(\frac{\pi \nu}{2}\right)\right) \left(1 + \frac{3+\nu}{4 z^2} + \frac{(4+\nu)(5+\nu)}{32 z^4} + O\left(\frac{1}{z^6}\right)\right)$$

07.05.06.0072.01

$$U_\nu(z) = 2^\nu z^\nu \sum_{k=0}^{\infty} \frac{\left(-\frac{\nu}{2}\right)_k \left(\frac{1-\nu}{2}\right)_k}{(-\nu)_k\, k!} z^{-2k} - 2^{-\nu-2} z^{-\nu-2} \sum_{k=0}^{\infty} \frac{\left(\frac{\nu+3}{2}\right)_k \left(\frac{\nu+2}{2}\right)_k}{(\nu+2)_k\, k!} z^{-2k} /;\ |z| > 1 \land \nu \notin \mathbb{Z}$$

07.05.06.0073.01

$$U_\nu(z) = 2^\nu z^\nu\, {}_2F_1\left(-\frac{\nu}{2}, \frac{1-\nu}{2}; -\nu; \frac{1}{z^2}\right) - 2^{-\nu-2} z^{-\nu-2}\, {}_2F_1\left(\frac{\nu+3}{2}, \frac{\nu+2}{2}; \nu+2; \frac{1}{z^2}\right) /;\ z \notin (-1, 0) \land \nu \notin \mathbb{Z}$$

07.05.06.0074.01

$$U_\nu(z) = \frac{1}{2\sqrt{z^2-1}} \left(z^\nu \left(\sqrt{z^2} + \sqrt{z^2-1}\right) \left(\frac{\sqrt{z^2}\sqrt{z^2-1}}{z^2} + 1\right)^\nu - \frac{1}{z^\nu \left(\sqrt{z^2} + \sqrt{z^2-1}\right)} \left(\frac{\sqrt{z^2}\sqrt{z^2-1}}{z^2} + 1\right)^{-\nu}\right) /;\ z \notin (-1, 0) \land \nu \notin \mathbb{Z}$$

07.05.06.0075.01

$$U_n(z) = 2^n z^n \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{\left(\frac{1-n}{2}\right)_k \left(-\frac{n}{2}\right)_k z^{-2k}}{k!\,(-n)_k} /;\ n \in \mathbb{N}^+$$

07.05.06.0076.01

$$U_n(z) = 2^n z^n\, {}_2F_1\left(-\frac{n}{2}, \frac{1-n}{2}; -n; \frac{1}{z^2}\right) /;\ n \in \mathbb{N}^+$$

07.05.06.0025.02

$$U_\nu(z) \propto -2^{-\nu-2} z^{-\nu-2} \left(1 + O\left(\frac{1}{z}\right)\right) + 2^\nu z^\nu \left(1 + O\left(\frac{1}{z}\right)\right) /;\ \nu \notin \mathbb{Z}$$

07.05.06.0027.02

$$U_n(z) \propto 2^n z^n \left(1 + O\left(\frac{1}{z}\right)\right) /;\ n \in \mathbb{N}$$

07.05.06.0077.01

$$U_\nu(z) = F_\infty(z, \nu) /; \left(F_m(z, \nu) = 2^\nu z^\nu \sum_{k=0}^{m} \frac{\left(-\frac{\nu}{2}\right)_k \left(\frac{1-\nu}{2}\right)_k}{(-\nu)_k \, k!} z^{-2k} - 2^{-\nu-2} z^{-\nu-2} \sum_{k=0}^{m} \frac{\left(\frac{\nu+3}{2}\right)_k \left(\frac{\nu+2}{2}\right)_k}{(\nu+2)_k \, k!} z^{-2k} = U_\nu(z) + \frac{2^{-2m-\nu-4} z^{-2m-\nu-4} \Gamma(2m+\nu+4)}{\Gamma(m+2)\Gamma(m+\nu+3)} \, {}_3F_2\left(1, m+\frac{\nu}{2}+2, m+\frac{\nu}{2}+\frac{5}{2}; m+2, m+\nu+3; \frac{1}{z^2}\right) - \frac{2^{-2m+\nu-2} z^{-2m+\nu-2} \Gamma(2m-\nu+2)}{\Gamma(m+2)\Gamma(m-\nu+1)} \, {}_3F_2\left(1, m-\frac{\nu}{2}+1, m-\frac{\nu}{2}+\frac{3}{2}; m+2, m-\nu+1; \frac{1}{z^2}\right) \bigwedge m \in \mathbb{N} \right) \bigwedge \nu \notin \mathbb{Z}$$

Summed form of the truncated series expansion.

Expansions in $1/(1-z)$

07.05.06.0022.02

$$U_\nu(z) \propto 2^{-\nu-2} (z-1)^{-\nu-2} \left(1 + \frac{\nu+2}{1-z} + \frac{(\nu+3)(2\nu+5)}{4(1-z)^2} + \ldots\right) + 2^\nu (z-1)^\nu \left(1 - \frac{\nu}{1-z} + \frac{(1-\nu)(1-2\nu)}{4(1-z)^2} + \ldots\right) /; (|z| \to \infty) \bigwedge 2\nu \notin \mathbb{Z}$$

07.05.06.0078.01

$$U_\nu(z) \propto 2^{-\nu-2} (z-1)^{-\nu-2} \left(1 + \frac{\nu+2}{1-z} + \frac{(\nu+3)(2\nu+5)}{4(1-z)^2} + O\left(\frac{1}{z^3}\right)\right) + 2^\nu (z-1)^\nu \left(1 - \frac{\nu}{1-z} + \frac{(1-\nu)(1-2\nu)}{4(1-z)^2} + O\left(\frac{1}{z^3}\right)\right) /; 2\nu \notin \mathbb{Z}$$

07.05.06.0023.02

$$U_\nu(z) = 2^\nu (z-1)^\nu \sum_{k=0}^{\infty} \frac{(-\nu)_k \left(-\nu-\frac{1}{2}\right)_k}{(-2\nu-1)_k \, k!} \left(\frac{2}{1-z}\right)^k - 2^{-\nu-2} (z-1)^{-\nu-2} \sum_{k=0}^{\infty} \frac{(\nu+2)_k \left(\nu+\frac{3}{2}\right)_k}{(2\nu+3)_k \, k!} \left(\frac{2}{1-z}\right)^k /; \left|\frac{1-z}{2}\right| > 1 \bigwedge 2\nu \notin \mathbb{Z}$$

07.05.06.0024.01

$$U_\nu(z) = 2^\nu (z-1)^\nu \, {}_2F_1\left(-\nu, -\nu-\frac{1}{2}; -2\nu-1; \frac{2}{1-z}\right) - 2^{-\nu-2} (z-1)^{-\nu-2} \, {}_2F_1\left(\nu+2, \nu+\frac{3}{2}; 2\nu+3; \frac{2}{1-z}\right) /; z \notin (-1, 1) \bigwedge 2\nu \notin \mathbb{Z}$$

07.05.06.0079.01

$$U_\nu(z) = \frac{2^{-\nu-2}\sqrt{1-z}\,(z-1)^\nu}{\sqrt{-z-1}} \left(\sqrt{\frac{z+1}{z-1}} + 1\right)^{2\nu+2} - \frac{2^\nu \sqrt{1-z}\,(z-1)^{-\nu-2}}{\sqrt{-z-1}} \left(\sqrt{\frac{z+1}{z-1}} + 1\right)^{-2\nu-2} /; z \notin (-1, 1)$$

