ClebschGordan

Notations

Traditional name

Clebsch-Gordan coefficient

Traditional notation

$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle$

Mathematica StandardForm notation

ClebschGordan[{j_1, m_1}, {j_2, m_2}, {j, m}]

Primary definition

07.38.02.0001.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle = \delta_{m,m_1+m_2} \frac{\sqrt{2\,j+1}\,\sqrt{(j+j_1-j_2)!}\,\sqrt{(j-j_1+j_2)!}\,\sqrt{(j-m)!}\,\sqrt{(j+m)!}\,\sqrt{(j_1+m_1)!}\,\sqrt{(j_2-m_2)!}}{\sqrt{(-j+j_1+j_2)!}\,\sqrt{(j+j_1+j_2+1)!}\,\sqrt{(j_1-m_1)!}\,\sqrt{(j_2+m_2)!}} \, {}_3\tilde{F}_2(j-j_1-j_2,\, m_1-j_1,\, -j_2-m_2;\, j-j_2+m_1+1,\, j-j_1-m_2+1;\, 1) \,/;\, \mathcal{P}hysical\mathcal{Q}(\{j_1, m_1\}, \{j_2, m_2\}, \{j, m\})$$

07.38.02.0002.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle = 0 \,/;\, \neg\, \mathcal{P}hysical\mathcal{Q}(\{j_1, m_1\}, \{j_2, m_2\}, \{j, m\})$$

$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle$ is the Clebsch-Gordan coefficient for the decomposition of $\left|j\, m\right\rangle$ in terms of $\left|j_1\, m_1\right\rangle \otimes \left|j_2\, m_2\right\rangle$. The Clebsch-Gordan coefficients appear in the quantum mechanical treatment of angular momentum, where j is the full angular momentum and m is its projection onto a given axis. The values of the Clebsch-Gordan coefficients which are physically realizable (in a Minkowski space-time) are obtained under the additional restrictions:

07.38.02.0003.01

$$\mathcal{P}hysical\mathcal{Q}(\{j_1, m_1\}, \{j_2, m_2\}, \{j, m\]) = \mathcal{T}riangular\mathcal{Q}(j_1, j_2, j) \wedge j_1 - m_1 \in \mathbb{Z} \wedge j_2 - m_2 \in \mathbb{Z} \wedge j - m \in \mathbb{Z} \wedge -j_1 \le m_1 \le j_1 \wedge -j_2 \le m_2 \le j_2 \wedge -j \le m \le j$$

where

$$\mathcal{T}riangular\mathcal{Q}[j_1, j_2, j] = 2\, j_1 \in \text{Integers} \wedge j_1 \ge 0 \wedge 2\, j_2 \in \text{Integers} \wedge j_2 \ge 0 \wedge 2\, j \in \text{Integers} \wedge j \ge 0 \wedge j_1 + j_2 + j \in \text{Integers} \wedge \text{Abs}[j_1 - j_2] \le j \le j_1 + j_2$$

07.38.02.0004.01

$$\mathcal{T}riangular\mathcal{Q}(j_1, j_2, j) = 2\, j_1 \in \mathbb{N} \wedge 2\, j_2 \in \mathbb{N} \wedge 2\, j \in \mathbb{N} \wedge j_1 + j_2 + j \in \mathbb{N} \wedge |j_1 - j_2| \le j \le j_1 + j_2$$

Specific values

Specialized values

Nonphysical cases

07.38.03.0001.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle = 0 \;/;\; m_1 + m_2 \neq m$$

Fixed j_1, j_2, m_1, m_2

07.38.03.0002.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, 0\, 0\rangle = \frac{(-1)^{j_1-m_1}}{\sqrt{2\, j_1+1}}\, \delta_{j_1,j_2}\, \delta_{m_1,-m_2} \;/;\; \mathcal{P}hysical\mathcal{Q}(\{j_1, m_1\}, \{j_2, m_2\}, \{0, 0\})$$

07.38.03.0003.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j_1+j_2\, m_1+m_2\rangle = \frac{\sqrt{(2\, j_1)!}\, \sqrt{(2\, j_2)!}\, \sqrt{(j_1+j_2+m_1+m_2)!}\, \sqrt{(j_1+j_2-m_1-m_2)!}}{\sqrt{(2\, j_1+2\, j_2)!}\, \sqrt{(j_1+m_1)!}\, \sqrt{(j_1-m_1)!}\, \sqrt{(j_2+m_2)!}\, \sqrt{(j_2-m_2)!}} \;/;\; \mathcal{P}hysical\mathcal{Q}(\{j_1, m_1\}, \{j_2, m_2\}, \{j_1+j_2, m_1+m_2\})$$

07.38.03.0004.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j_1+j_2-1\, m_1+m_2\rangle = 2\,(j_2\, m_1 - j_1\, m_2)\left(\sqrt{2\, j_1+2\, j_2-1}\, \sqrt{(2\, j_1-1)!}\, \sqrt{(2\, j_2-1)!}\, \sqrt{(j_1+j_2+m_1+m_2-1)!}\, \sqrt{(j_1+j_2-m_1-m_2-1)!}\right) \Big/ \left(\sqrt{(2\, j_1+2\, j_2)!}\, \sqrt{(j_1+m_1)!}\, \sqrt{(j_1-m_1)!}\, \sqrt{(j_2+m_2)!}\, \sqrt{(j_2-m_2)!}\right) /;\; \mathcal{P}hysical\mathcal{Q}(\{j_1, m_1\}, \{j_2, m_2\}, \{j_1+j_2-1, m_1+m_2\})$$

07.38.03.0005.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j_1+j_2-2\, m_1+m_2\rangle = \frac{\sqrt{j_1}\, \sqrt{j_2}\, \sqrt{2\, j_1-1}\, \sqrt{2\, j_2-1}}{\sqrt{j_1+j_2-1}\, \sqrt{2\, j_1+2\, j_2-1}} \left(\sqrt{(j_1-m_1)!}\, \sqrt{(j_1+m_1)!}\, \sqrt{(j_2-m_2)!}\, \sqrt{(j_2+m_2)!}\, \sqrt{(j_1+j_2-m_1-m_2-2)!}\, \sqrt{(j_1+j_2+m_1+m_2-2)!}\right) \Big/ \left(\sqrt{(2\, j_1)!}\, \sqrt{(2\, j_2)!}\, \sqrt{(2\, j_1+2\, j_2-4)!}\right) \left(\frac{(2\, j_1-2)!\,(2\, j_2-2)!}{(j_1-m_1)!\,(j_1+m_1-2)!\,(j_2+m_2)!\,(j_2-m_2-2)!} - \frac{2\,(2\, j_1-2)!\,(2\, j_2-2)!}{(j_1-m_1-1)!\,(j_1+m_1-1)!\,(j_2+m_2-1)!\,(j_2-m_2-1)!} + \frac{(2\, j_1-2)!\,(2\, j_2-2)!}{(j_1-m_1-2)!\,(j_1+m_1)!\,(j_2+m_2-2)!\,(j_2-m_2)!}\right) /;\; \mathcal{P}hysical\mathcal{Q}(\{j_1, m_1\}, \{j_2, m_2\}, \{j_1+j_2-2, m_1+m_2\})$$

07.38.03.0006.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j_1 - j_2 + 2\, m_1 + m_2\rangle = (-1)^{j_2+m_2} \frac{\sqrt{j_2}\,\sqrt{2\, j_2 - 1}\,\sqrt{2\, j_1 - 2\, j_2 + 5}\,\sqrt{2\, j_1 - 2\, j_2 + 4}\,\sqrt{2\, j_1 - 2\, j_2 + 3}}{\sqrt{2\, j_1 + 1}\,\sqrt{2\, j_1 + 2}\,\sqrt{2\, j_1 + 3}} \left(\sqrt{(j_1 - m_1)!}\,\sqrt{(j_1 + m_1)!}\,\sqrt{(j_2 - m_2)!}\,\sqrt{(j_2 + m_2)!}\,\sqrt{(j_1 - j_2 - m_1 - m_2 + 2)!}\,\sqrt{(j_1 - j_2 + m_1 + m_2 + 2)!}\right) \Big/ \left(\sqrt{(2\, j_1)!}\,\sqrt{(2\, j_2)!}\,\sqrt{(2\, j_1 - 2\, j_2 + 4)!}\right) \left(\frac{(2\, j_2 - 2)!\,(2\, j_1 - 2\, j_2 + 2)!}{(j_2 + m_2)!\,(j_2 - m_2 - 2)!\,(j_1 - j_2 - m_1 - m_2)!\,(j_1 - j_2 + m_1 + m_2 + 2)!} - \frac{2\,(2\, j_2 - 2)!\,(2\, j_1 - 2\, j_2 + 2)!}{(j_2 + m_2 - 1)!\,(j_2 - m_2 - 1)!\,(j_1 - j_2 - m_1 - m_2 + 1)!\,(j_1 - j_2 + m_1 + m_2 + 1)!} + \frac{(2\, j_2 - 2)!\,(2\, j_1 - 2\, j_2 + 2)!}{(j_2 + m_2 - 2)!\,(j_2 - m_2)!\,(j_1 - j_2 - m_1 - m_2 + 2)!\,(j_1 - j_2 + m_1 + m_2)!}\right) /;$$

$\mathcal{P}hysical\mathcal{Q}(\{j_1, m_1\}, \{j_2, m_2\}, \{j_1 - j_2 + 2, m_1 + m_2\})$

07.38.03.0007.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j_1 - j_2 + 1\, m_1 + m_2\rangle = (-1)^{j_2+m_2+1} \left(2\,(j_2\, m_1 + j_1\, m_2 + m_2)\,\sqrt{2\, j_1 - 2\, j_2 + 3}\,\sqrt{(2\, j_2 - 1)!}\,\sqrt{(2\, j_1 - 2\, j_2 + 1)!}\,\sqrt{(j_1 + m_1)!}\,\sqrt{(j_1 - m_1)!}\right) \Big/ \left(\sqrt{(2\, j_1 + 2)!}\,\sqrt{(j_2 + m_2)!}\,\sqrt{(j_2 - m_2)!}\,\sqrt{(j_1 - j_2 + m_1 + m_2 + 1)!}\,\sqrt{(j_1 - j_2 - m_1 - m_2 + 1)!}\right) /;$$

$\mathcal{P}hysical\mathcal{Q}(\{j_1, m_1\}, \{j_2, m_2\}, \{j_1 - j_2 + 1, m_1 + m_2\})$

07.38.03.0008.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j_1 - j_2\, m_1 + m_2\rangle = (-1)^{j_2+m_2} \frac{\sqrt{(j_1 + m_1)!}\,\sqrt{(j_1 - m_1)!}\,\sqrt{(2\, j_2)!}\,\sqrt{(2\, j_1 - 2\, j_2 + 1)!}}{\sqrt{(2\, j_1 + 1)!}\,\sqrt{(j_2 + m_2)!}\,\sqrt{(j_2 - m_2)!}\,\sqrt{(j_1 - j_2 + m_1 + m_2)!}\,\sqrt{(j_1 - j_2 - m_1 - m_2)!}} /;$$

$\mathcal{P}hysical\mathcal{Q}(\{j_1, m_1\}, \{j_2, m_2\}, \{j_1 - j_2, m_1 + m_2\})$

07.38.03.0009.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, m_1 + m_2\, m_1 + m_2\rangle = (-1)^{j_1-m_1} \left(\sqrt{(2\, m_1 + 2\, m_2 + 1)!}\,\sqrt{(j_1 + j_2 - m_1 - m_2)!}\,\sqrt{(j_1 + m_1)!}\,\sqrt{(j_2 + m_2)!}\right) \Big/ \left(\sqrt{(j_1 + j_2 + m_1 + m_2 + 1)!}\,\sqrt{(j_1 - j_2 + m_1 + m_2)!}\,\sqrt{(-j_1 + j_2 + m_1 + m_2)!}\,\sqrt{(j_1 - m_1)!}\,\sqrt{(j_2 - m_2)!}\right) /;$$

$\mathcal{P}hysical\mathcal{Q}(\{j_1, m_1\}, \{j_2, m_2\}, \{m_1 + m_2, m_1 + m_2\})$

07.38.03.0010.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, m_1 + m_2 + 1\, m_1 + m_2\rangle = (-1)^{j_1-m_1} \left((j_2 - m_2)\,(j_2 + m_2 + 1) - (j_1 - m_1)\,(j_1 + m_1 + 1)\right) \left(\sqrt{2\, m_1 + 2\, m_2 + 3}\,\sqrt{(2\, m_1 + 2\, m_2 + 1)!}\,\sqrt{(j_1 + j_2 - m_1 - m_2 - 1)!}\,\sqrt{(j_1 + m_1)!}\,\sqrt{(j_2 + m_2)!}\right) \Big/ \left(\sqrt{(j_1 + j_2 + m_1 + m_2 + 2)!}\,\sqrt{(j_1 - j_2 + m_1 + m_2 + 1)!}\,\sqrt{(-j_1 + j_2 + m_1 + m_2 + 1)!}\,\sqrt{(j_1 - m_1)!}\,\sqrt{(j_2 - m_2)!}\right) /;$$

$\mathcal{P}hysical\mathcal{Q}(\{j_1, m_1\}, \{j_2, m_2\}, \{m_1 + m_2 + 1, m_1 + m_2\})$

Fixed j_1, j_2, j, m_2

07.38.03.0011.01

$$\langle j_1\, j_2\, j_1\, m_2 \mid j_1\, j_2\, j\, j_1 + m_2\rangle = \frac{\sqrt{2\, j + 1}\,\sqrt{(2\, j_1)!}\,\sqrt{(-j_1 + j_2 + j)!}\,\sqrt{(j_2 - m_2)!}\,\sqrt{(j + j_1 + m_2)!}}{\sqrt{(j_1 + j_2 + j + 1)!}\,\sqrt{(j_1 - j_2 + j)!}\,\sqrt{(j_1 + j_2 - j)!}\,\sqrt{(j_2 + m_2)!}\,\sqrt{(j - j_1 - m_2)!}} /;$$

$\mathcal{P}hysical\mathcal{Q}(\{j_1, j_1\}, \{j_2, m_2\}, \{j, j_1 + m_2\})$

07.38.03.0012.01

$$\langle j_1\, j_2\, j_1-1\, m_2 \mid j_1\, j_2\, j\, j_1+m_2-1\rangle = ((j-j_1-m_2+1)(j+j_1+m_2)-(j_2+m_2)(j_2-m_2+1)) \frac{\sqrt{2j+1}\sqrt{(2j_1-1)!}\sqrt{(-j_1+j_2+j)!}\sqrt{(j_2-m_2)!}\sqrt{(j+j_1+m_2-1)!}}{\sqrt{(j_1+j_2+j+1)!}\sqrt{(j_1-j_2+j)!}\sqrt{(j_1+j_2-j)!}\sqrt{(j_2+m_2)!}\sqrt{(j-j_1-m_2+1)!}} /; \mathcal{P}hysical\mathcal{Q}(\{j_1, j_1-1\}, \{j_2, m_2\}, \{j, j_1+m_2-1\})$$

Fixed j_1, j, m_1

07.38.03.0013.01

$$\langle j_1\, 0\, m_1\, 0 \mid j_1\, 0\, j\, m\rangle = \delta_{j_1,j}\,\delta_{m_1,m} /; \mathcal{P}hysical\mathcal{Q}(\{j_1, m_1\}, \{0, 0\}, \{j, m_1\})$$

07.38.03.0014.01

$$\langle j_1\, j_1\, m_1\, m_1 \mid j_1\, j_1\, j\, 2m_1\rangle = 0 /; \frac{2j_1+j-1}{2} \in \mathbb{Z}$$

07.38.03.0015.01

$$\langle j_1\, j_1\, m_1\, m_1 \mid j_1\, j_1\, j\, 2m_1\rangle = (-1)^{\frac{2j_1-j}{2}} \frac{\left(\frac{2j_1+j}{2}\right)!\sqrt{2j+1}\sqrt{(j+2m_1)!}\sqrt{(j-2m_1)!}\sqrt{(2j_1-j)!}}{\left(\frac{j+2m_1}{2}\right)!\left(\frac{j-2m_1}{2}\right)!\left(\frac{2j_1-j}{2}\right)!\sqrt{(2j_1+j+1)!}} /; \frac{2j_1+j}{2} \in \mathbb{Z} \bigwedge \mathcal{P}hysical\mathcal{Q}(\{j_1, m_1\}, \{j_1, m_1\}, \{j, 2m_1\})$$

07.38.03.0016.01

$$\left\langle j_1\, j_1+\frac{1}{2}\, m_1\, m_1+\frac{1}{2} \,\middle|\, j_1\, j_1+\frac{1}{2}\, j+\frac{1}{2}\, 2m_1+\frac{1}{2}\right\rangle = (-1)^{\frac{j-2j_1}{2}} \frac{\left(\frac{2j_1+j}{2}\right)!\sqrt{j+2m_1+1}\sqrt{2j_1+j+2}\sqrt{(2j_1-j)!}\sqrt{(j+2m_1)!}\sqrt{(j-2m_1)!}}{\left(\frac{2j_1-j}{2}\right)!\left(\frac{j+2m_1}{2}\right)!\left(\frac{j-2m_1}{2}\right)!\sqrt{2}\sqrt{j_1+m_1+1}\sqrt{(2j_1+j+1)!}} /; \frac{2j_1+j}{2} \in \mathbb{Z} \bigwedge \mathcal{P}hysical\mathcal{Q}\left(\{j_1, m_1\}, \left\{j_1+\frac{1}{2}, m_1+\frac{1}{2}\right\}, \left\{j+\frac{1}{2}, 2m_1+\frac{1}{2}\right\}\right)$$

07.38.03.0017.01

$$\left\langle j_1\, j_1+\frac{1}{2}\, m_1\, m_1-\frac{1}{2} \,\middle|\, j_1\, j_1+\frac{1}{2}\, j+\frac{1}{2}\, 2m_1-\frac{1}{2}\right\rangle = (-1)^{\frac{j-2j_1}{2}} \frac{\left(\frac{2j_1+j}{2}\right)!\sqrt{j-2m_1+1}\sqrt{2j_1+j+2}\sqrt{(2j_1-j)!}\sqrt{(j+2m_1)!}\sqrt{(j-2m_1)!}}{\left(\frac{2j_1-j}{2}\right)!\left(\frac{j+2m_1}{2}\right)!\left(\frac{j-2m_1}{2}\right)!\sqrt{2}\sqrt{j_1-m_1+1}\sqrt{(2j_1+j+1)!}} /; \frac{2j_1+j}{2} \in \mathbb{Z} \bigwedge \mathcal{P}hysical\mathcal{Q}\left(\{j_1, m_1\}, \left\{j_1+\frac{1}{2}, m_1-\frac{1}{2}\right\}, \left\{j+\frac{1}{2}, 2m_1-\frac{1}{2}\right\}\right)$$

07.38.03.0018.01

$$\left\langle j_1\, j_1+\frac{1}{2}\, m_1\, m_1+\frac{1}{2} \,\middle|\, j_1\, j_1+\frac{1}{2}\, j+\frac{1}{2}\, 2\, m_1+\frac{1}{2}\right\rangle = (-1)^{\frac{j-2 j_1-1}{2}} \frac{\left(\frac{2 j_1+j+1}{2}\right)!\, \sqrt{j-2 m_1+1}\, \sqrt{(2 j_1-j)!}\, \sqrt{(j+2 m_1+1)!}\, \sqrt{(j-2 m_1+1)!}}{\left(\frac{2 j_1-j-1}{2}\right)!\left(\frac{j+2 m_1+1}{2}\right)!\left(\frac{j-2 m_1+1}{2}\right)!\, \sqrt{2}\, \sqrt{j_1+m_1+1}\, \sqrt{(2 j_1+j+2)!}} \;/;\; \frac{2 j_1+j-1}{2} \in \mathbb{Z} \bigwedge \mathcal{P}hysical\mathcal{Q}\left(\{j_1, m_1\}, \left\{j_1+\frac{1}{2}, m_1+\frac{1}{2}\right\}, \left\{j+\frac{1}{2}, 2 m_1+\frac{1}{2}\right\}\right)$$

07.38.03.0019.01

$$\left\langle j_1\, j_1+\frac{1}{2}\, m_1\, m_1-\frac{1}{2} \,\middle|\, j_1\, j_1+\frac{1}{2}\, j+\frac{1}{2}\, 2\, m_1-\frac{1}{2}\right\rangle = -(-1)^{\frac{j-2 j_1-1}{2}} \frac{\left(\frac{2 j_1+j+1}{2}\right)!\, \sqrt{j+2 m_1+1}\, \sqrt{(2 j_1-j)!}\, \sqrt{(j+2 m_1+1)!}\, \sqrt{(j-2 m_1+1)!}}{\left(\frac{2 j_1-j-1}{2}\right)!\left(\frac{j+2 m_1+1}{2}\right)!\left(\frac{j-2 m_1+1}{2}\right)!\, \sqrt{2}\, \sqrt{j_1-m_1+1}\, \sqrt{(2 j_1+j+2)!}} \;/;\; \frac{2 j_1+j-1}{2} \in \mathbb{Z} \bigwedge \mathcal{P}hysical\mathcal{Q}\left(\{j_1, m_1\}, \left\{j_1+\frac{1}{2}, m_1-\frac{1}{2}\right\}, \left\{j+\frac{1}{2}, 2 m_1-\frac{1}{2}\right\}\right)$$

