DivisorSigma

Notations

Traditional name

Sum of divisor powers

Traditional notation

$\sigma_k(n)$

Mathematica StandardForm notation

DivisorSigma[k, n]

Primary definition

13.05.02.0001.01

$$\sigma_k(n) = \sum_{d_j \mid n} d_j^k \;/;\; d_j \in \text{divisors}(n) \land n \in \mathbb{N}^+$$

13.05.02.0002.01

$$\sigma_k(-n) = \sigma_k(n) \;/;\; n \in \mathbb{N}^+$$

For positive integer n, the function $\sigma_k(n)$ is the sum of the kth powers of the positive divisors of n.
In particular, $\sigma_0(n)$ is the total number of divisors of n, $\sigma_1(n)$ is the sum of divisors of n.

Examples: The divisors of 4 are 1, 2, and 4, so $\sigma_k(4) = 1 + 2^k + 4^k$; the divisors of 5 are 1, and 5, so $\sigma_k(5) = 1 + 5^k$; the divisors of 6 are 1, 2, 3, and 6, so $\sigma_k(6) = 1 + 2^k + 3^k + 6^k$.

Specific values

Specialized values

For fixed *k*

13.05.03.0001.01

$$\sigma_k(p) = p^k + 1 \;/;\; p \in \mathbb{P}$$

13.05.03.0002.01

$$\sigma_k(2^n) = \frac{2^{(n+1)k} - 1}{2^k - 1} \;/;\; n \in \mathbb{N} \land k \neq 1$$