07.05.06.0026.01

$$U_n(z) = 2^n (z-1)^n \sum_{k=0}^{n} \frac{(-n)_k \left(-n-\frac{1}{2}\right)_k}{k! \, (-2n-1)_k} \left(\frac{2}{1-z}\right)^k /; n \in \mathbb{N}$$

07.05.06.0080.01

$$T_n(z) = 2^{n-1} (z-1)^n \, {}_2F_1\left(-n, \frac{1}{2}-n; 1-2n; \frac{2}{1-z}\right) /; n \in \mathbb{N}$$

07.05.06.0028.01

$$U_\nu(z) = 2^\nu (z-1)^\nu \sum_{k=0}^{\nu+\frac{1}{2}} \frac{\left(-\nu-\frac{1}{2}\right)_k (-\nu)_k}{k! \, (-2\nu-1)_k} \left(\frac{2}{1-z}\right)^k - 2^{-\nu-1} (z-1)^{-\nu-2} \, {}_2F_1\left(\nu+2, \nu+\frac{3}{2}; 2\nu+3; \frac{2}{1-z}\right) /; \nu+\frac{1}{2} \in \mathbb{N}$$

07.05.06.0030.01

$$U_\nu(z) == 2^{\nu+1}(z-1)^\nu\, {}_2F_1\left(-\nu, -\nu-\frac{1}{2}; -2\nu-1; \frac{2}{1-z}\right) - 2^{-\nu-2}(z-1)^{-\nu-2} \sum_{k=0}^{-\nu-\frac{3}{2}} \frac{\left(\nu+\frac{3}{2}\right)_k (\nu+2)_k}{k!\,(2\nu+3)_k} \left(\frac{2}{1-z}\right)^k /; -\nu-\frac{3}{2} \in \mathbb{N}$$

07.05.06.0081.01

$$U_\nu(z) == F_\infty(z, \nu) /;$$

$$\Bigg(\Bigg(F_m(z, \nu) == 2^\nu (z-1)^\nu \sum_{k=0}^{m} \frac{(-\nu)_k \left(-\nu-\frac{1}{2}\right)_k}{(-2\nu-1)_k\, k!} \left(\frac{2}{1-z}\right)^k - 2^{-\nu-2}(z-1)^{-\nu-2} \sum_{k=0}^{m} \frac{(\nu+2)_k \left(\nu+\frac{3}{2}\right)_k}{(2\nu+3)_k\, k!} \left(\frac{2}{1-z}\right)^k == U_\nu(z) +$$

$$\frac{2^{m+\nu+1}(-1)^m \left(-\nu-\frac{1}{2}\right)_{m+1} (-\nu)_{m+1} (z-1)^{-m+\nu-1}}{(m+1)!\,(-2\nu-1)_{m+1}}\, {}_3F_2\left(1, m-\nu+\frac{1}{2}, m-\nu+1; m+2, m-2\nu; \frac{2}{1-z}\right) -$$

$$\frac{2^{m-\nu-1}(-1)^m \left(\nu+\frac{3}{2}\right)_{m+1} (\nu+2)_{m+1} (z-1)^{-m-\nu-3}}{(m+1)!\,(2\nu+3)_{m+1}}$$

$$ {}_3F_2\left(1, m+\nu+\frac{5}{2}, m+\nu+3; m+2, m+2\nu+4; \frac{2}{1-z}\right) \bigwedge m \in \mathbb{N}\Bigg) \bigwedge \neg\, 2\nu \in \mathbb{Z}\Bigg)$$

Summed form of the truncated series expansion.

Generic formulas for main term

07.05.06.0082.01

$$U_\nu(z) \propto \begin{cases} 2^\nu z^\nu & \operatorname{Re}(\nu) > -1 \\ -2^{-\nu-2} z^{-\nu-2} & \operatorname{Re}(\nu) < -1 \\ 2^\nu z^\nu - 2^{-\nu-2} z^{-\nu-2} & \text{True} \end{cases} /; (|z| \to \infty)$$

Asymptotic series expansions

Expansions at $\nu == \infty$

07.05.06.0083.01

$$U_\nu(z) \propto \begin{cases} \left(\frac{1}{2} - \frac{i z}{2\sqrt{1-z^2}}\right) e^{i\nu\cos^{-1}(z)} & -\pi < \arg\left((\nu+1)\cos^{-1}(z)\right) < 0 \\ \left(\frac{i z}{2\sqrt{1-z^2}} + \frac{1}{2}\right) e^{-i\nu\cos^{-1}(z)} & 0 < \arg\left((\nu+1)\cos^{-1}(z)\right) < \pi \\ \frac{\sin\left((\nu+1)\cos^{-1}(z)\right)}{\sqrt{1-z^2}} & \text{True} \end{cases} /; (|\nu| \to \infty)$$

Other series representations

07.05.06.0032.01

$$U_n(z) == \sum_{k=0}^{\left\lfloor \frac{n+1}{2} \right\rfloor} \binom{n+1}{2k} z^{n-2k} \left(z^2-1\right)^{k-1} \left(z^2 + \frac{2k}{n+1} - 1\right) /; n \in \mathbb{N}$$

Integral representations

On the real axis

Of the direct function

07.05.07.0001.01

$$U_\nu(z) = \frac{\nu+1}{2} \int_0^\pi \left(z + \sqrt{z^2-1}\,\cos(t)\right)^\nu \sin(t)\,dt \,/;\, z \notin (-\infty, -1)$$

Integral representations of negative integer order

Rodrigues-type formula.

07.05.07.0002.01

$$U_n(z) = \frac{(-1)^n \sqrt{\pi}\,(n+1)}{2^{n+1}\,\Gamma\left(n+\frac{3}{2}\right)\sqrt{1-z^2}} \frac{\partial^n \left(1-z^2\right)^{n+\frac{1}{2}}}{\partial z^n} \,/;\, n \in \mathbb{N}$$

Generating functions

07.05.11.0001.01

$$U_n(z) = \left([t^n]\, \frac{1}{t^2-2\,t\,z+1}\right) /;\, n \in \mathbb{N} \land -1 < z < 1$$

Differential equations

Ordinary linear differential equations and wronskians

For the direct function itself

With respect to ν

07.05.13.0005.01

$$w''(\nu) + \cos^{-1}(z)^2\, w(\nu) = 0 \,/;\, w(\nu) = U_\nu(z) \bigwedge w(0) = 1 \bigwedge w'(0) = \frac{z\cos^{-1}(z)}{\sqrt{1-z^2}}$$

07.05.13.0006.01

$$w''(\nu) + \cos^{-1}(z)^2\, w(\nu) = 0 \,/;\, w(\nu) = c_1\, U_\nu(z) + c_2\, T_\nu(z)$$

07.05.13.0007.01

$$W_\nu(T_\nu(z),\, U_\nu(z)) = -\frac{z\cos^{-1}(z)}{\sqrt{1-z^2}}$$

07.05.13.0008.01

$$w''(\nu) - \frac{g''(\nu)}{g'(\nu)}\, w'(\nu) + \cos^{-1}(z)^2\, g'(\nu)^2\, w(\nu) = 0 \,/;\, w(\nu) = c_1\, U_{g(\nu)}(z) + c_2\, T_{g(\nu)}(z)$$

07.05.13.0009.01

$$W_\nu\left(U_{g(\nu)}(z),\, T_{g(\nu)}(z)\right) == -\frac{z\cos^{-1}(z)\, g'(\nu)}{\sqrt{1-z^2}}$$

