DiracDelta

Notations

Traditional name

Dirac delta function

Traditional notation

$\delta(x)$

Mathematica StandardForm notation

`DiracDelta[x]`

Primary definition

14.03.02.0001.01

$$\delta(x) = \frac{1}{\pi} \lim_{\varepsilon \to 0} \frac{\varepsilon}{x^2 + \varepsilon^2} \,/;\, x \in \mathbb{R}$$

Specific values

Specialized values

14.03.03.0001.01

$$\delta(x) = 0 \,/;\, x \neq 0$$

Values at fixed points

14.03.03.0002.01

$$\delta(0) = \infty$$

General characteristics

Domain and analyticity

$\delta(x)$ is a non-analytical function; it is generalized function defined for $x \in \mathbb{R}$.

14.03.04.0001.01

$$x \longrightarrow \delta(x) :: \mathbb{R} \longrightarrow \{0, \infty\}$$

Symmetries and periodicities

Parity

$\delta(x)$ is an even generalized function.

14.03.04.0002.01

$$\delta(-x) == \delta(x)$$

Periodicity

No periodicity

Series representations

Exponential Fourier series

14.03.06.0001.01

$$\delta(x) == \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} e^{i k x} \;/;\; -\pi < x < \pi$$

14.03.06.0002.01

$$\delta(x) == \frac{1}{2\pi} + \frac{1}{\pi} \sum_{k=1}^{\infty} \cos(k x) \;/;\; -\pi < x < \pi$$

14.03.06.0003.01

$$\delta(x-y) = \frac{1}{L} \sum_{k=1}^{\infty} \sin\left(\frac{k \pi x}{L}\right) \sin\left(\frac{k \pi y}{L}\right) /;\; L \in \mathbb{R} \land L > 0$$

Integral representations

On the real axis

Of the direct function

14.03.07.0001.01

$$\delta(x) == \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i t x} \, dt$$

14.03.07.0002.01

$$\delta(x) == \frac{1}{\pi} \int_{-\infty}^{\infty} e^{i t x} \theta(t) \, dt - \frac{i}{\pi x}$$

14.03.07.0003.01

$$\delta(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \cos(x t) \, dt$$

14.03.07.0004.01

$$\delta(x) = \frac{2}{\pi} \int_{0}^{\infty} \cos(x t) \, dt$$

14.03.07.0005.01

$$\frac{\partial \delta(x)}{\partial x} = -\frac{1}{2 \pi i} \int_{-\infty}^{\infty} t \, e^{i x t} \, dt$$

14.03.07.0006.01

$$\frac{\partial \delta(x)}{\partial x} = \frac{1}{2\pi i}\int_{-\infty}^{\infty} t\, e^{-i x t}\, dt$$

Limit representations

14.03.09.0001.01

$$\delta(x) = \lim_{\varepsilon \to +0} \frac{1}{2}\theta(x+1)\,\theta(1-x)\,\varepsilon\,|x|^{\varepsilon-1}$$

14.03.09.0014.01

$$\delta(x) = \lim_{\varepsilon \to \infty} \frac{(2l+1)!!}{2^{l+1}\, l!}\left(1-z^2\right)^l$$

14.03.09.0002.01

$$\delta(x) = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} e^{-\frac{|x|}{\varepsilon}}$$

14.03.09.0003.01

$$\delta(x) = \lim_{\varepsilon \to +0} \frac{1}{2\sqrt{\pi\varepsilon}} e^{-\frac{x^2}{4\varepsilon}}$$

14.03.09.0013.01

$$\delta(x) = \lim_{\varepsilon \to +0} \frac{1}{\sqrt{2\pi i\varepsilon}} e^{\frac{i x^2}{2\varepsilon}}$$

14.03.09.0004.01

$$\delta(x) = \lim_{\varepsilon \to 0} \frac{1}{\pi\,\varepsilon} \sin\left(\frac{x}{\varepsilon}\right)$$