07.38.03.0020.01

$$\langle j_1\, j_1+1\, m_1\, m_1+1 \mid j_1\, j_1+1\, j\, 2\, m_1+1\rangle = -(-1)^{\frac{j-2 j_1}{2}} \left(\left(\frac{2 j_1+j}{2}\right)!\, \sqrt{j+2 m_1+1}\, \sqrt{j-2 m_1}\, \sqrt{2 j+1}\, \sqrt{2 j_1+j+2}\, \sqrt{(2 j_1-j+1)!}\, \sqrt{(j+2 m_1)!}\, \sqrt{(j-2 m_1)!}\right) \Big/ \left(2\left(\frac{2 j_1-j}{2}\right)!\left(\frac{j+2 m_1}{2}\right)!\left(\frac{j-2 m_1}{2}\right)!\, \sqrt{j_1+m_1+2}\, \sqrt{j_1+m_1+1}\, \sqrt{j}\, \sqrt{j+1}\, \sqrt{(2 j_1+j+1)!}\right) \;/;\; \frac{2 j_1+j}{2} \in \mathbb{Z} \bigwedge \mathcal{P}hysical\mathcal{Q}(\{j_1, m_1\}, \{j_1+1, m_1+1\}, \{j, 2 m_1+1\})$$

07.38.03.0021.01

$$\langle j_1\, j_1+1\, m_1\, m_1 \mid j_1\, j_1+1\, j\, 2\, m_1\rangle = (-1)^{\frac{j-2 j_1}{2}} \frac{m_1 \left(\frac{2 j_1+j}{2}\right)!\, \sqrt{2 j+1}\, \sqrt{2 j_1+j+2}\, \sqrt{(2 j_1-j+1)!}\, \sqrt{(j+2 m_1)!}\, \sqrt{(j-2 m_1)!}}{\left(\frac{2 j_1-j}{2}\right)!\left(\frac{j+2 m_1}{2}\right)!\left(\frac{j-2 m_1}{2}\right)!\, \sqrt{j_1+m_1+1}\, \sqrt{j_1-m_1+1}\, \sqrt{j}\, \sqrt{j+1}\, \sqrt{(2 j_1+j+1)!}} \;/;\; \frac{2 j_1+j}{2} \in \mathbb{Z} \bigwedge \mathcal{P}hysical\mathcal{Q}(\{j_1, m_1\}, \{j_1+1, m_1\}, \{j, 2 m_1\})$$

07.38.03.0022.01

$$\langle j_1\, j_1+1\, m_1\, m_1-1 \mid j_1\, j_1+1\, j\, 2\, m_1-1\rangle = (-1)^{\frac{j-2 j_1}{2}} \left(\left(\frac{2 j_1+j}{2}\right)!\, \sqrt{j-2 m_1+1}\, \sqrt{j+2 m_1}\, \sqrt{2 j+1}\, \sqrt{2 j_1+j+2}\, \sqrt{(2 j_1-j+1)!}\, \sqrt{(j+2 m_1)!}\, \sqrt{(j-2 m_1)!}\right) \Big/ \left(2\left(\frac{2 j_1-j}{2}\right)!\left(\frac{j+2 m_1}{2}\right)!\left(\frac{j-2 m_1}{2}\right)!\, \sqrt{j_1-m_1+2}\, \sqrt{j_1-m_1+1}\, \sqrt{j}\, \sqrt{j+1}\, \sqrt{(2 j_1+j+1)!}\right) \;/;\; \frac{2 j_1+j}{2} \in \mathbb{Z} \bigwedge \mathcal{P}hysical\mathcal{Q}(\{j_1, m_1\}, \{j_1+1, m_1-1\}, \{j, 2 m_1-1\})$$

07.38.03.0023.01

$$\langle j_1\, j_1+1\, m_1\, m_1+1 \mid j_1\, j_1+1\, j\, 2\,m_1+1\rangle = (-1)^{\frac{j-2 j_1-1}{2}} (j(j+1)+(2 m_1+1)(2 j_1+2)) \frac{\left(\frac{2 j_1+j+1}{2}\right)! \sqrt{(j+2 m_1+1)!} \sqrt{(j-2 m_1-1)!} \sqrt{2 j+1} \sqrt{(2 j_1-j+1)!}}{2 \left(\frac{j+2 m_1+1}{2}\right)! \left(\frac{j-2 m_1-1}{2}\right)! \left(\frac{2 j_1-j+1}{2}\right)! \sqrt{j_1+m_1+1} \sqrt{j_1+m_1+2} \sqrt{j} \sqrt{j+1} \sqrt{(2 j_1+j+2)!}} \,/;\, \frac{2 j_1+j-1}{2} \in \mathbb{Z} \bigwedge \mathcal{P}hysical\mathcal{Q}(\{j_1, m_1\}, \{j_1+1, m_1+1\}, \{j, 2 m_1+1\})$$

07.38.03.0024.01

$$\langle j_1\, j_1+1\, m_1\, m_1 \mid j_1\, j_1+1\, j\, 2\,m_1\rangle = (-1)^{\frac{j-2 j_1-1}{2}} \frac{(2 j_1+2) \left(\frac{2 j_1+j+1}{2}\right)! \sqrt{(j+2 m_1)!} \sqrt{(j-2 m_1)!} \sqrt{2 j+1} \sqrt{(2 j_1-j+1)!}}{\left(\frac{2 j_1-j+1}{2}\right)! \left(\frac{j+2 m_1-1}{2}\right)! \left(\frac{j-2 m_1-1}{2}\right)! \sqrt{j_1+m_1+1} \sqrt{j_1-m_1+1} \sqrt{j} \sqrt{j+1} \sqrt{(2 j_1+j+2)!}} \,/;\, \frac{2 j_1+j-1}{2} \in \mathbb{Z} \bigwedge \mathcal{P}hysical\mathcal{Q}(\{j_1, m_1\}, \{j_1+1, m_1\}, \{j, 2 m_1\})$$

07.38.03.0025.01

$$\langle j_1\, j_1+1\, m_1\, m_1-1 \mid j_1\, j_1+1\, j\, 2\,m_1-1\rangle = (-1)^{\frac{j-2 j_1-1}{2}} (j(j+1)-(2 m_1-1)(2 j_1+2)) \frac{\left(\frac{2 j_1+j+1}{2}\right)! \sqrt{(j+2 m_1-1)!} \sqrt{(j-2 m_1+1)!} \sqrt{2 j+1} \sqrt{(2 j_1-j+1)!}}{2 \left(\frac{j-2 m_1+1}{2}\right)! \left(\frac{j+2 m_1-1}{2}\right)! \left(\frac{2 j_1-j+1}{2}\right)! \sqrt{j_1-m_1+1} \sqrt{j_1-m_1+2} \sqrt{j} \sqrt{j+1} \sqrt{(2 j_1+j+2)!}} \,/;\, \frac{2 j_1+j-1}{2} \in \mathbb{Z} \bigwedge \mathcal{P}hysical\mathcal{Q}(\{j_1, m_1\}, \{j_1+1, m_1-1\}, \{j, 2 m_1-1\})$$

Fixed j_1, j_2, j

07.38.03.0026.01

$$\langle j_1\, j_2\, 0\, 0 \mid j_1\, j_2\, j\, 0\rangle = 0 \,/;\, \frac{j+j_1+j_2-1}{2} \in \mathbb{N}$$

07.38.03.0027.01

$$\langle j_1\, j_2\, 0\, 0 \mid j_1\, j_2\, j\, 0\rangle = (-1)^{\frac{j_1+j_2-j}{2}} \frac{\left(\frac{j_1+j_2+j}{2}\right)! \sqrt{2 j+1} \sqrt{(-j_1+j_2+j)!} \sqrt{(j_1-j_2+j)!} \sqrt{(j_1+j_2-j)!}}{\left(\frac{-j_1+j_2+j}{2}\right)! \left(\frac{j_1-j_2+j}{2}\right)! \left(\frac{j_1+j_2-j}{2}\right)! \sqrt{(j_1+j_2+j+1)!}} \,/;\, \frac{j_1+j_2+j}{2} \in \mathbb{Z} \bigwedge \mathcal{T}riangular\mathcal{Q}(j_1, j_2, j)$$

Fixed j_1, j_2

07.38.03.0028.01

$$\langle j_1\, j_2\, 0\, 0 \mid j_1\, j_2\, j_1+j_2\, 0\rangle = \frac{(j_1+j_2)! \sqrt{(2 j_1)!} \sqrt{(2 j_2)!}}{j_1!\, j_2! \sqrt{(2 j_1+2 j_2)!}} \,/;\, j_1 \in \mathbb{N} \bigwedge j_2 \in \mathbb{N}$$

07.38.03.0029.01

$$\langle j_1\, j_2\, 0\, 0 \mid j_1\, j_2\, j_1-j_2\, 0\rangle = (-1)^{j_2} \frac{j_1! \sqrt{(2 j_2)!} \sqrt{(2 j_1-2 j_2+1)!}}{j_2!\, (j_1-j_2)! \sqrt{(2 j_1+1)!}} \,/;\, j_1 \in \mathbb{N} \bigwedge j_2 \in \mathbb{N} \bigwedge j_1 \geq j_2$$

07.38.03.0030.01

$$\langle j_1\, j_2\, j_1\, j_2 \mid j_1\, j_2\, j_1+j_2\, j_1+j_2\rangle = 1 \;/;\; 2\, j_1 \in \mathbb{N} \bigwedge 2\, j_2 \in \mathbb{N}$$

07.38.03.0031.01

$$\langle j_1\, j_2\, j_1\, -j_2 \mid j_1\, j_2\, j_1+j_2\, j_1-j_2\rangle = \frac{\sqrt{(2\, j_1)!}\, \sqrt{(2\, j_2)!}}{\sqrt{(2\, j_1+2\, j_2)!}} \;/;\; 2\, j_1 \in \mathbb{N} \bigwedge 2\, j_2 \in \mathbb{N}$$

07.38.03.0032.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j_1+j_2-1\, m_1+m_2\rangle = 0 \;/;\; j_1\, m_2 = j_2\, m_1$$

07.38.03.0033.01

$$\langle j_1\, j_2\, j_1\, -j_2 \mid j_1\, j_2\, j_1+j_2-1\, j_1-j_2\rangle = \frac{\sqrt{(2\, j_1)!}\, \sqrt{(2\, j_2)!}\, \sqrt{2\, j_1+2\, j_2-1}}{\sqrt{(2\, j_1+2\, j_2)!}} \;/;\; 2\, j_1 \in \mathbb{N}^+ \bigwedge 2\, j_2 \in \mathbb{N}^+$$

07.38.03.0034.01

$$\langle j_1\, j_2\, j_1-1\, j_2 \mid j_1\, j_2\, j_1+j_2-1\, j_1+j_2-1\rangle = -\frac{\sqrt{j_2}}{\sqrt{j_1+j_2}} \;/;\; 2\, j_1 \in \mathbb{N}^+ \bigwedge 2\, j_2 \in \mathbb{N}$$

07.38.03.0035.01

$$\langle j_1\, j_2\, j_1\, j_2-1 \mid j_1\, j_2\, j_1+j_2-1\, j_1+j_2-1\rangle = \frac{\sqrt{j_1}}{\sqrt{j_1+j_2}} \;/;\; 2\, j_1 \in \mathbb{N} \bigwedge 2\, j_2 \in \mathbb{N}^+$$

07.38.03.0036.01

$$\langle j_1\, j_2\, j_1\, -j_2 \mid j_1\, j_2\, j_1-j_2+1\, j_1-j_2\rangle = \frac{\sqrt{j_2}\, \sqrt{2\, j_1-2\, j_2+3}}{\sqrt{2\, j_1+1}\, \sqrt{j_1+1}} \;/;\; 2\, j_1 \in \mathbb{N}^+ \bigwedge 2\, j_2 \in \mathbb{N}^+ \bigwedge j_1 \geq j_2$$

07.38.03.0037.01

$$\langle j_1\, j_2\, j_1\, -j_2 \mid j_1\, j_2\, j_1-j_2\, j_1-j_2\rangle = \frac{\sqrt{2\, j_1-2\, j_2+1}}{\sqrt{2\, j_1+1}} \;/;\; 2\, j_1 \in \mathbb{N} \bigwedge 2\, j_2 \in \mathbb{N} \bigwedge j_1 \geq j_2$$

07.38.03.0038.01

$$\langle j_1\, j_2\, j_1\, -j_2 \mid j_1\, j_2\, j_1+j_2-2\, j_1-j_2\rangle = \frac{\sqrt{(2\, j_1)!}\, \sqrt{(2\, j_2)!}\, \sqrt{2\, j_1+2\, j_2-3}}{\sqrt{2}\, \sqrt{(2\, j_1+2\, j_2-1)!}} \;/;\; 2\, j_1-2 \in \mathbb{N} \bigwedge 2\, j_2-2 \in \mathbb{N}$$

07.38.03.0039.01

$$\langle j_1\, j_2\, j_1\, -j_2 \mid j_1\, j_2\, j_1-j_2+2\, j_1-j_2\rangle = \frac{\sqrt{2}\, \sqrt{j_2}\, \sqrt{2\, j_2-1}\, \sqrt{2\, j_1-2\, j_2+5}}{\sqrt{2\, j_1+1}\, \sqrt{2\, j_1+2}\, \sqrt{2\, j_1+3}} \;/;\; 2\, j_1 \in \mathbb{N}^+ \bigwedge 2\, j_2-2 \in \mathbb{N} \bigwedge j_1 > j_2-1$$

Fixed j_1, m_1

07.38.03.0040.01

$$\langle j_1\, j_1\, m_1\, m_1 \mid j_1\, j_1\, 2\, m_1+1\, 2\, m_1\rangle = 0$$

General characteristics

Domain and analyticity

$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle$ is a six-argument function. The condition $\mathcal{P}hysical\mathcal{Q}$ restricts the arguments to integers or half-integers (interpretated as quantum-mechanical spin quantum numbers) that fulfill certain inequalities. Without the condition $\mathcal{P}hysical\mathcal{Q}$ and the prefactor ensuring $m_1 + m_2 == m$, $\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle$ would be an analytic function in all six arguments.

07.38.04.0001.01

$$(\{j_1 * m_1\} * \{j_2 * m_2\} * \{j * m\}) \longrightarrow \langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle :: (\{\mathbb{Q} \otimes \mathbb{Q}\} \otimes \{\mathbb{Q} \otimes \mathbb{Q}\} \otimes \{\mathbb{Q} \otimes \mathbb{Q}\}) \longrightarrow \mathbb{C}$$

Symmetries and periodicities

Reflection symmetry

07.38.04.0002.01

$$\langle j_2\, j_1\, -m_2\, -m_1 \mid j_2\, j_1\, j\, -m\rangle == \langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle$$

Series representations

Generalized power series

07.38.06.0001.02

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle == \frac{\delta_{m,m_1+m_2}}{\sqrt{(j+j_1+j_2+1)!}} \sqrt{(j_1+j_2-j)!}\, \sqrt{(j+j_1-j_2)!}\, \sqrt{(j-j_1+j_2)!}\, \sqrt{2\,j+1}\, \sqrt{(j_1+m_1)!}\, \sqrt{(j_1-m_1)!}\, \sqrt{(j_2+m_2)!}\, \sqrt{(j_2-m_2)!}\, \sqrt{(j+m)!}\, \sqrt{(j-m)!} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(j_1+j_2-j-k)!\,(j_1-m_1-k)!\,(j_2+m_2-k)!\,(j-j_2+m_1+k)!\,(j-j_1-m_2+k)!} \,/;\, \mathcal{P}hysical\mathcal{Q}(\{j_1,\, m_1\},\, \{j_2,\, m_2\},\, \{j,\, m\})$$

07.38.06.0002.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle == \frac{\delta_{m,m_1+m_2}}{\sqrt{(j+j_1+j_2+1)!}} \sqrt{2\,j+1}\, \sqrt{(j-m)!}\, \sqrt{(j+m)!}\, \sqrt{(j+j_1-j_2)!}\, \sqrt{(j-j_1+j_2)!}\, \sqrt{(j_1+j_2-j)!}\, \sqrt{(j_1-m_1)!}\, \sqrt{(j_1+m_1)!}\, \sqrt{(j_2-m_2)!}\, \sqrt{(j_2+m_2)!} \sum_{k=-\infty}^{\infty} \frac{\theta(k,\, j_1+j_2-j-k,\, j_1-m_1-k,\, j_2+m_2-k,\, j+k-j_2+m_1,\, j+k-j_1-m_2)}{k!\,(j_1+j_2-j-k)!\,(j_1-m_1-k)!\,(j_2+m_2-k)!\,(j+k-j_2+m_1)!\,(j+k-j_1-m_2)!} \,/;\, \mathcal{P}hysical\mathcal{Q}(\{j_1,\, m_1\},\, \{j_2,\, m_2\},\, \{j,\, m\})$$

07.38.06.0004.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle == \frac{\delta_{m,m_1+m_2}}{\sqrt{(j+j_1+j_2+1)!}} \sqrt{(-j+j_1+j_2)!}\, \sqrt{(j+j_1-j_2)!}\, \sqrt{(j-j_1+j_2)!}\, \sqrt{2\,j+1}\, \sqrt{(j_1+m_1)!}\, \sqrt{(j_1-m_1)!}\, \sqrt{(j_2+m_2)!}\, \sqrt{(j_2-m_2)!}\, \sqrt{(j+m)!}\, \sqrt{(j-m)!} \sum_{k=\max(-j+j_2-m_1,-j+j_1+m_2,0)}^{\min(j_1-m_1,j_2+m_2)} \frac{(-1)^k}{k!\,(-j-k+j_1+j_2)!\,(-k+j_1-m_1)!\,(-k+j_2+m_2)!\,(j+k-j_2+m_1)!\,(j+k-j_1-m_2)!} \,/;\, \mathcal{P}hysical\mathcal{Q}(\{j_1,\, m_1\},\, \{j_2,\, m_2\},\, \{j,\, m\})$$

07.38.06.0005.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle = \delta_{m,m_1+m_2} \frac{\sqrt{(j_1+j_2-j)!}}{\sqrt{(1+j_1+j_2+j)!}\,\sqrt{(j_1-j_2+j)!}\,\sqrt{(-j_1+j_2+j)!}} \frac{\sqrt{(j_1-m_1)!}\,\sqrt{(j_2-m_2)!}\,\sqrt{(j+m)!}\,\sqrt{(j-m)!}\,\sqrt{2\,j+1}}{\sqrt{(j_1+m_1)!}\,\sqrt{(j_2+m_2)!}} \sum_{k=\max(0,j-j_2-m_1)}^{\min(j_1-m_1,j-m)} \frac{(-1)^{j_1-m_1+k}\,(j_1+m_1+k)!\,(j+j_2-m_1-k)!}{k!\,(j_1-m_1-k)!\,(j-m-k)!\,(-j+j_2+m_1+k)!} \;/;\; \mathcal{P}hysical\mathcal{Q}(\{j_1, m_1\}, \{j_2, m_2\}, \{j, m\})$$

07.38.06.0006.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle = \delta_{m,m_1+m_2} \frac{\sqrt{(j_1+j_2-j)!}\,\sqrt{(j_1-j_2+j)!}\,\sqrt{(-j_1+j_2+j)!}}{\sqrt{(1+j_1+j_2+j)!}} \frac{\sqrt{(j+m)!}\,\sqrt{(j-m)!}\,\sqrt{2\,j+1}}{\sqrt{(j_1+m_1)!}\,\sqrt{(j_1-m_1)!}\,\sqrt{(j_2+m_2)!}\,\sqrt{(j_2-m_2)!}} \sum_{k=\max(0,-j+m+j_2)}^{\min(j-j_1+j_2,j+m)} \frac{(-1)^{j_2+m_2+k}\,(j+j_2+m_1-k)!\,(j_1-m_1+k)!}{k!\,(j-j_1+j_2-k)!\,(j+m-k)!\,(j_1-j_2-m+k)!} \;/;\; \mathcal{P}hysical\mathcal{Q}(\{j_1, m_1\}, \{j_2, m_2\}, \{j, m\})$$

07.38.06.0007.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle = \delta_{m,m_1+m_2} \frac{\sqrt{(-j_1+j_2+j)!}}{\sqrt{(j_1-j_2+j)!}\,\sqrt{(j_1+j_2-j)!}\,\sqrt{(1+j_1+j_2+j)!}} \frac{\sqrt{(j_1+m_1)!}\,\sqrt{(j_1-m_1)!}\,\sqrt{(j_2-m_2)!}\,\sqrt{(j+m)!}\,\sqrt{2\,j+1}}{\sqrt{(j_2+m_2)!}\,\sqrt{(j-m)!}} \sum_{k=\max(0,-j_1+j+m_2)}^{\min(j-j_1+j_2,j+m)} \frac{(-1)^{j_2+m_2+k}\,(2\,j-k)!\,(j_1+j_2-j+k)!}{k!\,(j-j_1+j_2-k)!\,(j+m-k)!\,(j_1-j-m_2+k)!} \;/;\; \mathcal{P}hysical\mathcal{Q}(\{j_1, m_1\}, \{j_2, m_2\}, \{j, m\})$$

07.38.06.0008.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle = \delta_{m,m_1+m_2} \frac{\sqrt{(-j_1+j_2+j)!}\,\sqrt{(j_1+j_2-j)!}\,\sqrt{(1+j_1+j_2+j)!}}{\sqrt{(j_1-j_2+j)!}} \frac{\sqrt{(j_1-m_1)!}\,\sqrt{(j+m)!}\,\sqrt{2\,j+1}}{\sqrt{(j_1+m_1)!}\,\sqrt{(j_2+m_2)!}\,\sqrt{(j_2-m_2)!}\,\sqrt{(j-m)!}} \sum_{k=0}^{\min(j-j_1+j_2,j+m)} \frac{(-1)^{j_2+m_2+k}\,(2\,j-k)!\,(j_2+j+m_1-k)!}{k!\,(j-j_1+j_2-k)!\,(j+m-k)!\,(j_1+j_2+j+1-k)!} \;/;\; \mathcal{P}hysical\mathcal{Q}(\{j_1, m_1\}, \{j_2, m_2\}, \{j, m\})$$