07.05.13.0010.01

$$w''(\nu) + \left(-\frac{2\,h'(\nu)}{h(\nu)} - \frac{g''(\nu)}{g'(\nu)}\right) w'(\nu) + \left(\cos^{-1}(z)^2\, g'(\nu)^2 + \frac{h'(\nu)\,g''(\nu)}{h(\nu)\,g'(\nu)} + \frac{2\,h'(\nu)^2 - h(\nu)\,h''(\nu)}{h(\nu)^2}\right) w(\nu) = 0 \,/;\, w(\nu) = c_1\, h(\nu)\, U_{g(\nu)}(z) + c_2\, h(\nu)\, T_{g(\nu)}(z)$$

07.05.13.0011.01

$$W_\nu\left(h(\nu)\, U_{g(\nu)}(z),\, h(\nu)\, T_{g(\nu)}(z)\right) == -\frac{z\cos^{-1}(z)\, h(\nu)^2\, g'(\nu)}{\sqrt{1-z^2}}$$

07.05.13.0012.01

$$\nu^2\, w''(\nu) - (r + 2\,s - 1)\,\nu\, w'(\nu) + \left(a^2\, r^2 \cos^{-1}(z)^2\, \nu^{2\,r} + s\,(r+s)\right) w(\nu) = 0 \,/;\, w(\nu) = c_1\, \nu^s\, U_{a\,\nu^r}(z) + c_2\, \nu^s\, T_{a\,\nu^r}(z)$$

07.05.13.0013.01

$$W_\nu\left(\nu^s\, U_{a\,\nu^r}(z),\, \nu^s\, T_{a\,\nu^r}(z)\right) == -\frac{a\, r\, z\, \nu^{r+2\,s-1} \cos^{-1}(z)}{\sqrt{1-z^2}}$$

07.05.13.0014.01

$$w''(\nu) - (\log(r) + 2\log(s))\, w'(\nu) + \left(a^2 \cos^{-1}(z)^2 \log^2(r)\, r^{2\,\nu} + \log(s)\,(\log(r) + \log(s))\right) w(\nu) = 0 \,/;\, w(\nu) = c_1\, s^\nu\, U_{a\,r^\nu}(z) + c_2\, s^\nu\, T_{a\,r^\nu}(z)$$

07.05.13.0015.01

$$W_\nu\left(s^\nu\, U_{a\,r^\nu}(z),\, s^\nu\, T_{a\,r^\nu}(z)\right) == -\frac{a\, r^\nu\, s^{2\,\nu}\, z\cos^{-1}(z)\log(r)}{\sqrt{1-z^2}}$$

With respect to *z*

07.05.13.0001.01

$$\left(1-z^2\right) w''(z) - 3\,z\, w'(z) + (\nu+2)\,\nu\, w(z) == 0 \,/;\, w(z) == c_1\, U_\nu(z) + c_2\, \frac{\cos\left((\nu+1)\cos^{-1}(z)\right)}{\sqrt{1-z^2}}$$

07.05.13.0002.01

$$W_z\left(U_\nu(z),\, \frac{\cos\left((\nu+1)\cos^{-1}(z)\right)}{\sqrt{1-z^2}}\right) == \frac{\nu+1}{\left(1-z^2\right)^{3/2}}$$

07.05.13.0003.01

$$\left(1-z^2\right) w''(z) - 3\,z\, w'(z) + (\nu+2)\,\nu\, w(z) == 0 \,/;\, w(z) == c_1\, U_\nu(z) + c_2\, \frac{\cosh\left((\nu+1)\cosh^{-1}(z)\right)}{\sqrt{1-z^2}}$$

07.05.13.0004.01

$$W_z\left(U_\nu(z),\ \frac{\cosh\left((\nu+1)\cosh^{-1}(z)\right)}{\sqrt{1-z^2}}\right) = \frac{\nu+1}{(z-1)^2\,(z+1)^{3/2}}\left(\sqrt{1-z}\,\cos\left((\nu+1)\cos^{-1}(z)\right)\cosh\left((\nu+1)\cosh^{-1}(z)\right)-\sqrt{z-1}\,\sin\left((\nu+1)\cos^{-1}(z)\right)\sinh\left((\nu+1)\cosh^{-1}(z)\right)\right)$$

07.05.13.0016.01

$$\left(1-z^2\right)w''(z)-3\,z\,w'(z)+(\nu+2)\,\nu\,w(z)=0\ /;\ w(z)=c_1\,U_\nu(z)\ +c_2\,\frac{1}{\sqrt{1-z^2}}\,T_{\nu+1}(z)$$

07.05.13.0017.01

$$W_z\left(U_\nu(z),\ \frac{1}{\sqrt{1-z^2}}\,T_{\nu+1}(z)\right) = \frac{\nu+1}{\left(1-z^2\right)^{3/2}}$$

07.05.13.0018.01

$$w''(z)-\left(\frac{3\,g(z)\,g'(z)}{1-g(z)^2}+\frac{g''(z)}{g'(z)}\right)w'(z)+\frac{\nu\,(\nu+2)\,g'(z)^2}{1-g(z)^2}\,w(z)=0\ /;\ w(z)=c_1\,U_\nu(g(z))\ +c_2\,\frac{1}{\sqrt{1-g(z)^2}}\,T_{\nu+1}(g(z))$$

07.05.13.0019.01

$$W_z\left(U_\nu(g(z)),\ \frac{1}{\sqrt{1-g(z)^2}}\,T_{\nu+1}(g(z))\right) = \frac{(\nu+1)\,g'(z)}{\left(1-g(z)^2\right)^{3/2}}$$

07.05.13.0020.01

$$w''(z)-\left(\frac{3\,g(z)\,g'(z)}{1-g(z)^2}+\frac{2\,h(z)\,h'(z)}{h(z)^2}+\frac{g''(z)}{g'(z)}\right)w'(z)+\left(\frac{\nu\,(\nu+2)\,g'(z)^2}{1-g(z)^2}+\frac{2\,h'(z)^2}{h(z)^2}+\frac{h'(z)}{h(z)}\left(\frac{3\,g(z)\,g'(z)}{1-g(z)^2}+\frac{g''(z)}{g'(z)}\right)-\frac{h''(z)}{h(z)}\right)w(z)=0\ /;\ w(z)=c_1\,h(z)\,U_\nu(g(z))\ +c_2\,h(z)\,\frac{1}{\sqrt{1-g(z)^2}}\,T_{\nu+1}(g(z))$$

07.05.13.0021.01

$$W_z\left(h(z)\,U_\nu(g(z)),\ \frac{h(z)}{\sqrt{1-g(z)^2}}\,T_{\nu+1}(g(z))\right) = \frac{(\nu+1)\,h(z)^2\,g'(z)}{\left(1-g(z)^2\right)^{3/2}}$$

07.05.13.0022.01

$$z^2\left(a^2\,z^{2\,r}-1\right)w''(z)\ +\left(a^2\,(2\,r-2\,s+1)\,z^{2\,r}+r+2\,s-1\right)z\,w'(z)+\left(a^2\,z^{2\,r}\,(s+r\,\nu)\,(s-r\,(\nu+2))-s\,(r+s)\right)w(z)=0\ /;\ w(z)=c_1\,z^s\,U_\nu(a\,z^r)\ +c_2\,\frac{z^s}{\sqrt{1-a^2\,z^{2\,r}}}\,T_{\nu+1}(a\,z^r)$$