13.05.03.0003.01

$$\sigma_k(1) = 1$$

13.05.03.0004.01
$\sigma_k(2) == 1 + 2^k$

13.05.03.0005.01
$\sigma_k(3) == 1 + 3^k$

13.05.03.0006.01
$\sigma_k(4) == 1 + 2^k + 4^k$

13.05.03.0007.01
$\sigma_k(5) == 1 + 5^k$

13.05.03.0008.01
$\sigma_k(6) == 1 + 2^k + 3^k + 6^k$

13.05.03.0009.01
$\sigma_k(7) == 1 + 7^k$

13.05.03.0010.01
$\sigma_k(8) == 1 + 2^k + 4^k + 8^k$

13.05.03.0011.01
$\sigma_k(9) == 1 + 3^k + 9^k$

13.05.03.0012.01
$\sigma_k(10) == 1 + 2^k + 5^k + 10^k$

13.05.03.0013.01
$\sigma_k(11) == 1 + 11^k$

13.05.03.0014.01
$\sigma_k(12) == 1 + 2^k + 3^k + 4^k + 6^k + 12^k$

13.05.03.0015.01
$\sigma_k(13) == 1 + 13^k$

13.05.03.0016.01
$\sigma_k(14) == 1 + 2^k + 7^k + 14^k$

13.05.03.0017.01
$\sigma_k(15) == 1 + 3^k + 5^k + 15^k$

13.05.03.0018.01
$\sigma_k(16) == 1 + 2^k + 4^k + 8^k + 16^k$

13.05.03.0019.01
$\sigma_k(17) == 1 + 17^k$

13.05.03.0020.01
$\sigma_k(18) == 1 + 2^k + 3^k + 6^k + 9^k + 18^k$

13.05.03.0021.01
$\sigma_k(19) == 1 + 19^k$

13.05.03.0022.01
$\sigma_k(20) == 1 + 2^k + 4^k + 5^k + 10^k + 20^k$

13.05.03.0023.01

$\sigma_k(21) == 1 + 3^k + 7^k + 21^k$

13.05.03.0024.01

$\sigma_k(22) == 1 + 2^k + 11^k + 22^k$

13.05.03.0025.01

$\sigma_k(23) == 1 + 23^k$

13.05.03.0026.01

$\sigma_k(24) == 1 + 2^k + 3^k + 4^k + 6^k + 8^k + 12^k + 24^k$

13.05.03.0027.01

$\sigma_k(25) == 1 + 5^k + 25^k$

13.05.03.0028.01

$\sigma_k(26) == 1 + 2^k + 13^k + 26^k$

13.05.03.0029.01

$\sigma_k(27) == 1 + 3^k + 9^k + 27^k$

13.05.03.0030.01

$\sigma_k(28) == 1 + 2^k + 4^k + 7^k + 14^k + 28^k$

13.05.03.0031.01

$\sigma_k(29) == 1 + 29^k$

13.05.03.0032.01

$\sigma_k(30) == 1 + 2^k + 3^k + 5^k + 6^k + 10^k + 15^k + 30^k$

13.05.03.0033.01

$\sigma_k(31) == 1 + 31^k$

13.05.03.0034.01

$\sigma_k(32) == 1 + 2^k + 4^k + 8^k + 16^k + 32^k$

13.05.03.0035.01

$\sigma_k(33) == 1 + 3^k + 11^k + 33^k$

13.05.03.0036.01

$\sigma_k(34) == 1 + 2^k + 17^k + 34^k$

13.05.03.0037.01

$\sigma_k(35) == 1 + 5^k + 7^k + 35^k$

13.05.03.0038.01

$\sigma_k(36) == 1 + 2^k + 3^k + 4^k + 6^k + 9^k + 12^k + 18^k + 36^k$

13.05.03.0039.01

$\sigma_k(37) == 1 + 37^k$

13.05.03.0040.01

$\sigma_k(38) == 1 + 2^k + 19^k + 38^k$

13.05.03.0041.01

$\sigma_k(39) == 1 + 3^k + 13^k + 39^k$

13.05.03.0042.01

$$\sigma_k(40) == 1 + 2^k + 4^k + 5^k + 8^k + 10^k + 20^k + 40^k$$

13.05.03.0043.01

$$\sigma_k(41) == 1 + 41^k$$

13.05.03.0044.01

$$\sigma_k(42) == 1 + 2^k + 3^k + 6^k + 7^k + 14^k + 21^k + 42^k$$

13.05.03.0045.01

$$\sigma_k(43) == 1 + 43^k$$

13.05.03.0046.01

$$\sigma_k(44) == 1 + 2^k + 4^k + 11^k + 22^k + 44^k$$

13.05.03.0047.01

$$\sigma_k(45) == 1 + 3^k + 5^k + 9^k + 15^k + 45^k$$

13.05.03.0048.01

$$\sigma_k(46) == 1 + 2^k + 23^k + 46^k$$

13.05.03.0049.01

$$\sigma_k(47) == 1 + 47^k$$

13.05.03.0050.01

$$\sigma_k(48) == 1 + 2^k + 3^k + 4^k + 6^k + 8^k + 12^k + 16^k + 24^k + 48^k$$

13.05.03.0051.01

$$\sigma_k(49) == 1 + 7^k + 49^k$$

13.05.03.0052.01

$$\sigma_k(50) == 1 + 2^k + 5^k + 10^k + 25^k + 50^k$$

13.05.03.0058.01

$$\sigma_k(100) == \left(1 + 2^k + 4^k\right)\left(1 + 5^k + 25^k\right)$$

13.05.03.0059.01

$$\sigma_k(1000) == \left(1 + 2^k + 4^k + 8^k\right)\left(1 + 5^k + 25^k + 125^k\right)$$

13.05.03.0060.01

$$\sigma_k(10\,000) == \frac{\left(-1 + 5^{5k}\right)\left(-1 + 32^k\right)}{\left(-1 + 2^k\right)\left(-1 + 5^k\right)}$$

For fixed *n*

13.05.03.0053.01

$$\sigma_0(n) == \sum_{k=1}^{n} \left(\left\lfloor \frac{n}{k} \right\rfloor - \left\lfloor \frac{n-1}{k} \right\rfloor \right)$$

13.05.03.0054.01

$$\sigma_0(n) == 2 \log_n\left(\prod_{d_j \mid n} d_j \right) /;\ d_j \in \text{divisors}(n) \bigwedge n \in \mathbb{N}^+$$

13.05.03.0055.01

$$\sigma_0(n) == \frac{1}{n} \sum_{d_j|n} \sigma_1\left(\frac{n}{d_j}\right) \phi(d_j) \;/;\; d_j \in \text{divisors}(n)$$

13.05.03.0056.01

$$\sigma_1(n) == \sum_{d_j|n} \phi(d_j)\, \sigma_0\left(\frac{n}{d_j}\right) \;/;\; d_j \in \text{divisors}(n)$$

13.05.03.0061.01

$$\sigma_0(n) = \sum_{k|n} 1$$

Values at fixed points

13.05.03.0057.01

$$\sigma_0(1) == 1$$

General characteristics

Domain and analyticity

$\sigma_k(n)$ is an analytical function of k. $\sigma_k(n)$ is defined in the whole complex k-plane and for $n \in \mathbb{Z}$.