07.38.06.0009.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle = \delta_{m,m_1+m_2} \frac{\sqrt{(1+j_1+j_2+j)!}}{\sqrt{(j_1+j_2-j)!}\,\sqrt{(j_1-j_2+j)!}\,\sqrt{(-j_1+j_2+j)!}} \frac{\sqrt{(j_1+m_1)!}\,\sqrt{(j_1-m_1)!}\,\sqrt{(j+m)!}\,\sqrt{(j-m)!}\,\sqrt{2\,j+1}}{\sqrt{(j_2+m_2)!}\,\sqrt{(j_2-m_2)!}} \sum_{k=0}^{\min(j-m,j_1-m_1)} \frac{(-1)^{j_1-m_1+k}\,(j_1+j_2-m-k)!\,(j_2+j-m_1-k)!}{k!\,(j_1-m_1-k)!\,(j-m-k)!\,(j_1+j_2+j+1-k)!} \;/;\; \mathcal{P}hysical\mathcal{Q}(\{j_1, m_1\}, \{j_2, m_2\}, \{j, m\})$$

Other series representations

Series of binomial coefficients

07.38.06.0003.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle = \delta_{m,m_1+m_2} \frac{\sqrt{\binom{2 j_1}{-j+j_1+j_2}} \sqrt{\binom{2 j_2}{-j+j_1+j_2}}}{\sqrt{\binom{j+j_1+j_2+1}{-j+j_1+j_2}} \sqrt{\binom{2 j_1}{j_1-m_1}} \sqrt{\binom{2 j_2}{j_2-m_2}} \sqrt{\binom{2 j}{j-m}}} \sum_{k=\max(0,-j+j_2-m_1,-j+j_1+m_2)}^{\min(-j+j_1+j_2,j_1-m_1,j_2+m_2)} (-1)^k \binom{-j+j_1+j_2}{k} \binom{j+j_1-j_2}{-k+j_1-m_1} \binom{j-j_1+j_2}{-k+j_2+m_2} /; \mathcal{P}hysical\mathcal{Q}(\{j_1, m_1\}, \{j_2, m_2\}, \{j, m\})$$

Integral representations

On the real axis

Of the direct function

07.38.07.0001.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle = \delta_{m,m_1+m_2} \frac{(-1)^{j_1-j+m_2}}{2^{j_1+j_2+j+1}} \left(\sqrt{(j+m)!}\, \sqrt{(j_1+j_2-j)!}\, \sqrt{(j_1+j_2+j+1)!}\, \sqrt{2j+1}\right) \Big/ \left(\sqrt{(j_1-m_1)!}\, \sqrt{(j_1+m_1)!}\, \sqrt{(j_2-m_2)!}\, \sqrt{(j_2+m_2)!}\, \sqrt{(j-m)!}\, \sqrt{(-j_1+j_2+j)!}\, \sqrt{(j_1-j_2+j)!}\right) \int_{-1}^{1} (1-t)^{j_1-m_1} (1+t)^{j_2-m_2} \frac{\partial^{j-m}\left((1-t)^{j-j_1+j_2} (1+t)^{j+j_1-j_2}\right)}{\partial t^{j-m}} \, dt /; \mathcal{P}hysical\mathcal{Q}(\{j_1, m_1\}, \{j_2, m_2\}, \{j, m\})$$

07.38.07.0002.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle = \delta_{m,m_1+m_2} \frac{(-1)^{j_1-j+m_2}}{2^{j_1+j_2+j+1}} \frac{\sqrt{(j_1-j_2+j)!}\, \sqrt{(j_1+j_2-j)!}\, \sqrt{(j_1+j_2+j+1)!}\, \sqrt{2j+1}}{\sqrt{(j_1-m_1)!}\, \sqrt{(j_1+m_1)!}\, \sqrt{(j_2-m_2)!}\, \sqrt{(j_2+m_2)!}\, \sqrt{(j-m)!}\, \sqrt{(j+m)!}\, \sqrt{(-j_1+j_2+j)!}} \int_{-1}^{1} (1-t)^{j_2+m_2} (1+t)^{j_2-m_2} \frac{\partial^{j-j_1+j_2}\left((1-t)^{j-m} (1+t)^{j+m}\right)}{\partial t^{j-j_1+j_2}} \, dt /; \mathcal{P}hysical\mathcal{Q}(\{j_1, m_1\}, \{j_2, m_2\}, \{j, m\})$$

07.38.07.0003.01

$$\langle j_1\, j_2\, 0\, 0 \mid j_1\, j_2\, j\, 0\rangle = \frac{(-1)^{j_1-j}}{2^{j_1+j_2+j+1}} \frac{\sqrt{(j_1-j_2+j)!}\, \sqrt{(j_1+j_2-j)!}\, \sqrt{(j_1+j_2+j+1)!}\, \sqrt{2j+1}}{j_1!\, j_2!\, j!\, \sqrt{(-j_1+j_2+j)!}} \int_{-1}^{1} \left(1-t^2\right)^{j_2} \frac{\partial^{j-j_1+j_2}\left(1-t^2\right)^{j}}{\partial t^{j-j_1+j_2}} \, dt /; \mathcal{P}hysical\mathcal{Q}(\{j_1, m_1\}, \{j_2, m_2\}, \{j, m\})$$

07.38.07.0004.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle = \delta_{m,m_1+m_2} \frac{(-1)^{j_1+j-m_1-m}}{2^{j_1+j_2+1}} \frac{\sqrt{(j_1+j_2-j)!}\,\sqrt{(j_1+j_2+j+1)!}\,\sqrt{2j+1}}{\sqrt{(j_1+m_1)!}\,\sqrt{(j_1-m_1)!}\,\sqrt{(j_2+m_2)!}\,\sqrt{(j_2-m_2)!}} \int_{-1}^{1}\left(1-t^2\right)^{\frac{j_1+j_2}{2}}\left(\frac{1-t}{1+t}\right)^{\frac{m_1-m_2}{2}} d^{j}_{j_2-j_1,m}\left(\cos^{-1}(t)\right) dt \;/;\; \mathcal{P}hysical\mathcal{Q}(\{j_1, m_1\}, \{j_2, m_2\}, \{j, m\})$$

Multiple integral representations

For the direct function itself

07.38.07.0005.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle = \delta_{m,m_1+m_2} \frac{(-1)^{j+m}(2i)^{j+j_1+j_2}}{\pi^2} \frac{\sqrt{2j+1}\,\sqrt{(j-m)!}\,\sqrt{(j+m)!}\,\sqrt{(j_1-m_1)!}\,\sqrt{(j_1+m_1)!}\,\sqrt{(j_2-m_2)!}\,\sqrt{(j_2+m_2)!}}{\sqrt{(j+j_1-j_2)!}\,\sqrt{(j-j_1+j_2)!}\,\sqrt{(j_1+j_2-j)!}\,\sqrt{(j+j_1+j_2+1)!}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{2i\varphi m_1+2i\vartheta m_2} \sin^{j-j_1+j_2}(\vartheta)\sin(\vartheta-\varphi)^{-j+j_1+j_2}\sin^{j+j_1-j_2}(\varphi)\, d\varphi\, d\vartheta \;/;\; \mathcal{P}hysical\mathcal{Q}(\{j_1, m_1\}, \{j_2, m_2\}, \{j, m\})$$

07.38.07.0006.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle = \frac{2\, i^{j_1+j_2-j}\sqrt{\pi}\left(\frac{j+j_1-j_2}{2}\right)!\left(\frac{j-j_1+j_2}{2}\right)!\left(\frac{j_1+j_2-j}{2}\right)!\sqrt{(j_1+j_2+j+1)!}}{\sqrt{(j+j_1-j_2)!}\,\sqrt{(j-j_1+j_2)!}\,\sqrt{(j_1+j_2-j)!}\left(\frac{j+j_1+j_2}{2}\right)!\sqrt{2j_1+1}\,\sqrt{2j_2+1}} \int_0^{\pi}\int_0^{2\pi}\sin(\vartheta)\, Y_{j_1}^{m_1}(\vartheta,\varphi)\, Y_{j_2}^{m_2}(\vartheta,\varphi)\, \overline{Y_j^m(\vartheta,\varphi)}\, d\varphi\, d\vartheta \;/;\; \frac{j+j_1+j_2}{2}\in\mathbb{N} \bigwedge m\in\mathbb{Z} \bigwedge m_1\in\mathbb{Z} \bigwedge m_2\in\mathbb{Z} \bigwedge \mathcal{P}hysical\mathcal{Q}(\{j_1, m_1\}, \{j_2, m_2\}, \{j, m\})$$

07.38.07.0007.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle = \frac{1}{8\pi^2} \frac{\sqrt{(j_1+j_2+j+1)!}\,\sqrt{(j_1+j_2-j)!}\,\sqrt{2j+1}}{\sqrt{(2j_1)!}\,\sqrt{(2j_2)!}} \int_0^{2\pi}\int_0^{\pi}\int_0^{2\pi}\sin(\beta)\, D^{j_1}_{m_1,j_1}(\alpha,\beta,\gamma)\, D^{j_2}_{m_2,-j_2}(\alpha,\beta,\gamma)\, \overline{D^{j}_{m,j_1-j_2}(\alpha,\beta,\gamma)}\, d\gamma\, d\beta\, d\alpha \;/;\; \mathcal{P}hysical\mathcal{Q}(\{j_1, m_1\}, \{j_2, m_2\}, \{j, m\})$$

Involving the direct function

07.38.07.0008.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle \langle j_1\, j_2\, n_1\, n_2 \mid j_1\, j_2\, j\, n\rangle = \frac{2j+1}{8\pi^2}\int_0^{2\pi}\int_0^{\pi}\int_0^{2\pi}\sin(\beta)\, D^{j_1}_{m_1,n_1}(\alpha,\beta,\gamma)\, D^{j_2}_{m_2,n_2}(\alpha,\beta,\gamma)\, \overline{D^{j}_{m,n}(\alpha,\beta,\gamma)}\, d\gamma\, d\beta\, d\alpha \;/;\; \mathcal{P}hysical\mathcal{Q}(\{j_1, m_1\}, \{j_2, m_2\}, \{j, m\}) \bigwedge \mathcal{P}hysical\mathcal{Q}(\{j_1, n_1\}, \{j_2, n_2\}, \{j, n\})$$

07.38.07.0009.01

$$|\langle j_1\, j_2\, 0\, 0 \mid j_1\, j_2\, j\, 0\rangle|^2 = \frac{2j+1}{2}\int_0^{\pi}\sin(\vartheta)\, P_{j_1}(\cos(\vartheta))\, P_{j_2}(\cos(\vartheta))\, P_j(\cos(\vartheta))\, d\vartheta \;/;\; j_1\in\mathbb{N} \bigwedge j_2\in\mathbb{N} \bigwedge j\in\mathbb{N} \bigwedge |j_1-j_2|\le j\le j_1+j_2$$

Integral representations of negative integer order

07.38.07.0010.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle = \delta_{m,m_1+m_2} (-1)^{j_1+j_2-j} \frac{\sqrt{(j_1-j_2+j)!}}{\sqrt{(j_1+j_2-j)!}\,\sqrt{(-j_1+j_2+j)!}\,\sqrt{(j_1+j_2+j+1)!}}$$
$$\frac{\sqrt{(j_1+m_1)!}\,\sqrt{(j_1-m_1)!}\,\sqrt{(j_2+m_2)!}\,\sqrt{(j+m)!}\,\sqrt{2\,j+1}}{(j_1-j_2+m)!\,\sqrt{(j_2-m_2)!}\,\sqrt{(j-m)!}}$$
$$\frac{\partial^{j_2-m_2}\left((1-t)^{-j+j_1+j_2}\,{}_2F_1(-j+j_1-j_2,\, m-j;\, m+j_1-j_2+1;\, t)\right)}{\partial t^{j_2-m_2}}\Big|_{t=0} /;\, \mathcal{PhysicalQ}(\{j_1,\, m_1\},\, \{j_2,\, m_2\},\, \{j,\, m\})$$

Generating functions

07.38.11.0001.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle = (-1)^{m+j_1-j_2}$$
$$\left(\sqrt{2\,j+1}\,\sqrt{(j_1+j_2+j+1)!}\,\sqrt{(j_1+m_1)!}\,\sqrt{(j_1-m_1)!}\,\sqrt{(j_2+m_2)!}\,\sqrt{(j_2-m_2)!}\,\sqrt{(j-m)!}\,\sqrt{(j+m)!}\right)\Big/$$
$$\left(\sqrt{(j_1+j_2-j)!}\,\sqrt{(j_2+j-j_1)!}\,\sqrt{(j+j_1-j_2)!}\right)$$
$$\left(\left[x^{j_1+m_1},\, x_1^{j_1-m_1},\, y^{j_2+m_2},\, y_1^{j_2-m_2},\, z^{j-m},\, z_1^{j+m}\right] e^{z\,x_1-y_1\,z-y\,x_1-xz_1+x\,y_1+y\,z_1}\right) /;\, \mathcal{PhysicalQ}(\{j_1,\, m_1\},\, \{j_2,\, m_2\},\, \{j,\, m\})$$

Identities

Recurrence identities

Consecutive neighbors

07.38.17.0001.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle =$$
$$\frac{\sqrt{j_1+m_1}\,\sqrt{j_1-m_1+1}}{\sqrt{j-m+1}\,\sqrt{j+m}} \langle j_1\, j_2\, m_1-1\, m_2 \mid j_1\, j_2\, j\, m-1\rangle + \frac{\sqrt{j_2+m_2}\,\sqrt{j_2-m_2+1}}{\sqrt{j-m+1}\,\sqrt{j+m}} \langle j_1\, j_2\, m_1\, m_2-1 \mid j_1\, j_2\, j\, m-1\rangle$$

07.38.17.0002.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle =$$
$$\frac{\sqrt{j_1-m_1}\,\sqrt{j_1+m_1+1}}{\sqrt{j-m}\,\sqrt{j+m+1}} \langle j_1\, j_2\, m_1+1\, m_2 \mid j_1\, j_2\, j\, m+1\rangle + \frac{\sqrt{j_2-m_2}\,\sqrt{j_2+m_2+1}}{\sqrt{j-m}\,\sqrt{j+m+1}} \langle j_1\, j_2\, m_1\, m_2+1 \mid j_1\, j_2\, j\, m+1\rangle$$

07.38.17.0003.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle = -\frac{(2\,j_1-1)\,(2\,m_2\,j_1\,(j_1-1)+m_1\,(j_1\,(j_1-1)+j_2\,(j_2+1)-j\,(j+1)))}{(j_1-1)\,\sqrt{j_1-m_1}\,\sqrt{j_1+m_1}\,\sqrt{-j_1+j_2+j+1}\,\sqrt{j_1-j_2+j}\,\sqrt{j_1+j_2-j}\,\sqrt{j_1+j_2+j+1}}$$
$$\langle j_1-1\, j_2\, m_1\, m_2 \mid j_1-1\, j_2\, j\, m\rangle -$$
$$\frac{j_1\,\sqrt{j_1+m_1-1}\,\sqrt{j_1-m_1-1}\,\sqrt{-j_1+j_2+j+2}\,\sqrt{j_1-j_2+j-1}\,\sqrt{j_1+j_2-j-1}\,\sqrt{j_1+j_2+j}}{(j_1-1)\,\sqrt{j_1+m_1}\,\sqrt{j_1-m_1}\,\sqrt{-j_1+j_2+j+1}\,\sqrt{j_1-j_2+j}\,\sqrt{j_1+j_2-j}\,\sqrt{j_1+j_2+j+1}}$$
$$\langle j_1-2\, j_2\, m_1\, m_2 \mid j_1-2\, j_2\, j\, m\rangle$$

07.38.17.0004.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle = -((2\, j_1+3)\,(2\, m_2\,(j_1+1)\,(j_1+2)+m_1\,((j_1+1)\,(j_1+2)+j_2\,(j_2+1)-j\,(j+1)))) \Big/ \left((j_1+2)\sqrt{j_1-m_1+1}\,\sqrt{j_1+m_1+1}\,\sqrt{-j_1+j_2+j}\,\sqrt{j_1-j_2+j+1}\,\sqrt{j_1+j_2-j+1}\,\sqrt{j_1+j_2+j+2}\right)\langle j_1+1\, j_2\, m_1\, m_2 \mid j_1+1\, j_2\, j\, m\rangle - \left((j_1+1)\sqrt{j_1-m_1+2}\,\sqrt{j_1+m_1+2}\,\sqrt{-j_1+j_2+j-1}\,\sqrt{j_1-j_2+j+2}\,\sqrt{j_1+j_2-j+2}\,\sqrt{j_1+j_2+j+3}\right) \Big/ \left((j_1+2)\sqrt{j_1-m_1+1}\,\sqrt{j_1+m_1+1}\,\sqrt{-j_1+j_2+j}\,\sqrt{j_1-j_2+j+1}\,\sqrt{j_1+j_2-j+1}\,\sqrt{j_1+j_2+j+2}\right)\langle j_1+2\, j_2\, m_1\, m_2 \mid j_1+2\, j_2\, j\, m\rangle$$

07.38.17.0005.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle = \frac{(2\, j_2-1)\,(2\, m_1\, j_2\,(j_2-1)+m_2\,(j_2\,(j_2-1)+j_1\,(j_1+1)-j\,(j+1)))}{(j_2-1)\sqrt{j_2-m_2}\,\sqrt{j_2+m_2}\,\sqrt{j_1-j_2+j+1}\,\sqrt{-j_1+j_2+j}\,\sqrt{j_1+j_2-j}\,\sqrt{j_1+j_2+j+1}}\langle j_1\, j_2-1\, m_1\, m_2 \mid j_1\, j_2-1\, j\, m\rangle - \frac{j_2\sqrt{j_2-m_2-1}\,\sqrt{j_2+m_2-1}\,\sqrt{j_1-j_2+j+2}\,\sqrt{-j_1+j_2+j-1}\,\sqrt{j_1+j_2-j-1}\,\sqrt{j_1+j_2+j}}{(j_2-1)\sqrt{j_2-m_2}\,\sqrt{j_2+m_2}\,\sqrt{j_1-j_2+j+1}\,\sqrt{-j_1+j_2+j}\,\sqrt{j_1+j_2-j}\,\sqrt{j_1+j_2+j+1}}\langle j_1\, j_2-2\, m_1\, m_2 \mid j_1\, j_2-2\, j\, m\rangle$$

07.38.17.0006.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle = ((2\, j_2+3)\,(2\, m_1\,(j_2+1)\,(j_2+2)+m_2\,((j_2+1)\,(j_2+2)+j_1\,(j_1+1)-j\,(j+1)))) \Big/ \left((j_2+2)\sqrt{j_2-m_2+1}\,\sqrt{j_2+m_2+1}\,\sqrt{j_1-j_2+j}\,\sqrt{-j_1+j_2+j+1}\,\sqrt{j_1+j_2-j+1}\,\sqrt{j_1+j_2+j+2}\right)\langle j_1\, j_2+1\, m_1\, m_2 \mid j_1\, j_2+1\, j\, m\rangle - \left((j_2+1)\sqrt{j_2-m_2+2}\,\sqrt{j_2+m_2+2}\,\sqrt{j_1-j_2+j-1}\,\sqrt{-j_1+j_2+j+2}\,\sqrt{j_1+j_2-j+2}\,\sqrt{j_1+j_2+j+3}\right) \Big/ \left((j_2+2)\sqrt{j_2-m_2+1}\,\sqrt{j_2+m_2+1}\,\sqrt{j_1-j_2+j}\,\sqrt{-j_1+j_2+j+1}\,\sqrt{j_1+j_2-j+1}\,\sqrt{j_1+j_2+j+2}\right)\langle j_1\, j_2+2\, m_1\, m_2 \mid j_1\, j_2+2\, j\, m\rangle$$

07.38.17.0007.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle = \frac{2\, j\sqrt{2\, j+1}\,\sqrt{2\, j-1}}{\sqrt{j-m}\,\sqrt{j+m}\,\sqrt{-j_1+j_2+j}\,\sqrt{j_1-j_2+j}\,\sqrt{j_1+j_2-j+1}\,\sqrt{j_1+j_2+j+1}} \left(\frac{j\,(j-1)\,(m_1-m_2)-m\, j_1\,(j_1+1)+m\, j_2\,(j_2+1)}{2\, j\,(j-1)}\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j-1\, m\rangle - \frac{\sqrt{j-m-1}\,\sqrt{j+m-1}\,\sqrt{-j_1+j_2+j-1}\,\sqrt{j_1-j_2+j-1}\,\sqrt{j_1+j_2-j+2}\,\sqrt{j_1+j_2+j}}{2\,(j-1)\sqrt{2\, j-3}\,\sqrt{2\, j-1}}\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j-2\, m\rangle\right)$$

