Fibonacci2General

Notations

Traditional name

Fibonacci function

Traditional notation

$F_\nu(z)$

Mathematica StandardForm notation

Fibonacci[ν, z]

Primary definition

07.06.02.0001.01

$$F_\nu(z) == \frac{2^{-\nu}\left(z+\sqrt{z^2+4}\right)^\nu - \cos(\nu\pi)\, 2^\nu \left(z+\sqrt{z^2+4}\right)^{-\nu}}{\sqrt{z^2+4}}$$

Specific values

Specialized values

For fixed ν

07.06.03.0001.01

$$F_\nu(0) == \sin^2\left(\frac{\pi\nu}{2}\right)$$

07.06.03.0002.01

$$F_\nu(1) == F_\nu$$

07.06.03.0003.01

$$F_\nu(-1) == F_{-\nu}$$

07.06.03.0004.01

$$F_n(-2\,i) == -(-i)^{n+1}\, n \,/;\, n \in \mathbb{Z}$$

07.06.03.0005.01

$$F_n(2\,i) == -\,i^{n+1}\, n \,/;\, n \in \mathbb{Z}$$

For fixed z

07.06.03.0006.01

$$F_0(z) = 0$$

07.06.03.0007.01

$$F_1(z) = 1$$

07.06.03.0008.01

$$F_2(z) = z$$

07.06.03.0009.01

$$F_3(z) = 1 + z^2$$

07.06.03.0010.01

$$F_4(z) = 2 z + z^3$$

07.06.03.0011.01

$$F_5(z) = 1 + 3 z^2 + z^4$$

07.06.03.0012.01

$$F_6(z) = 3 z + 4 z^3 + z^5$$

07.06.03.0013.01

$$F_7(z) = 1 + 6 z^2 + 5 z^4 + z^6$$

07.06.03.0014.01

$$F_8(z) = 4 z + 10 z^3 + 6 z^5 + z^7$$

07.06.03.0015.01

$$F_9(z) = 1 + 10 z^2 + 15 z^4 + 7 z^6 + z^8$$

07.06.03.0016.01

$$F_{10}(z) = 5 z + 20 z^3 + 21 z^5 + 8 z^7 + z^9$$

07.06.03.0017.01

$$F_n(z) = \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n-k-1}{k} z^{n-2k-1} \;/;\; n \in \mathbb{N}$$

07.06.03.0018.01

$$F_{-n}(z) = (-1)^{n-1} \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n-k-1}{k} z^{n-2k-1} \;/;\; n \in \mathbb{N}$$

General characteristics

Domain and analyticity

$F_\nu(z)$ is an analytical function of ν and z which is defined over $\mathbb{C}^2$. For integer ν, $F_\nu(z)$ degenerates to a polynomial in z. For fixed z, it is an entire function of ν.

07.06.04.0001.01

$$(\nu * z) \longrightarrow F_\nu(z) :: (\mathbb{C} \otimes \mathbb{C}) \longrightarrow \mathbb{C}$$

Symmetries and periodicities

Parity

07.06.04.0002.01

$F_n(-z) = (-1)^{n-1} F_n(z) /; n \in \mathbb{Z}$

07.06.04.0003.01

$F_{-n}(z) = (-1)^{n-1} F_n(z) /; n \in \mathbb{Z}$

Mirror symmetry

07.06.04.0004.01

$F_{\bar{\nu}}(\bar{z}) = \overline{F_\nu(z)} /; i z \notin (-\infty, -2) \wedge i z \notin (2, \infty)$

Periodicity

No periodicity

Poles and essential singularities

With respect to *z*

For fixed noninteger ν, the function $F_\nu(z)$ does not have poles and essential singular points.

07.06.04.0005.01

$\mathcal{Sing}_z(F_\nu(z)) = \{\} /; \nu \notin \mathbb{Z}$

For integer ν, the function $F_\nu(z)$ is polynomial and has pole of order $|\nu|$ at $z = \tilde{\infty}$.

