EllipticF

Notations

Traditional name

Incomplete elliptic integral of the first kind

Traditional notation

$F(z \mid m)$

Mathematica StandardForm notation

`EllipticF[z, m]`

Primary definition

08.05.02.0001.01

$$F(z \mid m) = \int_0^z \frac{1}{\sqrt{1 - m \sin^2(t)}} \, dt$$

Specific values

Specialized values

For fixed *z*

08.05.03.0001.01

$$F(z \mid 0) = z$$

08.05.03.0002.02

$$F(z \mid 1) = \log(\sec(z) + \tan(z)) \;/;\; |\mathrm{Re}(z)| \leq \frac{\pi}{2}$$

08.05.03.0010.01

$$F(z \mid 1) = \tanh^{-1}(\sin(z)) + \left(\left\lceil \frac{\mathrm{Re}(z)}{\pi} - 1 \right\rceil + 1\right) i \, \pi \, \theta(\mathrm{Im}(z)\,\mathrm{Re}(z)) \;/;\; |\mathrm{Re}(z)| \leq \frac{\pi}{2}$$

08.05.03.0011.01

$$F(z \mid 1) = \tilde{\infty} \;/;\; |\mathrm{Re}(z)| > \frac{\pi}{2}$$

08.05.03.0012.01

$$F(z \mid 1) = 2 \tanh^{-1}\left(\tan\left(\frac{z}{2}\right)\right) \;/;\; |\mathrm{Re}(z)| < \frac{\pi}{2}$$

08.05.03.0013.01

$$F(z \mid 1) = \tanh^{-1}(\sin(z)) \,/;\, |\mathrm{Re}(z)| < \frac{\pi}{2}$$

08.05.03.0014.01

$$F\left(\sin^{-1}(z) \mid -1\right) = z\,{}_2F_1\left(\frac{1}{4}, \frac{1}{2}; \frac{5}{4}; z^4\right)$$

08.05.03.0015.01

$$F\left(\sin^{-1}(z) \mid -1\right) = \frac{z}{4\sqrt[4]{z^4}}\, \mathrm{B}_{z^4}\left(\frac{1}{4}, \frac{1}{2}\right)$$

For fixed *m*

08.05.03.0003.01

$$F(0 \mid m) = 0$$

08.05.03.0004.01

$$F\left(\frac{\pi}{2} \,\middle|\, m\right) = K(m)$$

08.05.03.0005.01

$$F\left(\frac{k\,\pi}{2} \,\middle|\, m\right) = k\,K(m) \,/;\, k \in \mathbb{Z}$$

08.05.03.0016.01

$$F\left(\csc^{-1}\left(\sqrt{m}\right) \,\middle|\, m\right) = \frac{1}{\sqrt{m}}\, K\left(\frac{1}{m}\right)$$

08.05.03.0017.01

$$F\left(\csc^{-1}\left(\sqrt{m}\right) + \pi\,k \,\middle|\, m\right) = \frac{1}{\sqrt{m}}\, K\left(\frac{1}{m}\right) + 2\,k\,K(m) \,/;\, k \in \mathbb{Z}$$

08.05.03.0018.01

$$F(\mathrm{am}(z \mid m) \mid m) = z \,/;\, (m \leq 1 \land -2 \leq z \leq 2) \lor (|z| < 1 \land |m| \leq 2)$$

Values at infinities

08.05.03.0006.01

$$F(z \mid \infty) = 0$$

08.05.03.0007.01

$$F(z \mid -\infty) = 0$$

08.05.03.0008.01

$$F(i\,\infty \mid m) = K(m) - \frac{1}{\sqrt{m}}\, K\left(\frac{1}{m}\right) \,/;\, 0 < m < 1$$

08.05.03.0009.01

$$F(-i\,\infty \mid m) = \frac{1}{\sqrt{m}}\, K\left(\frac{1}{m}\right) - K(m) \,/;\, 0 < m < 1$$

General characteristics

Domain and analyticity

$F(z \mid m)$ is an analytical function of z and m which is defined over $\mathbb{C}^2$.

08.05.04.0001.01

$$(z * m) \longrightarrow F(z \mid m) :: (\mathbb{C} \otimes \mathbb{C}) \longrightarrow \mathbb{C}$$

Symmetries and periodicities

Parity

$F(z \mid m)$ is an odd function with respect to z.

08.05.04.0002.01

$$F(-z \mid m) == -F(z \mid m)$$

Mirror symmetry

08.05.04.0004.01

$$F(\bar{z} \mid \bar{m}) == \overline{F(z \mid m)} \;/;\; \neg\,(m \in \mathbb{R} \land m > 1)$$

Periodicity

$F(z \mid m)$ is a quasi-periodic function with respect to z.

08.05.04.0003.01

$$F(z + \pi k \mid m) == F(z \mid m) + 2\, k\, K(m) \;/;\; k \in \mathbb{Z}$$

08.05.04.0011.01

$$F(z \mid m) = F\left(z - \pi \left\lfloor \frac{\mathrm{Re}(z)}{\pi} \right\rceil \,\middle|\, m\right) + 2 \left\lfloor \frac{\mathrm{Re}(z)}{\pi} \right\rceil K(m)$$

Quasi-symmetry

08.05.04.0005.01

$$F(x + i\, y \mid m) == F\left(\pi \operatorname{frac}\left(\frac{x}{\pi}\right) + i\, y \,\middle|\, m\right) + 2 \operatorname{sgn}(x) \left\lfloor \left| \frac{x}{\pi} \right| \right\rfloor K(m) \;/;\; x \in \mathbb{R} \bigwedge y \in \mathbb{R}$$

Poles and essential singularities

With respect to *z*

The function $F(z \mid m)$ does not have poles and essential singularities with respect to z.

08.05.04.0006.01

$$\mathcal{Sing}_z(F(z \mid m)) == \{\}$$

With respect to *m*

The function $F(z \mid m)$ does not have poles and essential singularities with respect to m.

08.05.04.0007.01

$$\mathcal{Sing}_m(F(z \mid m)) == \{\}$$

Branch points

With respect to z

For fixed m, the function $F(z \mid m)$ has an infinite number of branch points at $z == \pm \sin^{-1}\left(\frac{1}{\sqrt{m}}\right) + \pi k /; k \in \mathbb{Z}$, and $z == \tilde{\infty}$. All these are square-root-type branch points.

If real $m > 1$, the function $F(z \mid m)$ additionally has an infinite number of special contact points at $z == \frac{\pi}{2} + \pi k /; k \in \mathbb{Z}$, which are formed in places where pairs of corresponding branch cuts touch one another. That points look like branch cut points but they are not branch cut points. The function $F(z \mid m)$ has rather complicated behaviour near that points.

08.05.04.0008.01

$$\mathcal{BP}_a(F(z \mid m)) == \left\{\left\{\sin^{-1}\left(\frac{1}{\sqrt{m}}\right) + \pi k /; k \in \mathbb{Z}\right\}, \left\{-\sin^{-1}\left(\frac{1}{\sqrt{m}}\right) + \pi k /; k \in \mathbb{Z}\right\}, \tilde{\infty}\right\}$$

08.05.04.0009.01

$$\mathcal{R}_z\left(F(z \mid m), \sin^{-1}\left(\frac{1}{\sqrt{m}}\right) + \pi k\right) == 2 /; k \in \mathbb{Z}$$

08.05.04.0010.01

$$\mathcal{R}_z\left(F(z \mid m), -\sin^{-1}\left(\frac{1}{\sqrt{m}}\right) + \pi k\right) == 2 /; k \in \mathbb{Z}$$

With respect to m

For fixed z, the function $F(z \mid m)$ has two branch points at $m == \csc^2(z)$ and $m == \tilde{\infty}$.

08.05.04.0012.01

$$\mathcal{BP}_m(F(z \mid m)) == \left\{\csc^2(z), \tilde{\infty}\right\}$$

Branch cuts

With respect to z

General description

For fixed m, the function $F(z \mid m)$ can have up to six infinite sets of branch cuts (it has at least four), which form very complicated curves in the case of generic m.

For fixed real $m < 1$, the function $F(z \mid m)$ does not have branch cuts on the real axis and on the vertical intervals $\left\{\csc^{-1}\left(\sqrt{m}\right) + \pi k, \pi - \csc^{-1}\left(\sqrt{m}\right) + \pi k\right\} /; k \in \mathbb{Z} \bigwedge m \in (-\infty, 1)$.

For fixed real $m < 1$, the function $F(z \mid m)$ has four infinite sets of branch cuts located on vertical intervals starting at the points $z == \pi k \pm \csc^{-1}\left(\sqrt{m}\right) /; k \in \mathbb{Z}$ and extending to imaginary infinity.

For fixed generic m, the function $F(z \mid m)$ has the following six infinite sets of branch cuts:

1) real intervals $\left\{\pi k + \csc^{-1}\left(\sqrt{m}\right), \pi k + \frac{\pi}{2}\right\} /; k \in \mathbb{Z} \bigwedge m > 1$, where $F(z \mid m)$ is continuous from below (for generic complex m, these branch cuts deform into complicated curves); in the case $m < 1$ these real intervals vanish

2) real intervals $\left\{\pi k + \frac{\pi}{2}, \pi (k+1) - \csc^{-1}\left(\sqrt{m}\right)\right\} /; k \in \mathbb{Z} \bigwedge m > 1$, where $F(z \mid m)$ is continuous from above (for generic complex m, these branch cuts deform into complicated curves); in the case $m < 1$ these real intervals vanish

3) vertical intervals $\left\{\frac{\pi}{2} + 2\pi k, \frac{\pi}{2} + 2\pi k + i\infty\right\} /; k \in \mathbb{Z} \bigwedge m \notin (0, 1)$, or $\left\{\pi - \csc^{-1}\left(\sqrt{m}\right) + 2\pi k, \frac{\pi}{2} + 2\pi k + i\infty\right\} /; k \in \mathbb{Z} \bigwedge m \in (0, 1)$, where $F(z \mid m)$ is continuous from the left

4) vertical intervals $\left\{\frac{3\pi}{2} + 2\pi k, \frac{3\pi}{2} + 2\pi k + i\infty\right\} /; k \in \mathbb{Z} \bigwedge m \notin (0, 1)$, or $\left\{2\pi - \csc^{-1}\left(\sqrt{m}\right) + 2\pi k, \frac{3\pi}{2} + 2\pi k + i\infty\right\} /; k \in \mathbb{Z} \bigwedge m \in (0, 1)$, where $F(z \mid m)$ is continuous from the right

5) vertical intervals $\left\{\frac{\pi}{2} + 2\pi k - i\infty, \frac{\pi}{2} + 2\pi k\right\} /; k \in \mathbb{Z} \bigwedge m \notin (0, 1)$, or $\left\{\frac{\pi}{2} + 2\pi k - i\infty, 2\pi k + \csc^{-1}\left(\sqrt{m}\right)\right\} /; k \in \mathbb{Z} \bigwedge m \in (0, 1)$, where $F(z \mid m)$ is continuous from the left

6) vertical intervals $\left\{\frac{3\pi}{2} + 2\pi k - i\infty, \frac{3\pi}{2} + 2\pi k\right\} /; k \in \mathbb{Z} \bigwedge m \notin (0, 1)$, or $\left\{\frac{3\pi}{2} + 2\pi k - i\infty, 2\pi k + \pi + \csc^{-1}\left(\sqrt{m}\right)\right\} /; k \in \mathbb{Z} \bigwedge m \in (0, 1)$, where $F(z \mid m)$ is continuous from the right.

08.05.04.0013.01

$$
\begin{aligned}
\mathcal{BC}_z(F(z \mid m)) = \Big\{ & \Big\{\Big\{\Big(\pi k + \csc^{-1}\left(\sqrt{m}\right), \pi k + \frac{\pi}{2}\Big), i\Big\} /; k \in \mathbb{Z} \bigwedge m \in \mathbb{R} \bigwedge m > 1\Big\}, \\
& \Big\{\Big\{\Big(\pi k + \frac{\pi}{2}, \pi (k+1) - \csc^{-1}\left(\sqrt{m}\right)\Big), -i\Big\} /; k \in \mathbb{Z} \bigwedge m \in \mathbb{R} \bigwedge m > 1\Big\}, \\
& \Big\{\Big\{\Big(2\pi k + \frac{\pi}{2}, 2k\pi + \frac{\pi}{2} + i\infty\Big), 1\Big\} /; k \in \mathbb{Z} \bigwedge m \notin (0, 1)\Big\} \bigvee \\
& \quad \Big\{\Big\{\Big(2\pi k + \pi - \csc^{-1}\left(\sqrt{m}\right), 2k\pi + \frac{\pi}{2} + i\infty\Big), 1\Big\} /; k \in \mathbb{Z} \bigwedge m \in (0, 1)\Big\}, \\
& \Big\{\Big\{\Big(2\pi k + \frac{3\pi}{2}, 2k\pi + \frac{3\pi}{2} + i\infty\Big), -1\Big\} /; k \in \mathbb{Z} \bigwedge m \notin (0, 1)\Big\} \bigvee \\
& \quad \Big\{\Big\{\Big(2\pi k + 2\pi - \csc^{-1}\left(\sqrt{m}\right), 2k\pi + \frac{3\pi}{2} + i\infty\Big), -1\Big\} /; k \in \mathbb{Z} \bigwedge m \in (0, 1)\Big\}, \\
& \Big\{\Big\{\Big(2\pi k + \frac{\pi}{2} - i\infty, 2k\pi + \frac{\pi}{2}\Big), 1\Big\} /; k \in \mathbb{Z} \bigwedge m \notin (0, 1)\Big\} \bigvee \\
& \quad \Big\{\Big\{\Big(2\pi k + \frac{\pi}{2} - i\infty, 2\pi k + \csc^{-1}\left(\sqrt{m}\right)\Big), 1\Big\} /; k \in \mathbb{Z} \bigwedge m \in (0, 1)\Big\}, \\
& \Big\{\Big\{\Big(2\pi k + \frac{\pi}{2} - i\infty, 2k\pi + \frac{\pi}{2}\Big), -1\Big\} /; k \in \mathbb{Z} \bigwedge m \notin (0, 1)\Big\} \bigvee \\
& \quad \Big\{\Big\{\Big(2\pi k + \frac{\pi}{2} - i\infty, 2\pi k + \pi + \csc^{-1}\left(\sqrt{m}\right)\Big), -1\Big\} /; k \in \mathbb{Z} \bigwedge m \in (0, 1)\Big\}
\end{aligned}
$$

Formulas on real axis for real *m*

For $m < 1$

For fixed real $m < 1$, the function $F(z \mid m)$ does not have branch cuts on the real axis.