07.38.17.0008.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle = \frac{2\,(j+1)\sqrt{2\,j+1}\,\sqrt{2\,j+3}}{\sqrt{j-m+1}\,\sqrt{j+m+1}\,\sqrt{-j_1+j_2+j+1}\,\sqrt{j_1-j_2+j+1}\,\sqrt{j_1+j_2-j}\,\sqrt{j_1+j_2+j+2}} \left(\frac{(j+1)\,(j+2)\,(m_1-m_2)-m\,j_1\,(j_1+1)+m\,j_2\,(j_2+1)}{2\,(j+1)\,(j+2)}\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j+1\, m\rangle - \left(\sqrt{j-m+2}\,\sqrt{j+m+2}\,\sqrt{-j_1+j_2+j+2}\,\sqrt{j_1-j_2+j+2}\,\sqrt{j_1+j_2-j-1}\,\sqrt{j_1+j_2+j+3}\right)\Big/ \left(2\,(j+2)\sqrt{2\,j+3}\,\sqrt{2\,j+5}\right)\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j+2\, m\rangle\right)$$

Functional identities

General relations

07.38.17.0068.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle = \frac{\sqrt{(j_2+m_2-2\,n)!}\,\sqrt{(j+m)!}\,\sqrt{(j_1+j_2-j)!}\,\sqrt{(-j_1+j_2+j)!}\,\sqrt{(j_1+j_2+j+1)!}\,\sqrt{2\,j+1}}{\sqrt{(j_2+m_2)!}\,\sqrt{(j-m)!}\,\sqrt{(j_1-j_2+j)!}} \sum_{k=j-n}^{j+n}(-1)^{-j+k+n}\left((j+k-n)!\,(2\,n)!\,\sqrt{(k-m+n)!}\,\sqrt{(j_1-j_2+k+n)!}\,\sqrt{2\,k+1}\right)\Big/\left((j-k+n)!\,(j+k+n+1)!\,(-j+k+n)!\sqrt{(k+m-n)!}\,\sqrt{(-j_1+j_2+k-n)!}\,\sqrt{(j_1+j_2-k-n)!}\,\sqrt{(j_1+j_2+k-n+1)!}\right) \langle j_1\, j_2-n\, m_1\, m_2-n \mid j_1\, j_2-n\, k\, m-n\rangle \;/;\; 2\,n\in\mathbb{N}\bigwedge n\le\frac{j_2+m_2}{2}\bigwedge \mathcal{P}hysical\mathcal{Q}(\{j_1,\,m_1\},\,\{j_2,\,m_2\},\,\{j,\,m\})$$

07.38.17.0069.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle = \left(\sqrt{(j_1+j_2+j+1)!}\,\sqrt{(j_1-j_2+j)!}\,\sqrt{(j_1+j_2-j)!}\,\sqrt{(j_2+m_2)!}\,\sqrt{(j_2-m_2)!}\,\sqrt{(j_1-j_2+m_1-m_2)!}\,\sqrt{(j_1-j_2-m_1+m_2)!}\right)\Big/\left(\sqrt{(-j_1+j_2+j)!}\,\sqrt{(j_1+m_1)!}\,\sqrt{(j_1-m_1)!}\right) \sum_{k=|m_2|}^{j_2}(-1)^{j_2-k}\left((4\,k+1)\,(j_2+k)!\sqrt{(-j_1+j_2+j+2\,k)!}\,\sqrt{(2\,k+2\,m_2)!}\,\sqrt{(2\,k-2\,m_2)!}\right)\Big/ \left((2\,j_2+2\,k+1)!\,(j_2-k)!\,(k+m_2)!\,(k-m_2)!\sqrt{(j_1-j_2+j+2\,k+1)!}\,\sqrt{(j_1-j_2+j-2\,k)!}\,\sqrt{(j_1-j_2-j+2\,k)!}\right) \langle j_1-j_2\, 2\,k\, m_1-m_2\, 2\,m_2 \mid j_1-j_2\, 2\,k\, j\, m\rangle \;/;\; |m_1-m_2|\le j_1-j_2\land \mathcal{P}hysical\mathcal{Q}(\{j_1,\,m_1\},\,\{j_2,\,m_2\},\,\{j,\,m\})$$

Arguments changing by 1/2

07.38.17.0011.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle = -\frac{\sqrt{j_1-m_1}\,\sqrt{j_1+j_2-j}\,\sqrt{j_1+j_2+j+1}}{(2\,j_1+1)\sqrt{j_2+m_2}}\left\langle j_1-\frac{1}{2}\; j_2-\frac{1}{2}\; m_1+\frac{1}{2}\; m_2-\frac{1}{2}\;\middle|\; j_1-\frac{1}{2}\; j_2-\frac{1}{2}\; j\, m\right\rangle + \frac{\sqrt{j_1+m_1+1}\,\sqrt{-j_1+j_2+j}\,\sqrt{j_1-j_2+j+1}}{(2\,j_1+1)\sqrt{j_2+m_2}}\left\langle j_1+\frac{1}{2}\; j_2-\frac{1}{2}\; m_1+\frac{1}{2}\; m_2-\frac{1}{2}\;\middle|\; j_1+\frac{1}{2}\; j_2-\frac{1}{2}\; j\, m\right\rangle$$

07.38.17.0012.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle = \frac{\sqrt{j_1+m_1}\,\sqrt{j_1+j_2-j}\,\sqrt{j_1+j_2+j+1}}{(2\,j_1+1)\sqrt{j_2-m_2}}\left\langle j_1-\frac{1}{2}\,j_2-\frac{1}{2}\,m_1-\frac{1}{2}\,m_2+\frac{1}{2}\;\middle|\;j_1-\frac{1}{2}\,j_2-\frac{1}{2}\,j\,m\right\rangle + \frac{\sqrt{j_1-m_1+1}\,\sqrt{-j_1+j_2+j}\,\sqrt{j_1-j_2+j+1}}{(2\,j_1+1)\sqrt{j_2-m_2}}\left\langle j_1+\frac{1}{2}\,j_2-\frac{1}{2}\,m_1-\frac{1}{2}\,m_2+\frac{1}{2}\;\middle|\;j_1+\frac{1}{2}\,j_2-\frac{1}{2}\,j\,m\right\rangle$$

07.38.17.0013.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle = \frac{\sqrt{j_1-m_1}\,\sqrt{j_1-j_2+j}\,\sqrt{-j_1+j_2+j+1}}{(2\,j_1+1)\sqrt{j_2-m_2+1}}\left\langle j_1-\frac{1}{2}\,j_2+\frac{1}{2}\,m_1+\frac{1}{2}\,m_2-\frac{1}{2}\;\middle|\;j_1-\frac{1}{2}\,j_2+\frac{1}{2}\,j\,m\right\rangle + \frac{\sqrt{j_1+m_1+1}\,\sqrt{j_1+j_2-j+1}\,\sqrt{j_1+j_2+j+2}}{(2\,j_1+1)\sqrt{j_2-m_2+1}}\left\langle j_1+\frac{1}{2}\,j_2+\frac{1}{2}\,m_1+\frac{1}{2}\,m_2-\frac{1}{2}\;\middle|\;j_1+\frac{1}{2}\,j_2+\frac{1}{2}\,j\,m\right\rangle$$

07.38.17.0014.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle = \frac{\sqrt{j_1+m_1}\,\sqrt{j_1-j_2+j}\,\sqrt{-j_1+j_2+j+1}}{(2\,j_1+1)\sqrt{j_2+m_2+1}}\left\langle j_1-\frac{1}{2}\,j_2+\frac{1}{2}\,m_1-\frac{1}{2}\,m_2+\frac{1}{2}\;\middle|\;j_1-\frac{1}{2}\,j_2+\frac{1}{2}\,j\,m\right\rangle - \frac{\sqrt{j_1-m_1+1}\,\sqrt{j_1+j_2-j+1}\,\sqrt{j_1+j_2+j+2}}{(2\,j_1+1)\sqrt{j_2+m_2+1}}\left\langle j_1+\frac{1}{2}\,j_2+\frac{1}{2}\,m_1-\frac{1}{2}\,m_2+\frac{1}{2}\;\middle|\;j_1+\frac{1}{2}\,j_2+\frac{1}{2}\,j\,m\right\rangle$$

07.38.17.0015.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle = \frac{\sqrt{j_1-m_1+1}\,\sqrt{j_2-m_2}}{\sqrt{-j_1+j_2+j}\,\sqrt{j_1-j_2+j+1}}\left\langle j_1+\frac{1}{2}\,j_2-\frac{1}{2}\,m_1-\frac{1}{2}\,m_2+\frac{1}{2}\;\middle|\;j_1+\frac{1}{2}\,j_2-\frac{1}{2}\,j\,m\right\rangle + \frac{\sqrt{j_1+m_1+1}\,\sqrt{j_2+m_2}}{\sqrt{-j_1+j_2+j}\,\sqrt{j_1-j_2+j+1}}\left\langle j_1+\frac{1}{2}\,j_2-\frac{1}{2}\,m_1+\frac{1}{2}\,m_2-\frac{1}{2}\;\middle|\;j_1+\frac{1}{2}\,j_2-\frac{1}{2}\,j\,m\right\rangle$$

07.38.17.0016.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle = \frac{\sqrt{j_1-m_1}\,\sqrt{j_2-m_2+1}}{\sqrt{j_1-j_2+j}\,\sqrt{-j_1+j_2+j+1}}\left\langle j_1-\frac{1}{2}\,j_2+\frac{1}{2}\,m_1+\frac{1}{2}\,m_2-\frac{1}{2}\;\middle|\;j_1-\frac{1}{2}\,j_2+\frac{1}{2}\,j\,m\right\rangle + \frac{\sqrt{j_1+m_1}\,\sqrt{j_2+m_2+1}}{\sqrt{j_1-j_2+j}\,\sqrt{-j_1+j_2+j+1}}\left\langle j_1-\frac{1}{2}\,j_2+\frac{1}{2}\,m_1-\frac{1}{2}\,m_2+\frac{1}{2}\;\middle|\;j_1-\frac{1}{2}\,j_2+\frac{1}{2}\,j\,m\right\rangle$$

07.38.17.0017.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle = \frac{\sqrt{j_2-m_2}\,\sqrt{j-m}\,\sqrt{2\,j+1}}{\sqrt{2}\,\sqrt{j}\,\sqrt{-j_1+j_2+j}\,\sqrt{j_1+j_2+j+1}}\left\langle j_1\, j_2-\frac{1}{2}\,m_1\, m_2+\frac{1}{2}\;\middle|\;j_1\, j_2-\frac{1}{2}\,j-\frac{1}{2}\,m+\frac{1}{2}\right\rangle + \frac{\sqrt{j_2+m_2}\,\sqrt{j+m}\,\sqrt{2\,j+1}}{\sqrt{2}\,\sqrt{j}\,\sqrt{-j_1+j_2+j}\,\sqrt{j_1+j_2+j+1}}\left\langle j_1\, j_2-\frac{1}{2}\,m_1\, m_2-\frac{1}{2}\;\middle|\;j_1\, j_2-\frac{1}{2}\,j-\frac{1}{2}\,m-\frac{1}{2}\right\rangle$$

07.38.17.0018.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle = \frac{\sqrt{j_1-m_1}\,\sqrt{j_1-j_2+j}\,\sqrt{2\,j+1}}{\sqrt{2}\,\sqrt{j}\,\sqrt{j-m}\,\sqrt{j_1+j_2+j+1}}\left\langle j_1-\frac{1}{2}\,j_2\, m_1+\frac{1}{2}\,m_2\;\middle|\;j_1-\frac{1}{2}\,j_2\, j-\frac{1}{2}\,m+\frac{1}{2}\right\rangle + \frac{\sqrt{j_2-m_2}\,\sqrt{-j_1+j_2+j}\,\sqrt{2\,j+1}}{\sqrt{2}\,\sqrt{j}\,\sqrt{j-m}\,\sqrt{j_1+j_2+j+1}}\left\langle j_1\, j_2-\frac{1}{2}\,m_1\, m_2+\frac{1}{2}\;\middle|\;j_1\, j_2-\frac{1}{2}\,j-\frac{1}{2}\,m+\frac{1}{2}\right\rangle$$

07.38.17.0019.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle = \frac{\sqrt{j_1+m_1}\,\sqrt{j_1-j_2+j}\,\sqrt{2\,j+1}}{\sqrt{2}\,\sqrt{j}\,\sqrt{j+m}\,\sqrt{j_1+j_2+j+1}} \left\langle j_1-\frac{1}{2}\, j_2\, m_1-\frac{1}{2}\, m_2 \;\middle|\; j_1-\frac{1}{2}\, j_2\, j-\frac{1}{2}\, m-\frac{1}{2}\right\rangle + \frac{\sqrt{j_2+m_2}\,\sqrt{-j_1+j_2+j}\,\sqrt{2\,j+1}}{\sqrt{2}\,\sqrt{j}\,\sqrt{j+m}\,\sqrt{j_1+j_2+j+1}} \left\langle j_1\, j_2-\frac{1}{2}\, m_1\, m_2-\frac{1}{2} \;\middle|\; j_1\, j_2-\frac{1}{2}\, j-\frac{1}{2}\, m-\frac{1}{2}\right\rangle$$

07.38.17.0020.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle = \frac{\sqrt{j_1+m_1+1}\,\sqrt{-j_1+j_2+j}\,\sqrt{2\,j+1}}{\sqrt{2}\,\sqrt{j}\,\sqrt{j-m}\,\sqrt{j_1+j_2-j+1}} \left\langle j_1+\frac{1}{2}\, j_2\, m_1+\frac{1}{2}\, m_2 \;\middle|\; j_1+\frac{1}{2}\, j_2\, j-\frac{1}{2}\, m+\frac{1}{2}\right\rangle - \frac{\sqrt{j_2+m_2+1}\,\sqrt{j_1-j_2+j}\,\sqrt{2\,j+1}}{\sqrt{2}\,\sqrt{j}\,\sqrt{j-m}\,\sqrt{j_1+j_2-j+1}} \left\langle j_1\, j_2+\frac{1}{2}\, m_1\, m_2+\frac{1}{2} \;\middle|\; j_1\, j_2+\frac{1}{2}\, j-\frac{1}{2}\, m+\frac{1}{2}\right\rangle$$

07.38.17.0021.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle = -\frac{\sqrt{j_1-m_1+1}\,\sqrt{-j_1+j_2+j}\,\sqrt{2\,j+1}}{\sqrt{2}\,\sqrt{j}\,\sqrt{j+m}\,\sqrt{j_1+j_2-j+1}} \left\langle j_1+\frac{1}{2}\, j_2\, m_1-\frac{1}{2}\, m_2 \;\middle|\; j_1+\frac{1}{2}\, j_2\, j-\frac{1}{2}\, m-\frac{1}{2}\right\rangle + \frac{\sqrt{j_2-m_2+1}\,\sqrt{j_1-j_2+j}\,\sqrt{2\,j+1}}{\sqrt{2}\,\sqrt{j}\,\sqrt{j+m}\,\sqrt{j_1+j_2-j+1}} \left\langle j_1\, j_2+\frac{1}{2}\, m_1\, m_2-\frac{1}{2} \;\middle|\; j_1\, j_2+\frac{1}{2}\, j-\frac{1}{2}\, m-\frac{1}{2}\right\rangle$$

07.38.17.0022.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle = -\frac{\sqrt{j_1-m_1}\,\sqrt{-j_1+j_2+j+1}\,\sqrt{2\,j+1}}{\sqrt{2}\,\sqrt{j+1}\,\sqrt{j+m+1}\,\sqrt{j_1+j_2-j}} \left\langle j_1-\frac{1}{2}\, j_2\, m_1+\frac{1}{2}\, m_2 \;\middle|\; j_1-\frac{1}{2}\, j_2\, j+\frac{1}{2}\, m+\frac{1}{2}\right\rangle + \frac{\sqrt{j_2-m_2}\,\sqrt{j_1-j_2+j+1}\,\sqrt{2\,j+1}}{\sqrt{2}\,\sqrt{j+1}\,\sqrt{j+m+1}\,\sqrt{j_1+j_2-j}} \left\langle j_1\, j_2-\frac{1}{2}\, m_1\, m_2+\frac{1}{2} \;\middle|\; j_1\, j_2-\frac{1}{2}\, j+\frac{1}{2}\, m+\frac{1}{2}\right\rangle$$

07.38.17.0023.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle = \frac{\sqrt{j_1+m_1}\,\sqrt{-j_1+j_2+j+1}\,\sqrt{2\,j+1}}{\sqrt{2}\,\sqrt{j+1}\,\sqrt{j-m+1}\,\sqrt{j_1+j_2-j}} \left\langle j_1-\frac{1}{2}\, j_2\, m_1-\frac{1}{2}\, m_2 \;\middle|\; j_1-\frac{1}{2}\, j_2\, j+\frac{1}{2}\, m-\frac{1}{2}\right\rangle - \frac{\sqrt{j_2+m_2}\,\sqrt{j_1-j_2+j+1}\,\sqrt{2\,j+1}}{\sqrt{2}\,\sqrt{j+1}\,\sqrt{j-m+1}\,\sqrt{j_1+j_2-j}} \left\langle j_1\, j_2-\frac{1}{2}\, m_1\, m_2-\frac{1}{2} \;\middle|\; j_1\, j_2-\frac{1}{2}\, j+\frac{1}{2}\, m-\frac{1}{2}\right\rangle$$

07.38.17.0024.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle = \frac{\sqrt{j-m}\,\sqrt{-j_1+j_2+j}\,\sqrt{j_1+j_2+j+1}}{\sqrt{2}\,\sqrt{j}\,\sqrt{2\,j+1}\,\sqrt{j_2-m_2}} \left\langle j_1\, j_2-\frac{1}{2}\, m_1\, m_2+\frac{1}{2} \;\middle|\; j_1\, j_2-\frac{1}{2}\, j-\frac{1}{2}\, m+\frac{1}{2}\right\rangle + \frac{\sqrt{j+m+1}\,\sqrt{j_1-j_2+j+1}\,\sqrt{j_1+j_2-j}}{\sqrt{2}\,\sqrt{j+1}\,\sqrt{2\,j+1}\,\sqrt{j_2-m_2}} \left\langle j_1\, j_2-\frac{1}{2}\, m_1\, m_2+\frac{1}{2} \;\middle|\; j_1\, j_2-\frac{1}{2}\, j+\frac{1}{2}\, m+\frac{1}{2}\right\rangle$$

07.38.17.0025.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle = \frac{\sqrt{j+m}\,\sqrt{-j_1+j_2+j}\,\sqrt{j_1+j_2+j+1}}{\sqrt{2}\,\sqrt{j}\,\sqrt{2\,j+1}\,\sqrt{j_2+m_2}} \left\langle j_1\, j_2-\frac{1}{2}\, m_1\, m_2-\frac{1}{2} \;\middle|\; j_1\, j_2-\frac{1}{2}\, j-\frac{1}{2}\, m-\frac{1}{2}\right\rangle - \frac{\sqrt{j-m+1}\,\sqrt{j_1-j_2+j+1}\,\sqrt{j_1+j_2-j}}{\sqrt{2}\,\sqrt{j+1}\,\sqrt{2\,j+1}\,\sqrt{j_2+m_2}} \left\langle j_1\, j_2-\frac{1}{2}\, m_1\, m_2-\frac{1}{2} \;\middle|\; j_1\, j_2-\frac{1}{2}\, j+\frac{1}{2}\, m-\frac{1}{2}\right\rangle$$

Arguments j_1, j_2, j changing by 1

07.38.17.0026.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle = \frac{m_1 (2 j_1 - 1)\sqrt{-j_1+j_2+j+1}\,\sqrt{j_1-j_2+j}}{(j_1-1)\sqrt{j_1-m_1}\,\sqrt{j_1+m_1}\,\sqrt{j_1+j_2-j}\,\sqrt{j_1+j_2+j+1}}\langle j_1-1\, j_2\, m_1\, m_2 \mid j_1-1\, j_2\, j\, m\rangle - \frac{2 j_1 (2 j_1-1)\sqrt{j_2-m_2}\,\sqrt{j_2+m_2}}{\sqrt{j_1-m_1}\,\sqrt{j_1+m_1}\,\sqrt{j_1+j_2-j-1}\,\sqrt{j_1+j_2-j}\,\sqrt{j_1+j_2+j}\,\sqrt{j_1+j_2+j+1}} \langle j_1-1\, j_2-1\, m_1\, m_2 \mid j_1-1\, j_2-1\, j\, m\rangle + \left(j_1\sqrt{j_1-m_1-1}\,\sqrt{j_1+m_1-1}\,\sqrt{-j_1+j_2+j+1}\,\sqrt{-j_1+j_2+j+2}\,\sqrt{j_1-j_2+j-1}\,\sqrt{j_1-j_2+j}\right) \Big/ \left((j_1-1)\sqrt{j_1-m_1}\,\sqrt{j_1+m_1}\,\sqrt{j_1+j_2-j-1}\,\sqrt{j_1+j_2-j}\,\sqrt{j_1+j_2+j}\,\sqrt{j_1+j_2+j+1}\right) \langle j_1-2\, j_2\, m_1\, m_2 \mid j_1-2\, j_2\, j\, m\rangle$$

07.38.17.0027.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle = \frac{\sqrt{j_1-m_1}\,\sqrt{j_1+m_1}\,\sqrt{-j_1+j_2+j+1}\,\sqrt{-j_1+j_2+j+2}\,\sqrt{j_1-j_2+j-1}\,\sqrt{j_1-j_2+j}}{2 j_1 (2 j_1+1)\sqrt{j_2-m_2+1}\,\sqrt{j_2+m_2+1}} \langle j_1-1\, j_2+1\, m_1\, m_2 \mid j_1-1\, j_2+1\, j\, m\rangle + \frac{m_1\sqrt{-j_1+j_2+j+1}\,\sqrt{j_1-j_2+j}\,\sqrt{j_1+j_2-j+1}\,\sqrt{j_1+j_2+j+2}}{2 j_1 (j_1+1)\sqrt{j_2-m_2+1}\,\sqrt{j_2+m_2+1}}\langle j_1\, j_2+1\, m_1\, m_2 \mid j_1\, j_2+1\, j\, m\rangle - \frac{\sqrt{j_1-m_1+1}\,\sqrt{j_1+m_1+1}\,\sqrt{j_1+j_2-j+1}\,\sqrt{j_1+j_2-j+2}\,\sqrt{j_1+j_2+j+2}\,\sqrt{j_1+j_2+j+3}}{2 (j_1+1)(2 j_1+1)\sqrt{j_2-m_2+1}\,\sqrt{j_2+m_2+1}} \langle j_1+1\, j_2+1\, m_1\, m_2 \mid j_1+1\, j_2+1\, j\, m\rangle$$

