DedekindEta

Notations

Traditional name

Dedekind eta modular function

Traditional notation

$\eta(z)$

Mathematica StandardForm notation

`DedekindEta[z]`

Primary definition

09.49.02.0001.01

$$\eta(z) = e^{\frac{\pi i z}{12}} \prod_{k=1}^{\infty} \left(1 - e^{2 \pi i k z}\right) /; \operatorname{Im}(z) > 0$$

Specific values

Values at fixed points

09.49.03.0003.01

$$\eta(i) = \frac{1}{2 \pi^{3/4}} \Gamma\left(\frac{1}{4}\right)$$

09.49.03.0001.01

$$\eta(i) = \sqrt[3]{\frac{1}{2} \vartheta_1'(0, e^{-\pi})}$$

Values at infinities

09.49.03.0002.01

$$\eta(i \infty) = 0$$

General characteristics

Domain and analyticity

$\eta(z)$ is an analytical function of z which is defined over the upper half of the complex z-plane.

09.49.04.0001.01

$z \longrightarrow \eta(z) :: \mathbb{C} \longrightarrow \mathbb{C}$

Symmetries and periodicities

Symmetry

No symmetry

Periodicity

09.49.04.0002.01

$\eta(z) = \eta(24\,k + z) \,/;\, k \in \mathbb{Z}$

Poles and essential singularities

On the boundary of analyticity (the real axis) the function $\eta(z)$ has a dense set of poles.

09.49.04.0003.01

$\mathcal{Sing}_z(\eta(z)) = \{\} \,/;\, \operatorname{Im}(z) > 0$

Branch points

The function $\eta(z)$ does not have branch points.

09.49.04.0004.01

$\mathcal{BP}_z(\eta(z)) = \{\}$

Branch cuts

The function $\lambda(z)$ does not have branch cuts.

09.49.04.0005.01

$\mathcal{BC}_z(\eta(z)) = \{\}$

Natural boundary of analyticity

The real axis $\operatorname{Im}(z) = 0$ is the natural boundary of the region of analyticity.

09.49.04.0006.01

$\mathcal{AB}_z(\eta(z)) = \{(-\infty,\ \infty)\}$

Series representations

Exponential Fourier series

09.49.06.0001.01

$$\eta(z) = e^{\frac{\pi i z}{12}} \sum_{k=-\infty}^{\infty} (-1)^k\, e^{\pi i k (3k-1) z}$$

09.49.06.0002.01

$$\eta(z) = \sum_{k=0}^{\infty} \left(\frac{3}{2\,k+1}\right) \exp\left(\frac{\pi i z}{12}\,(2\,k+1)^2\right)$$

q-series

Expansions at $z == 0$

09.49.06.0006.01

$$\eta(z) \propto \frac{1}{\sqrt{-i z}} e^{-\frac{i\pi}{12 z}} \left(1 - e^{-\frac{2 i \pi}{z}} - e^{-\frac{4 i \pi}{z}} + e^{-\frac{10 i \pi}{z}} + e^{-\frac{14 i \pi}{z}} - e^{-\frac{24 i \pi}{z}} - e^{-\frac{30 i \pi}{z}} + e^{-\frac{44 i \pi}{z}} + e^{-\frac{52 i \pi}{z}} - e^{-\frac{70 i \pi}{z}} + \ldots\right) /; (z \to 0) \land \operatorname{Re}(z) > 0$$

09.49.06.0007.01

$$\eta(z) = \frac{q}{\sqrt{-i z}} \sqrt[3]{\sum_{k=0}^{\infty} (-1)^k (2k+1) q^{12 k (k+1)}} /; q == e^{-\frac{\pi i}{12 z}} \bigwedge (z \to 0) \bigwedge \operatorname{Re}(z) > 0$$

09.49.06.0008.01

$$\eta(z) \propto \frac{1}{\sqrt{-i z}} e^{-\frac{i\pi}{12 z}} \left(1 + O\left(e^{-\frac{2 i \pi}{z}}\right)\right) /; (z \to 0) \land \operatorname{Re}(z) > 0$$