For $m > 1$

08.05.04.0014.01

$$\lim_{\epsilon \to +0} F(x + i\epsilon \mid m) = -F(z \mid m) + \frac{2}{\sqrt{m}} K\left(\frac{1}{m}\right) + 4\left(\left\lfloor \frac{z}{\pi} - \frac{1}{2} \right\rfloor + 1\right) K(m) /;$$
$$x \in \mathbb{R} \bigwedge m \in \mathbb{R} \bigwedge m > 1 \bigwedge \pi k + \csc^{-1}\left(\sqrt{m}\right) < x < \pi k + \frac{\pi}{2} \bigwedge k \in \mathbb{Z}$$

08.05.04.0015.01

$$\lim_{\epsilon \to +0} F(x - i\epsilon \mid m) = F(x \mid m) /; x \in \mathbb{R} \bigwedge m \in \mathbb{R} \bigwedge m > 1 \bigwedge \pi k + \csc^{-1}\left(\sqrt{m}\right) < x < \pi k + \frac{\pi}{2} \bigwedge k \in \mathbb{Z}$$

08.05.04.0016.01

$$\lim_{\epsilon \to +0} F(x + i\epsilon \mid m) = F(x \mid m) /; x \in \mathbb{R} \bigwedge m \in \mathbb{R} \bigwedge m > 1 \bigwedge \frac{\pi}{2} + \pi k < x < \pi (k+1) - \csc^{-1}\left(\sqrt{m}\right) \bigwedge k \in \mathbb{Z}$$

08.05.04.0017.01

$$\lim_{\epsilon \to +0} F(x - i\epsilon \mid m) = -F(x \mid m) - \frac{2}{\sqrt{m}} K\left(\frac{1}{m}\right) + 4\left(\left\lfloor \frac{x}{\pi} - \frac{1}{2} \right\rfloor + 1\right) K(m) \;/;$$
$$x \in \mathbb{R} \bigwedge m \in \mathbb{R} \bigwedge m > 1 \bigwedge \pi k + \frac{\pi}{2} < x < \pi (k+1) - \csc^{-1}\left(\sqrt{m}\right) \bigwedge k \in \mathbb{Z}$$

Formulas for vertical intervals

For $m < 1$

For fixed real $m < 1$, the function $F(z \mid m)$ has branch points $\csc^{-1}\left(\sqrt{m}\right) + \pi k /; k \in \mathbb{Z}$ and $\pi - \csc^{-1}\left(\sqrt{m}\right) + \pi k /; k \in \mathbb{Z}$. In this case branch cuts lay at the vertical lines beginning from these points and going to imaginary infinity. By this reason for fixed real $m < 1$, the function $F(z \mid m)$ does not have branch cuts on the vertical intervals $\left\{\csc^{-1}\left(\sqrt{m}\right) + \pi k, \pi - \csc^{-1}\left(\sqrt{m}\right) + \pi k\right\} /; k \in \mathbb{Z} \wedge m \in (-\infty, 1)$.

For $m > 0$

08.05.04.0018.01

$$\lim_{\epsilon \to +0} F\left(2\pi k + i x + \frac{\pi}{2} - \epsilon \,\middle|\, m\right) = F\left(2\pi k + i x + \frac{\pi}{2} \,\middle|\, m\right) /; x \in \mathbb{R} \wedge k \in \mathbb{Z}$$

08.05.04.0019.01

$$\lim_{\epsilon \to +0} F\left(2\pi k + i x + \frac{\pi}{2} + \epsilon \,\middle|\, m\right) = -F\left(i x + \frac{\pi}{2} \,\middle|\, m\right) - \frac{2}{\sqrt{m}} K\left(\frac{1}{m}\right) + 4(k+1) K(m) /;$$
$$m \in \mathbb{R} \bigwedge x \in \mathbb{R} \bigwedge \left(0 < m < 1 \bigwedge x > -\mathrm{Im}\left(\csc^{-1}\left(\sqrt{m}\right)\right)\right) \bigvee (m > 1 \wedge x < 0) \wedge k \in \mathbb{Z}$$

08.05.04.0020.01

$$\lim_{\epsilon \to +0} F\left(2\pi k + i x + \frac{\pi}{2} + \epsilon \,\middle|\, m\right) = -F\left(i x + \frac{\pi}{2} \,\middle|\, m\right) + \frac{2}{\sqrt{m}} K\left(\frac{1}{m}\right) + 4 k K(m) /;$$
$$m \in \mathbb{R} \bigwedge x \in \mathbb{R} \bigwedge \left(0 < m < 1 \bigwedge x < \mathrm{Im}\left(\csc^{-1}\left(\sqrt{m}\right)\right) \bigvee m > 1 \wedge x > 0\right) \bigwedge k \in \mathbb{Z}$$

08.05.04.0021.01

$$\lim_{\epsilon\to+0} F\left(2\pi k + i x + \frac{3\pi}{2} - \epsilon \,\middle|\, m\right) = -F\left(i x + \frac{3\pi}{2} \,\middle|\, m\right) - \frac{2}{\sqrt{m}} K\left(\frac{1}{m}\right) + 4(k+2) K(m) \,/;$$
$$m \in \mathbb{R} \bigwedge x \in \mathbb{R} \bigwedge \left(0 < m < 1 \bigwedge x > -\operatorname{Im}\left(\csc^{-1}\left(\sqrt{m}\right)\right) \bigvee m > 1 \land x < 0\right) \bigwedge k \in \mathbb{Z}$$

08.05.04.0022.01

$$\lim_{\epsilon\to+0} F\left(2\pi k + i x + \frac{3\pi}{2} - \epsilon \,\middle|\, m\right) = -F\left(i x + \frac{3\pi}{2} \,\middle|\, m\right) + \frac{2}{\sqrt{m}} K\left(\frac{1}{m}\right) + 4(k+1) K(m) \,/;$$
$$m \in \mathbb{R} \bigwedge x \in \mathbb{R} \bigwedge \left(0 < m < 1 \bigwedge x < \operatorname{Im}\left(\csc^{-1}\left(\sqrt{m}\right)\right) \bigvee m > 1 \land x > 0\right) \bigwedge k \in \mathbb{Z}$$

08.05.04.0023.01

$$\lim_{\epsilon\to+0} F\left(2\pi k + i x + \frac{3\pi}{2} + \epsilon \,\middle|\, m\right) = F\left(2\pi k + i x + \frac{3\pi}{2} \,\middle|\, m\right) /;\; x \in \mathbb{R} \land k \in \mathbb{Z}$$

With respect to *m*

Branch cut locations: complicated.

Series representations

Generalized power series

Expansions at generic point $z == z_0$

For the function itself

08.05.06.0015.01

$$F(z \mid m) \propto F(z_0 \mid m) + \frac{1}{\sqrt{1 - m\sin^2(z_0)}} (z - z_0) + \frac{m \sin(2 z_0)}{4\left(1 - m \sin^2(z_0)\right)^{3/2}} (z - z_0)^2 + \ldots \,/;\, (z \to z_0)$$

08.05.06.0016.01

$$F(z \mid m) \propto F(z_0 \mid m) + \frac{1}{\sqrt{1 - m\sin^2(z_0)}} (z - z_0) + \frac{m \sin(2 z_0)}{4\left(1 - m \sin^2(z_0)\right)^{3/2}} (z - z_0)^2 + O\left((z - z_0)^3\right)$$

08.05.06.0017.01

$$F(z \mid m) == F(z_0 \mid m) + \frac{z - z_0}{\sqrt{1 - m\sin^2(z_0)}} + \frac{1}{\sqrt{1 - m\sin^2(z_0)}} \sum_{k=2}^{\infty} \frac{1}{k!} \left(i^{k-1} \binom{k - \frac{1}{2}}{k-1}\right)$$
$$\sum_{q=1}^{k-1} \frac{(-1)^q}{2q+1} \binom{k-1}{q} \left(1 - m\sin^2(z_0)\right)^{-q} \sum_{j=0}^{q} \binom{q}{j} m^j (2-m)^{q-j} 2^{-j+k-q-1} \sum_{i=0}^{j} \binom{j}{i} (2i - j)^{k-1} e^{2(2i-j) i z_0} (z - z_0)^k$$

08.05.06.0018.01

$$F(z \mid m) == F(z_0 \mid m) + \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{j=1}^{k} \frac{1}{j!} \sum_{q=0}^{j-1} \binom{j}{q} \sum_{p=0}^{j-q} (-1)^q 2^{q-j} \sin^q(z_0) (-j + 2p + q)^k e^{-\frac{1}{2} i \left(\pi (j+k-2p-q) + 2(-j+2p+q) z_0\right)} \binom{j-q}{p}$$
$$\sum_{i=0}^{j-1} \frac{(1-j)_{2(j-i)-2}}{(-i+j-1)!\,(2\sin(z_0))^{-2i+j-1}} \sum_{s=0}^{i} \binom{i}{s} \left(\frac{1}{2}\right)_s \left(\frac{1}{2}\right)_{i-s} m^{i-s} \cos^{-2s-1}(z_0) \left(1 - m\sin^2(z_0)\right)^{-i+s-\frac{1}{2}} (z - z_0)^k$$

08.05.06.0019.01

$$F(z \mid m) = \sum_{k=0}^{\infty} \frac{1}{k!} F^{(k,0)}(z_0 \mid m)\,(z - z_0)^k$$

08.05.06.0020.01

$$F(z \mid m) \propto F(z_0 \mid m)\,(1 + O(z - z_0))$$

Expansions on branch cuts

Formulas on real axis for real m

For $m > 1,\ \csc^{-1}\left(\sqrt{m}\right) + \pi u < x < \pi\left(u + \frac{1}{2}\right) /;\ u \in \mathbb{Z}$

08.05.06.0021.01

$$F(z \mid m) \propto \left(\frac{1}{\sqrt{m}} K\left(\frac{1}{m}\right) + 2\left(\left\lfloor \frac{x}{\pi} - \frac{1}{2} \right\rfloor + 1\right) K(m)\right)\left(1 - e^{-\pi i \left\lfloor \frac{\arg(x-z)}{2\pi} \right\rfloor}\right) + F(x \mid m)\, e^{-\pi i \left\lfloor \frac{\arg(x-z)}{2\pi} \right\rfloor} + e^{-\pi i \left\lfloor \frac{\arg(x-z)}{2\pi} \right\rfloor}\left(\frac{1}{\sqrt{1 - m\sin^2(x)}}(z - x) + \frac{m\sin(2x)}{4\left(1 - m\sin^2(x)\right)^{3/2}}(z - x)^2 + \ldots\right) /;$$
$$(z \to x) \bigwedge x \in \mathbb{R} \bigwedge m \in \mathbb{R} \bigwedge m > 1 \bigwedge \pi u + \csc^{-1}\left(\sqrt{m}\right) < x < \pi u + \frac{\pi}{2} \bigwedge u \in \mathbb{Z}$$

08.05.06.0022.01

$$F(z \mid m) = \left(\frac{1}{\sqrt{m}} K\left(\frac{1}{m}\right) + 2\left(\left\lfloor \frac{x}{\pi} - \frac{1}{2} \right\rfloor + 1\right) K(m)\right)\left(1 - e^{-\pi i \left\lfloor \frac{\arg(x-z)}{2\pi} \right\rfloor}\right) + F(x \mid m)\, e^{-\pi i \left\lfloor \frac{\arg(x-z)}{2\pi} \right\rfloor} + e^{-\pi i \left\lfloor \frac{\arg(x-z)}{2\pi} \right\rfloor}\left(\frac{z - x}{\sqrt{1 - m\sin^2(x)}} + \frac{1}{\sqrt{1 - m\sin^2(x)}} \sum_{k=2}^{\infty} \frac{i^{k-1}}{k!} \binom{k - \frac{1}{2}}{k - 1} \sum_{q=1}^{k-1} \frac{(-1)^q}{2q + 1} \binom{k - 1}{q} \left(1 - m\sin^2(x)\right)^{-q} \sum_{j=0}^{q} \binom{q}{j} m^j (2 - m)^{q-j}\, 2^{-j+k-q-1} \sum_{i=0}^{j} \binom{j}{i} (2i - j)^{k-1} e^{2(2i-j)\,i\,x} (z - x)^k\right) /;$$
$$x \in \mathbb{R} \bigwedge m \in \mathbb{R} \bigwedge m > 1 \bigwedge \pi u + \csc^{-1}\left(\sqrt{m}\right) < x < \pi u + \frac{\pi}{2} \bigwedge u \in \mathbb{Z}$$

08.05.06.0023.01

$$F(z \mid m) \propto \left(\left(\frac{1}{\sqrt{m}} K\left(\frac{1}{m}\right) + 2\left(\left\lfloor \frac{x}{\pi} - \frac{1}{2} \right\rfloor + 1\right) K(m)\right)\left(1 - e^{-\pi i \left\lfloor \frac{\arg(x-z)}{2\pi} \right\rfloor}\right) + F(x \mid m)\, e^{-\pi i \left\lfloor \frac{\arg(x-z)}{2\pi} \right\rfloor}\right)(1 + O(z - x)) /;$$
$$(z \to x) \bigwedge x \in \mathbb{R} \bigwedge m \in \mathbb{R} \bigwedge m > 1 \bigwedge \pi u + \csc^{-1}\left(\sqrt{m}\right) < x < \pi u + \frac{\pi}{2} \bigwedge u \in \mathbb{Z}$$

For $m > 1,\ \pi\left(u + \frac{1}{2}\right) < x < \pi(u + 1) - \csc^{-1}\left(\sqrt{m}\right) /;\ u \in \mathbb{Z}$

