# EulerE



## Notations

### Traditional name

Euler number

### Traditional notation

$E_n$

### *Mathematica* StandardForm notation

EulerE[*n*]

## Primary definition

04.12.02.0001.01

$$E_n = 2^{n+1}\, n! \left( [t^n] \frac{e^{t/2}}{e^t + 1} \right) /;\ n \in \mathbb{N}$$

## Specific values

### Specialized values

04.12.03.0001.01

$$E_{2n+1} = 0 /;\ n \in \mathbb{N}$$

### Values at fixed points

04.12.03.0002.01

$$E_0 = 1$$

04.12.03.0003.01

$$E_1 = 0$$

04.12.03.0004.01

$$E_2 = -1$$

04.12.03.0005.01

$$E_3 = 0$$

04.12.03.0006.01

$$E_4 = 5$$

04.12.03.0007.01

$$E_5 = 0$$

04.12.03.0008.01

$$E_6 = -61$$

04.12.03.0009.01

$$E_7 = 0$$

04.12.03.0010.01

$$E_8 = 1385$$

04.12.03.0011.01

$$E_9 = 0$$

04.12.03.0012.01

$$E_{10} = -50\,521$$

# General characteristics

## Domain and analyticity

$E_n$ is a nonanalytical function which is defined only for nonnegative integer $n$.

04.12.04.0001.01

$$n \longrightarrow E_n :: \mathbb{N} \longrightarrow \mathbb{Z}$$

## Symmetries and periodicities

### Symmetry

No symmetry

### Periodicity

No periodicity

# Series representations

## Generalized power series

04.12.06.0001.01

$$E_n = \frac{2^{n+2}\, n!}{\pi^{n+1}} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^{n+1}} \cos\left(k\pi - \frac{n\pi}{2}\right) /; n \in \mathbb{N}^+$$

04.12.06.0002.01

$$E_{2n} = \frac{(-1)^n\, 2^{2n+2}\, (2n)!}{\pi^{2n+1}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2n+1}} /; n \in \mathbb{N}$$

04.12.06.0005.01

$$E_n = \sum_{j=0}^{n} \sum_{k=0}^{j} 2^{j-k} (-1)^k\, k!\, \mathcal{S}_j^{(k)} \binom{n}{j} /; n \in \mathbb{N}$$

Victor Adamchik

### Asymptotic series expansions

04.12.06.0003.01

$$E_{2n} \propto \frac{4\sqrt{\pi}}{2n+1} \sum_{k=1}^{n} (-1)^k \left(2^{2k}-1\right) \sqrt{k} \left(\frac{2k}{\pi e}\right)^{2k} \binom{2n+1}{2k} \left(1+O\left(\frac{1}{n}\right)\right) /; (n \to \infty)$$

04.12.06.0006.01

$$E_{2n} \propto (-1)^n \, 8 \sqrt{\frac{n}{\pi}} \left(\frac{4n}{\pi e}\right)^{2n} \left(1+O\left(\frac{1}{n}\right)\right) /; (n \to \infty)$$

### Residue representations

04.12.06.0004.01

$$E_n = n! \operatorname{res}_z\left(\operatorname{sech}(z) z^{-n-1}\right)(0)$$

## Integral representations

### On the real axis

#### Of the direct function

04.12.07.0001.01

$$E_{2n} = (-1)^n 2^{2n+1} \int_0^\infty t^{2n} \operatorname{sech}(\pi t) \, dt /; n \in \mathbb{N}$$

04.12.07.0002.01

$$E_{2n} = (-1)^n \left(\frac{2}{\pi}\right)^{2n+1} \int_0^\infty \frac{\log^{2n}(t)}{t^2+1} \, dt /; n \in \mathbb{N}$$

04.12.07.0003.01

$$E_{2n} = 2(-1)^n \left(\frac{2}{\pi}\right)^{2n+1} \int_0^1 \frac{\log^{2n}(t)}{t^2+1} \, dt /; n \in \mathbb{N}$$

### Contour integral representations

04.12.07.0004.01

$$E_n = \frac{n!}{2\pi i} \int_{|z|=1} \operatorname{sech}(z) z^{-n-1} \, dz /; n \in \mathbb{N}^+$$

## Limit representations

04.12.09.0001.01

$$E_n = \lim_{z \to 0} \frac{\partial^n \operatorname{sech}(z)}{\partial z^n}$$

## Generating functions

04.12.11.0001.01

$$E_n = 2^{n+1} n! \left([t^n] \frac{e^{t/2}}{e^t+1}\right) /; n \in \mathbb{N}$$

04.12.11.0002.01

$$E_n = 2\,n!\left([t^n]\,\frac{e^t}{e^{2t}+1}\right)/;\; n\in\mathbb{N}$$

04.12.11.0003.01

$$E_n = n!\,([t^n]\,\operatorname{sech}(t))\;/;\; n\in\mathbb{N}$$

04.12.11.0004.01

$$E_n = 2^n\,n!\left([t^n]\,\operatorname{sech}\left(\frac{t}{2}\right)\right)/;\; n\in\mathbb{N}$$