07.06.04.0006.01

$\mathcal{Sing}_z(F_\nu(z)) = \{\{\tilde{\infty}, |\nu|\}\} /; \nu \in \mathbb{Z}$

With respect to ν

For fixed z, the function $F_\nu(z)$ has one singular point at $\nu = \tilde{\infty}$. It is an essential singular point.

07.06.04.0007.01

$\mathcal{Sing}_\nu(F_\nu(z)) = \{\{\tilde{\infty}, \infty\}\}$

Branch points

With respect to *z*

For fixed noninteger ν, the function $F_\nu(z)$ has three branch points: $z = \pm 2 i$, $z = \tilde{\infty}$.

07.06.04.0008.01

$\mathcal{BP}_z(F_\nu(z)) = \{2 i, -2 i, \tilde{\infty}\} /; \nu \notin \mathbb{Z}$

07.06.04.0009.01

$\mathcal{R}_z(F_\nu(z), 2 i) = 2 /; \nu \notin \mathbb{Z}$

07.06.04.0010.01

$\mathcal{R}_z(F_\nu(z), -2 i) = 2 /; \nu \notin \mathbb{Z}$

07.06.04.0011.01

$\mathcal{R}_z(F_\nu(z), \tilde{\infty}) = \log /; \nu \notin \mathbb{Q}$

07.06.04.0012.01

$\mathcal{R}_z\left(F_{\frac{p}{q}}(z), \tilde{\infty}\right) = 2 q /; p \in \mathbb{Z} \wedge q - 1 \in \mathbb{N}^+ \wedge \gcd(p, q) = 1$

With respect to ν

For fixed z, the function $F_\nu(z)$ does not have branch points.

07.06.04.0013.01

$$\mathcal{BP}_\nu(F_\nu(z)) = \{\}$$

Branch cuts

With respect to z

For fixed noninteger ν, the function $F_\nu(z)$ is a single-valued function on the z-plane cut along the intervals $(-i\infty, -2i]$ and $[2i, i\infty)$.

The function $F_\nu(z)$ is continuous from the left on the interval $(-i\infty, -2i]$ and from the right on the interval $[2i, i\infty)$.

07.06.04.0014.01

$$\mathcal{BC}_z(F_\nu(z)) = \{\{(-i\infty, -2i], 1\}, \{[2i, i\infty), -1\}\} /; \nu \notin \mathbb{Z}$$

07.06.04.0015.01

$$\mathcal{BC}_z(F_\nu(z)) = \{\} /; \nu \in \mathbb{Z}$$

07.06.04.0016.01

$$\lim_{\epsilon\to+0} F_\nu(x-\epsilon) = F_\nu(x) /; i\,x > 2$$

07.06.04.0017.01

$$\lim_{\epsilon\to+0} F_\nu(x+\epsilon) = \frac{i\sin(\pi\nu)\,2^{\nu+1}}{\sqrt{x^2+4}}\left(x+\sqrt{x^2+4}\right)^{-\nu} - e^{\pi i\nu} F_{-\nu}(x) /; i\,x > 2$$

07.06.04.0018.01

$$\lim_{\epsilon\to+0} F_\nu(x+\epsilon) = F_\nu(x) /; i\,x < -2$$

07.06.04.0019.01

$$\lim_{\epsilon\to+0} F_\nu(x-\epsilon) = -\frac{i\sin(2\pi\nu)\,2^{-\nu}}{\sqrt{x^2+4}}\left(x+\sqrt{x^2+4}\right)^{\nu} - e^{\pi i\nu} F_{-\nu}(x) /; i\,x < -2$$

With respect to ν

For fixed z, the function $F_\nu(z)$ does not have branch cuts.