08.05.06.0024.01

$$F(z\mid m)\propto\left(2\left(\left\lfloor\frac{x}{\pi}-\frac{1}{2}\right\rfloor+1\right)K(m)-\frac{1}{\sqrt{m}}K\left(\frac{1}{m}\right)\right)\left(1-e^{-\pi i\left\lfloor\frac{\arg(z-x)}{2\pi}\right\rfloor}\right)+ F(x\mid m)\,e^{-\pi i\left\lfloor\frac{\arg(z-x)}{2\pi}\right\rfloor}+e^{-\pi i\left\lfloor\frac{\arg(z-x)}{2\pi}\right\rfloor}\left(\frac{1}{\sqrt{1-m\sin^2(x)}}(z-x)+\frac{m\sin(2x)}{4\left(1-m\sin^2(x)\right)^{3/2}}(z-x)^2+\ldots\right)/;$$
$$(z\to x)\bigwedge x\in\mathbb{R}\bigwedge m\in\mathbb{R}\bigwedge m>1\bigwedge \pi u+\frac{\pi}{2}<x<\pi(u+1)-\csc^{-1}\left(\sqrt{m}\right)\bigwedge u\in\mathbb{Z}$$

08.05.06.0025.01

$$F(z\mid m)=\left(2\left(\left\lfloor\frac{x}{\pi}-\frac{1}{2}\right\rfloor+1\right)K(m)-\frac{1}{\sqrt{m}}K\left(\frac{1}{m}\right)\right)\left(1-e^{-\pi i\left\lfloor\frac{\arg(z-x)}{2\pi}\right\rfloor}\right)+ F(x\mid m)\,e^{-\pi i\left\lfloor\frac{\arg(z-x)}{2\pi}\right\rfloor}+e^{-\pi i\left\lfloor\frac{\arg(z-x)}{2\pi}\right\rfloor}\left(\frac{z-x}{\sqrt{1-m\sin^2(x)}}+\frac{1}{\sqrt{1-m\sin^2(x)}}\sum_{k=2}^{\infty}\frac{i^{k-1}}{k!}\binom{k-\frac{1}{2}}{k-1}\sum_{q=1}^{k-1}\frac{(-1)^q}{2q+1}\binom{k-1}{q}\left(1-m\sin^2(x)\right)^{-q}\sum_{j=0}^{q}\binom{q}{j}m^j(2-m)^{q-j}\,2^{-j+k-q-1}\sum_{i=0}^{j}\binom{j}{i}(2i-j)^{k-1}e^{2(2i-j)ix}(z-x)^k\right)/;$$
$$x\in\mathbb{R}\bigwedge m\in\mathbb{R}\bigwedge m>1\bigwedge \pi u+\frac{\pi}{2}<x<\pi(u+1)-\csc^{-1}\left(\sqrt{m}\right)\bigwedge u\in\mathbb{Z}$$

08.05.06.0026.01

$$F(z\mid m)\propto\left(\left(2\left(\left\lfloor\frac{x}{\pi}-\frac{1}{2}\right\rfloor+1\right)K(m)-\frac{1}{\sqrt{m}}K\left(\frac{1}{m}\right)\right)\left(1-e^{-\pi i\left\lfloor\frac{\arg(z-x)}{2\pi}\right\rfloor}\right)+F(x\mid m)\,e^{-\pi i\left\lfloor\frac{\arg(z-x)}{2\pi}\right\rfloor}\right)(1+O(z-x))/;$$
$$(z\to x)\bigwedge x\in\mathbb{R}\bigwedge m\in\mathbb{R}\bigwedge m>1\bigwedge \pi u+\frac{\pi}{2}<x<\pi(u+1)-\csc^{-1}\left(\sqrt{m}\right)\bigwedge u\in\mathbb{Z}$$

Formulas for vertical intervals

For $\operatorname{Re}\left(\frac{z_0}{2\pi}-\frac{1}{4}\right)\in\mathbb{Z}$

08.05.06.0027.01

$$F(z\mid m)\propto \left(2\left(2\operatorname{Re}\left(\frac{z_0}{2\pi}-\frac{1}{4}\right)+1\right)K(m)-\frac{1}{\sqrt{m}}K\left(\frac{1}{m}\right)\right)\left(e^{-\pi i\left(\left\lfloor\frac{\arg(z-z_0)}{2\pi}+\frac{3}{4}\right\rfloor+\left\lfloor\frac{3}{4}-\frac{\arg(z-z_0)}{2\pi}\right\rfloor\right)}+1\right)-e^{-\pi i\left(\left\lfloor\frac{\arg(z-z_0)}{2\pi}+\frac{3}{4}\right\rfloor+\left\lfloor\frac{3}{4}-\frac{\arg(z-z_0)}{2\pi}\right\rfloor\right)}F(z_0\mid m)+ \frac{1}{\sqrt{1-m\sin^2(z_0)}}(z-z_0)+\frac{m\sin(2z_0)}{4\left(1-m\sin^2(z_0)\right)^{3/2}}(z-z_0)^2+\ldots/;(z\to z_0)\bigwedge\operatorname{Re}\left(\frac{z_0}{2\pi}-\frac{1}{4}\right)\in\mathbb{Z}$$

08.05.06.0028.01

$$F(z \mid m) = \left(2\left(2\operatorname{Re}\left(\frac{z_0}{2\pi}-\frac{1}{4}\right)+1\right)K(m)-\frac{1}{\sqrt{m}}K\left(\frac{1}{m}\right)\right)\left(e^{-\pi i\left(\left\lfloor \frac{\arg(z-z_0)}{2\pi}+\frac{3}{4}\right\rfloor+\left\lfloor \frac{3}{4}-\frac{\arg(z-z_0)}{2\pi}\right\rfloor\right)}+1\right)- e^{-\pi i\left(\left\lfloor \frac{\arg(z-z_0)}{2\pi}+\frac{3}{4}\right\rfloor+\left\lfloor \frac{3}{4}-\frac{\arg(z-z_0)}{2\pi}\right\rfloor\right)}F(z_0 \mid m)+\sum_{k=1}^{\infty}\frac{1}{k!} \sum_{j=1}^{k}\frac{1}{j!}\sum_{q=0}^{j-1}\binom{j}{q}\sum_{p=0}^{j-q}(-1)^q\,2^{q-j}\sin^q(z_0)\,(2p+q-j)^k\,e^{-\frac{1}{2}i(\pi(j+k-2p-q)+2(2p+q-j)z_0)}\binom{j-q}{p}\sum_{i=0}^{j-1}\frac{(1-j)_{2(j-i)-2}}{(j-i-1)!\,(2\sin(z_0))^{j-2i-1}} \sum_{s=0}^{i}\binom{i}{s}\left(\frac{1}{2}\right)_s\left(\frac{1}{2}\right)_{i-s}m^{i-s}\cos^{-2s-1}(z_0)\left(1-m\sin^2(z_0)\right)^{s-i-\frac{1}{2}}(z-z_0)^k\;/;\;\operatorname{Re}\left(\frac{z_0}{2\pi}-\frac{1}{4}\right)\in\mathbb{Z}$$

08.05.06.0029.01

$$F(z \mid m) \propto \left(\left(2\left(2\operatorname{Re}\left(\frac{z_0}{2\pi}-\frac{1}{4}\right)+1\right)K(m)-\frac{1}{\sqrt{m}}K\left(\frac{1}{m}\right)\right)\left(e^{-\pi i\left(\left\lfloor \frac{\arg(z-z_0)}{2\pi}+\frac{3}{4}\right\rfloor+\left\lfloor \frac{3}{4}-\frac{\arg(z-z_0)}{2\pi}\right\rfloor\right)}+1\right)-e^{-\pi i\left(\left\lfloor \frac{\arg(z-z_0)}{2\pi}+\frac{3}{4}\right\rfloor+\left\lfloor \frac{3}{4}-\frac{\arg(z-z_0)}{2\pi}\right\rfloor\right)}F(z_0 \mid m)\right) (1+O(z-z_0))\;/;\;(z\to z_0)\bigwedge \operatorname{Re}\left(\frac{z_0}{2\pi}-\frac{1}{4}\right)\in\mathbb{Z}$$

For $\operatorname{Re}\left(\frac{z_0}{2\pi}-\frac{3}{4}\right)\in\mathbb{Z}$

08.05.06.0030.01

$$F(z \mid m) \propto \left(\frac{1}{\sqrt{m}}K\left(\frac{1}{m}\right)+2\left(2\operatorname{Re}\left(\frac{z_0}{2\pi}-\frac{3}{4}\right)+1\right)K(m)\right)\left(1-e^{-\pi i\left(\left\lfloor \frac{\arg(z-z_0)}{2\pi}+\frac{1}{4}\right\rfloor+\left\lfloor \frac{1}{4}-\frac{\arg(z-z_0)}{2\pi}\right\rfloor\right)}\right)+F(z_0 \mid m)\,e^{-\pi i\left(\left\lfloor \frac{\arg(z-z_0)}{2\pi}+\frac{1}{4}\right\rfloor+\left\lfloor \frac{1}{4}-\frac{\arg(z-z_0)}{2\pi}\right\rfloor\right)}+ \frac{1}{\sqrt{1-m\sin^2(z_0)}}(z-z_0)+\frac{m\sin(2z_0)}{4\left(1-m\sin^2(z_0)\right)^{3/2}}(z-z_0)^2+\ldots\;/;\;(z\to z_0)\bigwedge\operatorname{Re}\left(\frac{z_0}{2\pi}-\frac{3}{4}\right)\in\mathbb{Z}$$

08.05.06.0031.01

$$F(z \mid m) = \left(\frac{1}{\sqrt{m}}K\left(\frac{1}{m}\right)+2\left(2\operatorname{Re}\left(\frac{z_0}{2\pi}-\frac{3}{4}\right)+1\right)K(m)\right)\left(1-e^{-\pi i\left(\left\lfloor \frac{\arg(z-z_0)}{2\pi}+\frac{1}{4}\right\rfloor+\left\lfloor \frac{1}{4}-\frac{\arg(z-z_0)}{2\pi}\right\rfloor\right)}\right)+ F(z_0 \mid m)\,e^{-\pi i\left(\left\lfloor \frac{\arg(z-z_0)}{2\pi}+\frac{1}{4}\right\rfloor+\left\lfloor \frac{1}{4}-\frac{\arg(z-z_0)}{2\pi}\right\rfloor\right)}+\sum_{k=1}^{\infty}\frac{1}{k!} \sum_{j=1}^{k}\frac{1}{j!}\sum_{q=0}^{j-1}\binom{j}{q}\sum_{p=0}^{j-q}(-1)^q\,2^{q-j}\sin^q(z_0)\,(2p+q-j)^k\,e^{-\frac{1}{2}i(\pi(j+k-2p-q)+2(2p+q-j)z_0)}\binom{j-q}{p}\sum_{i=0}^{j-1}\frac{(1-j)_{2(j-i)-2}}{(j-i-1)!\,(2\sin(z_0))^{j-2i-1}} \sum_{s=0}^{i}\binom{i}{s}\left(\frac{1}{2}\right)_s\left(\frac{1}{2}\right)_{i-s}m^{i-s}\cos^{-2s-1}(z_0)\left(1-m\sin^2(z_0)\right)^{s-i-\frac{1}{2}}(z-z_0)^k\;/;\;\operatorname{Re}\left(\frac{z_0}{2\pi}-\frac{3}{4}\right)\in\mathbb{Z}$$

08.05.06.0032.01

$$F(z\mid m)\propto \left(\left(\frac{1}{\sqrt{m}}K\left(\frac{1}{m}\right)+2\left(2\operatorname{Re}\left(\frac{z_0}{2\pi}-\frac{3}{4}\right)+1\right)K(m)\right)\left(1-e^{-\pi i\left(\left\lfloor\frac{\arg(z-z_0)}{2\pi}+\frac{1}{4}\right\rfloor+\left\lfloor\frac{1}{4}-\frac{\arg(z-z_0)}{2\pi}\right\rfloor\right)}\right)+F(z_0\mid m)\,e^{-\pi i\left(\left\lfloor\frac{\arg(z-z_0)}{2\pi}+\frac{1}{4}\right\rfloor+\left\lfloor\frac{1}{4}-\frac{\arg(z-z_0)}{2\pi}\right\rfloor\right)}\right)(1+O(z-z_0))\;/;\;(z\to z_0)\bigwedge \operatorname{Re}\left(\frac{z_0}{2\pi}-\frac{3}{4}\right)\in\mathbb{Z}$$

Expansions at $z == 0$

08.05.06.0033.01

$$F(z\mid m)\propto z+\frac{m z^3}{6}+\frac{m(-4+9m)z^5}{120}+\frac{m\left(16-180m+225m^2\right)z^7}{5040}+\frac{m\left(-64+3024m-12\,600m^2+11\,025m^3\right)z^9}{362\,880}\ldots\;/;\;(z\to 0)$$

08.05.06.0001.02

$$F(z\mid m)\propto z+\frac{m z^3}{6}+\frac{m(-4+9m)z^5}{120}+\frac{m\left(16-180m+225m^2\right)z^7}{5040}+\frac{m\left(-64+3024m-12\,600m^2+11\,025m^3\right)z^9}{362\,880}+O\left(z^{11}\right)$$

08.05.06.0034.01

$$F(z\mid m)==z+\sum_{k=1}^{\infty}\frac{i^{k-1}}{k!}\binom{k-\frac{1}{2}}{k-1}\sum_{q=1}^{k-1}\frac{(-1)^q}{2q+1}\binom{k-1}{q}\sum_{j=0}^{q}\binom{q}{j}m^j(2-m)^{q-j}\,2^{k-j-q-1}\sum_{i=0}^{j}\binom{j}{i}(2i-j)^{k-1}z^k$$

08.05.06.0035.01

$$F(z\mid m)==z+\sum_{i=1}^{\infty}\sum_{k=1}^{\infty}\sum_{j=0}^{k-1}\frac{\left(\frac{1}{2}\right)_k\binom{2k}{j}(-1)^{i-j+k}\,2^{2i-2k+1}\,(j-k)^{2i}\,m^k}{k!\,(2i+1)!}z^{2i+1}$$