13.05.04.0001.01

$$(k * n) \longrightarrow \sigma_k(n) :: (\mathbb{C} \otimes \mathbb{Z}) \longrightarrow \mathbb{C}$$

Symmetries and periodicities

Parity

$\sigma_k(n)$ is an even function with respect to n.

13.05.04.0002.01

$$\sigma_k(-n) == \sigma_k(n)$$

Periodicity

No periodicity

Series representations

Other series representations

13.05.06.0005.01

$$\sigma_k(n) = \sum_{m=1}^{n} m^{k-1} \sum_{j=1}^{m} \cos\left(\frac{2\pi j n}{m}\right)$$

13.05.06.0004.01

$$\sigma_k(n) == \sum_{d_j|n} d_j^k \;/;\; d_j \in \text{divisors}(n) \bigwedge n \in \mathbb{N}^+$$

13.05.06.0006.01

$$\sigma_0(n) = \sum_{m=1}^{\infty} (-1)^{m+1} (2\pi n)^{2m} \sum_{j=1}^{m} \frac{(-1)^j (2\pi)^{2j} B_{2j}^2}{2((2j)!)^2 (2m-2j+1)!}$$

Jose Sousa (2007)

Product representations

13.05.08.0001.01

$$\sigma_k(n) == \prod_{j=1}^{m} \frac{p_j^{(n_j+1)k} - 1}{p_j^k - 1} \;/;\; n == \prod_{j=1}^{m} p_j^{n_j} \bigwedge p_j \in \mathbb{P} \bigwedge n_j \in \mathbb{N}^+$$

13.05.08.0002.01

$$\sigma_k(n) == \prod_{j=1}^{m} \sigma_k\left(p_j^{n_j}\right) \;/;\; \text{factors}(n) == \{\{p_1, n_1\}, \ldots, \{p_m, n_m\}\} \land p_j \in \mathbb{P} \land k \in \mathbb{Z} \land n \in \mathbb{N}^+$$

Transformations

Multiple arguments

13.05.16.0001.01

$$\sigma_k(m\,n) == \sum_{d_j \mid \gcd(n,m)} d_j^k \, \mu(d_j) \, \sigma_k\left(\frac{m}{d_j}\right) \sigma_k\left(\frac{n}{d_j}\right)$$

Products, sums, and powers of the direct function

Products of the direct function

13.05.16.0002.01

$$\sigma_k(m)\,\sigma_k(n) == \sum_{d_j \mid \gcd(n,m)} d_j^k \, \sigma_k\left(\frac{m\,n}{d_j^2}\right)$$

Identities

Functional identities

13.05.17.0006.01

$$\sigma_1(n) == \sum_{d_j \mid n} \sigma_0\left(\frac{n}{d_j}\right) \phi(d_j) \;/;\; d_j \in \text{divisors}(n) \land n \in \mathbb{N}^+$$

13.05.17.0001.01

$$\sum_{d_j \mid n} \left(\sigma_0(d_j)\right)^3 == \left(\sum_{d_j \mid n} \sigma_0(d_j)\right)^2 \;/;\; d_j \in \text{divisors}(n)$$

13.05.17.0002.01

$$\sum_{d_j|n} \sigma_1(d_j) = \sum_{d_j|n} \frac{n}{d_j}\, \sigma_0(d_j) \;/;\; d_j \in \text{divisors}(n)$$

13.05.17.0003.01

$$\sum_{d_j|n} \sigma_1(d_j)\, \sigma_1\!\left(\frac{n}{d_j}\right) = \sum_{d_j|n} d_j\, \sigma_0(d_j)\, \sigma_0\!\left(\frac{n}{d_j}\right) \;/;\; d_j \in \text{divisors}(n)$$