07.06.04.0020.01

$$\mathcal{BC}_\nu(F_\nu(z)) = \{\}$$

Series representations

Generalized power series

Expansions at generic point $\nu = \nu_0$

For the function itself

07.06.06.0028.01

$$F_\nu(z) \propto F_{\nu_0}(z) + \frac{1}{\sqrt{z^2+4}}\left(2^{-\nu_0}\left(z+\sqrt{z^2+4}\right)^{-\nu_0}\left(\sinh^{-1}\left(\frac{z}{2}\right)\left(\left(z+\sqrt{z^2+4}\right)^{2\nu_0}+4^{\nu_0}\cos(\pi\nu_0)\right)+4^{\nu_0}\pi\sin(\pi\nu_0)\right)\right)(\nu-\nu_0)+ \frac{1}{2}\left(\frac{2^{\nu_0}\pi}{\sqrt{z^2+4}}\left(\pi\cos(\pi\nu_0)-2\sinh^{-1}\left(\frac{z}{2}\right)\sin(\pi\nu_0)\right)\left(z+\sqrt{z^2+4}\right)^{-\nu_0}+\sinh^{-1}\left(\frac{z}{2}\right)^2 F_{\nu_0}(z)\right)(\nu-\nu_0)^2+\ldots\ /;\ (\nu\to\nu_0)$$

07.06.06.0029.01

$$F_\nu(z) \propto F_{\nu_0}(z) + \frac{1}{\sqrt{z^2+4}}\left(2^{-\nu_0}\left(z+\sqrt{z^2+4}\right)^{-\nu_0}\left(\sinh^{-1}\left(\frac{z}{2}\right)\left(\left(z+\sqrt{z^2+4}\right)^{2\nu_0}+4^{\nu_0}\cos(\pi\nu_0)\right)+4^{\nu_0}\pi\sin(\pi\nu_0)\right)\right)(\nu-\nu_0)+ \frac{1}{2}\left(\frac{2^{\nu_0}\pi}{\sqrt{z^2+4}}\left(\pi\cos(\pi\nu_0)-2\sinh^{-1}\left(\frac{z}{2}\right)\sin(\pi\nu_0)\right)\left(z+\sqrt{z^2+4}\right)^{-\nu_0}+\sinh^{-1}\left(\frac{z}{2}\right)^2 F_{\nu_0}(z)\right)(\nu-\nu_0)^2+O\left((\nu-\nu_0)^3\right)$$

07.06.06.0030.01

$$F_\nu(z) = \sum_{k=0}^{\infty}\frac{1}{k!}\left(\sinh^{-1}\left(\frac{z}{2}\right)^k F_{\nu_0}(z)-\frac{2^{\nu_0-1}}{\sqrt{z^2+4}}\left(z+\sqrt{z^2+4}\right)^{-\nu_0} \left(-2\cos(\pi\nu_0)\sinh^{-1}\left(\frac{z}{2}\right)^k+(-1)^k e^{i\pi\nu_0}\left(-i\pi+\sinh^{-1}\left(\frac{z}{2}\right)\right)^k+(-1)^k e^{-i\pi\nu_0}\left(i\pi+\sinh^{-1}\left(\frac{z}{2}\right)\right)^k\right)\right)(\nu-\nu_0)^k$$

07.06.06.0031.01

$$F_\nu(z) \propto F_{\nu_0}(z)\,(1+O(\nu-\nu_0))$$

Expansions at $\nu == 0$

07.06.06.0001.01

$$F_\nu(z) \propto \frac{1}{\sqrt{z^2+4}}\left(2\log(w)\,\nu+\frac{\pi^2\nu^2}{2}+\frac{\log(w)\left(2\log^2(w)-3\pi^2\right)}{6}\nu^3+\ldots\right)/;\ w == \frac{1}{2}\left(z+\sqrt{z^2+4}\right)\bigwedge(\nu\to 0)$$