14.03.09.0005.01

$$\delta(x) = \lim_{n \to \infty} \frac{1}{2\pi\sin\left(\frac{x}{2}\right)} \sin\left(x\left(n+\frac{1}{2}\right)\right)$$

14.03.09.0006.01

$$\delta(x) = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon\cosh^2\left(\frac{x}{\varepsilon}\right)}$$

14.03.09.0007.01

$$\delta(x) = \lim_{a \to \infty} \log(|\coth(a x)|)$$

14.03.09.0008.01

$$\delta(x) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \operatorname{Ai}\left(\frac{x}{\varepsilon}\right)$$

14.03.09.0009.01

$$\delta(x) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} J_{\frac{1}{\varepsilon}}\left(\frac{x+1}{\varepsilon}\right)$$

14.03.09.0010.01

$$\delta(x) = \lim_{\varepsilon \to 0} \left|\frac{1}{\varepsilon} e^{-\frac{x^2}{\varepsilon}} L_n\left(\frac{2x}{\varepsilon}\right)\right| /;\ n \in \mathbb{N}$$

14.03.09.0011.01

$$\delta'(x) = \lim_{a\to\infty} \frac{a}{\sinh(a\,x)}$$

14.03.09.0012.01

$$\delta^{(n)}(x) = \lim_{\varepsilon\to+0} \frac{1}{\sqrt{\pi\,\varepsilon}}\, e^{-\frac{x^2}{\varepsilon}} \left(-\frac{1}{\sqrt{\varepsilon}}\right)^n H_n\!\left(\frac{x}{\sqrt{\varepsilon}}\right) /;\ n \in \mathbb{N}^+$$

Transformations

Transformations and argument simplifications

14.03.16.0001.01

$$\delta(a\,x) = \frac{\delta(x)}{|a|} /;\ a \in \mathbb{R}$$

Identities

Functional identities

14.03.17.0001.01

$$\delta(f(x)) = \sum_{j=1}^{n} \frac{\delta(x - x_j)}{\left|f'(x_j)\right|} /;\ f(x_j) = 0 \land f'(x_j) \neq 0$$

14.03.17.0002.01

$$\delta(x_1 - x_0)\,\delta(x_2 - x_0) = \delta(x_1 - x_0)\,\delta(x_2 - x_1)$$

Differentiation

Symbolic differentiation

In a distributional sense, for $x \in \mathbb{R}$:

14.03.20.0001.01

$$\delta^{(n)}(-x) = (-1)^n\,\delta^{(n)}(x) /;\ n \in \mathbb{N}$$

14.03.20.0002.01

$$x^n\,\delta^{(m)}(x) = 0 /;\ m \in \mathbb{Z} \land n \in \mathbb{Z} \land 0 \leq m < n$$

14.03.20.0003.01

$$x^n\,\delta^{(n)}(x) = (-1)^n\,n!\,\delta(x) /;\ n \in \mathbb{N}$$

14.03.20.0004.01

$$x^n\,\delta^{(m)}(x) = \frac{(-1)^n\,m!}{(m-n)!}\,\delta^{(m-n)}(x) /;\ m \in \mathbb{Z} \land n \in \mathbb{Z} \land 0 \leq n < m$$

14.03.20.0005.01

$$f(x)\,\delta^{(m)}(x-\xi) = (-1)^m \sum_{k=0}^{m} (-1)^k \binom{m}{k} f^{(m-k)}(\xi)\,\delta^{(k)}(x-\xi) /;\ m \in \mathbb{N}$$

14.03.20.0006.01

$$\delta^{(n)}(a\,x) = a^{-n-1}\,\delta^{(n)}(x)\ /;\ n \in \mathbb{N}^+$$

Integration

Indefinite integration

Involving only one direct function

14.03.21.0001.02

$$\int \delta(x)\,dx = \theta(x)$$

Definite integration

For the direct function itself

14.03.21.0002.01

$$\int_{-\infty}^{\infty} \delta(t)\,dt == 1$$

Involving the direct function

In the following formulas $a \in \mathbb{R}$.