07.05.13.0023.01

$$W_z\left(z^s\,U_\nu(a\,z^r),\ \frac{z^s}{\sqrt{1-a^2\,z^{2\,r}}}\,T_{\nu+1}(a\,z^r)\right) = \frac{a\,r\,z^{r+2\,s-1}\,(\nu+1)}{\left(1-a^2\,z^{2\,r}\right)^{3/2}}$$

07.05.13.0024.01

$$\left(a^2 r^{2z}-1\right) w''(z)+\left(2 a^2 (\log(r)-\log(s)) r^{2z}+\log(r)+2\log(s)\right) w'(z)+ \left(-a^2 ((\nu+2)\log(r)-\log(s))(\nu\log(r)+\log(s)) r^{2z}-\log(s)(\log(r)+\log(s))\right) w(z)=0 /; w(z)=c_1 s^z U_\nu(a r^z)+c_2 \frac{s^z}{\sqrt{1-a^2 r^{2z}}} T_{\nu+1}(a r^z)$$

07.05.13.0025.01

$$W_z\left(s^z U_\nu(a r^z), \frac{s^z}{\sqrt{1-a^2 r^{2z}}} T_{\nu+1}(a r^z)\right) == \frac{a r^z s^{2z} (\nu+1)\log(r)}{\left(1-a^2 r^{2z}\right)^{3/2}}$$

Transformations

Transformations and argument simplifications

Argument involving basic arithmetic operations

07.05.16.0001.01

$$U_{-\nu-2}(z) == -U_\nu(z)$$

07.05.16.0002.01

$$U_n(-z) == (-1)^n U_n(z) /; n \in \mathbb{N}$$

Multiple arguments

07.05.16.0003.01

$$U_{2\nu}\left(\sqrt{\frac{z+1}{2}}\right) = U_{\nu-1}(z)+U_\nu(z)$$

07.05.16.0004.01

$$U_{2n}(z) = (-1)^n \left(U_n\left(1-2z^2\right)-U_{n-1}\left(1-2z^2\right)\right) /; n \in \mathbb{Z}$$

Identities

Recurrence identities

Consecutive neighbors

07.05.17.0001.01

$$U_\nu(z) == 2 z\, U_{\nu+1}(z)-U_{\nu+2}(z)$$

07.05.17.0002.01

$$U_\nu(z) == 2 z\, U_{\nu-1}(z)-U_{\nu-2}(z)$$

Distant neighbors

07.05.17.0008.01

$$U_\nu(z) == C_m(\nu, z)\, U_{\nu+m}(z)-C_{m-1}(\nu, z)\, U_{\nu+m+1}(z) /; C_0(\nu, z) == 1 \land C_1(\nu, z) == 2z \land C_m(\nu, z) == 2z\, C_{m-1}(\nu, z)-C_{m-2}(\nu, z) \land m \in \mathbb{N}^+$$

07.05.17.0009.01

$$U_\nu(z) = C_m(\nu, z)\, U_{\nu-m}(z) - C_{m-1}(\nu, z)\, U_{\nu-m-1}(z) \;/;\; C_0(\nu, z) = 1 \land C_1(\nu, z) = 2\,z \land C_m(\nu, z) = 2\,z\, C_{m-1}(\nu, z) - C_{m-2}(\nu, z) \land m \in \mathbb{N}^+$$

07.05.17.0003.01

$$U_\nu(z) = 2\,(-1)^{2\left\lfloor \frac{m}{2} \right\rfloor} z^{m-2\left\lfloor \frac{m}{2} \right\rfloor} \left(z^2\right)^{\frac{1-m}{2}+\left\lfloor \frac{m}{2} \right\rfloor} U_{\frac{m-1}{2}}\left(2\,z^2-1\right) U_{\nu+m}(z) - 2\,z^{1-m+2\left\lfloor \frac{m}{2} \right\rfloor} \left(z^2\right)^{\left\lfloor \frac{m+1}{2} \right\rfloor - \frac{m}{2}} U_{\frac{m}{2}-1}\left(2\,z^2-1\right) U_{\nu+m+1}(z) \;/;\; m \in \mathbb{N}^+$$

07.05.17.0004.01

$$U_\nu(z) = 2\,(-1)^{2\left\lfloor \frac{m}{2} \right\rfloor} z^{m-2\left\lfloor \frac{m}{2} \right\rfloor} \left(z^2\right)^{\frac{1-m}{2}+\left\lfloor \frac{m}{2} \right\rfloor} U_{\frac{m-1}{2}}\left(2\,z^2-1\right) U_{\nu-m}(z) - 2\,z^{1-m+2\left\lfloor \frac{m}{2} \right\rfloor} \left(z^2\right)^{\left\lfloor \frac{m+1}{2} \right\rfloor - \frac{m}{2}} U_{\frac{m}{2}-1}\left(2\,z^2-1\right) U_{\nu-m-1}(z) \;/;\; m \in \mathbb{N}^+$$

Functional identities

Relations between contiguous functions

Recurrence relations

07.05.17.0005.01

$$U_{\nu-1}(z) + U_{\nu+1}(z) = 2\,z\, U_\nu(z)$$

07.05.17.0006.01

$$U_\nu(z) = \frac{1}{2\,z}\left(U_{\nu-1}(z) + U_{\nu+1}(z)\right)$$

Normalized recurrence relation

07.05.17.0007.01

$$z\, p(\nu, z) = \frac{1}{4}\, p(\nu-1, z) + p(\nu+1, z) \;/;\; p(\nu, z) = 2^{-\nu}\, U_\nu(z)$$

Complex characteristics

Real part

07.05.19.0001.01

$$\operatorname{Re}(U_n(x + i\,y)) = \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} (-1)^j\, 2^{2\,j}\, C_{n-2\,j}^{2\,j+1}(x)\, y^{2\,j} \;/;\; x \in \mathbb{R} \land y \in \mathbb{R} \land n \in \mathbb{N}$$

Imaginary part

07.05.19.0002.01

$$\operatorname{Im}(U_n(x + i\,y)) = \sum_{j=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} (-1)^j\, 2^{2\,j+1}\, C_{-2\,j+n-1}^{2\,j+2}(x)\, y^{2\,j+1} \;/;\; x \in \mathbb{R} \land y \in \mathbb{R} \land n \in \mathbb{N}$$

Differentiation

Low-order differentiation

With respect to ν

07.05.20.0001.01

$$\frac{\partial U_\nu(z)}{\partial \nu} = \frac{\cos^{-1}(z)}{\sqrt{1-z^2}} T_{\nu+1}(z)$$

07.05.20.0002.01

$$\frac{\partial^2 U_\nu(z)}{\partial \nu^2} = -\cos^{-1}(z)^2 U_\nu(z)$$

With respect to z

Forward shift operator:

07.05.20.0003.01

$$\frac{\partial U_\nu(z)}{\partial z} = \frac{(\nu+1) T_{\nu+1}(z) - z U_\nu(z)}{z^2-1}$$

07.05.20.0004.01

$$\frac{\partial^2 U_\nu(z)}{\partial z^2} = \frac{\left(z^2 \left(\nu^2+2\nu+3\right) - \nu(\nu+2)\right) U_\nu(z) - 3 z (\nu+1) T_{\nu+1}(z)}{\left(z^2-1\right)^2}$$

Backward shift operator:

07.05.20.0005.01

$$\left(1-z^2\right) \frac{\partial U_\nu(z)}{\partial z} - z U_\nu(z) = -(\nu+1) T_{\nu+1}(z)$$