08.05.06.0036.01

$$F(z\mid m)\propto z+O\left(z^3\right)$$

Expansions at $z == \csc^{-1}\left(\sqrt{m}\right)+\pi u\;/;\;u\in\mathbb{Z}$

08.05.06.0037.01

$$F(z\mid m)\propto \frac{1}{\sqrt{m}}K\left(\frac{1}{m}\right)+2u\,K(m)-\frac{\sqrt{2}\sqrt{-(z-z_0)\sqrt{m-1}}}{\sqrt{m-1}}\left(1-\frac{m-2}{12\sqrt{m-1}}(z-z_0)+\frac{9m^2-4m+4}{480(m-1)}(z-z_0)^2-\frac{15m^3-26m^2-12m+8}{2688(m-1)^{3/2}}(z-z_0)^3+\ldots\right)/;\;(z\to z_0)\bigwedge z_0=\csc^{-1}\left(\sqrt{m}\right)+\pi u\bigwedge u\in\mathbb{Z}$$

08.05.06.0038.01

$$F(z\mid m)==\frac{1}{\sqrt{m}}K\left(\frac{1}{m}\right)+2u\,K(m)-\frac{\sqrt{2}}{\sqrt{m-1}}\sqrt{-(z-z_0)\sqrt{m-1}}\sum_{k=0}^{\infty}\frac{1}{2k+1}\binom{k+\frac{1}{2}}{k}\sum_{j=0}^{k}\frac{(-1)^j}{2j+1}\binom{k}{j}p_{j,k}(z-z_0)^k\;/;$$

$$z_0==\csc^{-1}\left(\sqrt{m}\right)+\pi u\bigwedge u\in\mathbb{Z}\bigwedge a_0==1\bigwedge a_{2k}==\frac{(-1)^k\,2^{2k}}{(2k+1)!}\bigwedge$$

$$a_{2k+1}==\frac{(-1)^{k-1}\,2^{2k}\,(2-m)}{\sqrt{m-1}\,(2k+2)!}\bigwedge k\in\mathbb{N}\bigwedge p_{u,0}==1\bigwedge p_{u,v}==\frac{1}{v}\sum_{j=1}^{v}(u\,j+j-v)\,a_j\,p_{u,v-j}$$

08.05.06.0039.01

$$F(z\mid m)\propto\frac{1}{\sqrt{m}}K\left(\frac{1}{m}\right)+2\,u\,K(m)-\frac{\sqrt{2}\,\sqrt{-(z-z_0)\sqrt{m-1}}}{\sqrt{m-1}}\,(1+O(z-z_0))\;/;\;(z\to z_0)\bigwedge z_0=\csc^{-1}\left(\sqrt{m}\right)+\pi\,u\bigwedge u\in\mathbb{Z}$$

Expansions at $z == -\csc^{-1}\left(\sqrt{m}\right)+\pi\,u\;/;\;u\in\mathbb{Z}$

08.05.06.0040.01

$$F(z\mid m)\propto-\frac{1}{\sqrt{m}}K\left(\frac{1}{m}\right)+2\,u\,K(m)+\frac{\sqrt{2}\,\sqrt{(z-z_0)\sqrt{m-1}}}{\sqrt{m-1}}\left(1+\frac{m-2}{12\sqrt{m-1}}(z-z_0)+\frac{9\,m^2-4\,m+4}{480\,(m-1)}(z-z_0)^2+\frac{15\,m^3-26\,m^2-12\,m+8}{2688\,(m-1)^{3/2}}(z-z_0)^3+\ldots\right)/;$$

$$(z\to z_0)\bigwedge z_0=-\csc^{-1}\left(\sqrt{m}\right)+\pi\,u\bigwedge u\in\mathbb{Z}$$

08.05.06.0041.01

$$F(z\mid m)==-\frac{1}{\sqrt{m}}K\left(\frac{1}{m}\right)+2\,u\,K(m)+\frac{\sqrt{2}}{\sqrt{m-1}}\sqrt{(z-z_0)\sqrt{m-1}}\sum_{k=0}^{\infty}\frac{(-1)^k}{2\,k+1}\binom{k+\frac{1}{2}}{k}\sum_{j=0}^{k}\frac{(-1)^j}{2\,j+1}\binom{k}{j}p_{j,k}\,(z-z_0)^k\;/;$$

$$z_0==-\csc^{-1}\left(\sqrt{m}\right)+\pi\,u\bigwedge u\in\mathbb{Z}\bigwedge a_0==1\bigwedge a_{2\,k}==\frac{(-1)^k\,2^{2\,k}}{(2\,k+1)!}\bigwedge$$

$$a_{2\,k+1}==\frac{(-1)^{k-1}\,2^{2\,k}\,(2-m)}{\sqrt{m-1}\,(2\,k+2)!}\bigwedge k\in\mathbb{N}\bigwedge p_{u,0}==1\bigwedge p_{u,v}==\frac{1}{v}\sum_{j=1}^{v}(u\,j+j-v)\,a_j\,p_{u,v-j}$$

08.05.06.0042.01

$$F(z\mid m)\propto-\frac{1}{\sqrt{m}}K\left(\frac{1}{m}\right)+2\,u\,K(m)-\frac{\sqrt{2}\,\sqrt{(z-z_0)\sqrt{m-1}}}{\sqrt{m-1}}\,(1+O(z-z_0))\;/;\;(z\to z_0)\bigwedge z_0=-\csc^{-1}\left(\sqrt{m}\right)+\pi\,u\bigwedge u\in\mathbb{Z}$$

Expansions at $z == \pi/2+2\,\pi\,u\;/;\;u\in\mathbb{Z}\bigwedge m>1$

08.05.06.0043.01

$$F(z\mid m)\propto\left(2\left\lfloor\frac{\mathrm{Re}(z_0)}{\pi}\right\rfloor+\sqrt{-\frac{i}{z-z_0}}\,\sqrt{i\,(z-z_0)}+\frac{\sqrt{(z-z_0)^2}}{z-z_0}+i\,(z-z_0)\sqrt{-\frac{1}{(z-z_0)^2}}\right)K(m)-$$

$$\frac{1}{\sqrt{m}}\left(i\sqrt{-\frac{1}{(z-z_0)^2}}\,(z-z_0)+\frac{\sqrt{(z-z_0)^2}}{z-z_0}\right)K\left(\frac{1}{m}\right)+\frac{z-z_0}{\sqrt{1-m}}-\frac{m\,(z-z_0)^3}{6\,(1-m)^{3/2}}+\ldots\;/;$$

$$(z\to z_0)\bigwedge z_0==\frac{\pi}{2}+2\,\pi\,u\bigwedge u\in\mathbb{Z}\bigwedge m\in\mathbb{R}\bigwedge m>1$$

08.05.06.0044.01

$$F(z \mid m) = \left(2\left\lfloor \frac{\mathrm{Re}(z_0)}{\pi} \right\rceil + (-1)^{\left\lfloor \frac{1}{2} - \frac{\arg(z-z_0)}{\pi} \right\rfloor} - (-1)^{\left\lfloor \frac{\arg(z-z_0)}{\pi} \right\rfloor} + (-1)^{\left\lfloor \frac{\arg(z-z_0)}{2\pi} + \frac{3}{4} \right\rfloor + \left\lfloor \frac{1}{4} - \frac{\arg(z-z_0)}{2\pi} \right\rfloor}\right) K(m) - \frac{1}{\sqrt{m}}\left(-(-1)^{\left\lfloor \frac{\arg(z-z_0)}{\pi} \right\rfloor} + (-1)^{\left\lfloor \frac{1}{2} - \frac{\arg(z-z_0)}{\pi} \right\rfloor}\right) K\left(\frac{1}{m}\right) + \frac{z - z_0}{\sqrt{1-m}} + \frac{1}{\sqrt{1-m}} \sum_{k=2}^{\infty} \frac{i^{k-1}}{k!} \binom{k-\frac{1}{2}}{k-1} \sum_{q=1}^{k-1} \frac{(-1)^q}{2q+1} \binom{k-1}{q} (1-m)^{-q} \sum_{j=0}^{q} \binom{q}{j} (-m)^j (2-m)^{q-j} 2^{k-j-q-1} \sum_{i=0}^{j} \binom{j}{i} (2i-j)^{k-1} (z-z_0)^k \;/;\; (z \to z_0) \bigwedge z_0 = \frac{\pi}{2} + 2\pi u \bigwedge u \in \mathbb{Z} \bigwedge m \in \mathbb{R} \bigwedge m > 1$$

08.05.06.0045.01

$$F(z \mid m) = \left(2\left\lfloor \frac{\mathrm{Re}(z_0)}{\pi} \right\rceil + \sqrt{-\frac{i}{z-z_0}} \sqrt{i(z-z_0)} + \frac{\sqrt{(z-z_0)^2}}{z-z_0} + i(z-z_0)\sqrt{-\frac{1}{(z-z_0)^2}}\right) K(m) - \frac{1}{\sqrt{m}}\left(i\sqrt{-\frac{1}{(z-z_0)^2}}(z-z_0) + \frac{\sqrt{(z-z_0)^2}}{z-z_0}\right) K\left(\frac{1}{m}\right) + \frac{z-z_0}{\sqrt{1-m}} + \frac{1}{\sqrt{1-m}} \sum_{k=2}^{\infty} \frac{i^{k-1}}{k!} \binom{k-\frac{1}{2}}{k-1} \sum_{q=1}^{k-1} \frac{(-1)^q}{2q+1} \binom{k-1}{q} (1-m)^{-q} \sum_{j=0}^{q} \binom{q}{j} (-m)^j (2-m)^{q-j} 2^{k-j-q-1} \sum_{i=0}^{j} \binom{j}{i} (2i-j)^{k-1} (z-z_0)^k \;/;\; (z \to z_0) \bigwedge z_0 = \frac{\pi}{2} + 2\pi u \bigwedge u \in \mathbb{Z} \bigwedge m \in \mathbb{R} \bigwedge m > 1$$

08.05.06.0046.01

$$F(z \mid m) \propto \left(\left(2\left\lfloor \frac{\mathrm{Re}(z_0)}{\pi} \right\rceil + \sqrt{-\frac{i}{z-z_0}} \sqrt{i(z-z_0)} + \frac{\sqrt{(z-z_0)^2}}{z-z_0} + i(z-z_0)\sqrt{-\frac{1}{(z-z_0)^2}}\right) K(m) - \frac{1}{\sqrt{m}}\left(i\sqrt{-\frac{1}{(z-z_0)^2}}(z-z_0) + \frac{\sqrt{(z-z_0)^2}}{z-z_0}\right) K\left(\frac{1}{m}\right)\right)(1 + O(z-z_0)) \;/;\; z_0 = \frac{\pi}{2} + 2\pi u \bigwedge u \in \mathbb{Z} \bigwedge m \in \mathbb{R} \bigwedge m > 1$$

Expansions at $z = 3\pi/2 + 2\pi u \;/;\; u \in \mathbb{Z} \bigwedge m > 1$

08.05.06.0047.01

$$F(z \mid m) \propto \left(2\left\lfloor \frac{\mathrm{Re}(z_0)}{\pi} \right\rceil - \sqrt{\frac{i}{z-z_0}} \sqrt{-i(z-z_0)} + \frac{\sqrt{(z-z_0)^2}}{z-z_0} + i(z-z_0)\sqrt{-\frac{1}{(z-z_0)^2}}\right) K(m) - \frac{1}{\sqrt{m}}\left(i\sqrt{-\frac{1}{(z-z_0)^2}}(z-z_0) + \frac{\sqrt{(z-z_0)^2}}{z-z_0}\right) K\left(\frac{1}{m}\right) + \frac{z-z_0}{\sqrt{1-m}} - \frac{m(z-z_0)^3}{6(1-m)^{3/2}} + \ldots \;/;\; (z \to z_0) \bigwedge z_0 = \frac{3\pi}{2} + 2\pi u \bigwedge u \in \mathbb{Z} \bigwedge m \in \mathbb{R} \bigwedge m > 1$$

08.05.06.0048.01

$$F(z\mid m)=\left(2\left\lceil\frac{\operatorname{Re}(z_0)}{\pi}\right\rceil+(-1)^{\left\lfloor\frac{1}{2}-\frac{\arg(z-z_0)}{\pi}\right\rfloor}-(-1)^{\left\lfloor\frac{\arg(z-z_0)}{\pi}\right\rfloor}-(-1)^{\left\lfloor\left\lfloor\frac{\arg(z-z_0)}{2\pi}+\frac{1}{4}\right\rfloor+\left\lfloor\frac{3}{4}-\frac{\arg(z-z_0)}{2\pi}\right\rfloor\right\rfloor}\right)K(m)-\frac{1}{\sqrt{m}}\left(-(-1)^{\left\lfloor\frac{\arg(z-z_0)}{\pi}\right\rfloor}+(-1)^{\left\lfloor\frac{1}{2}-\frac{\arg(z-z_0)}{\pi}\right\rfloor}\right)K\left(\frac{1}{m}\right)+\frac{z-z_0}{\sqrt{1-m}}+\frac{1}{\sqrt{1-m}}\sum_{k=2}^{\infty}\frac{i^{k-1}}{k!}\binom{k-\frac{1}{2}}{k-1}\sum_{q=1}^{k-1}\frac{(-1)^q}{2q+1}\binom{k-1}{q}(1-m)^{-q}\sum_{j=0}^{q}\binom{q}{j}(-m)^j(2-m)^{q-j}2^{k-j-q-1}\sum_{i=0}^{j}\binom{j}{i}(2i-j)^{k-1}(z-z_0)^k\;/;\;(z\to z_0)\bigwedge z_0=\frac{3\pi}{2}+2\pi u\bigwedge u\in\mathbb{Z}\bigwedge m\in\mathbb{R}\bigwedge m>1$$

08.05.06.0049.01

$$F(z\mid m)=\left(2\left\lceil\frac{\operatorname{Re}(z_0)}{\pi}\right\rceil-\sqrt{\frac{i}{z-z_0}}\sqrt{-i(z-z_0)}+\frac{\sqrt{(z-z_0)^2}}{z-z_0}+i(z-z_0)\sqrt{-\frac{1}{(z-z_0)^2}}\right)K(m)-\frac{1}{\sqrt{m}}\left(i\sqrt{-\frac{1}{(z-z_0)^2}}(z-z_0)+\frac{\sqrt{(z-z_0)^2}}{z-z_0}\right)K\left(\frac{1}{m}\right)+\frac{z-z_0}{\sqrt{1-m}}+\frac{1}{\sqrt{1-m}}\sum_{k=2}^{\infty}\frac{i^{k-1}}{k!}\binom{k-\frac{1}{2}}{k-1}\sum_{q=1}^{k-1}\frac{(-1)^q}{2q+1}\binom{k-1}{q}(1-m)^{-q}\sum_{j=0}^{q}\binom{q}{j}(-m)^j(2-m)^{q-j}2^{k-j-q-1}\sum_{i=0}^{j}\binom{j}{i}(2i-j)^{k-1}(z-z_0)^k\;/;\;(z\to z_0)\bigwedge z_0=\frac{3\pi}{2}+2\pi u\bigwedge u\in\mathbb{Z}\bigwedge m\in\mathbb{R}\bigwedge m>1$$