07.38.17.0028.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle = -\frac{(j_1+1)\sqrt{j_1-m_1}\,\sqrt{j_1+m_1}\,\sqrt{-j_1+j_2+j+1}\,\sqrt{j_1-j_2+j}\,\sqrt{j_1+j_2-j}\,\sqrt{j_1+j_2+j+1}}{(2 j_1+1)(2 m_2 j_1 (j_1+1) + m_1 (j_1 (j_1+1) + j_2 (j_2+1) - j (j+1)))} \langle j_1-1\, j_2\, m_1\, m_2 \mid j_1-1\, j_2\, j\, m\rangle - \frac{j_1\sqrt{j_1-m_1+1}\,\sqrt{j_1+m_1+1}\,\sqrt{-j_1+j_2+j}\,\sqrt{j_1-j_2+j+1}\,\sqrt{j_1+j_2-j+1}\,\sqrt{j_1+j_2+j+2}}{(2 j_1+1)(2 m_2 j_1 (j_1+1) + m_1 (j_1 (j_1+1) + j_2 (j_2+1) - j (j+1)))} \langle j_1+1\, j_2\, m_1\, m_2 \mid j_1+1\, j_2\, j\, m\rangle$$

07.38.17.0029.01

$$\langle j_1\, j_2+1\, m_1\, m_2 \mid j_1\, j_2+1\, j\, m\rangle = -\frac{\sqrt{j_1-m_1}\,\sqrt{j_1+m_1}\,\sqrt{j_1+j_2-j}\,\sqrt{j_1+j_2-j+1}\,\sqrt{j_1+j_2+j+1}\,\sqrt{j_1+j_2+j+2}}{2 j_1 (2 j_1+1)\sqrt{j_2-m_2+1}\,\sqrt{j_2+m_2+1}} \langle j_1-1\, j_2\, m_1\, m_2 \mid j_1-1\, j_2\, j\, m\rangle + \frac{m_1\sqrt{-j_1+j_2+j+1}\,\sqrt{j_1-j_2+j}\,\sqrt{j_1+j_2-j+1}\,\sqrt{j_1+j_2+j+2}}{2 j_1 (j_1+1)\sqrt{j_2-m_2+1}\,\sqrt{j_2+m_2+1}}\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle + \frac{\sqrt{j_1-m_1+1}\,\sqrt{j_1+m_1+1}\,\sqrt{-j_1+j_2+j}\,\sqrt{-j_1+j_2+j+1}\,\sqrt{j_1-j_2+j}\,\sqrt{j_1-j_2+j+1}}{2 (j_1+1)(2 j_1+1)\sqrt{j_2-m_2+1}\,\sqrt{j_2+m_2+1}} \langle j_1+1\, j_2\, m_1\, m_2 \mid j_1+1\, j_2\, j\, m\rangle$$

Arguments j_1, j_2, m_1, m_2 changing by 1

07.38.17.0030.01

$$\langle j_1\, j_2-1\, m_1\, m_2 \mid j_1\, j_2-1\, j\, m\rangle = \frac{\sqrt{j_1+m_1}\,\sqrt{j_1+m_1-1}\,\sqrt{-j_1+j_2+j}\,\sqrt{-j_1+j_2+j+1}\,\sqrt{j_1-j_2+j}\,\sqrt{j_1-j_2+j+1}}{2\, j_1\,(2\, j_1+1)\,\sqrt{j_2+m_2}\,\sqrt{j_2+m_2+1}} \langle j_1-1\, j_2\, m_1-1\, m_2+1 \mid j_1-1\, j_2\, j\, m\rangle - \frac{\sqrt{j_1+m_1}\,\sqrt{j_1-m_1+1}\,\sqrt{-j_1+j_2+j}\,\sqrt{j_1-j_2+j+1}\,\sqrt{j_1+j_2-j}\,\sqrt{j_1+j_2+j+1}}{2\, j_1\,(j_1+1)\,\sqrt{j_2+m_2}\,\sqrt{j_2+m_2+1}} \langle j_1\, j_2\, m_1-1\, m_2+1 \mid j_1\, j_2\, j\, m\rangle + \frac{\sqrt{j_1-m_1+1}\,\sqrt{j_1-m_1+2}\,\sqrt{j_1+j_2-j}\,\sqrt{j_1+j_2-j+1}\,\sqrt{j_1+j_2+j+1}\,\sqrt{j_1+j_2+j+2}}{2\,(j_1+1)\,(2\, j_1+1)\,\sqrt{j_2+m_2}\,\sqrt{j_2+m_2+1}} \langle j_1+1\, j_2\, m_1-1\, m_2+1 \mid j_1+1\, j_2\, j\, m\rangle$$

07.38.17.0031.01

$$\langle j_1\, j_2-1\, m_1\, m_2 \mid j_1\, j_2-1\, j\, m\rangle = \frac{\sqrt{j_1-m_1}\,\sqrt{j_1-m_1-1}\,\sqrt{-j_1+j_2+j}\,\sqrt{-j_1+j_2+j+1}\,\sqrt{j_1-j_2+j}\,\sqrt{j_1-j_2+j+1}}{2\, j_1\,(2\, j_1+1)\,\sqrt{j_2-m_2}\,\sqrt{j_2-m_2+1}} \langle j_1-1\, j_2\, m_1+1\, m_2-1 \mid j_1-1\, j_2\, j\, m\rangle + \frac{\sqrt{j_1-m_1}\,\sqrt{j_1+m_1+1}\,\sqrt{-j_1+j_2+j}\,\sqrt{j_1-j_2+j+1}\,\sqrt{j_1+j_2-j}\,\sqrt{j_1+j_2+j+1}}{2\, j_1\,(j_1+1)\,\sqrt{j_2-m_2}\,\sqrt{j_2-m_2+1}} \langle j_1\, j_2\, m_1+1\, m_2-1 \mid j_1\, j_2\, j\, m\rangle + \frac{\sqrt{j_1+m_1+1}\,\sqrt{j_1+m_1+2}\,\sqrt{j_1+j_2-j}\,\sqrt{j_1+j_2-j+1}\,\sqrt{j_1+j_2+j+1}\,\sqrt{j_1+j_2+j+2}}{2\,(j_1+1)\,(2\, j_1+1)\,\sqrt{j_2-m_2}\,\sqrt{j_2-m_2+1}} \langle j_1+1\, j_2\, m_1+1\, m_2-1 \mid j_1+1\, j_2\, j\, m\rangle$$

07.38.17.0032.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle = \frac{\sqrt{j_1+m_1}\,\sqrt{j_1+m_1-1}\,\sqrt{-j_1+j_2+j+1}\,\sqrt{j_1-j_2+j}\,\sqrt{j_1+j_2-j}\,\sqrt{j_1+j_2+j+1}}{2\, j_1\,(2\, j_1+1)\,\sqrt{j_2-m_2}\,\sqrt{j_2+m_2+1}} \langle j_1-1\, j_2\, m_1-1\, m_2+1 \mid j_1-1\, j_2\, j\, m\rangle - \frac{(j_1\,(j_1+1)+j_2\,(j_2+1)-j\,(j+1))\,\sqrt{j_1+m_1}\,\sqrt{j_1-m_1+1}}{2\, j_1\,(j_1+1)\,\sqrt{j_2-m_2}\,\sqrt{j_2+m_2+1}} \langle j_1\, j_2\, m_1-1\, m_2+1 \mid j_1\, j_2\, j\, m\rangle - \frac{\sqrt{j_1-m_1+1}\,\sqrt{j_1-m_1+2}\,\sqrt{-j_1+j_2+j}\,\sqrt{j_1-j_2+j+1}\,\sqrt{j_1+j_2-j+1}\,\sqrt{j_1+j_2+j+2}}{2\,(j_1+1)\,(2\, j_1+1)\,\sqrt{j_2-m_2}\,\sqrt{j_2+m_2+1}} \langle j_1+1\, j_2\, m_1-1\, m_2+1 \mid j_1+1\, j_2\, j\, m\rangle$$

07.38.17.0033.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle = -\frac{\sqrt{j_1-m_1}\,\sqrt{j_1-m_1-1}\,\sqrt{-j_1+j_2+j+1}\,\sqrt{j_1-j_2+j}\,\sqrt{j_1+j_2-j}\,\sqrt{j_1+j_2+j+1}}{2\, j_1\,(2\, j_1+1)\,\sqrt{j_2+m_2}\,\sqrt{j_2-m_2+1}} \langle j_1-1\, j_2\, m_1+1\, m_2-1 \mid j_1-1\, j_2\, j\, m\rangle - \frac{(j_1\,(j_1+1)+j_2\,(j_2+1)-j\,(j+1))\,\sqrt{j_1-m_1}\,\sqrt{j_1+m_1+1}}{2\, j_1\,(j_1+1)\,\sqrt{j_2+m_2}\,\sqrt{j_2-m_2+1}} \langle j_1\, j_2\, m_1+1\, m_2-1 \mid j_1\, j_2\, j\, m\rangle + \frac{\sqrt{j_1+m_1+1}\,\sqrt{j_1+m_1+2}\,\sqrt{-j_1+j_2+j}\,\sqrt{j_1-j_2+j+1}\,\sqrt{j_1+j_2-j+1}\,\sqrt{j_1+j_2+j+2}}{2\,(j_1+1)\,(2\, j_1+1)\,\sqrt{j_2+m_2}\,\sqrt{j_2-m_2+1}} \langle j_1+1\, j_2\, m_1+1\, m_2-1 \mid j_1+1\, j_2\, j\, m\rangle$$

07.38.17.0034.01

$$\langle j_1\, j_2+1\, m_1\, m_2 \mid j_1\, j_2+1\, j\, m\rangle = \frac{\sqrt{j_1+m_1}\,\sqrt{j_1+m_1-1}\,\sqrt{j_1+j_2-j}\,\sqrt{j_1+j_2-j+1}\,\sqrt{j_1+j_2+j+1}\,\sqrt{j_1+j_2+j+2}}{2\,j_1\,(2\,j_1+1)\sqrt{j_2-m_2}\,\sqrt{j_2-m_2+1}}\langle j_1-1\, j_2\, m_1-1\, m_2+1 \mid j_1-1\, j_2\, j\, m\rangle + \frac{\sqrt{j_1+m_1}\,\sqrt{j_1-m_1+1}\,\sqrt{-j_1+j_2+j+1}\,\sqrt{j_1-j_2+j}\,\sqrt{j_1+j_2-j+1}\,\sqrt{j_1+j_2+j+2}}{2\,j_1\,(j_1+1)\sqrt{j_2-m_2}\,\sqrt{j_2-m_2+1}}\langle j_1\, j_2\, m_1-1\, m_2+1 \mid j_1\, j_2\, j\, m\rangle + \frac{\sqrt{j_1-m_1+1}\,\sqrt{j_1-m_1+2}\,\sqrt{-j_1+j_2+j}\,\sqrt{-j_1+j_2+j+1}\,\sqrt{j_1-j_2+j}\,\sqrt{j_1-j_2+j+1}}{2\,(j_1+1)\,(2\,j_1+1)\sqrt{j_2-m_2}\,\sqrt{j_2-m_2+1}}\langle j_1+1\, j_2\, m_1-1\, m_2+1 \mid j_1+1\, j_2\, j\, m\rangle$$

07.38.17.0035.01

$$\langle j_1\, j_2+1\, m_1\, m_2 \mid j_1\, j_2+1\, j\, m\rangle = \frac{\sqrt{j_1-m_1}\,\sqrt{j_1-m_1-1}\,\sqrt{j_1+j_2-j}\,\sqrt{j_1+j_2-j+1}\,\sqrt{j_1+j_2+j+1}\,\sqrt{j_1+j_2+j+2}}{2\,j_1\,(2\,j_1+1)\sqrt{j_2+m_2}\,\sqrt{j_2+m_2+1}}\langle j_1-1\, j_2\, m_1+1\, m_2-1 \mid j_1-1\, j_2\, j\, m\rangle - \frac{\sqrt{j_1-m_1}\,\sqrt{j_1+m_1+1}\,\sqrt{-j_1+j_2+j+1}\,\sqrt{j_1-j_2+j}\,\sqrt{j_1+j_2-j+1}\,\sqrt{j_1+j_2+j+2}}{2\,j_1\,(j_1+1)\sqrt{j_2+m_2}\,\sqrt{j_2+m_2+1}}\langle j_1\, j_2\, m_1+1\, m_2-1 \mid j_1\, j_2\, j\, m\rangle + \frac{\sqrt{j_1+m_1+1}\,\sqrt{j_1+m_1+2}\,\sqrt{-j_1+j_2+j}\,\sqrt{-j_1+j_2+j+1}\,\sqrt{j_1-j_2+j}\,\sqrt{j_1-j_2+j+1}}{2\,(j_1+1)\,(2\,j_1+1)\sqrt{j_2+m_2}\,\sqrt{j_2+m_2+1}}\langle j_1+1\, j_2\, m_1+1\, m_2-1 \mid j_1+1\, j_2\, j\, m\rangle$$

Arguments j_2, j, m_2, m changing by 1

07.38.17.0036.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle = \frac{\sqrt{2\,j+1}}{\sqrt{-j_1+j_2+j}\,\sqrt{j_1-j_2+j}\,\sqrt{j_1+j_2-j+1}\,\sqrt{j_1+j_2+j+1}\,\sqrt{2\,j-1}}\Big(\sqrt{j_2+m_2}\,\sqrt{j_2-m_2+1}\,\sqrt{j+m}\,\sqrt{j+m-1}\,\langle j_1\, j_2\, m_1\, m_2-1 \mid j_1\, j_2\, j-1\, m-1\rangle - 2\,m_2\sqrt{j-m}\,\sqrt{j+m}\,\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j-1\, m\rangle - \sqrt{j_2-m_2}\,\sqrt{j_2+m_2+1}\,\sqrt{j-m}\,\sqrt{j-m-1}\,\langle j_1\, j_2\, m_1\, m_2+1 \mid j_1\, j_2\, j-1\, m+1\rangle\Big)$$

07.38.17.0037.01

$$\langle j_1\, j_2-1\, m_1\, m_2 \mid j_1\, j_2-1\, j\, m\rangle = \frac{\sqrt{j+m}\,\sqrt{j+m-1}\,\sqrt{j_1-j_2+j}\,\sqrt{j_1-j_2+j+1}\,\sqrt{j_1+j_2-j}\,\sqrt{j_1+j_2-j+1}}{2\,j\sqrt{2\,j-1}\,\sqrt{2\,j+1}\,\sqrt{j_2-m_2}\,\sqrt{j_2-m_2+1}}\langle j_1\, j_2\, m_1\, m_2-1 \mid j_1\, j_2\, j-1\, m-1\rangle + \frac{\sqrt{j+m}\,\sqrt{j-m+1}\,\sqrt{-j_1+j_2+j}\,\sqrt{j_1-j_2+j+1}\,\sqrt{j_1+j_2-j}\,\sqrt{j_1+j_2+j+1}}{2\,j\,(j+1)\sqrt{j_2-m_2}\,\sqrt{j_2-m_2+1}}\langle j_1\, j_2\, m_1\, m_2-1 \mid j_1\, j_2\, j\, m-1\rangle + \frac{\sqrt{j-m+1}\,\sqrt{j-m+2}\,\sqrt{-j_1+j_2+j}\,\sqrt{-j_1+j_2+j+1}\,\sqrt{j_1+j_2+j+1}\,\sqrt{j_1+j_2+j+2}}{2\,(j+1)\sqrt{2\,j+1}\,\sqrt{2\,j+3}\,\sqrt{j_2-m_2}\,\sqrt{j_2-m_2+1}}\langle j_1\, j_2\, m_1\, m_2-1 \mid j_1\, j_2\, j+1\, m-1\rangle$$

07.38.17.0038.01

$$\langle j_1\, j_2-1\, m_1\, m_2 \mid j_1\, j_2-1\, j\, m\rangle = \frac{\sqrt{j-m}\,\sqrt{j-m-1}\,\sqrt{j_1-j_2+j}\,\sqrt{j_1-j_2+j+1}\,\sqrt{j_1+j_2-j}\,\sqrt{j_1+j_2-j+1}}{2\,j\,\sqrt{2\,j-1}\,\sqrt{2\,j+1}\,\sqrt{j_2+m_2}\,\sqrt{j_2+m_2+1}}\langle j_1\, j_2\, m_1\, m_2+1 \mid j_1\, j_2\, j-1\, m+1\rangle - \frac{\sqrt{j-m}\,\sqrt{j+m+1}\,\sqrt{-j_1+j_2+j}\,\sqrt{j_1-j_2+j+1}\,\sqrt{j_1+j_2-j}\,\sqrt{j_1+j_2+j+1}}{2\,j\,(j+1)\,\sqrt{j_2+m_2}\,\sqrt{j_2+m_2+1}}\langle j_1\, j_2\, m_1\, m_2+1 \mid j_1\, j_2\, j\, m+1\rangle + \frac{\sqrt{j+m+1}\,\sqrt{j+m+2}\,\sqrt{-j_1+j_2+j}\,\sqrt{-j_1+j_2+j+1}\,\sqrt{j_1+j_2+j+1}\,\sqrt{j_1+j_2+j+2}}{2\,(j+1)\,\sqrt{2\,j+1}\,\sqrt{2\,j+3}\,\sqrt{j_2+m_2}\,\sqrt{j_2+m_2+1}}\langle j_1\, j_2\, m_1\, m_2+1 \mid j_1\, j_2\, j+1\, m+1\rangle$$

07.38.17.0039.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle = \frac{\sqrt{j+m}\,\sqrt{j+m-1}\,\sqrt{-j_1+j_2+j}\,\sqrt{j_1-j_2+j}\,\sqrt{j_1+j_2-j+1}\,\sqrt{j_1+j_2+j+1}}{2\,j\,\sqrt{2\,j-1}\,\sqrt{2\,j+1}\,\sqrt{j_2+m_2}\,\sqrt{j_2-m_2+1}}\langle j_1\, j_2\, m_1\, m_2-1 \mid j_1\, j_2\, j-1\, m-1\rangle + \frac{(-j_1\,(j_1+1)+j_2\,(j_2+1)+j\,(j+1))\,\sqrt{j+m}\,\sqrt{j-m+1}}{2\,j\,(j+1)\,\sqrt{j_2+m_2}\,\sqrt{j_2-m_2+1}}\langle j_1\, j_2\, m_1\, m_2-1 \mid j_1\, j_2\, j\, m-1\rangle - \frac{\sqrt{j-m+1}\,\sqrt{j-m+2}\,\sqrt{-j_1+j_2+j+1}\,\sqrt{j_1-j_2+j+1}\,\sqrt{j_1+j_2-j}\,\sqrt{j_1+j_2+j+2}}{2\,(j+1)\,\sqrt{2\,j+1}\,\sqrt{2\,j+3}\,\sqrt{j_2+m_2}\,\sqrt{j_2-m_2+1}}\langle j_1\, j_2\, m_1\, m_2-1 \mid j_1\, j_2\, j+1\, m-1\rangle$$

07.38.17.0040.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle = -\frac{\sqrt{j-m}\,\sqrt{j-m-1}\,\sqrt{-j_1+j_2+j}\,\sqrt{j_1-j_2+j}\,\sqrt{j_1+j_2-j+1}\,\sqrt{j_1+j_2+j+1}}{2\,j\,\sqrt{2\,j-1}\,\sqrt{2\,j+1}\,\sqrt{j_2-m_2}\,\sqrt{j_2+m_2+1}}\langle j_1\, j_2\, m_1\, m_2+1 \mid j_1\, j_2\, j-1\, m+1\rangle + \frac{(-j_1\,(j_1+1)+j_2\,(j_2+1)+j\,(j+1))\,\sqrt{j-m}\,\sqrt{j+m+1}}{2\,j\,(j+1)\,\sqrt{j_2-m_2}\,\sqrt{j_2+m_2+1}}\langle j_1\, j_2\, m_1\, m_2+1 \mid j_1\, j_2\, j\, m+1\rangle + \frac{\sqrt{j+m+1}\,\sqrt{j+m+2}\,\sqrt{-j_1+j_2+j+1}\,\sqrt{j_1-j_2+j+1}\,\sqrt{j_1+j_2-j}\,\sqrt{j_1+j_2+j+2}}{2\,(j+1)\,\sqrt{2\,j+1}\,\sqrt{2\,j+3}\,\sqrt{j_2-m_2}\,\sqrt{j_2+m_2+1}}\langle j_1\, j_2\, m_1\, m_2+1 \mid j_1\, j_2\, j+1\, m+1\rangle$$