13.05.17.0004.01

$$\sum_{d_j|n} d_j^{\mu-\nu}\, \sigma_{\nu+\tau}(d_j)\, \sigma_{\mu+\rho}\!\left(\frac{n}{d_j}\right) = \sum_{d_j|n} d_j^{\mu-\nu}\, \sigma_{\nu+\rho}(d_j)\, \sigma_{\mu+\tau}\!\left(\frac{n}{d_j}\right) \;/;\; d_j \in \text{divisors}(n) \land \mu \in \mathbb{Z} \land \nu \in \mathbb{Z} \land \rho \in \mathbb{Z} \land \tau \in \mathbb{Z}$$

13.05.17.0005.01

$$\sum_{d_j|n} d_j^{\mu}\, \sigma_\mu(d_j) = \sum_{d_j|n} \left(\frac{n}{d_j}\right)^{2\mu} \sigma_\mu(d_j) \;/;\; d_j \in \text{divisors}(n) \land \mu \in \mathbb{Z}$$

Summation

Finite summation

13.05.23.0001.01

$$\sum_{n=1}^{m} \sigma_k(n) = \sum_{n=1}^{m} n^k \left\lfloor \frac{m}{n} \right\rfloor$$

13.05.23.0002.01

$$\sum_{k=1}^{\left\lfloor \frac{n}{2} \right\rfloor} \sigma_1(k)\, \sigma_1(n-2k) = \frac{1}{12}\sigma_3(n) - \frac{3n-1}{24}\sigma_1(n) \;/;\; \frac{n-1}{2} \in \mathbb{N}$$

13.05.23.0003.01

$$\sum_{k=1}^{n-1} \sigma_r(k)\, \sigma_s(n-k) = \frac{\Gamma(r+1)\,\Gamma(s+1)\,\zeta(r+1)\,\zeta(s+1)}{\Gamma(r+s+2)\,\zeta(r+s+2)}\, \sigma_{r+s+1}(n) - \frac{1}{2}\sigma_s(n)\,\zeta(-r) + \frac{n}{r+s}\sigma_{r+s-1}(n)\,\zeta(1-r) + \zeta(1-s) - \frac{1}{2}\sigma_r(n)\,\zeta(-s) \;/;$$
$$r = 1 \land s = 3 \lor r = 1 \land s = 5 \lor r = 1 \land s = 7 \lor r = 1 \land s = 11 \lor r = 3 \land s = 3 \lor r = 3 \land s = 5 \lor r = 3 \land s = 9 \lor r = 5 \land s = 7$$

13.05.23.0004.01

$$\sum_{l=1}^{n} l^m\, \mu(l) \sum_{k=1}^{\frac{n}{l}} \sigma_m(k) = n \;/;\; n \in \mathbb{N}^+ \land m \in \mathbb{N}^+$$

13.05.23.0005.01

$$\sum_{d_j|n} \phi(d_j)\, \sigma_0\!\left(\frac{n}{d_j}\right) = \sigma_1(n) \;/;\; d_j \in \text{divisors}(n)$$

13.05.23.0006.01

$$\sum_{d_j|n} \phi\!\left(\frac{n}{d_j}\right) \sigma_k(d_j) = n\, \sigma_{k-1}(n) \;/;\; d_j \in \text{divisors}(n) \land k \in \mathbb{Z}$$

Infinite summation

13.05.23.0007.01

$$\sum_{n=1}^{\infty} \frac{\sigma_k(n)}{n^s} == \zeta(s)\,\zeta(s-k) \,/;\, s > 1 \bigwedge s > k+1$$

Asymptotic finite summation

13.05.23.0008.01

$$\sum_{j=1}^{n} \sigma_0(j) \propto (2\,\gamma - 1)\,n + \log(n)\,n + O\!\left(\sqrt[3]{n}\;\log(n)\right) /;\, (n \to \infty)$$

13.05.23.0009.01

$$\sum_{j=1}^{n} \sigma_1(j) \propto \frac{\pi^2}{12}\,n^2 + O(n\log(n)) \,/;\, (n \to \infty)$$