Expansions at $z == i \infty$

09.49.06.0009.01

$$\eta(z) \propto \frac{1}{\sqrt{-i z}} e^{-\frac{i\pi}{12 z}} \left(1 - e^{-\frac{2 i \pi}{z}} - e^{-\frac{4 i \pi}{z}} + e^{-\frac{10 i \pi}{z}} + e^{-\frac{14 i \pi}{z}} - e^{-\frac{24 i \pi}{z}} - e^{-\frac{30 i \pi}{z}} + e^{-\frac{44 i \pi}{z}} + e^{-\frac{52 i \pi}{z}} - e^{-\frac{70 i \pi}{z}} + \ldots\right) /;$$

$$(|z| \to \infty) \bigwedge \arg(z) = \frac{\pi}{2}$$

09.49.06.0010.01

$$\eta(z) = \frac{e^{-\frac{\pi i}{12 z}}}{\sqrt{-i z}} \sqrt[3]{\sum_{k=0}^{\infty} (-1)^k e^{-\frac{k(k+1)\pi i}{z}} (2k+1)} /; (|z| \to \infty) \bigwedge \arg(z) = \frac{\pi}{2}$$

09.49.06.0011.01

$$\eta(z) \propto \frac{1}{\sqrt{-i z}} e^{-\frac{i\pi}{12 z}} \left(1 + O\left(e^{-\frac{2 i \pi}{z}}\right)\right) /; (|z| \to \infty) \bigwedge \arg(z) = \frac{\pi}{2}$$

Other series representations

09.49.06.0012.01

$$\eta(z) = \frac{e^{-\frac{\pi i}{12 z}}}{\sqrt{-i z}} \sqrt[3]{\sum_{k=0}^{\infty} (-1)^k e^{-\frac{k(k+1)\pi i}{z}} (2k+1)} /; |z| < 2\pi \land \operatorname{Re}(z) > 0$$

09.49.06.0013.01

$$\eta(z)^3 = \frac{\pi^{3/2}}{\left(-\log\left(e^{\pi i z}\right)\right)^{3/2}} e^{\frac{\pi^2}{4 \log\left(e^{\pi i z}\right)}} \sum_{k=0}^{\infty} (-1)^k e^{\frac{k(k+1)\pi^2}{\log\left(e^{\pi i z}\right)}} (2k+1)$$

09.49.06.0003.01

$$\eta(z) == \sum_{k=1}^{\infty} \left(\delta_{1, k \bmod 12} - \delta_{5, k \bmod 12} - \delta_{7, k \bmod 12} + \delta_{11, k \bmod 12}\right) \exp\left(\frac{\pi i z}{12} k^2\right)$$

09.49.06.0004.01

$$\eta(z) == \frac{1}{\sum_{k=0}^{\infty} p(k) \exp\left(2 \pi i \left(k - \frac{1}{24}\right) z\right)}$$

09.49.06.0005.01

$$\eta(z) = \exp\left(\frac{\pi i z}{12} - \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{e^{2\pi i z n k}}{k}\right)$$

Integral representations

On the real axis

Of the direct function

09.49.07.0001.01

$$\eta(z) = \frac{1}{2\pi^{3/4}}\Gamma\left(\frac{1}{4}\right)e^{\frac{i}{\pi}\int_i^z \zeta(1; g_2(1,t), g_3(1,t))\,dt}$$

Product representations

09.49.08.0001.01

$$\eta(z) = e^{\frac{\pi i z}{12}}\prod_{k=1}^{\infty}\left(1 - e^{2\pi i k z}\right) /; \operatorname{Im}(z) > 0$$

Differential equations

Ordinary nonlinear differential equations

09.49.13.0001.01

$$w(z)^2\left(33\,w''(z)^2 + w(z)\,w^{(4)}(z)\right) - 18\,w'(z)^4 + 12\,w(z)\,w''(z)\,w'(z)^2 - 28\,w(z)^2\,w^{(3)}(z)\,w'(z) = 0 /; w(z) = \eta(z)$$

09.49.13.0002.01

$$36\,w'(z)^2 - 24\,w''(z)\,w(z) + w'''[z] = 0 /; w(z) = \frac{\eta'(z)}{\eta(z)}$$

Transformations

Transformations and argument simplifications

Argument involving basic arithmetic operations

09.49.16.0001.01

$$\eta(z+1) = e^{\frac{\pi i}{12}}\,\eta(z)$$

09.49.16.0002.01

$$\eta(z-1) = e^{-\frac{\pi i}{12}}\,\eta(z)$$

09.49.16.0003.01

$$\eta\left(-\frac{1}{z}\right) = \sqrt{z}\;e^{-\frac{\pi i}{4}}\,\eta(z)$$

Identities

Functional identities

09.49.17.0001.01

$$\eta(z) == e^{-\frac{\pi i}{12}} \eta(z+1)$$

09.49.17.0002.01

$$\eta(z) == e^{\frac{\pi i}{12}} \eta(z-1)$$

09.49.17.0003.01

$$\eta(z) == \frac{1}{\sqrt{z}} e^{\frac{\pi i}{4}} \eta\left(-\frac{1}{z}\right)$$