08.05.06.0050.01

$$F(z\mid m)\propto\left(\left(2\left\lceil\frac{\operatorname{Re}(z_0)}{\pi}\right\rceil-\sqrt{\frac{i}{z-z_0}}\sqrt{-i(z-z_0)}+\frac{\sqrt{(z-z_0)^2}}{z-z_0}+i(z-z_0)\sqrt{-\frac{1}{(z-z_0)^2}}\right)K(m)-\frac{1}{\sqrt{m}}\left(i\sqrt{-\frac{1}{(z-z_0)^2}}(z-z_0)+\frac{\sqrt{(z-z_0)^2}}{z-z_0}\right)K\left(\frac{1}{m}\right)\right)(1+O(z-z_0))\;/;\;z_0=\frac{3\pi}{2}+2\pi u\bigwedge u\in\mathbb{Z}\bigwedge m\in\mathbb{R}\bigwedge m>1$$

Expansions at $z = \infty$

08.05.06.0051.01

$$F(z\mid m)\propto 2\left\lceil\frac{\operatorname{Re}(z)}{\pi}\right\rfloor K(m)+(-1)^{\left\lceil\frac{\operatorname{Re}(z)}{\pi}\right\rfloor}\left(-\frac{\sqrt{-\sin^2(z)}}{\sin(z)}\left(K(1-m)+\frac{i}{2}\left(1-\sqrt{m}\sqrt{\frac{1}{m}}\right)\left(1-\frac{i\sqrt{-\sin^4(z)}}{\sin^2(z)}\right)K(m)\right)+\frac{\sqrt{-\sin^2(z)}}{\sin(z)\sqrt{-m\sin^2(z)}}\left(1+\frac{m+1}{6m}\csc^2(z)+\frac{3m^2+2m+3}{40m^2}\csc^4(z)+O\left(\csc^6(z)\right)\right)\right)\;/;\;(|z|\to\infty)$$

08.05.06.0052.01

$$F(z \mid m) = 2\left\lfloor \frac{\operatorname{Re}(z)}{\pi} \right\rceil K(m) + (-1)^{\left\lfloor \frac{\operatorname{Re}(z)}{\pi} \right\rceil} \left(-\frac{\sqrt{-\sin^2(z)}}{\sin(z)} \left(K(1-m) + \frac{i}{2}\left(1 - \sqrt{m}\sqrt{\frac{1}{m}}\right)\left(1 - \frac{i\sqrt{-\sin^4(z)}}{\sin^2(z)}\right) K(m) \right) + \frac{\sqrt{-\sin^2(z)}}{\sin(z)\sqrt{-m\sin^2(z)}} \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k \sin^{-2k}(z)}{(2k+1)\,k!}\, {}_2F_1\!\left(\frac{1}{2}, -k; \frac{1}{2}-k; \frac{1}{m}\right) \right)$$

08.05.06.0053.01

$$F(z \mid m) \propto 2\left\lfloor \frac{\operatorname{Re}(z)}{\pi} \right\rceil K(m) + (-1)^{\left\lfloor \frac{\operatorname{Re}(z)}{\pi} \right\rceil} \left(-\frac{\sqrt{-\sin^2(z)}}{\sin(z)} \left(K(1-m) + \frac{i}{2}\left(1 - \sqrt{m}\sqrt{\frac{1}{m}}\right)\left(1 - \frac{i\sqrt{-\sin^4(z)}}{\sin^2(z)}\right) K(m) \right) + \frac{\sqrt{-\sin^2(z)}}{\sin(z)\sqrt{-m\sin^2(z)}} \left(1 + O\!\left(\csc^2(z)\right)\right) \right) /;\ (|z| \to \infty)$$

Expansions at generic point $m == m_0$

For the function itself

08.05.06.0054.01

$$F(z \mid m) \propto F(z \mid m_0) + \frac{1}{4}\left(\frac{2\,E(z \mid m_0)}{m_0\,(1-m_0)} - \frac{2\,F(z \mid m_0)}{m_0} - \frac{\sin(2\,z)}{(1-m_0)\sqrt{1-m_0\sin^2(z)}} \right)(m-m_0) + \frac{1}{16\,(m_0-1)^2\,m_0^2\left(1-\sin^2(z)\,m_0\right)^{3/2}} \left(2\,(2\,E(z \mid m_0)\,(2\,m_0-1) + F(z \mid m_0)\,(m_0-1)\,(3\,m_0-2))\left(1-\sin^2(z)\,m_0\right)^{3/2} + \left(4\sin^2(z)\,m_0^2 - \left(2\sin^2(z)+3\right)m_0 + 1\right)\sin(2\,z)\,m_0 \right)(m-m_0)^2 + \ldots /;\ (m \to m_0)$$

08.05.06.0055.01

$$F(z \mid m) \propto F(z \mid m_0) + \frac{1}{4}\left(\frac{2\,E(z \mid m_0)}{m_0\,(1-m_0)} - \frac{2\,F(z \mid m_0)}{m_0} - \frac{\sin(2\,z)}{(1-m_0)\sqrt{1-m_0\sin^2(z)}} \right)(m-m_0) + \frac{1}{16\,(m_0-1)^2\,m_0^2\left(1-\sin^2(z)\,m_0\right)^{3/2}} \left(2\,(2\,E(z \mid m_0)\,(2\,m_0-1) + F(z \mid m_0)\,(m_0-1)\,(3\,m_0-2))\left(1-\sin^2(z)\,m_0\right)^{3/2} + \left(4\sin^2(z)\,m_0^2 - \left(2\sin^2(z)+3\right)m_0 + 1\right)\sin(2\,z)\,m_0 \right)(m-m_0)^2 + O\!\left((m-m_0)^3\right)$$

08.05.06.0056.01

$$F(z \mid m) == \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k \sin^{2k+1}(z)}{k!\,(2\,k+1)}\, F_1\!\left(k+\frac{1}{2}; \frac{1}{2}, k+\frac{1}{2}; k+\frac{3}{2}; \sin^2(z), m_0\sin^2(z)\right)(m-m_0)^k$$

08.05.06.0057.01

$$F(z \mid m) = \sum_{k=0}^{\infty} \frac{1}{k!}\, F^{(0,k)}(z \mid m_0)\,(m-m_0)^k$$

08.05.06.0058.01

$$F(z \mid m) \propto F(z \mid m_0)\,(1 + O(m - m_0))$$

Expansions at $m = 0$

08.05.06.0002.02

$$F(z \mid m) \propto z + \frac{2z - \sin(2z)}{8} m + \frac{3}{256}(12z - 8\sin(2z) + \sin(4z))\, m^2 + \ldots \;/;\; (m \to 0)$$

08.05.06.0059.01

$$F(z \mid m) \propto z + \frac{2z - \sin(2z)}{8} m + \frac{3}{256}(12z - 8\sin(2z) + \sin(4z))\, m^2 + O\big(m^3\big)$$

08.05.06.0004.02

$$F(z \mid m) = \sum_{k=0}^{\infty} \frac{1}{2^{2k}\, k!} \left(\frac{1}{2}\right)_k \left(z \binom{2k}{k} + \sum_{j=1}^{k} \frac{(-1)^j}{j} \binom{2k}{k-j} \sin(2jz) \right) m^k \;/;\; |m| < 1$$

08.05.06.0060.01

$$F(z \mid m) = -K(m) + \cos(z) \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{1}{2}\right)_k {}_2F_1\!\left(\frac{1}{2}, \frac{1}{2} - k; \frac{3}{2}; \cos^2(z)\right) m^k \;/;\; |m| < 1 \bigwedge -\frac{\pi}{2} \leq z \leq \frac{\pi}{2}$$

08.05.06.0061.01

$$F(z \mid m) = z + z \sum_{k=1}^{\infty} \left(\frac{\left(\frac{1}{2}\right)_k^2}{k!^2} + \frac{(-1)^k}{4^k\, k} \binom{k - \frac{1}{2}}{k} {}_2F_1\!\left(k + \frac{1}{2}, k + \frac{1}{2}; 2k + 1; m\right) \sin(2kz) \right) m^k \;/;\; |m| < 1$$

08.05.06.0003.02

$$F(z \mid m) = \frac{2z}{\pi} K(m) + \sum_{k=1}^{\infty} \frac{(-1)^k}{2^{2k}\, k} \binom{k - \frac{1}{2}}{k} {}_2F_1\!\left(k + \frac{1}{2}, k + \frac{1}{2}; 2k + 1; m\right) \sin(2kz)\, m^k$$

08.05.06.0005.02

$$F(z \mid m) = \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k \sin^{2k+1}(z)}{(2k+1)\, k!} {}_2F_1\!\left(\frac{1}{2}, k + \frac{1}{2}; k + \frac{3}{2}; \sin^2(z)\right) m^k + 2\left\lfloor \frac{\mathrm{Re}(z)}{\pi} \right\rceil K(m) \;/;\; |m| < 1$$

08.05.06.0006.01

$$F(z \mid m) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_j \left(\frac{1}{2}\right)_k \sin^{2j+2k+1}(z)}{(2j + 2k + 1)\, j!\, k!} m^k$$

08.05.06.0007.01

$$F(z \mid m) = \sin(z)\, F_{1\times 0\times 0}^{1\times 1\times 1}\!\left(\begin{matrix} \frac{1}{2}; \frac{1}{2}; \frac{1}{2}; \\ \frac{3}{2};;; \end{matrix} \; m \sin^2(z), \sin^2(z) \right)$$

08.05.06.0062.01

$$F(z \mid m) \propto z + O(m)$$

Expansions at $m = 1$

08.05.06.0063.01

$$F(z\mid m)\propto(-1)^{\left\lceil\frac{\operatorname{Re}(z)}{\pi}\right\rfloor}\left(\log(\sec(z)+\tan(z))+\frac{1}{4}(\sec(z)\tan(z)-\log(\sec(z)+\tan(z)))(m-1)+\frac{3}{256}\left(-5\sin(3z)\sec^4(z)+3\tan(z)\sec^3(z)+12\log(\sec(z)+\tan(z))\right)(m-1)^2+\ldots\right)-\left\lceil\frac{\operatorname{Re}(z)}{\pi}\right\rfloor\left(\left(1-\frac{m-1}{4}+\frac{9}{64}(m-1)^2+\ldots\right)\log(1-m)-4\log(2)+\frac{\log(4)-1}{2}(m-1)-\frac{9}{32}\left(\log(4)-\frac{7}{6}\right)(m-1)^2+\ldots\right)\;/;\;(m\to1)\bigwedge\neg\frac{2\operatorname{Re}(z)+\pi}{4\pi}\in\mathbb{Z}$$

08.05.06.0064.01

$$F(z\mid m)=(-1)^{\left\lceil\frac{\operatorname{Re}(z)}{\pi}\right\rfloor}\sum_{k=0}^{\infty}\frac{(-1)^k\left(\frac{1}{2}\right)_k^2}{(k!)^2}\left(\log(\sec(z)+\tan(z))+\frac{1}{2}\csc(z)\sum_{j=1}^{k}\frac{\left((-1)^j(j-1)!\right)\tan^{2j}(z)}{\left(\frac{1}{2}\right)_j}\right)(m-1)^k-\left\lceil\frac{\operatorname{Re}(z)}{\pi}\right\rfloor\left(\log(1-m)\sum_{k=0}^{\infty}\frac{(-1)^k\left(\frac{1}{2}\right)_k^2}{(k!)^2}(m-1)^k-2\sum_{k=0}^{\infty}\frac{(-1)^k\left(\frac{1}{2}\right)_k^2\left(\psi(k+1)-\psi\left(k+\frac{1}{2}\right)\right)}{(k!)^2}(m-1)^k\right)\;/;\;\neg\frac{2\operatorname{Re}(z)+\pi}{4\pi}\in\mathbb{Z}$$

08.05.06.0065.01

$$F(z\mid m)=2\left\lceil\frac{\operatorname{Re}(z)}{\pi}\right\rfloor K(m)+(-1)^{\operatorname{Round}\left(\frac{\operatorname{Re}(z)}{\pi}\right)}\sin(z)\sum_{k=0}^{\infty}\frac{\sin^{2k}(z)\left(\frac{1}{2}\right)_k}{(2k+1)\,k!}\,{}_2F_1\left(k+\frac{1}{2},k+1;k+\frac{3}{2};\sin^2(z)\right)(m-1)^k\;/;\;\neg\frac{2\operatorname{Re}(z)+\pi}{4\pi}\in\mathbb{Z}$$

08.05.06.0066.01

$$F(z\mid m)=\sum_{k=0}^{\infty}\frac{(-1)^k\left(\frac{1}{2}\right)_k^2}{(k!)^2}\left(\log(\sec(z)+\tan(z))+\frac{1}{2}\csc(z)\sum_{j=1}^{k}\frac{\left((-1)^j(j-1)!\right)\tan^{2j}(z)}{\left(\frac{1}{2}\right)_j}\right)(m-1)^k\;/;\;|\operatorname{Re}(z)|\leq\frac{\pi}{2}$$

08.05.06.0067.01

$$F(z\mid m)=\sin(z)\sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}\right)_k\sin^{2k}(z)}{(2k+1)\,k!}\,{}_2F_1\left(k+\frac{1}{2},k+1;k+\frac{3}{2};\sin^2(z)\right)(m-1)^k\;/;\;|\operatorname{Re}(z)|\leq\frac{\pi}{2}$$