07.38.17.0041.01

$$\langle j_1\, j_2+1\, m_1\, m_2 \mid j_1\, j_2+1\, j\, m\rangle = \frac{\sqrt{j+m}\,\sqrt{j+m-1}\,\sqrt{-j_1+j_2+j}\,\sqrt{-j_1+j_2+j+1}\,\sqrt{j_1+j_2+j+1}\,\sqrt{j_1+j_2+j+2}}{2\,j\,\sqrt{2\,j-1}\,\sqrt{2\,j+1}\,\sqrt{j_2+m_2}\,\sqrt{j_2+m_2+1}}\langle j_1\, j_2\, m_1\, m_2-1 \mid j_1\, j_2\, j-1\, m-1\rangle - \frac{\sqrt{j+m}\,\sqrt{j-m+1}\,\sqrt{-j_1+j_2+j+1}\,\sqrt{j_1-j_2+j}\,\sqrt{j_1+j_2-j+1}\,\sqrt{j_1+j_2+j+2}}{2\,j\,(j+1)\,\sqrt{j_2+m_2}\,\sqrt{j_2+m_2+1}}\langle j_1\, j_2\, m_1\, m_2-1 \mid j_1\, j_2\, j\, m-1\rangle + \frac{\sqrt{j-m+1}\,\sqrt{j-m+2}\,\sqrt{j_1-j_2+j}\,\sqrt{j_1-j_2+j+1}\,\sqrt{j_1+j_2-j}\,\sqrt{j_1+j_2-j+1}}{2\,(j+1)\,\sqrt{2\,j+1}\,\sqrt{2\,j+3}\,\sqrt{j_2+m_2}\,\sqrt{j_2+m_2+1}}\langle j_1\, j_2\, m_1\, m_2-1 \mid j_1\, j_2\, j+1\, m-1\rangle$$

07.38.17.0042.01

$$\langle j_1\, j_2+1\, m_1\, m_2 \mid j_1\, j_2+1\, j\, m\rangle = \frac{\sqrt{j-m}\,\sqrt{j-m-1}\,\sqrt{-j_1+j_2+j}\,\sqrt{-j_1+j_2+j+1}\,\sqrt{j_1+j_2+j+1}\,\sqrt{j_1+j_2+j+2}}{2\,j\,\sqrt{2\,j-1}\,\sqrt{2\,j+1}\,\sqrt{j_2-m_2}\,\sqrt{j_2-m_2+1}} \langle j_1\, j_2\, m_1\, m_2+1 \mid j_1\, j_2\, j-1\, m+1\rangle + \frac{\sqrt{j-m}\,\sqrt{j+m+1}\,\sqrt{-j_1+j_2+j+1}\,\sqrt{j_1-j_2+j}\,\sqrt{j_1+j_2-j+1}\,\sqrt{j_1+j_2+j+2}}{2\,j\,(j+1)\,\sqrt{j_2-m_2}\,\sqrt{j_2-m_2+1}} \langle j_1\, j_2\, m_1\, m_2+1 \mid j_1\, j_2\, j\, m+1\rangle + \frac{\sqrt{j+m+1}\,\sqrt{j+m+2}\,\sqrt{j_1-j_2+j}\,\sqrt{j_1-j_2+j+1}\,\sqrt{j_1+j_2-j}\,\sqrt{j_1+j_2-j+1}}{2\,(j+1)\,\sqrt{2\,j+1}\,\sqrt{2\,j+3}\,\sqrt{j_2-m_2}\,\sqrt{j_2-m_2+1}} \langle j_1\, j_2\, m_1\, m_2+1 \mid j_1\, j_2\, j+1\, m+1\rangle$$

Arguments m_1, m_2, m equal to zero

07.38.17.0043.01

$$\langle n+j_1\, j_2-n\, 0\, 0 \mid n+j_1\, j_2-n\, j\, 0\rangle = \frac{\left(\frac{-j_1+j_2+j}{2}\right)!\left(\frac{j_1-j_2+j}{2}\right)!\sqrt{(-j_1+j_2+j-2\,n)!}\,\sqrt{(j_1-j_2+j+2\,n)!}}{\left(\frac{-j_1+j_2+j}{2}-n\right)!\left(\frac{j_1-j_2+j}{2}+n\right)!\sqrt{(-j_1+j_2+j)!}\,\sqrt{(j_1-j_2+j)!}} \langle j_1\, j_2\, 0\, 0 \mid j_1\, j_2\, j\, 0\rangle \,/;$$

$$n\in\mathbb{Z}\bigwedge \max\left(-j_1,\, \frac{-j-j_1+j_2}{2}\right)\le n\le \min\left(j_2,\, \frac{j-j_1+j_2}{2}\right)\bigwedge j_1\in\mathbb{N}\bigwedge j_2\in\mathbb{N}\bigwedge j\in\mathbb{N}\bigwedge |j_1-j_2|\le j\le j_1+j_2$$

Involving two Clebsch Gordan coefficients

07.38.17.0044.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle = \left\langle \frac{1}{2}(m+j_1+j_2)\, \frac{1}{2}(-m+j_1+j_2)\, \frac{1}{2}(j_1-j_2+m_1-m_2)\, \frac{1}{2}(j_1-j_2-m_1+m_2) \;\middle|\; \frac{1}{2}(m+j_1+j_2)\, \frac{1}{2}(-m+j_1+j_2)\, j\, j_1-j_2\right\rangle$$

07.38.17.0045.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle = \left\langle \frac{1}{2}(-m+j_1+j_2)\, \frac{1}{2}(m+j_1+j_2)\, \frac{1}{2}(-j_1+j_2+m_1-m_2)\, \frac{1}{2}(-j_1+j_2-m_1+m_2) \;\middle|\; \frac{1}{2}(-m+j_1+j_2)\, \frac{1}{2}(m+j_1+j_2)\, j\, j_2-j_1\right\rangle$$

07.38.17.0046.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle = \frac{\sqrt{2\,j+1}}{\sqrt{j+j_2+m_1+1}} \left\langle j_1\, \frac{1}{2}(j+j_2-m_1)\, j-j_2\, \frac{1}{2}(-j+m+j_2+m_2) \;\middle|\; j_1\, \frac{1}{2}(j+j_2-m_1)\, \frac{1}{2}(j+j_2+m_1)\, \frac{1}{2}(j+m-j_2+m_2)\right\rangle$$

07.38.17.0047.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle = \frac{\sqrt{2\,j+1}}{\sqrt{j+j_2+m_1+1}} \left\langle \frac{1}{2}(j+j_1+m_2)\, \frac{1}{2}(-m+j_1+j_2)\, \frac{1}{2}(j+j_1-2\,j_2-m_2)\, \frac{1}{2}(-2\,j+m+j_1+j_2) \;\middle|\; \frac{1}{2}(j+j_1+m_2)\, \frac{1}{2}(-m+j_1+j_2)\, \frac{1}{2}(j+j_2+m_1)\, \frac{1}{2}(-j+2\,j_1-j_2+m_1)\right\rangle$$

07.38.17.0048.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle = \frac{\sqrt{2\,j+1}}{\sqrt{j+j_1-m_2+1}} \left\langle \frac{1}{2}(j+j_1+m_2)\, j_2\, \frac{1}{2}(j+m-j_1+m_1)\, j_1-j \;\middle|\; \frac{1}{2}(j+j_1+m_2)\, j_2\, \frac{1}{2}(j+j_1-m_2)\, \frac{1}{2}(-j+m+j_1+m_1)\right\rangle$$

07.38.17.0049.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle = \frac{\sqrt{2\,j+1}}{\sqrt{j+j_1-m_2+1}} \left\langle \frac{1}{2}(m+j_1+j_2)\; \frac{1}{2}(j+j_2-m_1)\; \frac{1}{2}(2\,j+m-j_1-j_2)\; \frac{1}{2}(-j+2\,j_1-j_2-m_1) \;\middle|\; \frac{1}{2}(m+j_1+j_2)\; \frac{1}{2}(j+j_2-m_1)\; \frac{1}{2}(j+j_1-m_2)\; \frac{1}{2}(j+j_1-2\,j_2+m_2)\right\rangle$$

07.38.17.0050.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle = \frac{\sqrt{2\,j+1}}{\sqrt{j+j_2+m_1+1}} \left\langle \frac{1}{2}(-m+j_1+j_2)\; \frac{1}{2}(j+j_1+m_2)\; \frac{1}{2}(2\,j-m-j_1-j_2)\; \frac{1}{2}(-j-j_1+2\,j_2+m_2) \;\middle|\; \frac{1}{2}(-m+j_1+j_2)\; \frac{1}{2}(j+j_1+m_2)\; \frac{1}{2}(j+j_2+m_1)\; \frac{1}{2}(j-2\,j_1+j_2-m_1)\right\rangle$$

07.38.17.0051.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle = \frac{\sqrt{2\,j+1}}{\sqrt{j+j_2+m_1+1}} \left\langle \frac{1}{2}(j+j_2-m_1)\; j_1\; \frac{1}{2}(j-m-j_2-m_2)\; j_2-j \;\middle|\; \frac{1}{2}(j+j_2-m_1)\; j_1\; \frac{1}{2}(j+j_2+m_1)\; \frac{1}{2}(-j-m+j_2-m_2)\right\rangle$$

07.38.17.0052.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle = \frac{\sqrt{2\,j+1}}{\sqrt{j+j_1-m_2+1}} \left\langle \frac{1}{2}(j+j_2-m_1)\; \frac{1}{2}(m+j_1+j_2)\; \frac{1}{2}(j-2\,j_1+j_2+m_1)\; \frac{1}{2}(-2\,j-m+j_1+j_2) \;\middle|\; \frac{1}{2}(j+j_2-m_1)\; \frac{1}{2}(m+j_1+j_2)\; \frac{1}{2}(j+j_1-m_2)\; \frac{1}{2}(-j-j_1+2\,j_2-m_2)\right\rangle$$

07.38.17.0053.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle = \frac{\sqrt{2\,j+1}}{\sqrt{j+j_1-m_2+1}} \left\langle j_2\; \frac{1}{2}(j+j_1+m_2)\; j-j_1\; \frac{1}{2}(-j-m+j_1-m_1) \;\middle|\; j_2\; \frac{1}{2}(j+j_1+m_2)\; \frac{1}{2}(j+j_1-m_2)\; \frac{1}{2}(j-m-j_1-m_1)\right\rangle$$

07.38.17.0054.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle = (-1)^{-j+j_1+j_2} \langle j_2\, j_1\, m_2\, m_1 \mid j_2\, j_1\, j\, m\rangle \;/;\; -j+j_1+j_2 \in \mathbb{Z}$$

07.38.17.0055.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle = (-1)^{-j+j_1+j_2} \langle j_1\, j_2\, -m_1\, -m_2 \mid j_1\, j_2\, j\, -m\rangle \;/;\; -j+j_1+j_2 \in \mathbb{Z}$$

07.38.17.0056.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle = \frac{(-1)^{j_2+m_2}\sqrt{2\,j+1}}{\sqrt{2\,j_1+1}} \langle j\, j_2\, -m\, m_2 \mid j\, j_2\, j_1\, -m_1\rangle \;/;\; j_2+m_2 \in \mathbb{N}$$

07.38.17.0057.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle = \frac{(-1)^{j_1-m_1}\sqrt{2\,j+1}}{\sqrt{2\,j_2+1}} \langle j_1\, j\, m_1\, -m \mid j_1\, j\, j_2\, -m_2\rangle \;/;\; j_1-m_1 \in \mathbb{N}$$

07.38.17.0058.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle = \frac{(-1)^{j_1-m_1}\sqrt{2\,j+1}}{\sqrt{2\,j_2+1}} \langle j\, j_1\, m\, -m_1 \mid j\, j_1\, j_2\, m_2\rangle \;/;\; j_1-m_1 \in \mathbb{N}$$

07.38.17.0059.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle = \frac{(-1)^{j_1-m_1}\sqrt{2\,j+1}}{\sqrt{2\,j_2+1}} \langle j\, j_1\, m\, -m_1 \mid j\, j_1\, j_2\, m_2\rangle \;/;\; j_1-m_1 \in \mathbb{N}$$

07.38.17.0060.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle = \frac{(-1)^{j-m+j_1-m_1}\sqrt{2\,j+1}}{\sqrt{2\,j_1+1}}\,\langle j\, j_2\, m\, -m_2 \mid j\, j_2\, j_1\, m_1\rangle\, /;\, j-j_1-m_2\in\mathbb{N}$$

07.38.17.0061.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle = \frac{(-1)^{j-m+j_1-m_1}\sqrt{2\,j+1}}{\sqrt{2\,j_1+1}}\,\langle j_2\, j\, m_2\, -m \mid j_2\, j\, j_1\, -m_1\rangle\, /;\, j-j_1-m_2\in\mathbb{N}$$

07.38.17.0062.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle = \left(\sqrt{2\,j+1}\,\left\langle \frac{1}{2}(-j+j_2+m_1-1)\; j_1\; \frac{1}{2}(j-m+j_2-m_2+1)\; -j-j_2-1 \;\middle|\; \frac{1}{2}(-j+j_2+m_1-1)\; j_1\; \frac{1}{2}(j-j_2+m_1-1)\; \frac{1}{2}(-j-m-j_2-m_2-1)\right\rangle \sqrt{(-j+j_1-j_2-1)!}\,\sqrt{(j-j_1+j_2)!}\,\sqrt{(j-m)!}\,\sqrt{(-j+m-1)!}\right) \Big/ \left(\sqrt{(j_1-m_1)!}\,\sqrt{(m_1-j_1)!}\,\sqrt{(-j_2-m_2-1)!}\,\sqrt{(j_2+m_2)!}\,\sqrt{(j-j_2+m_1-1)!}\right)$$

07.38.17.0063.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle = \left(\sqrt{2\,j+1}\,\left\langle \frac{1}{2}(j-j_2-m_1-1)\; -j_1-1\; \frac{1}{2}(j-m+j_2-m_2+1)\; -j-j_2-1 \;\middle|\; \frac{1}{2}(j-j_2-m_1-1)\; -j_1-1\; \frac{1}{2}(j-j_2+m_1-1)\; \frac{1}{2}(-j-m-j_2-m_2-1)\right\rangle \sqrt{(-j-j_1-j_2-2)!}\,\sqrt{(j+j_1+j_2+1)!}\,\sqrt{(-j_1-m_1-1)!}\,\sqrt{(j_1+m_1)!}\,\sqrt{(j_2-m_2)!}\,\sqrt{(-j_2+m_2-1)!}\right) \Big/ \left(\sqrt{(j-j_1-j_2+1)!}\,\sqrt{(-j+j_1+j_2)!}\,\sqrt{(j_1-m_1)!}\,\sqrt{(-j_1+m_1-1)!}\,\sqrt{(-j_2-m_2-1)!}\,\sqrt{(j_2+m_2)!}\,\sqrt{j-j_2+m_1}\right)$$

Involving three Clebsch Gordan coefficients

07.38.17.0064.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle = \langle j_2\, j_1\, m_2\, m_1 \mid j_2\, j_1\, j\, m\rangle\,\csc(\pi\,(j-j_1+m_2))\sin(\pi\,(j_2+m_2)) + \frac{\sqrt{2\,j+1}\,\csc(\pi\,(j-j_1+m_2))\sin(\pi\,(j-j_1-j_2))}{\sqrt{j+j_1-m_2+1}}\left\langle j_2\; \frac{1}{2}(j+j_1+m_2)\; j_1-j\; \frac{1}{2}(j+m-j_1+m_1) \;\middle|\; j_2\; \frac{1}{2}(j+j_1+m_2)\; \frac{1}{2}(j+j_1-m_2)\; \frac{1}{2}(-j+m+j_1+m_1)\right\rangle\, /;\, \operatorname{Re}(j+j_1+j_2) > -2$$

07.38.17.0065.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle = -\frac{1}{\sqrt{j+j_2-m_1+1}} \left(\sqrt{2j+1}\, \csc((j+m)\pi)\csc(\pi(j+j_1-j_2))\csc(\pi(j_1+m_1))\sin(\pi(j-j_1-j_2))\sin(\pi(j_1-m_1))\sin(\pi(j_2+m_2))\right) \left\langle j_1\, \frac{1}{2}(j+j_2+m_1)\, j_2-j\, \frac{1}{2}(j+m-j_2+m_2) \,\middle|\, j_1\, \frac{1}{2}(j+j_2+m_1)\, \frac{1}{2}(j+j_2-m_1)\, \frac{1}{2}(-j+m+j_2+m_2)\right\rangle - \left(\sqrt{2j+1}\, \sqrt{\Gamma(-j-m)}\, \sqrt{\Gamma(j-m+1)}\, \sqrt{\Gamma(m-j)}\, \sqrt{\Gamma(j+m+1)}\, \sqrt{\Gamma(j+j_1-j_2+1)}\, \sqrt{\Gamma(-j-j_1+j_2)}\, \sqrt{\Gamma(-j_1-m_1)}\, \sqrt{\Gamma(j_1+m_1+1)}\, \sin(\pi(j-j_2+m_1))\right) \Big/ \left(\pi \sqrt{\Gamma(-j-j_1-j_2-1)}\, \sqrt{\Gamma(j+j_1+j_2+2)}\, \sqrt{\Gamma(j_1-m_1+1)}\, \sqrt{\Gamma(m_1-j_1)}\, \sqrt{-2j_1-1}\right) \left\langle \frac{1}{2}(-j+j_2-m_1-1)\, \frac{1}{2}(-j+j_2+m_1-1)\, \frac{1}{2}(-j-m-j_2-m_2-1)\, \frac{1}{2}(-j+m-j_2+m_2-1) \,\middle|\, \frac{1}{2}(-j+j_2-m_1-1)\, \frac{1}{2}(-j+j_2+m_1-1) - j_1-1-j-j_2-1\right\rangle \;/;\; \mathrm{Re}(j+j_1+j_2) > -2 \bigwedge \mathrm{Re}(j+j_1-j_2) < 0$$

07.38.17.0066.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle = \left(\sqrt{2j+1}\, \csc(\pi(j_1+m_1))\sqrt{\Gamma(j+j_1-j_2+1)}\, \sqrt{\Gamma(-j-j_1+j_2)}\, \sqrt{\Gamma(-j-m)}\, \sqrt{\Gamma(j+m+1)}\, \sin(\pi(j-j_2+m_1))\right) \Big/ \left(\sqrt{\Gamma(j_1-m_1+1)}\, \sqrt{\Gamma(m_1-j_1)}\, \sqrt{\Gamma(-j_2-m_2)}\, \sqrt{\Gamma(j_2+m_2+1)}\, \sqrt{j-j_1-m_2}\right) \left\langle \frac{1}{2}(-j+j_2+m_1-1)\, \frac{1}{2}(-m+j_1+j_2)\, \frac{1}{2}(-j-2j_1+j_2-m_1-1)\, \frac{1}{2}(2j+m+j_1+j_2+2) \,\middle|\, \frac{1}{2}(-j+j_2+m_1-1)\, \frac{1}{2}(-m+j_1+j_2)\, \frac{1}{2}(j-j_1-m_2-1)\, \frac{1}{2}(j-j_1+2j_2+m_2+1)\right\rangle - \frac{1}{\sqrt{j+j_2-m_1+1}} \left(\sqrt{2j+1}\, \csc(\pi(j+j_1-j_2))\csc(\pi(j_1+m_1))\csc(\pi(j+m))\sin(\pi(j-j_1-j_2))\sin(\pi(j_1-m_1))\sin(\pi(j_2+m_2))\right) \left\langle j_1\, \frac{1}{2}(j+j_2+m_1)\, j_2-j\, \frac{1}{2}(j+m-j_2+m_2) \,\middle|\, j_1\, \frac{1}{2}(j+j_2+m_1)\, \frac{1}{2}(j+j_2-m_1)\, \frac{1}{2}(-j+m+j_2+m_2)\right\rangle \;/;\; \mathrm{Re}(j+j_1+j_2) > -2 \bigwedge \mathrm{Re}(j_2-m_2) > -1$$

07.38.17.0067.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle = -\Big(\sqrt{2j+1}\;\pi\,\csc(2\pi j_1)\sqrt{\Gamma(-j+j_1-j_2)}\;\sqrt{\Gamma(j-j_1+j_2+1)}\;\sqrt{\Gamma(-j-m)}\;\sqrt{\Gamma(j+m+1)}\Big)\Big/ \Big(\sqrt{\Gamma(-j-j_1-j_2-1)}\;\sqrt{\Gamma(j+j_1-j_2+1)}\;\sqrt{\Gamma(-j-j_1+j_2)}\;\sqrt{\Gamma(j+j_1+j_2+2)}\;\sqrt{\Gamma(-j_1-m_1)}\;\sqrt{\Gamma(j_1+m_1+1)}\;\sqrt{\Gamma(-j_2-m_2)}\;\sqrt{\Gamma(j_2+m_2+1)}\;\sqrt{-m-j_1-j_2-1}\Big) \Big\langle \frac{1}{2}(-j+j_2-m_1-1)\;\frac{1}{2}(-j+j_1-m_2-1)\;\frac{1}{2}(-j-2j_1+j_2+m_1-1)\;\frac{1}{2}(-j+j_1-2j_2+m_2-1)\;\Big|\; \frac{1}{2}(-j+j_2-m_1-1)\;\frac{1}{2}(-j+j_1-m_2-1)\;\frac{1}{2}(-m-j_1-j_2-2)\;\frac{1}{2}(-2j+m-j_1-j_2-2)\Big\rangle - \Big(\sqrt{2j+1}\;\csc(2\pi j_1)\sqrt{\Gamma(-j-j_1-j_2-1)}\;\sqrt{\Gamma(j+j_1-j_2+1)}\;\sqrt{\Gamma(-j-j_1+j_2)}\;\sqrt{\Gamma(j+j_1+j_2+2)}\;\sqrt{\Gamma(-j_1-m_1)}\;\sqrt{\Gamma(j_1+m_1+1)}\;\sqrt{\Gamma(-j-m)}\;\sqrt{\Gamma(j+m+1)}\;\sin(\pi(j-j_1-j_2))\sin(\pi(j_1-m_1))\Big)\Big/ \Big(\pi\sqrt{\Gamma(-j+j_1-j_2)}\;\sqrt{\Gamma(j-j_1+j_2+1)}\;\sqrt{\Gamma(-j_2-m_2)}\;\sqrt{\Gamma(j_2+m_2+1)}\;\sqrt{-j+j_1-m_2}\Big) \Big\langle \frac{1}{2}(-j+j_2-m_1-1)\;\frac{1}{2}(-m-j_1-j_2-2)\;\frac{1}{2}(j+2j_1-j_2-m_1+1)\;\frac{1}{2}(-2j+m-j_1-j_2-2)\;\Big|\;\frac{1}{2}(-j+j_2-m_1-1)\;\frac{1}{2}(-m-j_1-j_2-2)\;\frac{1}{2}(-j+j_1-m_2-1)\;\frac{1}{2}(-j+j_1-2j_2+m_2-1)\Big\rangle \;/;\; \mathrm{Re}(j+j_1+j_2) > -2 \land \mathrm{Re}(j+m) < 0$$