# Identities

## Identities involving determinants

04.12.17.0001.01

$$\left|(E_{k+l+2})_{0\le k\le n,\;0\le l\le n}\right| = \prod_{k=1}^{n}(2\,k)!^2$$

# Complex characteristics

## Real part

04.12.19.0001.01

$$\operatorname{Re}(E_n) = E_n$$

## Imaginary part

04.12.19.0002.01

$$\operatorname{Im}(E_n) = 0$$

## Absolute value

04.12.19.0003.01

$$|E_n| = \sqrt{E_n^2}$$

## Argument

04.12.19.0004.01

$$\arg(E_n) = \tan^{-1}(E_n,\,0)$$

## Conjugate value

04.12.19.0005.01

$$\overline{E_n} = E_n$$

## Signum value

04.12.19.0006.01

$$\operatorname{sgn}(E_n) = (-1)^{n/2}\left(1-n+2\left\lfloor\frac{n}{2}\right\rfloor\right)$$

# Summation

## Finite summation

04.12.23.0001.01

$$\sum_{k=0}^{n} \binom{2n}{2k} E_{2k} == 0 \;/;\; n \in \mathbb{N}^+$$

04.12.23.0002.01

$$\sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} E_k == -\frac{2^n}{n} (1-2^n) B_n \;/;\; n-1 \in \mathbb{N}^+$$

04.12.23.0003.01

$$\sum_{k=0}^{n} \binom{n}{k} E_k z^k == 2^n z^n \; E_n\left(\frac{z+1}{2z}\right) /;\; n \in \mathbb{N}$$

04.12.23.0005.01

$$\sum_{k=0}^{n} \binom{2n}{2k} E_{2k} E_{2n-2k} == \frac{\left(\left(2^{2n+2}-1\right) 2^{2n+2}\right)}{2n+2} B_{2n+2}$$

## Infinite summation

04.12.23.0006.01

$$\sum_{n=0}^{\infty} \frac{E_n z^n}{n!} == \operatorname{sech}(z) \;/;\; |z| < \frac{\pi}{2}$$

04.12.23.0007.01

$$\sum_{k=0}^{\infty} \frac{4^k E_{2k} z^{2k}}{(2k)!} == \operatorname{sech}(2z) \;/;\; |z| < \frac{\pi}{4}$$

# Representations through more general functions

## Through other functions

### Involving Stirling numbers

04.12.26.0001.01

$$E_n == -\frac{2^n}{\sqrt{\pi}} \sum_{m=0}^{n} (-1)^m \,\Gamma\left(m-\frac{1}{2}\right) {}_2F_1\left(1, \frac{3}{2}; \frac{3}{2}-m; -1\right) \mathcal{S}_n^{(m)}$$

### Involving zeta functions

04.12.26.0002.01

$$E_n == 2\,(2^n-1)\,\zeta(-n) + 4^{n+1}\,\zeta\left(-n, \frac{1}{4}\right) /;\; n \in \mathbb{N}$$

04.12.26.0003.01

$$E_n == \frac{(-1)^{n/2}\,\pi^{-n-1}\,n!}{2^n} \left(\zeta\left(n+1, \frac{1}{4}\right) - \zeta\left(n+1, \frac{3}{4}\right)\right) /;\; \frac{n}{2} \in \mathbb{N}^+$$

# Representations through equivalent functions

## With related functions

04.12.27.0001.01

$$E_n = 2^n E_n\left(\frac{1}{2}\right)$$

04.12.27.0002.01

$$E_n = \frac{2^{n+1}}{n+1}\left(B_{n+1}\left(\frac{1}{2}\right) - 2^{n+1} B_{n+1}\left(\frac{1}{4}\right)\right)$$

04.12.27.0003.01

$$E_n = \frac{2^{2n+1}}{n+1}\left(B_{n+1}\left(\frac{3}{4}\right) - B_{n+1}\left(\frac{1}{4}\right)\right)$$

04.12.27.0004.01

$$E_n = -\frac{1}{n+1}\sum_{k=0}^{n+1} 2^k \left(2^k - 1\right)\binom{n+1}{k} B_k \; /; \; n \in \mathbb{N}$$

04.12.27.0005.01

$$E_n = -\frac{2^{n+2}}{(n+1)(n+2)}\sum_{k=0}^{n}\binom{n+2}{k}\left(2^{n-k+2} - 1\right)\left(1 - 2^{1-k}\right) B_{n-k+2} B_k$$

# Inequalities

04.12.29.0001.01

$$\frac{4^{n+1}(2n)!}{\pi^{2n+1}\left(1 + 3^{-3n-1}\right)} < (-1)^n E_{2n} < \frac{4^{n+1}(2n)!}{\pi^{2n+1}} \; /; \; n \in \mathbb{Z} \land n \geq 0$$

# Zeros

04.12.30.0001.01

$$E_{2n+1} = 0 \; /; \; n \in \mathbb{N}$$

# History

– L. Euler (1755) use these numbers in series expansion for $\sec(z)$ near $z = 0$

– H. F. Scherk (1825) suggested the name and calculated the first 30 numbers

– G. Chrystal (1889) introduced modern notations

– L. Saalschütz (1893) found the relation between Euler and Bernoulli numbers

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