08.05.06.0068.01

$$F(z\mid m)\propto\left((-1)^{\left\lceil\frac{\operatorname{Re}(z)}{\pi}\right\rfloor}\log(\sec(z)+\tan(z))-4\log(2)\left\lceil\frac{\operatorname{Re}(z)}{\pi}\right\rfloor\right)(1+O(m-1))-\log(1-m)\left\lceil\frac{\operatorname{Re}(z)}{\pi}\right\rfloor(1+O(m-1))\;/;\;(m\to1)\bigwedge\neg\frac{2\operatorname{Re}(z)+\pi}{4\pi}\in\mathbb{Z}$$

08.05.06.0069.01

$$F(z\mid m)\propto\log(\sec(z)+\tan(z))+O(m-1)\;/;\;|\operatorname{Re}(z)|\leq\frac{\pi}{2}$$

Expansions at $m = \infty$

08.05.06.0070.01

$$F(z\mid m)\propto\left\lfloor\frac{\operatorname{Re}(z)}{\pi}\right\rceil\left(\frac{\log(-m)}{\sqrt{-m}}\left(1+\frac{1}{4m}+\frac{9}{64m^2}+\ldots\right)+\frac{2}{\sqrt{-m}}\left(\log(4)+\frac{\log(4)-1}{4m}+\frac{3(6\log(4)-7)}{128m^2}+\ldots\right)\right)+(-1)^{\left\lfloor\frac{\operatorname{Re}(z)}{\pi}\right\rceil}\frac{\sqrt{-m\sin^2(z)}}{2m\sin(z)}\left(-\log\left(-4m\sin^2(z)\right)+2\log\left(\cos^2\left(\frac{z}{2}\right)\right)+\frac{2\cot(z)\csc(z)+2\log\left(\cos^2\left(\frac{z}{2}\right)\right)-\log\left(-4m\sin^2(z)\right)+2}{4m}+\frac{3\left(2\cos(z)\left(2\csc^2(z)+3\right)\csc^2(z)+6\log\left(\cos^2\left(\frac{z}{2}\right)\right)-3\log\left(-4m\sin^2(z)\right)+7\right)}{64m^2}+\ldots\right)/;\ (|m|\to\infty)$$

08.05.06.0071.01

$$F(z\mid m)=2\left\lfloor\frac{\operatorname{Re}(z)}{\pi}\right\rceil K(m)+(-1)^{\left\lfloor\frac{\operatorname{Re}(z)}{\pi}\right\rceil}\frac{\sqrt{-m\sin^2(z)}}{2m\sin(z)}\left(-\frac{2\log\left(-4m\sin^2(z)\right)}{\pi}K\left(\frac{1}{m}\right)+\frac{1}{4m}\,{}_4F_3\left(1,1,\frac{3}{2},\frac{3}{2};2,2,2;\frac{1}{m}\right)+\log(4)-2\log\left(1+\sqrt{1-\frac{\csc^2(z)}{m}}\right)-\frac{1}{2}\sin^2(z)\sum_{k=0}^{\infty}\frac{m^{-k}\left(\frac{1}{2}\right)_k\left(\frac{3}{2}\right)_k}{k!\,(k+1)!}\,{}_3F_2\left(1,1,k+\frac{3}{2};2,k+2;\sin^2(z)\right)+\frac{3\csc^2(z)}{8m^2}\sum_{k=0}^{\infty}\frac{m^{-k}\left(\frac{3}{2}\right)_k\left(\frac{5}{2}\right)_k}{(k+1)!}\,{}_3\tilde{F}_2\left(1,1,k+\frac{5}{2};k+3,2;\frac{\csc^2(z)}{m}\right)+\frac{9}{8m^2}\sum_{k=0}^{\infty}\frac{m^{-k}\left(\frac{5}{2}\right)_k^2}{((k+2)!)^2}\sum_{i=0}^{k}\frac{2}{i+k+3}\right)$$

08.05.06.0072.01

$$F(z\mid m)=2\left\lfloor\frac{\operatorname{Re}(z)}{\pi}\right\rceil K(m)+(-1)^{\left\lfloor\frac{\operatorname{Re}(z)}{\pi}\right\rceil}\frac{\sqrt{-m\sin^2(z)}}{2m\sin(z)}\left(-2\log\left(1+\sqrt{1-\frac{\csc^2(z)}{m}}\right)-\log\left(-m\sin^2(z)\right)+\sum_{k=1}^{\infty}\frac{\left(\frac{1}{2}\right)_k m^{-k}}{k!}\left(2m\sin^2(z)\sum_{i=0}^{k-1}\frac{m^i\left(-\frac{1}{2}\right)_{k-i}}{(i+1)\,(k-i-1)!}\sin^{2i}(z)-\frac{1}{k!}\left(\frac{1}{2}\right)_k\left(\log\left(-4m\sin^2(z)\right)-\sum_{i=0}^{k-2}\frac{2}{i+k+1}-\frac{1}{k}\right)+\frac{\left(\frac{3}{2}\right)_k}{2m\sin^2(z)}\,{}_3\tilde{F}_2\left(1,1,k+\frac{3}{2};k+2,2;\frac{\csc^2(z)}{m}\right)\right)\right)$$

08.05.06.0073.01

$$F(z\mid m)=2\left\lfloor\frac{\operatorname{Re}(z)}{\pi}\right\rceil K(m)+(-1)^{\left\lfloor\frac{\operatorname{Re}(z)}{\pi}\right\rceil}\frac{\sqrt{-m\sin^2(z)}}{2m\sin(z)}\sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}\right)_k m^{-k}}{k!}\left(2m\sin^2(z)\sum_{i=0}^{k-1}\frac{m^i\,i!\left(-\frac{1}{2}\right)_{k-i}\sin^{2i}(z)}{(i+1)!\,(k-i-1)!}+\frac{\left(\frac{3}{2}\right)_k}{2m\sin^2(z)}\,{}_3\tilde{F}_2\left(1,1,k+\frac{3}{2};k+2,2;\frac{\csc^2(z)}{m}\right)-\frac{\left(\frac{1}{2}\right)_k}{k!}\left(\log\left(-m\sin^2(z)\right)+\psi(k+1)-\psi\left(k+\frac{1}{2}\right)\right)\right)$$

08.05.06.0074.01

$$F(z\mid m)=2\left\lfloor\frac{\operatorname{Re}(z)}{\pi}\right\rceil K(m)+(-1)^{\left\lfloor\frac{\operatorname{Re}(z)}{\pi}\right\rceil}\sin(z)\sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}\right)_k\sin^{2k}(z)}{k!\,(2k+1)}\,{}_2F_1\left(k+\frac{1}{2},\frac{1}{2};k+\frac{3}{2};m\sin^2(z)\right)$$

08.05.06.0075.01

$$F(z\mid m)\propto\frac{(-1)^{\left\lfloor\frac{\mathrm{Re}(z)}{\pi}\right\rceil}\sqrt{-m\sin^2(z)}}{2\,m\sin(z)}\left(-\log\left(-4\,m\sin^2(z)\right)+2\log\left(\cos^2\left(\frac{z}{2}\right)\right)\left(1+O\left(\frac{1}{m}\right)\right)\right)+\left\lfloor\frac{\mathrm{Re}(z)}{\pi}\right\rceil\left(\frac{\log(-m)}{\sqrt{-m}}\left(1+O\left(\frac{1}{m}\right)\right)+\frac{2\log(4)}{\sqrt{-m}}\left(1+O\left(\frac{1}{m}\right)\right)\right)\;/;\;(|m|\to\infty)$$

Residue representations

08.05.06.0008.01

$$F(z\mid m)=\frac{\sin(z)}{2\,\pi}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\mathrm{res}_{s,t}\left(\frac{\Gamma\left(\frac{1}{2}-s-t\right)\Gamma(s)\,\Gamma\left(\frac{1}{2}-s\right)\Gamma(t)\,\Gamma\left(\frac{1}{2}-t\right)}{\Gamma\left(\frac{3}{2}-s-t\right)}\left(-\sin^2(z)\right)^{-s}\left(-m\sin^2(z)\right)^{-t}\right)(-j,-k)$$

Other series representations

Expansions $F\left(\sin^{-1}(z)\mid m\right)$ at $z=0$

08.05.06.0009.02

$$F\left(\sin^{-1}(z)\mid m\right)\propto z+\frac{m+1}{6}z^3+\frac{3+2\,m+3\,m^2}{40}z^5+\ldots\;/;\;(z\to 0)$$

08.05.06.0076.01

$$F\left(\sin^{-1}(z)\mid m\right)\propto z+\frac{m+1}{6}z^3+\frac{3+2\,m+3\,m^2}{40}z^5+O\left(z^7\right)$$

08.05.06.0010.01

$$F\left(\sin^{-1}(z)\mid m\right)=\sum_{k=0}^{\infty}\frac{m^k\left(\frac{1}{2}\right)_k}{(2\,k+1)\,k!}\,{}_2F_1\left(\frac{1}{2},-k;\frac{1}{2}-k;\frac{1}{m}\right)z^{2k+1}\;/;\;|z|<1$$

08.05.06.0077.01

$$F\left(\sin^{-1}(z)\mid m\right)=z\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{\left(\frac{1}{2}\right)_{j+k}\left(\frac{1}{2}\right)_k\left(\frac{1}{2}\right)_j m^j}{\left(\frac{3}{2}\right)_{j+k}k!\,j!}z^{2j+2k}\;/;\;|z|<1$$

08.05.06.0078.01

$$F\left(\sin^{-1}(z)\mid m\right)\propto z+O\left(z^3\right)$$

Expansions $F\left(\sin^{-1}(z)\mid m\right)$ at $z=\infty$

08.05.06.0079.01

$$F\left(\sin^{-1}(z)\mid m\right)\propto-\frac{\sqrt{-z^2}}{z}\left(K(1-m)+\frac{1}{2}\,i\left(1-\sqrt{\frac{1}{m}}\sqrt{m}\right)\left(1-\frac{i\sqrt{-z^4}}{z^2}\right)K(m)\right)-\frac{\sqrt{-z^2}\sqrt{-m\,z^2}}{m\,z^3}\left(1+\frac{m+1}{6\,m\,z^2}+\frac{3\,m^2+2\,m+3}{40\,m^2\,z^4}+\ldots\right)\;/;\;(|z|\to\infty)$$

08.05.06.0080.01

$$F\left(\sin^{-1}(z)\,\middle|\,m\right)\propto -\frac{\sqrt{-z^2}}{z}\left(K(1-m)+\frac{1}{2}\,i\left(1-\sqrt{\frac{1}{m}}\,\sqrt{m}\right)\left(1-\frac{i\sqrt{-z^4}}{z^2}\right)K(m)\right)-\frac{\sqrt{-z^2}\,\sqrt{-m\,z^2}}{m\,z^3}\left(1+\frac{m+1}{6\,m\,z^2}+\frac{3\,m^2+2\,m+3}{40\,m^2\,z^4}+O\left(\frac{1}{z^6}\right)\right)$$

08.05.06.0081.01

$$F\left(\sin^{-1}(z)\,\middle|\,m\right)=-\frac{\sqrt{-z^2}}{z}\left(K(1-m)+\frac{i}{2}\left(1-\sqrt{m}\,\sqrt{\frac{1}{m}}\right)\left(1-\frac{i\sqrt{-z^4}}{z^2}\right)K(m)\right)-\frac{\sqrt{1-z^2}\,\sqrt{1-m\,z^2}}{\sqrt{m}\,\sqrt{1-\frac{1}{z^2}}\,\sqrt{1-\frac{1}{m\,z^2}}\,z^2}\,F\left(\sin^{-1}\left(\frac{1}{\sqrt{m}\,z}\right)\,\middle|\,m\right)$$

08.05.06.0082.01

$$F\left(\sin^{-1}(z)\,\middle|\,m\right)=-\frac{\sqrt{-z^2}}{z}\left(K(1-m)+\frac{1}{2}\,i\left(1-\sqrt{m}\,\sqrt{\frac{1}{m}}\right)\left(1-\frac{i\sqrt{-z^4}}{z^2}\right)K(m)\right)-\frac{\sqrt{-z^2}\,\sqrt{-m\,z^2}}{m\,z^3}\sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}\right)_k}{(2\,k+1)\,k!}\,{}_2F_1\left(\frac{1}{2},-k;\frac{1}{2}-k;\frac{1}{m}\right)z^{-2\,k}\;/;\;|z|>1$$

08.05.06.0083.01

$$F\left(\sin^{-1}(z)\,\middle|\,m\right)\propto -\frac{\sqrt{-z^2}}{z}\left(K(1-m)+\frac{1}{2}\,i\left(1-\sqrt{\frac{1}{m}}\,\sqrt{m}\right)\left(1-\frac{i\sqrt{-z^4}}{z^2}\right)K(m)\right)-\frac{\sqrt{-z^2}\,\sqrt{-m\,z^2}}{m\,z^3}\left(1+O\left(\frac{1}{z^2}\right)\right)$$

Expansions $F\left(\sin^{-1}(z)\,\middle|\,m\right)$ at $m == 0$

08.05.06.0011.01

$$F\left(\sin^{-1}(z)\,\middle|\,m\right)\propto\sin^{-1}(z)-\frac{1}{4}\left(z\sqrt{1-z^2}-\sin^{-1}(z)\right)m-\frac{3}{64}\left(z\left(2\,z^2+3\right)\sqrt{1-z^2}-3\sin^{-1}(z)\right)m^2+\ldots\;/;\;|m|<1$$

08.05.06.0084.01

$$F\left(\sin^{-1}(z)\,\middle|\,m\right)\propto\sin^{-1}(z)-\frac{1}{4}\left(z\sqrt{1-z^2}-\sin^{-1}(z)\right)m-\frac{3}{64}\left(z\left(2\,z^2+3\right)\sqrt{1-z^2}-3\sin^{-1}(z)\right)m^2+O\left(m^3\right)$$

08.05.06.0012.01

$$F\left(\sin^{-1}(z)\,\middle|\,m\right)=\sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}\right)_k z^{2\,k+1}}{(2\,k+1)\,k!}\,{}_2F_1\left(\frac{1}{2},k+\frac{1}{2};k+\frac{3}{2};z^2\right)m^k\;/;\;|m|<1$$

08.05.06.0013.01

$$F\left(\sin^{-1}(z)\,\middle|\,m\right)=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{z^{2\,j+2\,k+1}}{(2\,j+2\,k+1)\,j!\,k!}\left(\frac{1}{2}\right)_j\left(\frac{1}{2}\right)_k m^k$$