07.05.20.0006.01

$$\frac{\partial\left(\sqrt{1-z^2}\, U_\nu(z)\right)}{\partial z} = -(\nu+1) \left(1-z^2\right)^{-\frac{1}{2}} T_{\nu+1}(z)$$

Symbolic differentiation

With respect to ν

07.05.20.0007.02

$$\frac{\partial^m U_\nu(z)}{\partial \nu^m} = \frac{1}{\sqrt{1-z^2}} \cos^{-1}(z)^m \sin\left(\frac{\pi m}{2} + (\nu+1)\cos^{-1}(z)\right) /; m \in \mathbb{N}$$

07.05.20.0008.02

$$\frac{\partial^m U_\nu(z)}{\partial \nu^m} = \frac{i^m \cos^{-1}(z)^m}{\sqrt{1-z^2}} \left(\sqrt{1-z^2}\, U_\nu(z) (1 - m \bmod 2) - i\, T_{\nu+1}(z) (m \bmod 2)\right) /; m \in \mathbb{N}$$

With respect to z

07.05.20.0009.02

$$\frac{\partial^m U_\nu(z)}{\partial z^m} = 2^m m!\, C_{\nu-m}^{m+1}(z) /; m \in \mathbb{N}$$

07.05.20.0010.02

$$\frac{\partial^m U_\nu(z)}{\partial z^m} = \frac{\sqrt{\pi}}{2} (\nu+1) (z-1)^{-m} \, _3\tilde{F}_2\left(1, -\nu, \nu+2; \frac{3}{2}, 1-m; \frac{1-z}{2}\right) /; m \in \mathbb{N}$$

07.05.20.0013.01

$$\frac{\partial^m U_n(z)}{\partial z^m} = 2^m m! \sum_{i_1=0}^{n-m} \ldots \sum_{i_{m+1}=0}^{n-m} \delta_{\sum_{j=1}^{m+1} i_j, n-m} \prod_{j=1}^{m+1} U_{i_j}(z) /; m \in \mathbb{N} \land n \in \mathbb{N}$$

Fractional integro-differentiation

With respect to ν

07.05.20.0011.01

$$\frac{\partial^\alpha U_\nu(z)}{\partial \nu^\alpha} = -\frac{i\, e^{-i(\nu+1)\cos^{-1}(z)} \nu^{-\alpha}}{2\sqrt{1-z^2}} \left(\left(-i \nu \cos^{-1}(z)\right)^\alpha \left(Q\left(-\alpha, -i \nu \cos^{-1}(z)\right) - 1\right) + e^{2i(\nu+1)\cos^{-1}(z)} \left(i \nu \cos^{-1}(z)\right)^\alpha \left(1 - Q\left(-\alpha, i \nu \cos^{-1}(z)\right)\right)\right)$$

With respect to z

07.05.20.0012.01

$$\frac{\partial^\alpha U_\nu(z)}{\partial z^\alpha} = \frac{\sqrt{\pi}}{2} (\nu+1) z^{-\alpha} \tilde{F}_{1\times1\times0}^{2\times1\times0}\left(\begin{matrix} -\nu, \nu+2; 1; ; \\ \frac{3}{2}; 1-\alpha; ; \end{matrix} -\frac{z}{2}, \frac{1}{2}\right)$$

Integration

Indefinite integration

Involving only one direct function

07.05.21.0001.01

$$\int U_\nu(a z)\, dz = \frac{1}{a(\nu+1)} T_{\nu+1}(a z)$$

07.05.21.0002.01

$$\int U_\nu(z)\, dz = \frac{1}{\nu+1} T_{\nu+1}(z)$$

Involving one direct function and elementary functions

Involving power function

07.05.21.0003.01

$$\int z^{\alpha-1} U_\nu(z)\, dz = \frac{1}{(2-\alpha+\nu)(\alpha+\nu)} \left(2^{-\alpha} z^\alpha \left(z\left(z + i\sqrt{1-z^2}\right)\right)^{-\alpha} \left(T_\nu(z) - i\sqrt{1-z^2}\; U_{\nu-1}(z)\right)\left((2-\alpha+\nu)\, _2F_1\left(-\frac{\alpha+\nu}{2}, 1-\alpha; 1-\frac{\alpha+\nu}{2}; -2z^2 - 2i\sqrt{1-z^2}\; z+1\right) + (\alpha+\nu)\left(T_{2\nu+2}(z) + i\sqrt{1-z^2}\; U_{2\nu+1}(z)\right) {}_2F_1\left(\frac{\nu-\alpha}{2}+1, 1-\alpha; \frac{\nu-\alpha}{2}+2; -2z^2 - 2i\sqrt{1-z^2}\; z+1\right)\right)\right)$$

Involving algebraic functions

07.05.21.0004.01

$$\int \left(1-z^2\right)^{\frac{1}{2}(-\nu-3)} U_\nu(z)\, dz = \frac{\left(1-z^2\right)^{\frac{1}{2}(-\nu-1)}}{\nu+2} U_{\nu+1}(z)$$

07.05.21.0005.01

$$\int \left(1-z^2\right)^{\frac{\nu-1}{2}} U_\nu(z)\, dz = -\frac{\left(1-z^2\right)^{\frac{\nu+1}{2}}}{\nu} U_{\nu-1}(z)$$

Involving only one direct function with respect to ν

07.05.21.0006.01

$$\int U_\nu(z)\, d\nu = -\frac{z\, T_\nu(z) + \left(z^2-1\right) U_{\nu-1}(z)}{\sqrt{1-z^2}\, \cos^{-1}(z)}$$

Involving one direct function and elementary functions with respect to ν

Involving power function

07.05.21.0007.01

$$\int \nu^{\alpha-1} U_\nu(z)\, d\nu = -\frac{\nu^\alpha}{2\sqrt{1-z^2}} \left(-\nu^2 \left(\operatorname{Im}\left(\cos^{-1}(z)\right) - i \operatorname{Re}\left(\cos^{-1}(z)\right)\right)^2\right)^{-\alpha}$$
$$\left(\left(\sqrt{1-z^2} - i z\right) \Gamma\left(\alpha,\, \nu \left(\operatorname{Im}\left(\cos^{-1}(z)\right) - i \operatorname{Re}\left(\cos^{-1}(z)\right)\right)\right) \left(-\nu \left(\operatorname{Im}\left(\cos^{-1}(z)\right) - i \operatorname{Re}\left(\cos^{-1}(z)\right)\right)\right)^\alpha + \right.$$
$$\left.\left(\sqrt{1-z^2} + i z\right) \Gamma\left(\alpha,\, -\nu \left(\operatorname{Im}\left(\cos^{-1}(z)\right) - i \operatorname{Re}\left(\cos^{-1}(z)\right)\right)\right) \left(\nu \left(\operatorname{Im}\left(\cos^{-1}(z)\right) - i \operatorname{Re}\left(\cos^{-1}(z)\right)\right)\right)^\alpha\right)$$

Definite integration

Involving the direct function

07.05.21.0008.01

$$\mathcal{P} \int_{-1}^{1} \frac{\sqrt{1-t^2}\, U_n(t)}{t-x}\, dt = -\pi\, T_{n+1}(x) \;/;\; n \in \mathbb{N} \land -1 < x < 1$$

Orthogonality:

07.05.21.0009.01

$$\int_{-1}^{1} \sqrt{1-t^2}\, U_m(t)\, U_n(t)\, dt = \frac{\pi}{2} \delta_{m,n} \;/;\; m \in \mathbb{N} \land n \in \mathbb{N}$$