Summation

Finite summation

Involving one Clebsch Gordan coefficient

07.38.23.0003.01

$$\sum_{m_1=-j_1}^{j_1} \langle j_1\, j_2\, m_1\, 0 \mid j_1\, j_2\, j_1\, m_1\rangle = (2j_1+1)\,\delta_{j_2,0} \;/;\; 2j_1 \in \mathbb{N}$$

07.38.23.0004.01

$$\sum_{m_1=-j_1}^{j_1} (-1)^{j_1-m_1} \langle j_1\, j_1\, m_1\, -m_1 \mid j_1\, j_1\, j\, 0\rangle = \sqrt{2j_1+1}\;\delta_{j,0} \;/;\; 2j_1 \in \mathbb{N}$$

07.38.23.0005.01

$$\sum_{m_1=-j_1}^{j_1}\sum_{m_2=-j_2}^{j_2}\sum_{m=-j}^{j} \frac{(-1)^{j+m}\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle}{\sqrt{(j_1+m_1)!}\;\sqrt{(j_1-m_1)!}\;\sqrt{(j_2+m_2)!}\;\sqrt{(j_2-m_2)!}\;\sqrt{(j+m)!}\;\sqrt{(j-m)!}} = 0$$

07.38.23.0006.01

$$\sum_{j_1=|j-j_2|}^{j+j_2} \frac{2j_1+1}{j_1(j_1+1)-n(n+1)} \langle j_1\, j_2\, 0\, 0 \mid j_1\, j_2\, j\, 0\rangle^2 = 0 \;/;\; n \in \mathbb{Z} \bigwedge |j-j_2| \le n \le j+j_2 \bigwedge \frac{n+j_2+j+1}{2} \in \mathbb{Z}$$

07.38.23.0007.01

$$\sum_{j_2=0}^{j_1} \frac{\langle j_1\, j+j_2\, 0\, 0 \mid j_1\, j+j_2\, j+j_1-j_2\, 0\rangle^2}{(2j_2-1)(2j+2j_1-2j_2+1)} = -\frac{\delta_{j_1,0}}{2j+1} \;/;\; j \in \mathbb{N}$$

07.38.23.0008.01

$$\sum_{j_2=0}^{j_1}\left(\frac{1}{2 j_2+3}-\frac{j_1+1}{(2 j_1+3)(2 j_2+1)}\right)\frac{\langle j_1\, j+j_2\, 0\, 0 \mid j_1\, j+j_2\, j+j_1-j_2\, 0\rangle^2}{2 j_1-2 j_2+2 j+1} == 0$$

Involving two Clebsch Gordan coefficients

07.38.23.0001.01

$$\sum_{m_1=-j_1}^{j_1}\sum_{m_2=-j_2}^{j_2}\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j'\, m'\rangle == \delta_{j,j'}\,\delta_{m,m'} /; \mathcal{TriangularQ}(j_1,\, j_2,\, j) \bigwedge j-m\in\mathbb{Z} \bigwedge -j\le m\le j$$

07.38.23.0002.01

$$\sum_{j=|j_1-j_2|}^{j_1+j_2}\sum_{m=-j}^{j}\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle\langle j_1\, j_2\, m_1'\, m_2' \mid j_1\, j_2\, j\, m\rangle == \delta_{m_1,m_1'}\,\delta_{m_2,m_2'} /; \mathcal{PhysicalQ}(\{j_1,\, m_1\},\, \{j_2,\, m_2\},\, \{j_1+j_2,\, m_1+m_2\})$$

07.38.23.0009.01

$$\sum_{m_1=-j_1}^{j_1}\sum_{m=-j}^{j}\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle\langle j_1\, j_2'\, m_1\, m_2' \mid j_1\, j_2'\, j\, m\rangle == \frac{2 j+1}{2 j_2+1}\,\delta_{j_2,j_2'}\,\delta_{m_2,m_2'} /;$$
$$\mathcal{TriangularQ}(j_1,\, j_2,\, j) \bigwedge j_2-m_2\in\mathbb{Z} \bigwedge -j_2\le m_2\le j_2$$

07.38.23.0010.01

$$\sum_{m_1=-j_1}^{j_1}\sum_{m_2=-j_2}^{j_2}(-1)^{j_2+m_2}\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle\langle j'\, j_1\, -m'\, m_1 \mid j'\, j_1\, j_2\, -m_2\rangle == (-1)^{j+m}\,\frac{\sqrt{2 j_2+1}}{\sqrt{2 j+1}}\,\delta_{j,j'}\,\delta_{m,m'} /;$$
$$\mathcal{TriangularQ}(j_1,\, j_2,\, j) \bigwedge j-m\in\mathbb{Z} \bigwedge -j\le m\le j$$

07.38.23.0011.01

$$\sum_{m_1=-j_1}^{j_1}\sum_{m_2=-j_2}^{j_2}(-1)^{j_1+m_1}\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle\langle j_1\, j'\, -m_1\, m' \mid j_1\, j'\, j_2\, m_2\rangle == \frac{\sqrt{2 j_2+1}}{\sqrt{2 j+1}}\,\delta_{j,j'}\,\delta_{m,m'} /;$$
$$\mathcal{TriangularQ}(j_1,\, j_2,\, j) \bigwedge j-m\in\mathbb{Z} \bigwedge -j\le m\le j$$

07.38.23.0012.01

$$\sum_{m_1=-j_1}^{j_1}\sum_{m_2=-j_2}^{j_2}\langle j\, j_2\, m\, m_2 \mid j\, j_2\, j_1\, m_1\rangle\langle j'\, j_1\, -m'\, m_1 \mid j'\, j_1\, j_2\, m_2\rangle == (-1)^{-j-j_1+j_2}\,\frac{\sqrt{2 j_1+1}\,\sqrt{2 j_2+1}}{2 j+1}\,\delta_{j,j'}\,\delta_{m,m'} /;$$
$$\mathcal{TriangularQ}(j_1,\, j_2,\, j) \bigwedge j-m\in\mathbb{Z} \bigwedge -j\le m\le j$$

07.38.23.0013.01

$$\sum_{j=|j_1-j_2|}^{j_1+j_2}\sum_{m=-j}^{j}(2 j+1)\langle j_1\, j\, m_1\, m \mid j_1\, j\, j_2\, m_2\rangle\langle j_1\, j\, m_1'\, m \mid j_1\, j\, j_2\, m_2'\rangle == (2 j_2+1)\,\delta_{m_1,m_1'}\,\delta_{m_2,m_2'} /;$$
$$\mathcal{PhysicalQ}(\{j_1,\, m_1\},\, \{j_2,\, m_2\},\, \{j_1+j_2,\, m_1+m_2\})$$

07.38.23.0014.01

$$\sum_{j_1=|j-j_2|}^{j+j_2}\sum_{m_1=-j_1}^{j_1}(-1)^{j_1-m_1}(2 j_1+1)\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle\langle j_1\, j\, m_1\, m' \mid j_1\, j\, j_2\, m_2'\rangle == \sqrt{2 j_2+1}\,\sqrt{2 j+1}\,\delta_{m,-m'}\,\delta_{m_2,-m_2'} /;$$
$$\mathcal{PhysicalQ}(\{j_2+j,\, m-m_2\},\, \{j_2,\, m_2\},\, \{j,\, m\})$$

Involving three Clebsch Gordan coefficients

07.38.23.0015.01

$$\sum_{m_1=-j_1}^{j_1}\sum_{m_2=-j_2}^{j_2}\sum_{m_6=-j_6}^{j_6}\langle j_1\,j_2\,m_1\,m_2\mid j_1\,j_2\,j_3\,m_3\rangle\langle j_6\,j_2\,m_6\,m_2\mid j_6\,j_2\,j_4\,m_4\rangle\langle j_1\,j_5\,m_1\,m_5\mid j_1\,j_5\,j_6\,m_6\rangle = (-1)^{j_2+j_3+j_5+j_6}\sqrt{2\,j_3+1}\,\sqrt{2\,j_6+1}\,\langle j_3\,j_5\,m_3\,m_5\mid j_3\,j_5\,j_4\,m_4\rangle\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix}$$

07.38.23.0016.01

$$\sum_{m_1=-j_1}^{j_1}\sum_{m_2=-j_2}^{j_2}\sum_{m_6=-j_6}^{j_6}\langle j_2\,j_3\,m_2\,m_3\mid j_2\,j_3\,j_1\,m_1\rangle\langle j_2\,j_4\,m_2\,m_4\mid j_2\,j_4\,j_6\,m_6\rangle\langle j_1\,j_5\,m_1\,m_5\mid j_1\,j_5\,j_6\,m_6\rangle = (-1)^{j_2+j_3+j_5+j_6}\frac{\sqrt{2\,j_1+1}\,(2\,j_6+1)}{\sqrt{2\,j_4+1}}\langle j_3\,j_5\,m_3\,m_5\mid j_3\,j_5\,j_4\,m_4\rangle\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix}$$

07.38.23.0017.01

$$\sum_{m_1=-j_1}^{j_1}\sum_{m_2=-j_2}^{j_2}\sum_{m_6=-j_6}^{j_6}\langle j_2\,j_1\,m_2\,m_1\mid j_2\,j_1\,j_3\,m_3\rangle\langle j_2\,j_6\,m_2\,m_6\mid j_2\,j_6\,j_4\,m_4\rangle\langle j_1\,j_5\,m_1\,m_5\mid j_1\,j_5\,j_6\,m_6\rangle = (-1)^{j_1+j_2+j_4+j_5}\sqrt{2\,j_3+1}\,\sqrt{2\,j_6+1}\,\langle j_3\,j_5\,m_3\,m_5\mid j_3\,j_5\,j_4\,m_4\rangle\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix}$$

07.38.23.0018.01

$$\sum_{m_1=-j_1}^{j_1}\sum_{m_2=-j_2}^{j_2}\sum_{m_6=-j_6}^{j_6}(-1)^{j_1-m_1}\langle j_1\,j_2\,m_1\,m_2\mid j_1\,j_2\,j_3\,m_3\rangle\langle j_6\,j_2\,m_6\,m_2\mid j_6\,j_2\,j_4\,m_4\rangle\langle j_6\,j_1\,m_6\,-m_1\mid j_6\,j_1\,j_5\,m_5\rangle = (-1)^{j_2+j_3+j_5+j_6}\sqrt{2\,j_3+1}\,\sqrt{2\,j_5+1}\,\langle j_3\,j_5\,m_3\,m_5\mid j_3\,j_5\,j_4\,m_4\rangle\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix}$$

07.38.23.0019.01

$$\sum_{m_1=-j_1}^{j_1}\sum_{m_2=-j_2}^{j_2}\sum_{m_6=-j_6}^{j_6}(-1)^{j_2+m_2}\langle j_1\,j_2\,m_1\,m_2\mid j_1\,j_2\,j_3\,m_3\rangle\langle j_2\,j_4\,-m_2\,m_4\mid j_2\,j_4\,j_6\,m_6\rangle\langle j_1\,j_5\,m_1\,m_5\mid j_1\,j_5\,j_6\,m_6\rangle = (-1)^{j_2+j_3+j_5+j_6}\frac{\sqrt{2\,j_3+1}\,(2\,j_6+1)}{\sqrt{2\,j_4+1}}\langle j_3\,j_5\,m_3\,m_5\mid j_3\,j_5\,j_4\,m_4\rangle\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix}$$

07.38.23.0020.01

$$\sum_{m_1=-j_1}^{j_1}\sum_{m_2=-j_2}^{j_2}\sum_{m_6=-j_6}^{j_6}(-1)^{j_1-m_1}\langle j_2\,j_1\,m_2\,m_1\mid j_2\,j_1\,j_3\,m_3\rangle\langle j_2\,j_6\,m_2\,m_6\mid j_2\,j_6\,j_4\,m_4\rangle\langle j_6\,j_1\,m_6\,-m_1\mid j_6\,j_1\,j_5\,m_5\rangle = (-1)^{j_1+j_2+j_4+j_5}\sqrt{2\,j_3+1}\,\sqrt{2\,j_5+1}\,\langle j_3\,j_5\,m_3\,m_5\mid j_3\,j_5\,j_4\,m_4\rangle\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix}$$

07.38.23.0021.01

$$\sum_{m_1=-j_1}^{j_1}\sum_{m_2=-j_2}^{j_2}\sum_{m_6=-j_6}^{j_6}(-1)^{j_2+m_2}\langle j_2\,j_3\,-m_2\,m_3\mid j_2\,j_3\,j_1\,m_1\rangle\langle j_6\,j_2\,m_6\,m_2\mid j_6\,j_2\,j_4\,m_4\rangle\langle j_1\,j_5\,m_1\,m_5\mid j_1\,j_5\,j_6\,m_6\rangle = (-1)^{j_2+j_3+j_5+j_6}\sqrt{2\,j_1+1}\,\sqrt{2\,j_6+1}\,\langle j_3\,j_5\,m_3\,m_5\mid j_3\,j_5\,j_4\,m_4\rangle\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix}$$

07.38.23.0022.01

$$\sum_{m_1=-j_1}^{j_1}\sum_{m_2=-j_2}^{j_2}\sum_{m_6=-j_6}^{j_6}(-1)^{j_1-m_1}\langle j_1\,j_3\,m_1\,-m_3 \mid j_1\,j_3\,j_2\,m_2\rangle\langle j_6\,j_2\,m_6\,-m_2 \mid j_6\,j_2\,j_4\,m_4\rangle\langle j_1\,j_5\,m_1\,m_5 \mid j_1\,j_5\,j_6\,m_6\rangle = (-1)^{j_2+j_3+j_5+j_6}\sqrt{2\,j_2+1}\,\sqrt{2\,j_6+1}\,\langle j_3\,j_5\,m_3\,m_5 \mid j_3\,j_5\,j_4\,m_4\rangle\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix}$$

Involving four Clebsch Gordan coefficients

07.38.23.0023.01

$$\sum_{\beta=-b}^{b}\sum_{\gamma=-c}^{c}\sum_{\epsilon=-e}^{e}\sum_{\varphi=-f}^{f}\langle b\,c\,\beta\,\gamma \mid b\,c\,a\,\alpha\rangle\langle e\,f\,\epsilon\,\varphi \mid e\,f\,d\,\delta\rangle\langle e\,b\,\epsilon\,\beta \mid e\,b\,g\,\eta\rangle\langle f\,c\,\varphi\,\gamma \mid f\,c\,j\,\mu\rangle = \sqrt{2\,a+1}\,\sqrt{2\,d+1}\,\sqrt{2\,g+1}\,\sqrt{2\,j+1}\sum_{k=\max(|g-j|,|a-d|)}^{\min(g+j,a+d)}\sum_{\kappa=-k}^{k}\langle g\,j\,\eta\,\mu \mid g\,j\,k\,\kappa\rangle\langle d\,a\,\delta\,\alpha \mid d\,a\,k\,\kappa\rangle\begin{Bmatrix} c & b & a \\ f & e & d \\ j & g & k \end{Bmatrix}$$

07.38.23.0024.01

$$\sum_{\beta=-b}^{b}\sum_{\gamma=-c}^{c}\sum_{\epsilon=-e}^{e}\sum_{\varphi=-f}^{f}\langle b\,c\,\beta\,-\gamma \mid b\,c\,a\,\alpha\rangle\langle e\,f\,\epsilon\,-\varphi \mid e\,f\,d\,\delta\rangle\langle b\,e\,\beta\,\epsilon \mid b\,e\,g\,\eta\rangle\langle c\,f\,\gamma\,\varphi \mid c\,f\,j\,\mu\rangle = (-1)^{b+e-g}\sqrt{2\,a+1}\,\sqrt{2\,d+1}\,\sqrt{2\,g+1}\,\sqrt{2\,j+1}\sum_{k=\max(|g-j|,|a-d|)}^{\min(g+j,a+d)}\sum_{\kappa=-k}^{k}\langle g\,j\,\eta\,-\mu \mid g\,j\,k\,\kappa\rangle\langle d\,a\,\delta\,\alpha \mid d\,a\,k\,\kappa\rangle\begin{Bmatrix} c & b & a \\ f & e & d \\ j & g & k \end{Bmatrix}$$

07.38.23.0025.01

$$\sum_{\beta=-b}^{b}\sum_{\gamma=-c}^{c}\sum_{\epsilon=-e}^{e}\sum_{\varphi=-f}^{f}\langle b\,a\,\beta\,\alpha \mid b\,a\,c\,\gamma\rangle\langle f\,j\,\varphi\,\mu \mid f\,j\,c\,\gamma\rangle\langle b\,g\,\beta\,\eta \mid b\,g\,e\,\epsilon\rangle\langle f\,d\,\varphi\,\delta \mid f\,d\,e\,\epsilon\rangle = (-1)^{a-b+f-j}(2\,c+1)(2\,e+1)\sum_{k=\max(|g-j|,|a-d|)}^{\min(g+j,a+d)}\sum_{\kappa=-k}^{k}\langle g\,j\,\eta\,\mu \mid g\,j\,k\,\kappa\rangle\langle d\,a\,\delta\,\alpha \mid d\,a\,k\,\kappa\rangle\begin{Bmatrix} c & b & a \\ f & e & d \\ j & g & k \end{Bmatrix}$$

07.38.23.0026.01

$$\sum_{\beta=-b}^{b}\sum_{\gamma=-c}^{c}\sum_{\epsilon=-e}^{e}\sum_{\varphi=-f}^{f}\langle a\,b\,\alpha\,\beta \mid a\,b\,c\,\gamma\rangle\langle j\,c\,\mu\,\gamma \mid j\,c\,f\,\varphi\rangle\langle g\,e\,\eta\,\epsilon \mid g\,e\,b\,\beta\rangle\langle d\,f\,\delta\,\varphi \mid d\,f\,e\,\epsilon\rangle = (-1)^{-c+d+e-j}\sqrt{2\,b+1}\,\sqrt{2\,c+1}\,\sqrt{2\,e+1}\,\sqrt{2\,f+1}\sum_{k=\max(|g-j|,|a-d|)}^{\min(g+j,a+d)}\sum_{\kappa=-k}^{k}(-1)^{k-\kappa}\langle g\,j\,\eta\,\mu \mid g\,j\,k\,-\kappa\rangle\langle d\,a\,\delta\,\alpha \mid d\,a\,k\,\kappa\rangle\begin{Bmatrix} c & b & a \\ f & e & d \\ j & g & k \end{Bmatrix}$$

07.38.23.0027.01

$$\sum_{\beta=-b}^{b}\sum_{\gamma=-c}^{c}\sum_{\epsilon=-e}^{e}\sum_{\varphi=-f}^{f}\langle b\,a\,\beta\,-\alpha \mid b\,a\,c\,\gamma\rangle\langle j\,c\,\mu\,-\gamma \mid j\,c\,f\,\varphi\rangle\langle g\,b\,\eta\,-\beta \mid g\,b\,e\,\epsilon\rangle\langle e\,d\,\epsilon\,-\delta \mid e\,d\,f\,\varphi\rangle = (-1)^{b-c-g-\alpha+\eta}\sqrt{2\,c+1}\,\sqrt{2\,e+1}\,(2\,f+1)\sum_{k=\max(|g-j|,|a-d|)}^{\min(g+j,a+d)}\sum_{\kappa=-k}^{k}\langle g\,j\,\eta\,-\mu \mid g\,j\,k\,\kappa\rangle\langle d\,a\,\delta\,\alpha \mid d\,a\,k\,\kappa\rangle\begin{Bmatrix} c & b & a \\ f & e & d \\ j & g & k \end{Bmatrix}$$

07.38.23.0028.01

$$\sum_{\beta=-b}^{b}\sum_{\gamma=-c}^{c}\sum_{\epsilon=-e}^{e}\sum_{\varphi=-f}^{f}\langle b\,c\,\beta-\gamma\mid b\,c\,a\,\alpha\rangle\langle f\,j\,\varphi\,\mu\mid f\,j\,c\,\gamma\rangle\langle e\,g-\epsilon\,\eta\mid e\,g\,b\,\beta\rangle\langle e\,f\,\epsilon\,\varphi\mid e\,f\,d\,\delta\rangle = (-1)^{b+f-g-\delta}\sqrt{2a+1}\sqrt{2b+1}\sqrt{2c+1}\sqrt{2d+1}\sum_{k=\max(|g-j|,|a-d|)}^{\min(g+j,a+d)}\sum_{\kappa=-k}^{k}\langle g\,j\,\eta-\mu\mid g\,j\,k\,\kappa\rangle\langle d\,a\,\delta\,\alpha\mid d\,a\,k\,\kappa\rangle\begin{Bmatrix}c&b&a\\f&e&d\\j&g&k\end{Bmatrix}$$