09.49.17.0004.01

$$\eta\left(z+\frac{1}{2}\right) == \frac{e^{\frac{\pi i}{24}} \eta(2 z)^3}{\eta(z) \eta(4 z)}$$

09.49.17.0005.01

$$\eta\left(z+\frac{1}{3}\right)^3 == e^{\frac{\pi i}{12}} \eta(z)^3 + 3\sqrt{3}\, e^{-\frac{1}{12}(\pi i)} \eta(9 z)^3$$

09.49.17.0006.01

$$\eta(z)^{48} == \frac{1}{256}\left(2 \eta\left(\frac{z+1}{2}\right)^{24} \eta\left(\frac{z}{2}\right)^{24} + \eta\left(\frac{z+1}{2}\right)^{16}\left(e^{\frac{2\pi i}{3}} \eta\left(\frac{z+1}{2}\right)^{16} + e^{-\frac{2\pi i}{3}} \eta\left(\frac{z}{2}\right)^{16}\right) \eta\left(\frac{z}{2}\right)^{16}\right)$$

09.49.17.0007.01

$$\frac{\eta(6 z)^7}{\eta(3 z)} == \frac{\eta(4 z)^3 \eta(18 z)^9}{\eta(9 z)^3 \eta(36 z)^3} - \frac{\eta(36 z)^3 \eta(2 z)^9}{\eta(z)^3 \eta(4 z)^3}$$

09.49.17.0008.01

$$\eta(z)^2 \eta(4 z) \eta(6 z)^9 - 2 \eta(z) \eta(2 z) \eta(3 z)^3 \eta(4 z)^3 \eta(12 z)^2 \eta(6 z)^2 + \eta(2 z)^3 \eta(3 z)^6 \eta(12 z)^3 == 0$$

09.49.17.0009.01

$$-\eta(2 z)^{12} \eta(3 z)^{12} + 9 \eta(z)^4 \eta(2 z)^4 \eta(6 z)^8 \eta(3 z)^8 + \eta(z)^8 \eta(2 z)^8 \eta(6 z)^4 \eta(3 z)^4 - \eta(z)^{12} \eta(6 z)^{12} == 0$$

09.49.17.0010.01

$$\eta\left(\frac{b+a z}{d+c z}\right) == \sqrt{i (d+c z)}\, \eta(z) \exp\left(\pi i \left(\frac{a+d}{12 c} + s(d, -c)\right)\right) \theta(-c) + \sqrt{-i (d+c z)}\, \eta(z) \exp\left(\pi i \left(\frac{a+d}{12 c} + s(-d, c)\right)\right) \theta(c) /;$$

$$a \in \mathbb{Z} \bigwedge b \in \mathbb{Z} \bigwedge c \in \mathbb{Z} \bigwedge d \in \mathbb{Z} \bigwedge a d - b c == 1 \bigwedge s(h, l) == \sum_{k=1}^{l-1} \frac{1}{l} k \left(\frac{h k}{l} - \left\lfloor \frac{k h}{l} \right\rfloor - \frac{1}{2}\right)$$

Differentiation

Low-order differentiation

09.49.20.0005.01

$$\eta'(i) = -\frac{i \sqrt[4]{\pi}}{\sqrt{2}\, \Gamma\left(-\frac{1}{4}\right)}$$

09.49.20.0006.01

$$\frac{\partial \eta(z)}{\partial z} = \frac{i}{\pi}\, \eta(z)\, \zeta(1;\, g_2(1,\, z),\, g_3(1,\, z))$$