13.05.23.0010.01

$$\sum_{j=1}^{n} \sigma_k(j) \propto \frac{\zeta(k+1)\,n^{k+1}}{k+1} + O\!\left(n^k\right) /;\, (n \to \infty) \bigwedge k-1 \in \mathbb{N}^+$$

13.05.23.0011.01

$$\sum_{j=1}^{n} \sigma_k(j)^2 \propto \frac{\zeta(2\,k+1)\,\zeta(k+1)^2}{(2\,k+1)\,\zeta(2\,k+2)}\,n^{2\,k+1} + O\!\left(n^{2\,k}\right) /;\, (n \to \infty) \bigwedge k \in \mathbb{N}^+$$

13.05.23.0012.01

$$\sum_{j=1}^{n} \sigma_k(j)\,\sigma_l(j+\delta) \propto \frac{\zeta(k+1)\,\zeta(l+1)}{(k+l+1)\,\zeta(k+l+2)}\,\sigma_{-k-l-1}(\delta)\,n^{k+l+1} + O\!\left(n^{k+1}\right) /;\, (n \to \infty) \bigwedge k \in \mathbb{N}^+ \bigwedge l \in k \in \mathbb{N}^+ \bigwedge \delta \in k \in \mathbb{N}$$

13.05.23.0013.01

$$\sum_{j=1}^{n} \frac{\sigma_1(j)}{j} \propto \frac{\pi^2}{6}\,n - \log(n) + O\!\left(\log^{\frac{2}{3}}(n)\right) /;\, (n \to \infty)$$

13.05.23.0014.01

$$\sum_{j=1}^{n} \frac{1}{\log(\sigma_k(j))} \propto \frac{n}{\log(k)\log(\log(n))} + O\!\left(\frac{n}{\log^2(\log(n))}\right) /;\, (n \to \infty) \bigwedge k \in \mathbb{N}$$

13.05.23.0015.01

$$\max\!\left(\{\sigma_0(k)\}_{k,1,n}\right) \propto 2^{\frac{\log(n)}{\log(\log(n))} + O\left(\frac{\log(n)\log(\log(\log(x)))}{\log^2(\log(n))}\right)} /;\, (n \to \infty)$$

13.05.23.0016.01

$$\sum_{k=1}^{x} \text{boole}\!\left(\frac{\sigma_0(k)}{p} \in \mathbb{N}\right) \propto x\left(1 - \frac{\zeta(p)}{\zeta(p-1)}\right) + O\!\left(x^{\frac{1}{p-1}}\right) /;\, (x \to \infty) \bigwedge p \in \mathbb{P}$$

13.05.23.0017.01

$$\sum_{n=1}^{x}\left(\sum_{k=1}^{n} \sigma_0(k) - (n\log(n) - (2\,\gamma - 1)\,n)\right) \propto \left(\gamma - \frac{1}{4}\right)x + \frac{1}{2}\,x\log(x) + O\!\left(x^{3/4}\right) /;\, (x \to \infty)$$

13.05.23.0018.01

$$\sum_{k=1}^{n} \sigma_0(p_k - 1) \propto \frac{315\,\zeta(3)}{2\,\pi^4}\, p_k \;/;\; (n \to \infty) \bigwedge p_k \in \mathbb{P}$$

Asymptotic infinite summation

13.05.23.0019.01

$$\sum_{n=1}^{\infty} \theta(\sigma_1(n) - m) \propto c\, m + o(m) \;/;\; m \in \mathbb{R} \bigwedge m > 0 \bigwedge \left(c = \prod_{j=1}^{\infty} \left(1 - \frac{1}{p_j}\right) \sum_{k=1}^{\infty} \frac{p_j - 1}{p_j^{k+1} - 1} \bigwedge p_j \in \mathbb{P} \right)$$

Operations

Limit operation

13.05.25.0001.01

$$\lim_{n\to\infty} \log\left(\max\left(\frac{\sigma_1(2)}{2\log(\log(2))}, \frac{\sigma_1(3)}{3\log(\log(3))}, \ldots, \frac{\sigma_1(n)}{n\log(\log(n))}\right)\right) = \gamma$$

13.05.25.0002.01

$$\lim_{n\to\infty} \frac{\sigma_n(m+n)}{\sigma_n(n)} = e^m \;/;\; n \in \mathbb{N}^+ \bigwedge m \in \mathbb{N}^+$$