07.38.23.0029.01

$$\sum_{\beta=-b}^{b}\sum_{\gamma=-c}^{c}\sum_{\epsilon=-e}^{e}\sum_{\varphi=-f}^{f}\langle b\,c\,\beta\,\gamma\mid b\,c\,a\,\alpha\rangle\langle f\,j\,\varphi\,\mu\mid f\,j\,c\,\gamma\rangle\langle e\,g\,\epsilon\,\eta\mid e\,g\,b\,\beta\rangle\langle e\,f\,\epsilon\,\varphi\mid e\,f\,d\,\delta\rangle = \sqrt{2b+1}\sqrt{2c+1}\sqrt{2d+1}\sum_{k=\max(|g-j|,|a-d|)}^{\min(g+j,a+d)}\sum_{\kappa=-k}^{k}\sqrt{2k+1}\,\langle g\,j\,\eta\,\mu\mid g\,j\,k\,\kappa\rangle\langle d\,k\,\delta\,\kappa\mid d\,k\,a\,\alpha\rangle\begin{Bmatrix}a&b&c\\d&e&f\\k&g&j\end{Bmatrix}$$

07.38.23.0030.01

$$\sum_{\beta=-b}^{b}\sum_{\gamma=-c}^{c}\sum_{\epsilon=-e}^{e}\sum_{\varphi=-f}^{f}\langle b\,c\,\beta-\gamma\mid b\,c\,a\,\alpha\rangle\langle j\,f\,\mu-\varphi\mid j\,f\,c\,\gamma\rangle\langle g\,b\,\eta-\beta\mid g\,b\,e\,\epsilon\rangle\langle e\,f\,\epsilon-\varphi\mid e\,f\,d\,\delta\rangle = (-1)^{c+e-g+j+\alpha-\mu}\sqrt{2a+1}\sqrt{2d+1}\sqrt{2e+1}\sqrt{2c+1}\sum_{k=\max(|g-j|,|a-d|)}^{\min(g+j,a+d)}\sum_{\kappa=-k}^{k}\langle g\,j\,\eta-\mu\mid g\,j\,k\,\kappa\rangle\langle d\,a\,\delta\,\alpha\mid d\,a\,k\,\kappa\rangle\begin{Bmatrix}c&b&a\\f&e&d\\j&g&k\end{Bmatrix}$$

07.38.23.0031.01

$$\sum_{\beta=-b}^{b}\sum_{\gamma=-c}^{c}\sum_{\epsilon=-e}^{e}\sum_{\varphi=-f}^{f}\langle b\,c\,\beta\,\gamma\mid b\,c\,a\,\alpha\rangle\langle f\,c\,\varphi\,\gamma\mid f\,c\,j\,\mu\rangle\langle b\,g\,\beta\,\eta\mid b\,g\,e\,\epsilon\rangle\langle f\,d\,\varphi\,\delta\mid f\,d\,e\,\epsilon\rangle = (-1)^{-a+j+\delta-\eta}\sqrt{2a+1}\sqrt{2j+1}\,(2e+1)\sum_{k=\max(|g-j|,|a-d|)}^{\min(g+j,a+d)}\sum_{\kappa=-k}^{k}\langle g\,j\,\eta-\mu\mid g\,j\,k\,\kappa\rangle\langle d\,a\,\delta-\alpha\mid d\,a\,k\,\kappa\rangle\begin{Bmatrix}c&b&a\\f&e&d\\j&g&k\end{Bmatrix}$$

07.38.23.0032.01

$$\sum_{\beta=-b}^{b}\sum_{\gamma=-c}^{c}\sum_{\epsilon=-e}^{e}\sum_{\varphi=-f}^{f}\langle b\,c\,\beta\,\gamma\mid b\,c\,a\,\alpha\rangle\langle f\,c\,\varphi\,\gamma\mid f\,c\,j\,\mu\rangle\langle g\,e\,\eta\,\epsilon\mid g\,e\,b\,\beta\rangle\langle f\,d\,\varphi\,\delta\mid f\,d\,e\,\epsilon\rangle = (-1)^{-a+g+j+\delta}\sqrt{2a+1}\sqrt{2b+1}\sqrt{2e+1}\sqrt{2j+1}\sum_{k=\max(|g-j|,|a-d|)}^{\min(g+j,a+d)}\sum_{\kappa=-k}^{k}\langle g\,j-\eta-\mu\mid g\,j\,k\,\kappa\rangle\langle d\,a\,\delta-\alpha\mid d\,a\,k\,\kappa\rangle\begin{Bmatrix}c&b&a\\f&e&d\\j&g&k\end{Bmatrix}$$

07.38.23.0033.01

$$\sum_{\beta=-b}^{b}\sum_{\gamma=-c}^{c}\sum_{\epsilon=-e}^{e}\sum_{\varphi=-f}^{f}(-1)^{c+e-\gamma-\epsilon}\langle a\,b\,\alpha\,\beta\mid a\,b\,c\,\gamma\rangle\langle c\,f\,\gamma\,\varphi\mid c\,f\,j\,\mu\rangle\langle e\,b\,\epsilon\,\beta\mid e\,b\,g\,\eta\rangle\langle d\,f\,\delta\,\varphi\mid d\,f\,e\,\epsilon\rangle = (-1)^{a+d-\alpha-\delta}\sqrt{2c+1}\sqrt{2e+1}\sqrt{2g+1}\sqrt{2j+1}\sum_{k=\max(|g-j|,|a-d|)}^{\min(g+j,a+d)}\sum_{\kappa=-k}^{k}\langle g\,j\,\eta-\mu\mid g\,j\,k\,\kappa\rangle\langle d\,a\,\delta-\alpha\mid d\,a\,k\,\kappa\rangle\begin{Bmatrix}c&b&a\\f&e&d\\j&g&k\end{Bmatrix}$$

Involving two Clebsch Gordan coefficients and one 6 j symbol

07.38.23.0034.01

$$\sum_{e=\max(|b-d|,|c-f|)}^{\min(b+d,c+f)} \sum_{\epsilon=-e}^{e} (-1)^{2e} \sqrt{2c+1} \sqrt{2d+1} \langle b\, d\, \beta\, \delta \mid b\, d\, e\, \epsilon\rangle \langle f\, c\, \varphi\, \gamma \mid f\, c\, e\, \epsilon\rangle \begin{Bmatrix} a & b & c \\ e & f & d \end{Bmatrix} = \langle a\, b\, \alpha\, \beta \mid a\, b\, c\, \gamma\rangle \langle a\, f\, \alpha\, \varphi \mid a\, f\, d\, \delta\rangle$$

07.38.23.0035.01

$$\sum_{f=\max(|a-e|,|c-d|)}^{\min(a+e,c+d)} \sum_{\varphi=-f}^{f} (-1)^{c+d+f} \sqrt{2c+1} \sqrt{2e+1} \langle e\, a\, \epsilon\, \alpha \mid e\, a\, f\, \varphi\rangle \langle d\, c\, \delta\, \gamma \mid d\, c\, f\, \varphi\rangle \begin{Bmatrix} b & a & c \\ f & d & e \end{Bmatrix} = \langle a\, b\, \alpha\, \beta \mid a\, b\, c\, \gamma\rangle \langle d\, b\, \delta\, \beta \mid d\, b\, e\, \epsilon\rangle$$

07.38.23.0036.01

$$\sum_{c=\max(|a-e|,|b-f|)}^{\min(a+e,b+f)} \sum_{\gamma=-c}^{c} (-1)^{-d+2e+\alpha+\varphi} \sqrt{2a+1} \sqrt{2e+1} \langle f\, b\, -\varphi\, \beta \mid f\, b\, c\, \gamma\rangle \langle e\, a\, \epsilon\, -\alpha \mid e\, a\, c\, \gamma\rangle \begin{Bmatrix} c & f & b \\ d & e & a \end{Bmatrix} = \langle b\, d\, \beta\, \delta \mid b\, d\, e\, \epsilon\rangle \langle f\, d\, \varphi\, \delta \mid f\, d\, a\, \alpha\rangle$$

07.38.23.0037.01

$$\sum_{c=\max(|a-b|,|e-f|)}^{\min(a+b,e+f)} \sum_{\gamma=-c}^{c} (-1)^{c+d-\beta-\varphi} (2d+1) \langle a\, b\, \alpha\, \beta \mid a\, b\, c\, \gamma\rangle \langle f\, e\, -\varphi\, \epsilon \mid f\, e\, c\, \gamma\rangle \begin{Bmatrix} a & b & c \\ e & f & d \end{Bmatrix} = \langle a\, f\, \alpha\, \varphi \mid a\, f\, d\, \delta\rangle \langle b\, e\, -\beta\, \epsilon \mid b\, e\, d\, \delta\rangle$$

07.38.23.0038.01

$$\sum_{c=\max(|a-b|,|e-f|)}^{\min(a+b,e+f)} \sum_{\gamma=-c}^{c} (-1)^{2e} \sqrt{2c+1} \sqrt{2d+1} \langle a\, b\, \alpha\, \beta \mid a\, b\, c\, \gamma\rangle \langle f\, c\, \varphi\, \gamma \mid f\, c\, e\, \epsilon\rangle \begin{Bmatrix} a & b & c \\ e & f & d \end{Bmatrix} = \langle b\, d\, \beta\, \delta \mid b\, d\, e\, \epsilon\rangle \langle a\, f\, \alpha\, \varphi \mid a\, f\, d\, \delta\rangle$$

07.38.23.0039.01

$$\sum_{f=\max(|a-c|,|b-d|)}^{\min(a+c,b+d)} \sum_{\varphi=-f}^{f} (-1)^{2c} \sqrt{2e+1} \sqrt{2f+1} \langle b\, d\, \beta\, \delta \mid b\, d\, f\, \varphi\rangle \langle a\, f\, \alpha\, \varphi \mid a\, f\, c\, \gamma\rangle \begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix} = \langle b\, a\, \beta\, \alpha \mid b\, a\, e\, \epsilon\rangle \langle d\, e\, \delta\, \epsilon \mid d\, e\, c\, \gamma\rangle$$

07.38.23.0040.01

$$\sum_{c=\max(|a-b|,|e-f|)}^{\min(a+b,e+f)} \sum_{\gamma=-c}^{c} (-1)^{d+e-\beta} \frac{\sqrt{2c+1}\,(2d+1)}{\sqrt{2e+1}} \langle a\, b\, \alpha\, \beta \mid a\, b\, c\, \gamma\rangle \langle f\, c\, \varphi\, \gamma \mid f\, c\, e\, \epsilon\rangle \begin{Bmatrix} a & b & c \\ e & f & d \end{Bmatrix} = \langle b\, e\, -\beta\, \epsilon \mid b\, e\, d\, \delta\rangle \langle a\, f\, \alpha\, \varphi \mid a\, f\, d\, \delta\rangle$$

07.38.23.0041.01

$$\sum_{c=\max(|a-e|,|b-f|)}^{\min(a+e,b+f)} \sum_{\gamma=-c}^{c} (-1)^{2e} \frac{\sqrt{2a+1}\,(2c+1)}{\sqrt{2b+1}} \langle f\, c\, \varphi\, \gamma \mid f\, c\, b\, \beta\rangle \langle c\, a\, \gamma\, \alpha \mid c\, a\, e\, \epsilon\rangle \begin{Bmatrix} c & f & b \\ d & e & a \end{Bmatrix} = \langle d\, b\, \delta\, \beta \mid d\, b\, e\, \epsilon\rangle \langle f\, d\, \varphi\, \delta \mid f\, d\, a\, \alpha\rangle$$

Representations through more general functions

Through hypergeometric functions

Involving ${}_p\tilde{F}_q$

07.38.26.0001.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle = \delta_{m,m_1+m_2}\left(\sqrt{2\,j+1}\,\sqrt{(j+j_1-j_2)!}\,\sqrt{(j-j_1+j_2)!}\,\sqrt{(j_1+m_1)!}\,\sqrt{(j_2-m_2)!}\,\sqrt{(j-m_1-m_2)!}\,\sqrt{(j+m_1+m_2)!}\right)\Big/ \left(\sqrt{(-j+j_1+j_2)!}\,\sqrt{(j+j_1+j_2+1)!}\,\sqrt{(j_1-m_1)!}\,\sqrt{(j_2+m_2)!}\right)\, {}_3\tilde{F}_2(j-j_1-j_2,\, m_1-j_1,\, -j_2-m_2;\, j-j_2+m_1+1,\, j-j_1-m_2+1;\, 1)\,/;\, \mathcal{P}hysical\mathcal{Q}(\{j_1, m_1\}, \{j_2, m_2\}, \{j, m\})$$

Involving ${}_pF_q$

07.38.26.0002.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle = \delta_{m,m_1+m_2}\frac{\sqrt{(j_1-j_2+j)!}\,\sqrt{(-j_1+j_2+j)!}}{\sqrt{(j_1+j_2-j)!}\,\sqrt{(j_1+j_2+j+1)!}}\,\frac{\sqrt{(j_1+m_1)!}\,\sqrt{(j_2-m_2)!}\,\sqrt{(j+m)!}\,\sqrt{(j-m)!}\,\sqrt{2\,j+1}}{(-j_2+j+m_1)!\,(-j_1+j-m_2)!\,\sqrt{(j_1-m_1)!}\,\sqrt{(j_2+m_2)!}}\, {}_3F_2(-j_1-j_2+j,\, -j_1+m_1,\, -j_2-m_2;\, -j_1+j-m_2+1,\, -j_2+j+m_1+1;\, 1)\,/;\,/;\, \mathcal{P}hysical\mathcal{Q}(\{j_1, m_1\}, \{j_2, m_2\}, \{j, m\})$$

07.38.26.0005.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle = \delta_{m,m_1+m_2}(-1)^{j_1-m_1}\frac{\sqrt{(j_1+j_2-j)!}}{\sqrt{(j_1-j_2+j)!}\,\sqrt{(-j_1+j_2+j)!}\,\sqrt{(j_1+j_2+j+1)!}}\,\frac{\sqrt{(j_1+m_1)!}\,\sqrt{(j_2-m_2)!}\,\sqrt{(j+m)!}\,\sqrt{2\,j+1}\,(j_2+j-m_1)!}{\sqrt{(j_1-m_1)!}\,\sqrt{(j_2+m_2)!}\,\sqrt{(j-m)!}\,(j_2-j+m_1)!}\, {}_3F_2(j_1+m_1+1,\, m_1-j_1,\, m-j;\, -j-j_2+m_1,\, -j+j_2+m_1+1;\, 1)\,/;\, \mathcal{P}hysical\mathcal{Q}(\{j_1, m_1\}, \{j_2, m_2\}, \{j, m\})$$

07.38.26.0006.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle = \delta_{m,m_1+m_2}\,(-1)^{j_2+m_2}\frac{\sqrt{(j_1+j_2-j)!}\,\sqrt{(j_1-j_2+j)!}}{\sqrt{(-j_1+j_2+j)!}\,\sqrt{(j_1+j_2+j+1)!}}\,\frac{\sqrt{(j_1-m_1)!}\,\sqrt{(j-m)!}\,\sqrt{2\,j+1}\,(j_2+j+m_1)!}{\sqrt{(j_1+m_1)!}\,\sqrt{(j_2+m_2)!}\,\sqrt{(j_2-m_2)!}\,\sqrt{(j+m)!}\,(j_1-j_2-m)!}\, {}_3F_2(-j+j_1-j_2,\, j_1-m_1+1,\, -j-m;\, -m+j_1-j_2+1,\, -j-j_2-m_1;\, 1)\,/;\, \mathcal{P}hysical\mathcal{Q}(\{j_1, m_1\}, \{j_2, m_2\}, \{j, m\})$$

07.38.26.0007.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle = \delta_{m,m_1+m_2}\,(-1)^{j_2+m_2}\frac{\sqrt{(j_1+j_2-j)!}}{\sqrt{(j_1-j_2+j)!}\,\sqrt{(-j_1+j_2+j)!}\,\sqrt{(j_1+j_2+j+1)!}}\,\frac{\sqrt{(j_1+m_1)!}\,\sqrt{(j_1-m_1)!}\,\sqrt{(j_2-m_2)!}\,\sqrt{2\,j+1}\,(2\,j)!}{\sqrt{(j_2+m_2)!}\,\sqrt{(j+m)!}\,\sqrt{(j-m)!}\,(j_1-j-m_2)!}\, {}_3F_2(-j+j_1-j_2,\, -j+j_1+j_2+1,\, -j-m;\, -2\,j,\, -j+j_1-m_2+1;\, 1)\,/;\, \mathcal{P}hysical\mathcal{Q}(\{j_1, m_1\}, \{j_2, m_2\}, \{j, m\})$$

07.38.26.0008.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle = \delta_{m,m_1+m_2} (-1)^{j_2+m_2} \frac{\sqrt{(j_1+j_2-j)!}}{\sqrt{(j_1-j_2+j)!}\,\sqrt{(-j_1+j_2+j)!}\,\sqrt{(j_1+j_2+j+1)!}} \frac{\sqrt{(j_1-m_1)!}\,\sqrt{2\,j+1}\,(2\,j)!\,(j_2+j+m_1)!}{\sqrt{(j_1+m_1)!}\,\sqrt{(j_2+m_2)!}\,\sqrt{(j_2-m_2)!}\,\sqrt{(j+m)!}\,\sqrt{(j-m)!}} \,{}_3F_2(-j+j_1-j_2,\,-j-j_1-j_2-1,\,-j-m;\,-2\,j,\,-j-j_2-m_1;\,1)\;/;\;\mathcal{P}hysical\mathcal{Q}(\{j_1,\,m_1\},\,\{j_2,\,m_2\},\,\{j,\,m\})$$

07.38.26.0009.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle = \delta_{m,m_1+m_2} (-1)^{j_1-m_1} \frac{(j_1+j_2-m)!\,(j_2+j-m_1)!}{\sqrt{(j_1+j_2-j)!}\,\sqrt{(j_1-j_2+j)!}\,\sqrt{(-j_1+j_2+j)!}\,\sqrt{(j_1+j_2+j+1)!}} \frac{\sqrt{(j_1+m_1)!}\,\sqrt{(j+m)!}\,\sqrt{2\,j+1}}{\sqrt{(j_1-m_1)!}\,\sqrt{(j_2+m_2)!}\,\sqrt{(j_2-m_2)!}\,\sqrt{(j-m)!}} \,{}_3F_2(-j-j_1-j_2-1,\,m_1-j_1,\,m-j;\,m-j_1-j_2,\,-j-j_2+m_1;\,1)\;/;\;\mathcal{P}hysical\mathcal{Q}(\{j_1,\,m_1\},\,\{j_2,\,m_2\},\,\{j,\,m\})$$

Through Meijer *G*

Classical cases

07.38.26.0003.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle = \delta_{m,m_1+m_2} \left(\sqrt{2\,j+1}\,\sqrt{(j-m)!}\,\sqrt{(j+m)!}\,\sqrt{(j+j_1-j_2)!}\,\sqrt{(j-j_1+j_2)!}\,\sqrt{(j_1+m_1)!}\,\sqrt{(j_2-m_2)!}\right) \Big/ \left(\sqrt{(-j+j_1+j_2)!}\,\sqrt{(j+j_1+j_2+1)!}\,\sqrt{(j_1-m_1)!}\,\sqrt{(j_2+m_2)!}\,(-j_1+m_1-1)!\,(-j_2-m_2-1)!\,(j-j_1-j_2-1)!\right) G_{3,3}^{1,3}\!\left(-1 \,\middle|\, \begin{matrix} -j+j_1+j_2+1,\, j_1-m_1+1,\, j_2+m_2+1 \\ 0,\, -j+j_2-m_1,\, -j+j_1+m_2 \end{matrix}\right) /;\, \mathcal{P}hysical\mathcal{Q}(\{j_1,\,m_1\},\,\{j_2,\,m_2\},\,\{j,\,m\})$$

Through other functions

Involving some hypergeometric-type functions

07.38.26.0004.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle = \delta_{m,m_1+m_2} \frac{(-1)^{4\,j_2}\,\sqrt{2\,j+1}\,\sqrt{\Gamma(-2\,j_1+2\,j_2+1)}\,\sqrt{\Gamma(-j_1+4\,j_2+m_1+2)}\,\sqrt{\Gamma(3\,j_2-m_2+2)}}{\sqrt{\Gamma(4\,j_2+2)}\,\sqrt{\Gamma(-j_1+2\,j_2+m_1+1)}\,\sqrt{\Gamma(-2\,j_1+3\,j_2-m_2+1)}} \begin{Bmatrix} 2\,j_2-j_1 & j_2 & j \\ -\frac{1}{2}(j+m+1) & \frac{1}{2}(-j+m-1) & \frac{1}{2}(-j_1+3\,j_2+m_1-m_2) \end{Bmatrix} /;\, j+j_1+j_2 == -1$$

Representations through equivalent functions

With related functions

07.38.27.0001.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m\rangle = (-1)^{m+j_1-j_2}\,\sqrt{2\,j+1} \begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & -m \end{pmatrix}$$

Theorems

Wigner-Eckhart theorem

The matrix element of an irreducible tensor operator in an angular momentum basis is the product of an angular momentum dependent factor containing the Clebsch-Gordan coefficients and a term which is rotationally invariant and independent of all projection quantum numbers m_1, m_2, m .

History

A. Clebsch (1872); P. Gordan (1875); H. Weyl (1928); E.P. Wigner (1928, 1931).

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