09.49.20.0001.02

$$\frac{\partial \eta(z)}{\partial z} = -\frac{4096\, i\, K(\lambda(z))^{13}}{847\,288\,609\,443\, \pi^{13}\, \eta(z)^{23}\, (J(z)-1)^4\, J(z)^5\, (\lambda(z)-1)^{14}\, \lambda(z)^{14}} \Big(K(\lambda(z)) \left(\lambda(z)^2-\lambda(z)+1\right)^{11} \left(2\,\lambda(z)^3-3\,\lambda(z)^2-3\,\lambda(z)+2\right)^5 \big(81\,(J(z)-1)\,J(z)\,(\lambda(z)-1)^2\,\lambda(z)^2 \left(4\,\lambda(z)^6-11\,\lambda(z)^5+6\,\lambda(z)^4+4\,\lambda(z)^3+13\,\lambda(z)^2-18\,\lambda(z)+6\right) - 2\,(7\,J(z)-4)\,(1-2\,\lambda(z))^2\,(\lambda(z)-2)^2\,(\lambda(z)+1)^2\,((\lambda(z)-1)\,\lambda(z)+1)^3\big) - 81\,E(\lambda(z))\,(J(z)-1)\,J(z)\,(\lambda(z)-2)^6\,(\lambda(z)-1)^2\,\lambda(z)^2\,(\lambda(z)+1)^6\,(2\,\lambda(z)-1)^6 \left(\lambda(z)^2-\lambda(z)+1\right)^{12}\Big)$$

09.49.20.0002.01

$$\frac{\partial \eta(z)}{\partial z} == \frac{\pi\, i}{12}\, \eta(z) + e^{\frac{\pi\, i\, z}{12}}\, i\, \pi \sum_{k=-\infty}^{\infty} (-1)^k\, k\, (3\,k-1)\, e^{\pi\, i\, k\, (3\,k-1)\, z}$$

09.49.20.0007.01

$$\frac{\partial^2 \eta(z)}{\partial z^2} = -\frac{i}{6\,\pi^2\,\sqrt{-z^2}}\, \eta(z) \left(z\, g_2(1,\, z) - 6\left(\left(2\,z + i\,\sqrt{-z^2}\right) \zeta(1;\, g_2(1,\, z),\, g_3(1,\, z))^2 + z\, \wp'(1;\, g_2(1,\, z),\, g_3(1,\, z))\right)\right)$$

09.49.20.0003.01

$$\frac{\partial^2 \eta(z)}{\partial z^2} == -\frac{\pi^2}{144}\, \eta(z) - \frac{\pi^2}{6} \sum_{k=-\infty}^{\infty} (-1)^k\, k\, (3\,k-1) \left(18\,k^2-6\,k+1\right) e^{\frac{\pi\, i}{12}\, i\, (6\,k-1)^2\, z}$$