Representations through equivalent functions

With related functions

13.05.27.0001.01

$$\sigma_k(n) = \prod_{j=1}^{m} \frac{p_j^{(n_j+1)k} - 1}{p_j^k - 1} \;/;\; \text{factors}(n) = \{\{p_1, n_1\}, \ldots, \{p_m, n_m\}\} \bigwedge p_k \in \mathbb{P} \bigwedge n > 0$$

13.05.27.0002.01

$$\sigma_k(n) = \sum_{d_j | n} d_j^k \;/;\; d_j \in \text{divisors}(n) \bigwedge n \in \mathbb{N}^+$$

Inequalities

13.05.29.0002.01

$$\sigma_k(n) \geq \phi(n)^k + \sigma_0(n)^2 \;/;\; k \geq 2 \bigwedge n \geq 3$$

13.05.29.0003.01

$$\sigma_0(n) < 2\sqrt{n}$$

13.05.29.0004.01

$$\sigma_1(n) \leq \sigma_0(n)\,(n - \phi(n)) + \phi(n)$$

13.05.29.0005.01

$$\sigma_1(n) \leq n\,\sigma_0(n) + \phi(n)$$

13.05.29.0032.01

$$\sigma_1(n) \leq n^2$$

Aardvark

13.05.29.0006.01

$$\sigma_0(n) < \frac{n^2}{\phi(n)}$$

13.05.29.0007.01

$$\sigma_0(n) \le 2^{\frac{1.5379 \log(n)}{\log(\log(n))}} /; n \ge 3$$

13.05.29.0008.01

$$\sigma_0(n) \le 2^{\frac{\log(n)}{\log(\log(n))} + 1.9349 \frac{\log(n)}{\log^2(\log(n))}}$$

13.05.29.0009.01

$$\sigma_0(n) \le 2^{\frac{\log(n)}{\log(\log(n))} + \frac{\log(n)}{\log^2(\log(n))} + 4.7624 \frac{\log(n)}{\log^3(\log(n))}} /; n \ge 3$$

13.05.29.0010.01

$$\sigma_0(n) \le 2^{\frac{\log(n)}{\log(\log(n)) - 1.39177}} /; n \ge 57$$

13.05.29.0011.01

$$\sigma_0(n) > \frac{n}{\phi(n)} /; n \ge 3$$

13.05.29.0012.01

$$\sigma_0(n) < \phi(n) /; n \ne 1 \land n \ne 3 \land n \ne 8 \land n \ne 10 \land n \ne 24 \land n \ne 30$$

13.05.29.0013.01

$$\sigma_0(n^2) \ge \sigma_0(n) \left(\frac{3}{2}\right)^m /; n \ge 2 \bigwedge n == \prod_{k=1}^{m} p_k^{n_k} \bigwedge p_k \in \mathbb{P} \bigwedge n_k \in \mathbb{N} \bigwedge p_k < p_{k+1} \bigwedge 1 \le k \le m - 1$$

13.05.29.0014.01

$$\sigma_1(n) \ge n + \sqrt{n} /; n \notin \mathbb{P}$$

13.05.29.0015.01

$$\sigma_1(n) < n \sqrt{n} /; n \ge 3$$

13.05.29.0016.01

$$\sigma_1(n) < \frac{6}{\pi^2} n \sqrt{n} /; n \ge 9$$

13.05.29.0017.01

$$\sigma_1(n) < 2.59\, n \log(\log(n)) /; n \ge 7$$

13.05.29.0018.01

$$\sigma_1(n) < e^{\gamma} n \log(\log(n)) /; n \ge 5041$$

This inequalities holds if the Riemann hypothesis holds.