Summation

Infinite summation

07.05.23.0001.01

$$\sum_{n=0}^{\infty} U_n(z)\, w^n == \frac{1}{w^2 - 2\, z\, w + 1} \;/;\; -1 < z < 1 \land |w| < 1$$

07.05.23.0002.01

$$\sum_{n=0}^{\infty} \frac{\left(\frac{3}{2}\right)_n}{(n+1)!} U_n(z)\, w^n == \frac{\sqrt{2}}{\sqrt{w^2 - 2\, w\, z + 1}\, \sqrt{1 + \sqrt{w^2 - 2\, w\, z + 1} - z\, w}} \;/;\; -1 < z < 1 \land |w| < 1$$

07.05.23.0003.01

$$\sum_{n=0}^{\infty} \frac{U_n(z)\, w^n}{\left(\frac{3}{2}\right)_n (n+1)!} == \frac{\sinh\left(\sqrt{2}\, \sqrt{w\,(z-1)}\right) \sinh\left(\sqrt{2}\, \sqrt{w\,(z+1)}\right)}{2\, \sqrt{w\,(z-1)}\, \sqrt{w\,(z+1)}} \;/;\; -1 < z < 1 \land |w| < 1$$

07.05.23.0004.01

$$\sum_{n=0}^{\infty} \frac{U_n(z)\, w^n}{(n+1)!} == e^{w\, z}\, {}_0F_1\left(; \frac{3}{2}; \frac{1}{4}\left(z^2 - 1\right) w^2\right) \;/;\; -1 < z < 1 \land |w| < 1$$

07.05.23.0005.01

$$\sum_{n=0}^{\infty} \frac{(\gamma)_n\, (2-\gamma)_n\, U_n(z)\, w^n}{\left(\frac{3}{2}\right)_n (n+1)!} == {}_2F_1\left(\gamma, 2-\gamma; \frac{3}{2}; \frac{1}{2}\left(1 - \sqrt{w^2 - 2\, z\, w + 1} - w\right)\right) {}_2F_1\left(\gamma, 2-\gamma; \frac{3}{2}; \frac{1}{2}\left(1 - \sqrt{w^2 - 2\, z\, w + 1} + w\right)\right) /;$$
$$-1 < z < 1 \land |w| < 1$$

07.05.23.0006.01

$$\sum_{n=0}^{\infty} \frac{(\gamma)_n}{(n+1)!} U_n(z)\, w^n == (1 - w\, z)^{-\gamma}\, {}_2F_1\left(\frac{\gamma}{2}, \frac{\gamma+1}{2}; \frac{3}{2}; \frac{\left(z^2 - 1\right) w^2}{(1 - w\, z)^2}\right) \;/;\; -1 < z < 1 \land |w| < 1$$

07.05.23.0007.01

$$\sum_{n=0}^{\infty} U_n(x)\, U_n(y) == \frac{\pi}{2} \frac{1}{\sqrt[4]{1 - x^2}} \frac{1}{\sqrt[4]{1 - x^2}}\, \delta(x - y) \;/;\; -1 < x < 1 \bigwedge -1 < y < 1$$

Operations

Orthogonality, completeness, and Fourier expansions

The set of functions $U_n(x)$, $n = 0, 1, \ldots$, forms a complete, orthogonal (with weight $\frac{2}{\pi} \sqrt{1 - x^2}$) system on the interval $(-1, 1)$.

07.05.25.0001.01

$$\sum_{n=0}^{\infty} \left(\sqrt{\frac{2}{\pi}}\, \sqrt[4]{1 - x^2}\, U_n(x)\right) \left(\sqrt{\frac{2}{\pi}}\, \sqrt[4]{1 - y^2}\, U_n(y)\right) == \delta(x - y) \;/;\; -1 < x < 1 \bigwedge -1 < y < 1$$

07.05.25.0002.01

$$\int_{-1}^{1} \left(\sqrt{\frac{2}{\pi}}\, \sqrt[4]{1 - t^2}\, U_m(t)\right) \left(\sqrt{\frac{2}{\pi}}\, \sqrt[4]{1 - t^2}\, U_n(t)\right) dt == \delta_{m,n}$$

Any sufficiently smooth function $f(x)$ can be expanded in the system $\{U_n(x)\}_{n=0,1,\ldots}$ as a generalized Fourier series, with its sum converging to $f(x)$ almost everywhere.

07.05.25.0003.01

$$f(x) = \sum_{n=0}^{\infty} c_n \psi_n(x) \,/;\, c_n = \int_{-1}^{1} \psi_n(t) f(t) \, dt \bigwedge \psi_n(x) = \sqrt{\frac{2}{\pi}} \sqrt[4]{1-x^2} \, U_n(x) \bigwedge -1 < x < 1$$

Representations through more general functions

Through hypergeometric functions

Involving ${}_2F_1$

07.05.26.0001.01

$$U_\nu(z) = (\nu+1) \, {}_2F_1\left(-\nu, \nu+2; \frac{3}{2}; \frac{1-z}{2}\right)$$

07.05.26.0002.01

$$U_\nu(z) = (\nu+1) \cos(\nu\pi) \, {}_2F_1\left(-\nu, \nu+2; \frac{3}{2}; \frac{z+1}{2}\right) - \frac{\sin(\nu\pi)}{\sqrt{2}\sqrt{z+1}} \, {}_2F_1\left(\nu+\frac{3}{2}, -\nu-\frac{1}{2}; \frac{1}{2}; \frac{z+1}{2}\right)$$

07.05.26.0003.01

$$U_\nu(z) = 2^\nu (z-1)^\nu \, {}_2F_1\left(-\nu, -\nu-\frac{1}{2}; -2\nu-1; \frac{2}{1-z}\right) - 2^{-\nu-2} (z-1)^{-\nu-2} \, {}_2F_1\left(\nu+2, \nu+\frac{3}{2}; 2\nu+3; \frac{2}{1-z}\right) /;$$
$$z \notin (-1, 1) \bigwedge 2\nu \notin \mathbb{Z}$$

Through hypergeometric functions of two variables

07.05.26.0004.01

$$U_\nu(z) = (\nu+1) \, F_{1\times0\times0}^{2\times0\times0}\left(\begin{matrix} -\nu, \nu+2;;; \\ \frac{3}{2};;; \end{matrix} \; \frac{1}{2}, -\frac{z}{2}\right)$$

Through Meijer G

Classical cases for the direct function itself

07.05.26.0005.01

$$U_\nu(z) = -\frac{\sin(\pi\nu)}{2\sqrt{\pi}} \, G_{2,2}^{1,2}\left(\frac{z-1}{2} \,\middle|\, \begin{matrix} \nu+1, -\nu-1 \\ 0, -\frac{1}{2} \end{matrix}\right) /; \nu \notin \mathbb{Z}$$

07.05.26.0006.01

$$U_\nu(z) = -\frac{1}{2\sqrt{\pi}} \lim_{m\to\nu} \sin(\pi m) \, G_{2,2}^{1,2}\left(\frac{z-1}{2} \,\middle|\, \begin{matrix} m+1, -m-1 \\ 0, -\frac{1}{2} \end{matrix}\right) /; \nu \in \mathbb{Z}$$

07.05.26.0007.01

$$U_\nu(2z+1) = -\frac{\sin(\pi\nu)}{2\sqrt{\pi}} \, G_{2,2}^{1,2}\left(z \,\middle|\, \begin{matrix} \nu+1, -\nu-1 \\ 0, -\frac{1}{2} \end{matrix}\right) /; \nu \notin \mathbb{Z}$$