08.05.06.0014.01

$$F\left(\sin^{-1}(z)\,\middle|\,m\right)=z\,F_{1\times 0\times 0}^{1\times 1\times 1}\left(\begin{matrix}\frac{1}{2};\frac{1}{2};\frac{1}{2};\\ \frac{3}{2};;;\end{matrix}\;m\,z^2,z^2\right)$$

Expansions $F\left(\sin^{-1}(z)\,\middle|\,m\right)$ at $m == 1$

08.05.06.0085.01

$$F\left(\sin^{-1}(z)\,\middle|\,m\right)\propto\tanh^{-1}(z)-\frac{z\left(z+(z-1)\tanh^{-1}(z)\right)}{4\,(z-1)}\,(m-1)-\frac{9\,z^2}{32}\left(3\,z+\left(z^2-3\right)\tanh^{-1}(z)\right)(m-1)^2+\ldots\;/;\;(m\to 1)$$

08.05.06.0086.01

$$F\left(\sin^{-1}(z)\,\middle|\,m\right)\propto\tanh^{-1}(z)-\frac{z\left(z+(z-1)\tanh^{-1}(z)\right)}{4\,(z-1)}\,(m-1)-\frac{9\,z^2}{32}\left(3\,z+\left(z^2-3\right)\tanh^{-1}(z)\right)(m-1)^2+O\left((m-1)^3\right)$$

08.05.06.0087.01

$$F\left(\sin^{-1}(z)\,\middle|\,m\right)=\sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}\right)_k}{(k!)^2}\sum_{p=0}^{k}\frac{(-1)^{k-p}\,(p)_{2(k-p)}}{2^{2k-p}\,(k-p)!}$$
$$\left((-1)^p\tanh^{-1}(z)\,p!+\frac{2\,z}{1-z^2}\sum_{j=0}^{p}(-1)^{p-j}\binom{p}{j}(p-j)!\sum_{q=0}^{j-1}\frac{2^{2q-j}\,q!\,(2\,q+2-j)_{2(j-q-1)}}{(j-q-1)!}\left(\frac{z^2}{1-z^2}\right)^q\right)(m-1)^k$$

08.05.06.0088.01

$$F\left(\sin^{-1}(z)\,\middle|\,m\right)=z\sum_{k=0}^{\infty}\frac{z^{2k}\left(\frac{1}{2}\right)_k}{(2\,k+1)\,k!}\,{}_2F_1\left(k+\frac{1}{2},k+1;k+\frac{3}{2};z^2\right)(m-1)^k$$

08.05.06.0089.01

$$F\left(\sin^{-1}(z)\,\middle|\,m\right)\propto\tanh^{-1}(z)\,(1+O(m-1))$$

Other expansions

08.05.06.0090.01

$$F(z\mid m)=\frac{\sin(z)}{\sqrt{\sin^2(z)}}\left(K(m)-\cos(z)\sum_{k=0}^{\infty}\frac{m^k\left(\frac{1}{2}\right)_k}{k!}\,{}_2F_1\left(\frac{1}{2},\frac{1}{2}-k;\frac{3}{2};\cos^2(z)\right)\right)+2\left\lfloor\frac{\operatorname{Re}(z)}{\pi}\right\rceil K(m)\;/;\;|\cos(z)|<1$$

08.05.06.0091.01

$$F(z\mid m)=\frac{2\,z}{\pi}K(m)+\sum_{k=1}^{\infty}\frac{\left(\frac{1}{2}\right)_k}{k!}\left(\frac{m}{4}\right)^k\sum_{j=0}^{k-1}\frac{(-1)^{k-j}\binom{2\,k}{j}}{j-k}\sin(2\,(j-k)\,z)$$

Integral representations

On the real axis

Of the direct function

08.05.07.0001.01

$$F(z\mid m)=\int_0^z\frac{1}{\sqrt{1-m\sin^2(t)}}\,dt\;/;\;-\frac{\pi}{2}\le z\le\frac{\pi}{2}$$

08.05.07.0004.01

$$F\,(z\mid m)=(-1)^{\left\lfloor\frac{\operatorname{Re}(z)}{\pi}\right\rceil}\int_0^z\frac{\cos(t)}{\sqrt{1-m\sin^2(t)}\,\sqrt{\cos^2(t)}}\,dt+2\left\lfloor\frac{\operatorname{Re}(z)}{\pi}\right\rceil K(m)$$

08.05.07.0002.01

$$F(z\mid m)=\int_0^{\sin(z)}\frac{1}{\sqrt{1-t^2}\,\sqrt{1-m\,t^2}}\,dt\,/;\,-\frac{\pi}{2}\leq z\leq\frac{\pi}{2}$$

08.05.07.0005.01

$$F(z\mid m)=(-1)^{\left\lfloor\frac{\mathrm{Re}(z)}{\pi}\right\rceil}\int_0^{\sin(z)}\frac{1}{\sqrt{1-t^2}\,\sqrt{1-m\,t^2}}\,dt+2\left\lfloor\frac{\mathrm{Re}(z)}{\pi}\right\rceil K(m)$$

Contour integral representations

08.05.07.0003.01

$$F(z\mid m)=\frac{\sin(z)}{2\pi\,(2\pi\,i)^2}\int_{\mathcal{L}^*}\int_{\mathcal{L}}\frac{\Gamma\left(\frac{1}{2}-s-t\right)\Gamma(s)\,\Gamma\left(\frac{1}{2}-s\right)\Gamma(t)\,\Gamma\left(\frac{1}{2}-t\right)}{\Gamma\left(\frac{3}{2}-s-t\right)}\left(-\sin^2(z)\right)^{-s}\left(-m\sin^2(z)\right)^{-t}\,ds\,dt$$

Differential equations

Ordinary linear differential equations and wronskians

For the direct function itself

08.05.13.0003.01

$$(1-m)\,m\,w''(m)+(1-m)\,w'(m)+\frac{1}{4}\,w(m)=\frac{\sin(2\,z)}{8\sqrt{1-m\sin^2(z)}}\,/;\,w(m)=c_1\,E(m)\,+c_2(K(1-m)-E(1-m))\,+E(z\mid m)$$

08.05.13.0001.01

$$(1-m)\,m\,\frac{\partial^2 w(m)}{\partial m^2}+(1-2\,m)\,\frac{\partial w(m)}{\partial m}-\frac{w(m)}{4}=-\frac{\sin(2\,z)}{8\left(1-m\sin^2(z)\right)^{3/2}}\,/;\,w(m)=F(z\mid m)$$

08.05.13.0004.01

$$W_m(K(m),\,K(1-m))=\frac{\pi}{4\,m\,(m-1)}$$

Ordinary nonlinear differential equations

08.05.13.0002.01

$$\left(\frac{\partial w(z)}{\partial z}-1\right)\left(\frac{\partial w(z)}{\partial z}\right)^2\left(\frac{\partial w(z)}{\partial z}+1\right)\left((m-1)\left(\frac{\partial w(z)}{\partial z}\right)^2+1\right)-\left(\frac{\partial^2 w(z)}{\partial z^2}\right)^2=0\,/;\,w(z)=F(z\mid m)$$

Transformations

Transformations and argument simplifications

Argument involving basic arithmetic operations

08.05.16.0001.01

$$F(-z\mid m)=-F(z\mid m)$$

08.05.16.0002.01

$$F(z+\pi\,k\mid m)=F(z\mid m)+2\,k\,K(m)\,/;\,k\in\mathbb{Z}$$

Argument involving inverse trigonometric functions

08.05.16.0003.01

$$F\left(\sin^{-1}(z) \mid m\right) = \mathrm{sn}^{-1}(z \mid m) \;/;\; -1 < z < 1 \land m < 1$$

08.05.16.0004.01

$$F\left(\cos^{-1}(z) \mid m\right) = \mathrm{cn}^{-1}(z \mid m) \;/;\; -1 < z < 1 \land m \in \mathbb{R}$$

08.05.16.0005.01

$$F\left(i \sinh^{-1}(z) \mid m\right) = i\, \mathrm{sc}^{-1}(z \mid 1 - m) \;/;\; m > 0 \land m \in \mathbb{R}$$

08.05.16.0006.01

$$F\left(\sin^{-1}\left(\sqrt{m}\, \sin(z)\right) \,\middle|\, \frac{1}{m}\right) = \sqrt{m}\, F(z \mid m) \;/;\; -\frac{\pi}{2} < z < \frac{\pi}{2}$$

08.05.16.0007.01

$$F\left(\frac{1}{2} \sin^{-1}\left(\frac{\sqrt{m}\, \tan(z) + \sqrt{(1-m)\tan^4(z) + \tan^2(z)}}{\tan^2(z) + 1}\right) \,\middle|\, \frac{4\sqrt{m}}{\left(\sqrt{m} + 1\right)^2}\right) = \frac{\sqrt{m} + 1}{2} F(z \mid m) \;/;\; 0 \le m < 1 \land 0 \le z < 1$$

Products, sums, and powers of the direct function

Sums of the direct function

08.05.16.0008.01

$$F(z_1 \mid m) + F(z_2 \mid m) = F\left(\sin^{-1}\left(\frac{\cos(z_2)\sqrt{1 - m\sin^2(z_2)}\, \sin(z_1) + \cos(z_1)\sqrt{1 - m\sin^2(z_1)}\, \sin(z_2)}{1 - m\sin^2(z_1)\sin^2(z_2)}\right) \,\middle|\, m\right) \;/;$$

$$0 \le m < 1 \bigwedge |z_1| < 1 \bigwedge |z_2| < 1$$

Identities

Functional identities

08.05.17.0001.01

$$F(z \mid m) = \frac{1}{\sqrt{m}} F\left(\sin^{-1}\left(\sqrt{m}\, \sin(z)\right) \,\middle|\, \frac{1}{m}\right) + 2\left\lfloor \frac{\mathrm{Re}(z)}{\pi} \right\rceil K(m)$$

08.05.17.0002.01

$$F(z \mid m) = \frac{2}{\sqrt{m} + 1} F\left(\frac{1}{2} \sin^{-1}\left(\frac{\sqrt{m}\, \tan(z) + \sqrt{(1-m)\tan^4(z) + \tan^2(z)}}{\tan^2(z) + 1}\right) \,\middle|\, \frac{4\sqrt{m}}{\left(\sqrt{m} + 1\right)^2}\right) \;/;\; 0 \le m < 1 \land 0 \le z < 1$$

Differentiation

Low-order differentiation

With respect to *z*

08.05.20.0001.01

$$\frac{\partial F(z \mid m)}{\partial z} = \frac{1}{\sqrt{1 - m \sin^2(z)}}$$

08.05.20.0002.01

$$\frac{\partial^2 F(z \mid m)}{\partial z^2} = \frac{m \sin(2 z)}{2 \left(1 - m \sin^2(z)\right)^{3/2}}$$

With respect to *m*

08.05.20.0003.01

$$\frac{\partial F(z \mid m)}{\partial m} = \frac{E(z \mid m)}{2 (1 - m) m} - \frac{F(z \mid m)}{2 m} - \frac{\sin(2 z)}{4 (1 - m) \sqrt{1 - m \sin^2(z)}}$$

08.05.20.0004.01

$$\frac{\partial^2 F(z \mid m)}{\partial m^2} = \frac{1}{16 (m - 1)^2 m^2} \left(8 (2 m - 1) E(z \mid m) + 4 (3 m - 2) (m - 1) F(z \mid m) + \frac{m \left(m^2 + 2 (3 m - 1) \sin^2(z) m - (m - 1) \cos(2 z) m - 7 m + 2\right) \sin(2 z)}{\left(1 - m \sin^2(z)\right)^{3/2}}\right)$$

Symbolic differentiation

With respect to *z*

08.05.20.0012.01

$$\frac{\partial^n F(z \mid m)}{\partial z^n} = \delta_n F(z \mid m) + \frac{\delta_{n-1}}{\sqrt{1 - m \sin^2(z)}} + \frac{2 i^{n-1} \left(\frac{1}{2}\right)_n}{(n - 1)! \sqrt{1 - m \sin^2(z)}} \sum_{q=1}^{n-1} \frac{(-1)^q \binom{n-1}{q}}{2 q + 1} \left(1 - m \sin^2(z)\right)^{-q} \sum_{j=0}^{q} \binom{q}{j} m^j (2 - m)^{q-j} 2^{-j+n-q-1} \sum_{i=0}^{j} \binom{j}{i} (2 i - j)^{n-1} e^{2 (2 i - j) i z} /; n \in \mathbb{N}$$

08.05.20.0013.01

$$\frac{\partial^n F(z \mid m)}{\partial z^n} = F(z \mid m) \delta_n + \sum_{j=1}^{n} \frac{1}{j!} \sum_{q=0}^{j-1} \binom{j}{q} \sum_{p=0}^{j-q} (-1)^q 2^{q-j} \sin^q(z) (q + 2 p - j)^n e^{-\frac{1}{2} i (\pi (j+n-q-2 p) + 2 (-j+q+2 p) z)} \binom{j-q}{p} \sum_{i=0}^{j-1} \frac{(1 - j)_{2 (j-i)-2}}{(j - i - 1)! (2 \sin(z))^{j-2 i-1}} \sum_{s=0}^{i} \binom{i}{s} \left(\frac{1}{2}\right)_s \left(\frac{1}{2}\right)_{i-s} m^{i-s} \cos^{-2 s-1}(z) \left(1 - m \sin^2(z)\right)^{-i+s-\frac{1}{2}} /; n \in \mathbb{N}$$

08.05.20.0005.02

$$\frac{\partial^n F(z\mid m)}{\partial z^n} == \delta_n F(z\mid m) + \frac{i^{n-1}\, 2^{n-2}}{\sqrt{\frac{e^{2iz}\, m}{m-2\left(\sqrt{1-m}+1\right)}}\,\sqrt{1-m\sin^2(z)}} \sqrt{\frac{\left(1-e^{2iz}\right)m+2\sqrt{1-m}-2}{\sqrt{1-m}}} \sum_{k=0}^{n-1} \frac{e^{2ikz}\, \mathcal{S}_{n-1}^{(k)}\,(-m)^k}{\left(\left(e^{2iz}-1\right)m+2\sqrt{1-m}+2\right)^k}$$

$$\left(\frac{1}{2}\right)_k F_1\left(\frac{1}{2}; -\frac{1}{2}, \frac{1}{2}; \frac{1}{2}-k; \frac{\left(e^{2iz}-1\right)m+2\sqrt{1-m}+2}{-m+2\sqrt{1-m}+2}, \frac{\left(e^{2iz}-1\right)m+2\sqrt{1-m}+2}{4\sqrt{1-m}}\right) /; n \in \mathbb{N}$$