13.05.29.0033.01

$$\sigma_1(n) < \chi\, n \log(\log(n)) /; \chi = e^{\gamma} + \frac{6482}{\log^2(\log(3))}$$

13.05.29.0019.01

$$\sigma_1(n) < \frac{7\,m+10}{6}\,n\;/;\;n == \prod_{k=1}^{m} p_k^{n_k} \bigwedge p_k \in \mathbb{P} \bigwedge n_k \in \mathbb{N}^+ \bigwedge p_k < p_{k+1} \bigwedge 1 \leq k \leq m-1$$

13.05.29.0020.01

$$\sigma_1(n) > \frac{n}{\phi(n)}\;/;\;n \geq 2$$

13.05.29.0021.01

$$\sigma_1(n) > \frac{n+\phi(n)-1}{\sigma_0(n)}$$

13.05.29.0022.01

$$\sigma_1(n) > \frac{n+\phi(n)-1}{\phi(n)}$$

13.05.29.0001.01

$$\sigma_1(n) \leq \frac{n\left(2\left(1-\sqrt{2}\right)-\log(4\,\pi)+\gamma\right)}{\log^{\frac{1}{2}}(n)} + e^{\gamma}\,n\log(\log(n))\;/;\;n > 18$$

13.05.29.0023.01

$$\sigma_1(n) \leq H_n + e^{H_n}\log(H_n)\;/;\;n \in \mathbb{N}^+$$

If the Riemann hypothesis holds.

13.05.29.0024.01

$$\frac{6}{\pi^2} < \frac{\sigma_1(n)\,\phi(n)}{n^2} < 1\;/;\;n \geq 2$$

13.05.29.0025.01

$$n^k \prod_{j=1}^{m} \frac{p_j^{2k}-1}{p_j^k\left(p_j^k-1\right)} \leq \sigma_k(n) < n^k \prod_{j=1}^{m} \frac{p_j^k}{p_j^k-1}\;/;\;k \in \mathbb{N}^+ \bigwedge n == \prod_{k=1}^{m} p_k^{n_k} \bigwedge p_k \in \mathbb{P} \bigwedge n_k \in \mathbb{N}^+ \bigwedge p_k < p_{k+1} \bigwedge 1 \leq k \leq m-1$$

13.05.29.0026.01

$$\sigma_0(m\,n)^2 \geq \sigma_0(m)\,\sigma_0(n)$$

13.05.29.0027.01

$$\frac{\sigma_0(m\,n)}{\sigma_0(m)\,\sigma_0(n)} \leq \frac{\sigma_1(m\,n)}{\sigma_1(m)\,\sigma_1(n)}$$

13.05.29.0028.01

$$\frac{\sigma_0\left(m^2\,n\right)\sigma_0\left(k^2\,n\right)}{\sigma_0(m\,n\,k)^2} \geq \frac{\sigma_0\left(m^2\right)\sigma_0\left(k^2\right)}{\sigma_0(m\,k)^2}$$

13.05.29.0029.01

$$\frac{\sigma_0(m\,n)}{\sigma_0(m)} \geq \frac{\sigma_1(m\,n)}{n\,\sigma_1(m)}$$

13.05.29.0030.01

$$\sigma_0(n)\sqrt{n^k} \leq \sigma_k(n) \leq \sigma_0(n)\,\frac{n^k+1}{2}\;/;\;k \in \mathbb{N}$$

13.05.29.0031.01

$$\sigma_k(n) \leq \frac{\sigma_{k/2}(n)^2}{\sigma_0(n)} \frac{\left(\frac{\sqrt{n^k}+1}{2}\right)^2}{\sqrt{n^k}} \;/;\; k \in \mathbb{R} \bigwedge k > 0$$

Other identities

Congruence properties

13.05.32.0001.01

$$(p\,\sigma_1(p)) \bmod (p-1) == 2 \;/;\; p \in \mathbb{P} \bigwedge p > 3$$

Theorems

Inversion of one sum transformation

$$f(n) == \sum_{d|n} g\left(\frac{n}{d}\right) \sigma_0(d) \Longleftrightarrow g(n) == \sum_{d|n} f\left(\frac{n}{d}\right) \delta(d) \;/;\; \left(\delta(d) == \prod_{d|n} \binom{2}{n_k} (-1)^{n_k} \;/;\; d == \prod_{k=1}^{r} {p_k}^{n_k}\right)$$

History

- R. Descartes (1638)
- J. Wallis (1658)
- E. Waring (1770)
- S. Ramanujan (1911) introduced the sum of the integer powers of divisors
- L. E. Dickson (1920) introduced symbol $\sigma_k(n)$