14.03.21.0003.01

$$\int_{-d}^{d} \delta(t-a)\,f(t)\,dt == f(a)\ /;\ -\infty \le -d < a < d \le \infty$$

14.03.21.0004.01

$$\int_{-d}^{d} \delta'(t-a)\,f(t)\,dt == -f'(a)\ /;\ -\infty \le -d < a < d \le \infty$$

14.03.21.0005.01

$$\int_{-\infty}^{\infty} \frac{\partial^m \delta(t)}{\partial t^m}\,\frac{\partial^n \delta(x-t)}{\partial t^n}\,dt = \frac{\partial^{m+n} \delta(x)}{\partial x^{m+n}}\ /;\ m \in \mathbb{N}^+ \bigwedge n \in \mathbb{N}^+$$

Integral transforms

Fourier exp transforms

14.03.22.0001.01

$$\mathcal{F}_t[\delta(t)](z) == \frac{1}{\sqrt{2\,\pi}}$$

14.03.22.0006.01

$$\mathcal{F}_t[\delta(t-a)](x) = \frac{e^{i\,a\,x}}{\sqrt{2\,\pi}}$$

14.03.22.0007.01

$$\mathcal{F}_t\left[\frac{\partial^n \delta(t)}{\partial t^n}\right](x) = \frac{(-i)^n\,x^n}{\sqrt{2\,\pi}}\ /;\ n \in \mathbb{N}$$

14.03.22.0008.01

$$\mathcal{F}_t[\text{DiracComb}(t)](x) = \frac{1}{\sqrt{2\pi}} \text{DiracComb}\left(\frac{x}{2\pi}\right)$$

Inverse Fourier exp transforms

14.03.22.0002.01

$$\mathcal{F}_t^{-1}[\delta(t)](z) == \frac{1}{\sqrt{2\pi}}$$

Fourier cos transforms

14.03.22.0003.01

$$\mathcal{F}c_t[\delta(t)](z) == \sqrt{\frac{2}{\pi}}$$

Fourier sin transforms

14.03.22.0004.01

$$\mathcal{F}s_t[\delta(t)](z) == 0$$

Laplace transforms

14.03.22.0005.01

$$\mathcal{L}_t[\delta(t)](z) == 1$$

14.03.22.0009.01

$$\mathcal{L}_t[\delta(t-a)](x) = e^{-a x}\, \theta(a)$$

14.03.22.0010.01

$$\mathcal{L}_t\left[\frac{\partial^n \delta(t-a)}{\partial t^n}\right](x) = e^{-a x}\, x^n\, \theta(a) \;/;\; n \in \mathbb{N}^+$$

Summation

Infinite summation

14.03.23.0001.01

$$\sum_{k=-\infty}^{\infty} \delta(x-k) = \text{DiracComb}(x)$$

14.03.23.0002.01

$$\sum_{k=-\infty}^{\infty} \delta(x-t\,k) = \frac{1}{t} \sum_{k=-\infty}^{\infty} e^{\frac{2 k \pi x i}{t}}$$

14.03.23.0003.01

$$\sum_{k=-\infty}^{\infty} \delta(x-t\,k) = \frac{1}{t}\left(1 + 2\sum_{k=1}^{\infty} \cos\left(\frac{2 k \pi x}{t}\right)\right)$$

14.03.23.0004.01

$$\sum_{k=-\infty}^{\infty} \frac{\partial^n \delta(x-k)}{\partial x^n} = 2\,(2\,\pi)^n \sum_{k=1}^{\infty} k^n \cos\left(\frac{\pi\,n}{2} + 2\,k\,\pi\,x\right) /;\; n \in \mathbb{N}^+$$

Representations through more general functions

Through Meijer *G*

Classical cases for the direct function itself

14.03.26.0001.01

$$\delta(x) == G_{0,0}^{0,0}\left(1-x \,\middle|\, \right) /;\; x \in \mathbb{R} \land x < 2$$