Classical cases involving algebraic functions

07.05.26.0008.01

$$(z+1)^{-\nu-2}\,U_\nu\!\left(\frac{1-z}{1+z}\right)=\frac{2^{2\nu}}{\Gamma(2\nu+2)}\,G_{2,2}^{1,2}\!\left(z\,\middle|\,\begin{matrix}-\nu-1,\,-\nu-\frac{1}{2}\\ 0,\,-\frac{1}{2}\end{matrix}\right)/;\,z\notin(-\infty,-1)$$

07.05.26.0009.01

$$(z+1)^{-\nu-2}\,U_\nu\!\left(\frac{z-1}{z+1}\right)=\frac{4^{\nu}}{\Gamma(2\nu+2)}\,G_{2,2}^{2,1}\!\left(z\,\middle|\,\begin{matrix}-\nu-1,\,-\nu-\frac{1}{2}\\ -\frac{1}{2},\,0\end{matrix}\right)/;\,z\notin(-1,0)$$

07.05.26.0010.01

$$(z+1)^{-\frac{\nu}{2}-1}\,U_\nu\!\left(\frac{1}{\sqrt{z+1}}\right)=\frac{2^{\nu}}{\Gamma(\nu+1)}\,G_{2,2}^{1,2}\!\left(z\,\middle|\,\begin{matrix}-\frac{\nu}{2},\,-\frac{\nu+1}{2}\\ 0,\,-\frac{1}{2}\end{matrix}\right)$$

07.05.26.0011.01

$$(z+1)^{-\frac{\nu}{2}-1}\,U_\nu\!\left(\sqrt{\frac{z}{z+1}}\right)=\frac{2^{\nu}}{\Gamma(\nu+1)}\,G_{2,2}^{2,1}\!\left(z\,\middle|\,\begin{matrix}-\frac{\nu}{2},\,\frac{1-\nu}{2}\\ 0,\,\frac{1}{2}\end{matrix}\right)/;\,z\notin(-1,0)$$

07.05.26.0012.01

$$(z+1)^{-\frac{\nu}{2}-1}\,U_\nu\!\left(\frac{z+2}{2\sqrt{z+1}}\right)=\frac{1}{\Gamma(\nu+1)}\,G_{2,2}^{1,2}\!\left(z\,\middle|\,\begin{matrix}0,\,-\nu-1\\ 0,\,-1\end{matrix}\right)$$

07.05.26.0013.01

$$(z+1)^{-\frac{\nu}{2}-1}\,U_\nu\!\left(\frac{2z+1}{2\sqrt{z}\,\sqrt{z+1}}\right)=G_{2,2}^{2,1}\!\left(z\,\middle|\,\begin{matrix}-\frac{\nu}{2},\,1-\frac{\nu}{2}\\ -\frac{\nu}{2},\,\frac{\nu}{2}+1\end{matrix}\right)/;\,z\notin(-1,0)$$

07.05.26.0014.01

$$(z+1)^{-\nu-2}\,(z+2)\,U_\nu\!\left(\frac{z^2+2z+2}{2(z+1)}\right)=\frac{1}{\Gamma(2\nu+2)}\,G_{2,2}^{1,2}\!\left(z\,\middle|\,\begin{matrix}0,\,-2\nu-2\\ 0,\,-1\end{matrix}\right)/;\,z\notin(-\infty,-1)$$

07.05.26.0015.01

$$(2z+1)\,(z+1)^{-\nu-2}\,U_\nu\!\left(\frac{2z^2+2z+1}{2z(z+1)}\right)=\frac{1}{\Gamma(2\nu+2)}\,G_{2,2}^{2,1}\!\left(z\,\middle|\,\begin{matrix}-\nu,\,1-\nu\\ -\nu,\,\nu+2\end{matrix}\right)/;\,z\notin(-1,0)$$

Classical cases involving unit step θ

07.05.26.0016.01

$$\sqrt{1-z}\;\theta(1-|z|)\,U_\nu(2z-1)=\frac{\sqrt{\pi}\,(\nu+1)}{2}\,G_{2,2}^{2,0}\!\left(z\,\middle|\,\begin{matrix}-\nu-\frac{1}{2},\,\nu+\frac{3}{2}\\ -\frac{1}{2},\,0\end{matrix}\right)/;\,z\notin(-1,0)$$

07.05.26.0017.01

$$\sqrt{z-1}\;\theta(|z|-1)\,U_\nu(2z-1)=\frac{1}{2}\sqrt{\pi}\,(\nu+1)\,G_{2,2}^{0,2}\!\left(z\,\middle|\,\begin{matrix}-\nu-\frac{1}{2},\,\nu+\frac{3}{2}\\ -\frac{1}{2},\,0\end{matrix}\right)$$

07.05.26.0018.01

$$\sqrt{1-z}\;\theta(1-|z|)\,U_\nu\!\left(\frac{2}{z}-1\right)=\frac{1}{2}\sqrt{\pi}\,(\nu+1)\,G_{2,2}^{2,0}\!\left(z\,\middle|\,\begin{matrix}\frac{3}{2},\,2\\ -\nu,\,\nu+2\end{matrix}\right)$$

07.05.26.0019.01

$$\sqrt{z-1}\;\theta(|z|-1)\,U_\nu\!\left(\frac{2}{z}-1\right)=\frac{1}{2}\sqrt{\pi}\,(\nu+1)\,G_{2,2}^{0,2}\!\left(z\,\middle|\,\begin{matrix}\frac{3}{2},\,2\\ -\nu,\,\nu+2\end{matrix}\right)/;\,z\notin(-\infty,-1)$$

07.05.26.0020.01

$$\left(1-z^2\right)\theta(1-|z|)\,U_\nu\left(\frac{z^2+1}{2\,z}\right)=2\,(\nu+1)\,G_{2,2}^{2,0}\left(z\,\middle|\,\begin{matrix}1-\nu,\,\nu+3\\-\nu,\,\nu+2\end{matrix}\right)$$

07.05.26.0021.01

$$\left(z^2-1\right)\theta(|z|-1)\,U_\nu\left(\frac{z^2+1}{2\,z}\right)=2\,(\nu+1)\,G_{2,2}^{0,2}\left(z\,\middle|\,\begin{matrix}1-\nu,\,\nu+3\\-\nu,\,\nu+2\end{matrix}\right)$$

07.05.26.0022.01

$$\sqrt{z-1}\,(2\,z-1)\,\theta(|z|-1)\,U_\nu\left(8\,z^2-8\,z+1\right)=\frac{1}{2}\sqrt{\pi}\,(\nu+1)\,G_{2,2}^{0,2}\left(z\,\middle|\,\begin{matrix}-2\,\nu-\frac{3}{2},\,2\,\nu+\frac{5}{2}\\-\frac{1}{2},\,0\end{matrix}\right)/;\operatorname{Re}(z)>0$$

07.05.26.0023.01

$$\sqrt{1-z}\,(2-z)\,\theta(1-|z|)\,U_\nu\left(\frac{8}{z^2}-\frac{8}{z}+1\right)=\frac{1}{2}\sqrt{\pi}\,(\nu+1)\,G_{2,2}^{2,0}\left(z\,\middle|\,\begin{matrix}\frac{5}{2},\,3\\-2\,\nu,\,2\,(\nu+2)\end{matrix}\right)/;\operatorname{Re}(z)>0$$