09.49.20.0008.01

$$\frac{\partial^2 \eta(z)}{\partial z^2} = \frac{8192\, K(\lambda(z))^{14} \left(\lambda(z)^2-\lambda(z)+1\right)^{10} (\lambda(z)-2)^4}{717\,897\,987\,691\,852\,588\,770\,249\, \pi^{26}\, \eta(z)^{47}\, (J(z)-1)^8\, J(z)^{10}\, (\lambda(z)-1)^{28}\, \lambda(z)^{28}}$$
$$\Big(-892\,194\,905\,743\,479\, \pi^{12}\, E(\lambda(z))^2\, (J(z)-1)^5\, J(z)^6\, (\lambda(z)-2)^2\, (\lambda(z)-1)^{16}\, \lambda(z)^{16}\, (\lambda(z)+1)^6\, (2\,\lambda(z)-1)^6 \left(\lambda(z)^2-\lambda(z)+1\right)^2 \eta(z)^{24} - 22\,029\,503\,845\,518\, \pi^{12}\, E(\lambda(z))\, K(\lambda(z))\, (J(z)-1)^4\, J(z)^5\, (\lambda(z)-2)\, (\lambda(z)-1)^{14}\, \lambda(z)^{14}\, (\lambda(z)+1)^5\, (2\,\lambda(z)-1)^5 \left(\lambda(z)^2-\lambda(z)+1\right) \big(-8\,(\lambda(z)-2)^2\,(\lambda(z)+1)^2\,(1-2\,\lambda(z))^2\,((\lambda(z)-1)\,\lambda(z)+1)^3 - 81\,J(z)^2\,(\lambda(z)-1)^2\,\lambda(z)^2\,(\lambda(z)\,(\lambda(z)\,(\lambda(z)\,((\lambda(z)-2)\,\lambda(z)\,(4\,\lambda(z)-3)+4)+13)-18)+6) + J(z)\,((\lambda(z)-1)\,\lambda(z)\,((\lambda(z)-1)\,\lambda(z)\,(\lambda(z)\,(\lambda(z)\,(\lambda(z)\,(\lambda(z)\,(\lambda(z)\,(2\,\lambda(z)\,(28\,(\lambda(z)-4)\,\lambda(z)+309)-989)+122)+954)+745)-1444)+948)+336)+56)\big)\, \eta(z)^{24} - 31\,381\,059\,609\, \pi^{12}\, K(\lambda(z))^2\, (J(z)-1)^3\, J(z)^4\, (\lambda(z)-1)^{12}\, \lambda(z)^{12}\, (\lambda(z)+1)^4\, (2\,\lambda(z)-1)^4 \big(320\,(\lambda(z)-2)^4\,(\lambda(z)+1)^4\,(1-2\,\lambda(z))^4\,((\lambda(z)-1)\,\lambda(z)+1)^6 - 16\,J(z)\,(1-2\,\lambda(z))^2\,(\lambda(z)-2)^2\,(\lambda(z)+1)^2\,((\lambda(z)-1)\,\lambda(z)\,((\lambda(z)-1)\,\lambda(z)\,(\lambda(z)\,(\lambda(z)\,(\lambda(z)\,(\lambda(z)\,(\lambda(z)\,(4\,\lambda(z)\,(64\,(\lambda(z)-4)\,\lambda(z)+687)-4309)+442)+4284)+3155)-6254)+4218)+1536)+256)\,((\lambda(z)-1)\,\lambda(z)+1)^3 + 2187\,J(z)^4\,(\lambda(z)-1)^4\,\lambda(z)^4\,(\lambda(z)\,(\lambda(z)\,(\lambda(z)\,(\lambda(z)\,(\lambda(z)\,(\lambda(z)\,(\lambda(z)\,(\lambda(z)\,(\lambda(z)\,(8\,\lambda(z)\,(\lambda(z)\,(24\,\lambda(z)-131)+248)-1115)+187)-3064)+5062)-1270)+772)-5275)+6027)-2712)+452) - 54\,J(z)^3\,(\lambda(z)-1)^2\,\lambda(z)^2\,(\lambda(z)\,(\lambda(z)\,(\lambda(z)\,(\lambda(z)\,(\lambda(z)\,(\lambda(z)\,(\lambda(z)\,(\lambda(z)\,(\lambda(z)\,(\lambda(z)\,(\lambda(z)\,(\lambda(z)\,(\lambda(z)\,(2\,\lambda(z)\,(\lambda(z)\,(4\,\lambda(z)\,(91\,\lambda(z)\,(4\,\lambda(z)-35)+13\,683)-150\,361)+268\,475)-548\,481)+256\,199)-288\,895)+959\,211)-1\,126\,144)+313\,748)-87\,915)+1\,046\,911)-1\,701\,035)+1\,308\,669)-604\,480)+186\,944)-39\,312)+4368) +$$

$$J(z)^2\,((\lambda(z)-1)\,\lambda(z)\,((\lambda(z)-1)\,\lambda(z)\,(\lambda(z)\,(\lambda(z)\,(\lambda(z)\,(\lambda(z)\,(\lambda(z)\,(\lambda(z)\,(\lambda(z)\,(\lambda(z)\,(\lambda(z)\,(\lambda(z)\,(\lambda(z)\,(\lambda(z)\,(\lambda(z)\,(4\,\lambda(z)\,(\lambda(z)\,(4\,\lambda(z)\,(\lambda(z)\,(4\,\lambda(z)\,(224\,(\lambda(z)-10)\,\lambda(z)+13605)-230559)+655185)-4868517)+5490353)-9582965)-7161009)-2091543)+37198791)-41766450)-3485674)+31562929)-700317)-42139839)+51756825)-34928928)+15939572)-4748000)+1123296)+172032)+14336)\big)$$

$$\eta(z)^{24}+309049344\,E(\lambda(z))^2\,K(\lambda(z))^{12}\,(J(z)-1)^2\,J(z)^2\,(\lambda(z)-2)^8\,(\lambda(z)-1)^4\,\lambda(z)^4\,(\lambda(z)+1)^{12}\,(2\,\lambda(z)-1)^{12}\,\left(\lambda(z)^2-\lambda(z)+1\right)^{14}+$$