08.05.20.0006.02

$$\frac{\partial^n F(z\mid m)}{\partial z^n} == \delta_n F(z\mid m) + 2\left(\frac{1}{2}\right)_n \sum_{k=0}^{n-1} \frac{(-1)^k \left(1-m\sin^2(z)\right)^{-k-\frac{1}{2}}}{k!\,(2k+1)\,(n-k-1)!} \frac{\partial^{n-1}\left(1-m\sin^2(z)\right)^k}{\partial z^{n-1}} /; n \in \mathbb{N}$$

With respect to *m*

08.05.20.0007.02

$$\frac{\partial^n F(z\mid m)}{\partial m^n} == \frac{\left(-\frac{1}{2}\right)_n \sin^{2n+1}(z)}{2n+1} F_1\left(n+\frac{1}{2}; \frac{1}{2}, n+\frac{1}{2}; n+\frac{3}{2}; \sin^2(z), m\sin^2(z)\right) /; n \in \mathbb{N}$$

Fractional integro-differentiation

With respect to *z*

08.05.20.0008.01

$$\frac{\partial^\alpha F(z\mid m)}{\partial z^\alpha} == 2^\alpha \sqrt{\pi}\, z^{1-\alpha} \sum_{k=1}^{\infty} \frac{(-1)^k \left(\frac{1}{2}\right)_k}{k!}\, {}_2F_1\left(k+\frac{1}{2}, k+\frac{1}{2}; 2k+1; m\right) {}_1\tilde{F}_2\left(1; 1-\frac{\alpha}{2}, \frac{3-\alpha}{2}; -k^2 z^2\right)\left(\frac{m}{4}\right)^k + \frac{2 z^{1-\alpha}}{\pi\,\Gamma(2-\alpha)} K(m)$$

08.05.20.0009.01

$$\frac{\partial^\alpha F(z\mid m)}{\partial z^\alpha} == \sqrt{\pi}\, z^{1-\alpha}\, 2^{\alpha-1} \sum_{j=0}^{\infty}\sum_{l=0}^{\infty} \frac{(-1)^{j+l}\, 2^{-2j-2l}\, m^l}{(2j+2l+1)\, j!\, l!} \left(\frac{1}{2}\right)_j \left(\frac{1}{2}\right)_l$$

$$\sum_{p=0}^{j+l} (-1)^p \binom{2j+2l+1}{p} (2j+2l-2p+1)\, {}_1\tilde{F}_2\left(1; 1-\frac{\alpha}{2}, \frac{3-\alpha}{2}; -\frac{1}{4}\left((2j+2l-2p+1)^2 z^2\right)\right)$$

With respect to *m*

08.05.20.0010.01

$$\frac{\partial^\alpha F(z\mid m)}{\partial m^\alpha} == \frac{m^{-\alpha}\sin(z)\sqrt{\pi}}{2} \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k^2 \sin^{2k}(z)}{k!}\, {}_3\tilde{F}_2\left(\frac{1}{2}, 1, k+\frac{1}{2}; k+\frac{3}{2}, 1-\alpha; m\sin^2(z)\right)$$

08.05.20.0011.01

$$\frac{\partial^\alpha F(z\mid m)}{\partial m^\alpha} == \frac{m^{-\alpha}\sqrt{\pi}\,\sin(z)}{2}\, \tilde{F}_{2\times1\times0}^{1\times3\times2}\left(\begin{matrix} \frac{1}{2}; \frac{1}{2}; \frac{1}{2}, 1; \\ \frac{3}{2};; 1-\alpha; \end{matrix}\ \sin^2(z), m\sin^2(z)\right)$$

Integration

Indefinite integration

Involving only one direct function

08.05.21.0001.01

$$\int F(z \mid m)\, dz = \frac{z^2}{\pi} K(m) + \frac{1}{2} \sum_{k=1}^{\infty} \frac{(-1)^k (1 - \cos(2 k z))}{k!\, k^2} \left(\frac{1}{2}\right)_k {}_2F_1\left(k + \frac{1}{2}, k + \frac{1}{2}; 2 k + 1; m\right) \left(\frac{m}{4}\right)^k$$

Involving one direct function and elementary functions

Involving trigonometric functions

Involving sin

08.05.21.0002.01

$$\int \sin(z)\, F(z \mid m)\, dz = \frac{1}{\sqrt{m}} \tan^{-1}\left(\frac{\sqrt{2} \sqrt{m} \sin(z)}{\sqrt{\cos(2 z)\, m - m + 2}}\right) - \cos(z)\, F(z \mid m)$$

08.05.21.0003.01

$$\int \sin(2 z)\, F(z \mid m)\, dz = \frac{2 E(z \mid m) + (-\cos(2 z)\, m + m - 2)\, F(z \mid m)}{2 m}$$

08.05.21.0004.01

$$\int \frac{F(z \mid m)}{\sqrt{1 - m \sin^2(z)}}\, dz = \frac{1}{2} F(z \mid m)^2$$

08.05.21.0005.01

$$\int \frac{\sin(2 z)\, F(z \mid m)}{\sqrt{1 - m \sin^2(z)}}\, dz = \frac{2 z - \sqrt{2 \cos(2 z)\, m - 2 m + 4}\; F(z \mid m)}{m}$$

08.05.21.0006.01

$$\int \frac{\sin(2 z)\, F(z \mid m)}{\left(1 - m \sin^2(z)\right)^{3/2}}\, dz = \frac{2}{m} \left(\frac{\sqrt{2}\; F(z \mid m)}{\sqrt{\cos(2 z)\, m - m + 2}} - \frac{\tanh^{-1}\left(\sqrt{m - 1} \tan(z)\right)}{\sqrt{m - 1}}\right)$$

Involving cos

08.05.21.0007.01

$$\int \cos(z)\, F(z \mid m)\, dz = \frac{\log\left(\sqrt{2} \sqrt{m} \cos(z) + \sqrt{\cos(2 z)\, m - m + 2}\right)}{\sqrt{m}} + F(z \mid m) \sin(z)$$

Involving sinand cos

08.05.21.0008.01

$$\int \frac{\cos(z)\, F(z \mid m)}{\left(1 - m \sin^2(z)\right)^{3/2}}\, dz = \frac{\sqrt{2}\, F(z \mid m) \sin(z)}{\sqrt{\cos(2z)\, m - m + 2}} - \frac{1}{\sqrt{m-1}\,\sqrt{m}} \tanh^{-1}\left(\frac{\cos(z)}{\sqrt{\frac{m-1}{m}}}\right)$$

Involving cotand csc

08.05.21.0009.01

$$\int \cot(z) \csc(z)\, F(z \mid m)\, dz = -\csc(z)\left(F(z \mid m) + \tanh^{-1}\left(\frac{\sqrt{2}\, \cos(z)}{\sqrt{\cos(2z)\, m - m + 2}}\right) \sin(z)\right)$$

Involving tanand sec

08.05.21.0010.01

$$\int \sec(z) \tan(z)\, F(z \mid m)\, dz = F(z \mid m) \sec(z) - \frac{1}{\sqrt{1-m}} \tanh^{-1}\left(\frac{\sqrt{2 - 2m}\, \sin(z)}{\sqrt{\cos(2z)\, m - m + 2}}\right)$$

08.05.21.0011.01

$$\int \sec^2(z) \tan(z)\, F(z \mid m)\, dz = \frac{1}{4(m-1)}\left(\tan(z)\left(2(m-1)\, F(z \mid m) \tan(z) + \sqrt{2 \cos(2z)\, m - 2m + 4}\right) - 2\, E(z \mid m)\right)$$

Involving different trigonometric functions

08.05.21.0012.01

$$\int \frac{\cot(z) \csc(z)\, F(z \mid m)}{\sqrt{1 - m \sin^2(z)}}\, dz = -\frac{\sqrt{\cos(2z)\, m - m + 2}\, \csc(z)\, F(z \mid m)}{\sqrt{2}} - \log\left(\cos\left(\frac{z}{2}\right)\right) + \log\left(\sin\left(\frac{z}{2}\right)\right)$$

08.05.21.0013.01

$$\int \frac{\sec(z) \tan(z)\, F(z \mid m)}{\sqrt{1 - m \sin^2(z)}}\, dz = \frac{1}{2m-2}\left(-2 \log\left(\cos\left(\frac{z}{2}\right) - \sin\left(\frac{z}{2}\right)\right) + 2 \log\left(\cos\left(\frac{z}{2}\right) + \sin\left(\frac{z}{2}\right)\right) - \sqrt{2 \cos(2z)\, m - 2m + 4}\, F(z \mid m) \sec(z)\right)$$

Involving only one direct function with respect to *m*

08.05.21.0014.01

$$\int F(z \mid m)\, dm = \sqrt{2 \cos(2z)\, m - 2m + 4}\, \cot(z) + 2\, E(z \mid m) + 2(m-1)\, F(z \mid m)$$

Involving one direct function and elementary functions with respect to *m*

Involving power function

08.05.21.0015.01

$$\int m\, F\left(z \mid m^2\right) dm = \sqrt{1 - m^2 \sin^2(z)}\, \cot(z) + E\left(z \mid m^2\right) + \left(m^2 - 1\right) F\left(z \mid m^2\right)$$

Representations through more general functions

Through hypergeometric functions of two variables

08.05.26.0001.01

$$F(z \mid m) = (-1)^{\left\lfloor \frac{\operatorname{Re}(z)}{\pi} \right\rceil} \sin(z)\, F_{1\times 0\times 0}^{1\times 1\times 1}\left(\begin{matrix} \frac{1}{2}; \frac{1}{2}; \frac{1}{2}; \\ \frac{3}{2};;; \end{matrix} \; m \sin^2(z), \sin^2(z) \right) + 2 \left\lfloor \frac{\operatorname{Re}(z)}{\pi} \right\rceil K(m)$$

08.05.26.0006.01

$$F(z \mid m) = \sin(z)\, F_1\left(\frac{1}{2}; \frac{1}{2}, \frac{1}{2}; \frac{3}{2}; \sin^2(z), m \sin^2(z)\right) /; |\operatorname{Re}(z)| < \frac{\pi}{2}$$

08.05.26.0007.01

$$F(z \mid m) = 2 \left\lfloor \frac{\operatorname{Re}(z)}{\pi} \right\rceil K(m) + (-1)^{\left\lfloor \frac{\operatorname{Re}(z)}{\pi} \right\rceil} \sin(z)\, F_1\left(\frac{1}{2}; \frac{1}{2}, \frac{1}{2}; \frac{3}{2}; \sin^2(z), m \sin^2(z)\right)$$

Through Meijer *G*

Classical cases involving $\tan^{-1}$ in the arguments

08.05.26.0002.01

$$F\left(\tan^{-1}\left(\sqrt[4]{z}\right) \,\middle|\, 1 - \frac{1}{z}\right) = \frac{1}{4\pi} G_{2,2}^{2,2}\left(z \,\middle|\, \begin{matrix} 1, 1 \\ \frac{1}{2}, \frac{1}{2} \end{matrix}\right) /; z \notin (-\infty, 0)$$

Classical cases involving $\cot^{-1}$ in the arguments

08.05.26.0003.01

$$F\left(\cot^{-1}\left(\sqrt[4]{z}\right) \,\middle|\, 1 - z\right) = \frac{1}{4\pi} G_{2,2}^{2,2}\left(z \,\middle|\, \begin{matrix} \frac{1}{2}, \frac{1}{2} \\ 0, 0 \end{matrix}\right)$$

Through other functions

Involving some elliptic-type functions

08.05.26.0004.01

$$F(z \mid m) = \Pi(0; z \mid m)$$

08.05.26.0005.01

$$F\left(\cot^{-1}\left(\sqrt{\frac{2 z_1}{a + \sqrt{a^2 - 4b}}}\right) \,\middle|\, \frac{2\sqrt{a^2 - 4b}}{a + \sqrt{a^2 - 4b}}\right) = -2 \frac{1}{\sqrt{\frac{2\left(a - \sqrt{a^2 - 4b}\right)}{b}}} \operatorname{elog}\left(z_1, \sqrt{z_1^3 + a z_1^2 + b z_1}; a, b\right)$$

Representations through equivalent functions

With inverse function

08.05.27.0001.02

$$F(\operatorname{am}(z \mid m) \mid m) = z /; (m \leq 1 \land -2 \leq z \leq 2) \lor (|z| < 1 \land |m| \leq 2)$$

08.05.27.0002.02

$$\operatorname{am}(F(z \mid m) \mid m) = z$$

With related functions

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$$F(\phi \mid m) = \frac{\left(\sqrt{1-n}\,\cot(\phi) + E(\phi \mid m)\right) K(m) - \sqrt{1-n}\,\cot(\phi)\,\Pi(n \mid m)}{E(m)} \;/;\; \phi = \sin^{-1}\left(\sqrt{\frac{n}{m}}\right) \bigwedge 0 < n < 1 \bigwedge 0 < m < 1$$

Theorems

The electrostatic potential of an ellipsoid

The electrostatic potential $V(x, y, z)$ of an ellipsoid with semi-axes a, b, c of charge Q at the point $\{x, y, z\}$ is given by

$$V(x, y, z) \propto Q\sqrt{c^2 - a^2}\; F\left(\sin^{-1}\left(\frac{\sqrt{c^2 - a^2}}{\sqrt{c^2 + \xi}}\right) \,\middle|\, \frac{b^2 - c^2}{a^2 - c^2}\right) /;\; \frac{x^2}{a^2 + \xi} + \frac{y^2}{b^2 + \xi} + \frac{z^2}{c^2 + \xi} = 1.$$

History

– L. Euler (1733, 1757, 1763, 1766)
– J.-L. Lagrange (1783)
– A. M. Legendre (1793, 1811, 1825–1828)
– C. F. Gauss (1799, 1818)
– C. G. J. Jacobi (1827)
– J. Liouville (1840)

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