Generalized cases involving algebraic functions

07.05.26.0024.01

$$\left(z^2+1\right)^{-\frac{\nu}{2}-1}U_\nu\left(\frac{z}{\sqrt{z^2+1}}\right)=\frac{2^\nu}{\Gamma(\nu+1)}\,G_{2,2}^{2,1}\left(z,\,\frac{1}{2}\,\middle|\,\begin{matrix}-\frac{\nu}{2},\,\frac{1-\nu}{2}\\0,\,\frac{1}{2}\end{matrix}\right)/;\operatorname{Re}(z)>0$$

07.05.26.0025.01

$$\left(z^2+1\right)^{-\frac{\nu}{2}-1}U_\nu\left(\frac{2\,z^2+1}{2\,z\sqrt{z^2+1}}\right)=\frac{1}{\Gamma(\nu+1)}\,G_{2,2}^{2,1}\left(z,\,\frac{1}{2}\,\middle|\,\begin{matrix}-\frac{\nu}{2},\,1-\frac{\nu}{2}\\-\frac{\nu}{2},\,\frac{\nu}{2}+1\end{matrix}\right)/;\operatorname{Re}(z)>0$$

Generalized cases involving unit step θ

07.05.26.0026.01

$$\sqrt{1-z^2}\,\theta(1-|z|)\,U_\nu(z)=\frac{1}{2}\sqrt{\pi}\,(\nu+1)\,G_{2,2}^{2,0}\left(z,\,\frac{1}{2}\,\middle|\,\begin{matrix}\frac{\nu+3}{2},\,\frac{1-\nu}{2}\\0,\,\frac{1}{2}\end{matrix}\right)/;\nu\notin\mathbb{Z}$$

07.05.26.0027.01

$$\sqrt{z^2-1}\,\theta(|z|-1)\,U_\nu(z)=\frac{1}{2}\sqrt{\pi}\,(\nu+1)\sqrt{z^2}\,G_{2,2}^{0,2}\left(z,\,\frac{1}{2}\,\middle|\,\begin{matrix}-\frac{\nu}{2},\,\frac{\nu+2}{2}\\-\frac{1}{2},\,0\end{matrix}\right)/;\nu\notin\mathbb{Z}$$

07.05.26.0028.01

$$\sqrt{1-z^2}\,\theta(1-|z|)\,U_\nu\left(\frac{1}{z}\right)=\frac{1}{2}\sqrt{\pi}\,(\nu+1)\,G_{2,2}^{2,0}\left(z,\,\frac{1}{2}\,\middle|\,\begin{matrix}1,\,\frac{3}{2}\\-\frac{\nu}{2},\,\frac{\nu+2}{2}\end{matrix}\right)/;\nu\notin\mathbb{Z}$$

07.05.26.0029.01

$$\sqrt{z^2-1}\,\theta(|z|-1)\,U_\nu\left(\frac{1}{z}\right)=\frac{1}{2}\sqrt{\pi}\,(\nu+1)\sqrt{z^2}\,G_{2,2}^{0,2}\left(z,\,\frac{1}{2}\,\middle|\,\begin{matrix}\frac{1}{2},\,1\\-\frac{\nu+1}{2},\,\frac{\nu+1}{2}\end{matrix}\right)/;\nu\notin\mathbb{Z}$$

Through other functions

Involving some hypergeometric-type functions

07.05.26.0030.01

$$U_\nu(z) = \sqrt{\frac{\pi}{2}}\, \frac{\nu+1}{\sqrt[4]{z+1}}\, \sqrt[4]{\frac{1}{1-z}}\, P_{\nu+\frac{1}{2}}^{-\frac{1}{2}}(z)$$

07.05.26.0031.01

$$U_\nu(z) = \sqrt{\frac{\pi}{2}}\, \frac{\nu+1}{\sqrt{z+1}}\, \sqrt[4]{\frac{z+1}{z-1}}\, P_{\nu+\frac{1}{2}}^{-\frac{1}{2}}(z)$$

07.05.26.0032.01

$$U_\nu(z) = \frac{(\nu+1)!}{\left(\frac{3}{2}\right)_\nu}\, P_\nu^{\left(\frac{1}{2},\frac{1}{2}\right)}(z)$$

07.05.26.0033.01

$$U_\nu(z) = \frac{(\nu+1)\, P_\nu^{\left(\frac{1}{2},\frac{1}{2}\right)}(z)}{P_\nu^{\left(\frac{1}{2},\frac{1}{2}\right)}(1)}$$

07.05.26.0034.01

$$U_\nu(z) = C_\nu^1(z)$$

Involving spheroidal functions

07.05.26.0035.01

$$U_\nu(z) = -\frac{\sqrt{2}}{\sqrt{\pi}\, \sqrt[4]{1-z^2}}\, QS_{\nu+\frac{1}{2},\frac{1}{2}}(0, z)$$

Representations through equivalent functions

With related functions

07.05.27.0001.01

$$U_\nu(z) = \frac{z\, T_{\nu+1}(z) - T_{\nu+2}(z)}{1-z^2}$$

07.05.27.0002.01

$$U_\nu(z) = \frac{1}{\nu+1}\, \frac{\partial T_{\nu+1}(z)}{\partial z}$$

07.05.27.0003.01

$$U_\nu(z) = \frac{1}{2}\, \frac{\partial C_{\nu+1}^{(0)}(z)}{\partial z}$$

07.05.27.0004.01

$$U_n(x) = \frac{1}{\pi}\, \mathcal{P} \int_{-1}^{1} \frac{T_{n+1}(t)}{\sqrt{1-t^2}\, (t-x)}\, dt \;/;\; n \in \mathbb{N}^+ \bigwedge -1 < x < 1$$

With elementary functions

07.05.27.0005.01

$$U_\nu(z) = \frac{1}{2\sqrt{1-z^2}} \left(e^{\frac{i\pi\nu}{2}} \left(i\,z + \sqrt{1-z^2} \right)^{-\nu-1} + e^{-\frac{i\pi\nu}{2}} \left(i\,z + \sqrt{1-z^2} \right)^{\nu+1} \right)$$

07.05.27.0006.01

$$U_\nu(z) = \frac{\sin\left((\nu+1)\cos^{-1}(z)\right)}{\sqrt{1-z^2}}$$

07.05.27.0007.01

$$U_\nu(z) = \frac{1}{\sqrt{1-z^2}} \sin\left(2(\nu+1)\sin^{-1}\left(\frac{\sqrt{1-z}}{\sqrt{2}}\right)\right)$$

Theorems

Expansions in generalized Fourier series

$$f(x) = \sum_{k=0}^{\infty} c_k\, \psi_k(x) \;/;\; c_k = \int_{-1}^{1} f(t)\, \psi_k(t)\, dt,\; \psi_k(x) = \sqrt{\frac{2}{\pi}} \left(1-x^2\right)^{\frac{1}{4}} U_k(x),\; k \in \mathbb{N}.$$

One property of a unimodular matrix

If $\mathbb{A}$ is a unimodular matrix, then $\mathbb{A}^n = \mathbb{A}\, U_{n-1}\left(\frac{\mathrm{Tr}(\mathbb{A})}{2}\right) - \mathbb{I}\, U_{n-2}\left(\frac{\mathrm{Tr}(\mathbb{A})}{2}\right) /;\, n \in \mathbb{Z}^+$.

The length of the hypotenuse of the *r*th Pythagorean triangle

The length of the hypotenuse of the r th Pythagorean triangle with consecutive integer legs is $U_r(3) - U_{r-1}(3)$.

History

– P. L. Chebyshev (1855, 1859)

Copyright