$$47104\,K(\lambda(z))^{14}\,(\lambda(z)-2)^6\,(\lambda(z)+1)^{10}\,(2\,\lambda(z)-1)^{10}\,\left(\lambda(z)^2-\lambda(z)+1\right)^{12}\,\big(8\,(\lambda(z)-2)^2\,(\lambda(z)+1)^2\,(1-2\,\lambda(z))^2\,((\lambda(z)-1)\,\lambda(z)+1)^3+81\,J(z)^2\,(\lambda(z)-1)^2\,\lambda(z)^2\,(\lambda(z)\,(\lambda(z)\,(\lambda(z)\,((\lambda(z)-2)\,\lambda(z)\,(4\,\lambda(z)-3)+4)+13)-18)+6)+J(z)\,(-(\lambda(z)-1)\,\lambda(z)\,((\lambda(z)-1)\,\lambda(z)\,(\lambda(z)\,(\lambda(z)\,(\lambda(z)\,(\lambda(z)\,(\lambda(z)\,(2\,\lambda(z)\,(28\,(\lambda(z)-4)\,\lambda(z)+309)-989)+122)+954)+745)-1444)+948)+336)-56)\big)^2+$$

$$7630848\,E(\lambda(z))\,K(\lambda(z))^{13}\,(J(z)-1)\,J(z)\,(\lambda(z)-2)^7\,(\lambda(z)-1)^2\,\lambda(z)^2\,(\lambda(z)+1)^{11}\,(2\,\lambda(z)-1)^{11}\,\left(\lambda(z)^2-\lambda(z)+1\right)^{13}\,\big(-8\,(\lambda(z)-2)^2\,(\lambda(z)+1)^2\,(1-2\,\lambda(z))^2\,((\lambda(z)-1)\,\lambda(z)+1)^3-81\,J(z)^2\,(\lambda(z)-1)^2\,\lambda(z)^2\,(\lambda(z)\,(\lambda(z)\,(\lambda(z)\,((\lambda(z)-2)\,\lambda(z)\,(4\,\lambda(z)-3)+4)+13)-18)+6)+J(z)\,((\lambda(z)-1)\,\lambda(z)\,((\lambda(z)-1)\,\lambda(z)\,(\lambda(z)\,(\lambda(z)\,(\lambda(z)\,(\lambda(z)\,(\lambda(z)\,(2\,\lambda(z)\,(28\,(\lambda(z)-4)\,\lambda(z)+309)-989)+122)+954)+745)-1444)+948)+336)+56)\big)\big)$$

Fractional integro-differentiation

09.49.20.0004.01

$$\frac{\partial^{\alpha}\eta(z)}{\partial z^{\alpha}}=\left(\frac{\pi}{12}\right)^{\alpha} z^{-\alpha}\sum_{k=-\infty}^{\infty}(-1)^k \exp\!\left(\frac{\pi\,i\,z}{12}\,(1-6\,k)^2\right)\left(i\,(1-6\,k)^2\,z\right)^{\alpha} Q\!\left(-\alpha,\,0,\,\frac{\pi\,i\,z}{12}\,(1-6\,k)^2\right)$$

Integration

Indefinite integration

Involving only one direct function

09.49.21.0001.01

$$\int \eta(z)\,dz=-\frac{12\,i}{\pi}\sum_{k=-\infty}^{\infty}\frac{(-1)^k}{(6\,k-1)^2}\left(\exp\!\left(\frac{\pi\,i\,z}{12}\,(6\,k-1)^2\right)-1\right)$$

Operations

Limit operation

09.49.25.0001.01

$$\lim_{\epsilon \to +0} \eta(i\,\epsilon) = 0$$

Representations through equivalent functions

With related functions

Involving Weierstrass functions

09.49.27.0001.01

$$\eta(z)^{24} = \frac{g_2^3 - 27\,g_3^2}{\pi^{12}} \;/;\; \{g_2, g_3\} = \{g_2(1, z), g_3(1, z)\} \bigwedge \operatorname{Im}(z) > 0$$

09.49.27.0002.01

$$\eta(z)^{24} = \frac{g_2^3 - 27\,g_3^2}{(2\,\pi)^{12}} \;/;\; \{g_2, g_3\} = \left\{g_2\left(\frac{1}{2}, \frac{z}{2}\right), g_3\left(\frac{1}{2}, \frac{z}{2}\right)\right\} \bigwedge \operatorname{Im}(z) > 0$$

Involving theta functions

09.49.27.0007.01

$$\eta(z) = \frac{1}{\sqrt{3}}\,\vartheta_2\left(\frac{\pi}{6}, e^{\frac{\pi i z}{3}}\right) /;\; \operatorname{Im}(z) > 0 \land |\operatorname{Re}(z)| < 3$$