14.03.26.0002.01

$$\delta(x) == G_{0,0}^{0,0}\left(x+1 \,\middle|\, \right) /;\; x \in \mathbb{R} \land x > -2$$

Representations through equivalent functions

14.03.27.0001.01

$$\delta(x) == \theta'(x) /;\; x \neq 0$$

14.03.27.0003.01

$$\delta(x) = \theta'(x)$$

14.03.27.0002.01

$$\delta(x) == \delta(x_1,\, x_2,\, \ldots,\, x_n) /;\; x_1 == x \bigwedge n == 1$$

Theorems

Sokhotskii's formulas

$$\lim_{\varepsilon \to 0} \frac{1}{x \pm i\,\varepsilon} == \mp i\,\pi\,\delta(x) + \mathcal{P}\left(\frac{1}{x}\right)$$

Green's function of one linear differential operator

Let $\hat{L}_{\boldsymbol{x}}$ be a linear differential operator. The fundamental solution $G(\boldsymbol{x}-\boldsymbol{\xi})$ (also called Green's function) of $\hat{L}_{\boldsymbol{x}}$ fulfills the equation $\hat{L}_{\boldsymbol{x}}\, G(\boldsymbol{x}-\boldsymbol{\xi}) == \delta(\boldsymbol{x}-\boldsymbol{\xi})$. Then the solution to the equation $\hat{L}_{\boldsymbol{x}}\, y(\boldsymbol{x}) == f(\boldsymbol{x})$ can be represented in the form $y(\boldsymbol{x}) == \int f(\boldsymbol{\xi})\, G(\boldsymbol{x}-\boldsymbol{\xi})\, d\boldsymbol{\xi}$.

The left eigenstates of the Fröbenius-Perron operator

The functional $\psi_n(x) == (-1)^{n-1} \left(\delta^{(n-1)}(x-1) - \delta^{n-1}(x)\right) /;\; n \in \mathbb{N}^+$ are the left eigenstates of the Fröbenius-Perron operator for the r-adic map $x_{n+1} == r\, x_n \bmod 1$.

Poincaré-Bertrand theorem

$$\lim_{\substack{\varepsilon_1\to 0\\ \varepsilon_2\to 0}} \frac{1}{x_1 - i\, s_1\, \varepsilon}\, \frac{1}{x_2 - i\, s_2\, \varepsilon} = \left(\mathcal{P}\left(\frac{1}{x_1}\right) + i\, \pi\, s_1\, \delta(x_1)\right)\left(\mathcal{P}\left(\frac{1}{x_2}\right) + i\, \pi\, s_2\, \delta(x_2)\right) + \pi^2\, \delta(x_1)\, \delta(x_2)\ /;\ s_1,\, s_2 = \pm 1$$

The generalized solutions of the linear differential operators with singular coefficients

Linear differential operators with singular coefficients can have generalized solutions. For the hypergeometric differential equation $x(1-x)\, y''(x) + (\gamma - (\alpha+\beta+1)\, x)\, y'(x) - \alpha\, \beta\, y(x) = 0$ for $\alpha,\ \beta,\ \gamma \in \mathbb{N}^+,\ \alpha \geq \gamma > \beta$ it can be presented in the form

$$y(x) = c_1 \sum_{k=0}^{\gamma-\beta-1} \frac{(-1)^k\, (\beta-\gamma-1)_k}{k!\, (\beta-\alpha-1)_k}\, \delta^{(\beta+k-1)}(x) + c_2 \sum_{k=0}^{\infty} \frac{(-1)^k\, (\alpha-\gamma-1)_k}{k!\, (\alpha-\beta-1)_k}\, \delta^{(\beta+k-1)}(x)$$

The functional derivative of a function

The functional derivative of a function $f(x)$ is $\frac{\delta f(x)}{\delta f(y)} = \delta(x-y)$.

History

– O. Heaviside (1893–1895)
– G. Kirchhoff (1891)
– P. A. M. Dirac (1926)
– L. Schwartz (1945).

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