09.49.27.0003.01

$$\eta(z) = e^{\frac{\pi i z}{12}}\,\vartheta_3\left(\frac{\pi\,(z+1)}{2}, e^{3\pi i z}\right) /;\; \operatorname{Im}(z) > 0$$

09.49.27.0004.01

$$\eta(z)^3 = \frac{1}{2}\,\vartheta_1'\left(0, e^{i\pi z}\right) /;\; \operatorname{Im}(z) > 0 \land |\operatorname{Re}(z)| \le 1$$

09.49.27.0005.01

$$\eta(z)^3 = \frac{1}{2}\,\vartheta_2\left(0, e^{\pi i z}\right)\vartheta_3\left(0, e^{\pi i z}\right)\vartheta_4\left(0, e^{\pi i z}\right) /;\; \operatorname{Im}(z) > 0 \land |\operatorname{Re}(z)| \le 1$$

Involving other related functions

09.49.27.0006.01

$$\eta(z)^{24} = \frac{1}{\left(48\,\pi^2\right)^3 J(z)^4\,(1 - J(z))^3}\left(\frac{\partial J(z)}{\partial z}\right)^6 /;\; \operatorname{Im}(z) > 0$$

Theorems

Modular form of order 12

The 24th power of the Dedekind Eta function $\eta(z)$ is a modular form of order 12. (This means that under all argument substitutions of the form $z \to z' = (a\,z + b) / (c\,z + d)$ where a, b, c, and d are integers with $a == 1 \bmod 12$, $b == 0 \bmod 12$, $c == 0 \bmod 12$, $d == 1 \bmod 12$, and $a\,d - b\,c == 1$ the identity $\eta(z')^{24} = (c\,z + d)^{12}\,\eta(z)^{24}$ holds.)

Kronecker's first limit formula

$$\sum_{\substack{k,l=0 \\ \{k,l\}\neq\{0,0\}}}^{\infty} \frac{y^s}{|\,k\,(x + i\,y) + l\,|^{2\,s}} = \frac{\pi}{s-1} + 2\,\pi\left(\gamma - \log(2) - \log\left(\sqrt{y}\,\left|\eta\,(x + i\,y)\right|\right)^2\right) + O(s-1) \;/;\; \mathrm{Re}(s) > 1 \bigwedge x,\, y \in \mathbb{R}$$

The number of representations of *n* as a sum of triangular numbers

The number ν_n of representations of n as a sum of triangular numbers is $\nu_n == \left([q^n]\ \left(q^{-1/8}\,\frac{\eta(-i\log(q)/\pi)^2}{\eta(-i\log(q)/(2\,\pi))}\right)\right)$.

One product-series representation

$$\eta\,(\tau)^{n^2+2} == (-1)^{\binom{n+1}{2}} \left(\prod_{k=1}^{n-1} \frac{1}{k!}\right) \sum_{\substack{k_1,k_2,\ldots,k_n=-\infty \\ k_i \bmod n==l}}^{\infty} \left(\left(\frac{\sum_{i=1}^{n} k_i}{n} - \frac{n+1}{2}\right)(-1)^{\sum_{i=1}^{n} k_i} \prod_{1\leq r<s\leq n} (k_s - k_r)\, q^{1/(2\,n)\sum_{i=1}^{n}\left(\nu(i) - \frac{n}{2}\right)^2}\right) /;\; q = e^{i\,2\,\pi\,\tau}$$

One series representation

$$\eta^{24}(z) == \sum_{k=1}^{\infty} \tau(k)\, e^{k\,\pi\,i\,z/12} \;/;$$

$$\tau(n) == \frac{1}{288} \sum_{\substack{k_1,k_2,k_3,k_4,k_5=-\infty\ \{k_1,k_2,k_3,k_4,k_5\} \bmod 5==\{1,2,3,4,5\} \\ k_1+k_2+k_3+k_4+k_5==1 \\ k_1^2+k_2^2+k_3^2+k_4^2+k_5^2==10\,n}}^{\infty} (k_1-k_2)(k_1-k_3)(k_1-k_4)(k_1-k_5)(k_2-k_3)(k_2-k_4)(k_2-k_5)(k_3-k_4)(k_3-k_5)(k_4-k_5)$$

History

– R. Dedekind (1877) introduced the name "elliptic modular function"

Copyright