07.06.06.0032.01

$$F_\nu(z) \propto \frac{1}{\sqrt{z^2+4}}\left(2\log(w)\,\nu+\frac{\pi^2\nu^2}{2}+\frac{\log(w)\left(2\log^2(w)-3\pi^2\right)}{6}\nu^3+O\left(\nu^4\right)\right)/;\ w == \frac{1}{2}\left(z+\sqrt{z^2+4}\right)$$

07.06.06.0002.01

$$F_\nu(z) == \frac{1}{\sqrt{z^2+4}}\sum_{k=1}^{\infty}\frac{\log^k(w)}{k!}\left(1-\frac{(-1)^k}{2}\left(\left(1+\frac{i\pi}{\log(w)}\right)^k+\left(1-\frac{i\pi}{\log(w)}\right)^k\right)\right)\nu^k\ /;\ w == \frac{1}{2}\left(z+\sqrt{z^2+4}\right)$$

07.06.06.0003.01

$$F_\nu(z) \propto \frac{2}{\sqrt{z^2+4}}\log\left(\frac{1}{2}\left(z+\sqrt{z^2+4}\right)\right)\nu\,(1+O[\nu])\ /;\ (\nu\to 0)$$

Expansions at generic point $z == z_0$

For the function itself

07.06.06.0033.01

$$F_n(z) \propto F_n(z_0) + \sum_{j=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n-j-1}{j} z_0^{n-2j-2} (z-z_0) \left(1 + \frac{z-z_0}{2 z_0} + \ldots\right) /; (z \to z_0)$$

07.06.06.0034.01

$$F_n(z) \propto F_n(z_0) + \sum_{j=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n-j-1}{j} z_0^{n-2j-2} (z-z_0) \left(1 + \frac{z-z_0}{2 z_0} + O\left((z-z_0)^2\right)\right)$$

07.06.06.0035.01

$$F_n(z) = \sqrt{\pi} \sum_{k=0}^{n} \frac{1}{k!} \left(2^{k-2} n \cos^2\left(\frac{\pi n}{2}\right) z_0^{1-k} \, {}_3\tilde{F}_2\left(1, 1-\frac{n}{2}, \frac{n}{2}+1; 1-\frac{k}{2}, \frac{3-k}{2}; -\frac{z_0^2}{4}\right) + 2^k \sin^2\left(\frac{\pi n}{2}\right) z_0^{-k} \, {}_3\tilde{F}_2\left(1, \frac{1-n}{2}, \frac{n+1}{2}; \frac{1-k}{2}, 1-\frac{k}{2}; -\frac{z_0^2}{4}\right)\right) (z-z_0)^k$$

07.06.06.0036.01

$$F_n(z) = \sum_{k=0}^{n} \frac{1}{k!} \sum_{j=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n-j-1}{j} (n-2j-k)_k \, z_0^{n-2j-k-1} (z-z_0)^k$$

07.06.06.0037.01

$$F_n(z) \propto F_n(z_0)\,(1 + O(z-z_0))$$

Expansions at $z = 0$

07.06.06.0004.01

$$F_\nu(z) \propto \sin^2\left(\frac{\pi \nu}{2}\right) + \frac{\nu}{2} \cos^2\left(\frac{\pi \nu}{2}\right) z + \frac{\nu^2-1}{8} \sin^2\left(\frac{\pi \nu}{2}\right) z^2 + \frac{(\nu^2-4)\,\nu}{48} \cos^2\left(\frac{\pi \nu}{2}\right) z^3 + \ldots /; (z \to 0)$$

07.06.06.0038.01

$$F_\nu(z) \propto \sin^2\left(\frac{\pi \nu}{2}\right) + \frac{1}{2} \nu \cos^2\left(\frac{\pi \nu}{2}\right) z + \frac{1}{8} (\nu^2-1) \sin^2\left(\frac{\pi \nu}{2}\right) z^2 + \frac{1}{48} \left((\nu^2-4)\,\nu\right) \cos^2\left(\frac{\pi \nu}{2}\right) z^3 + O(z^4)$$

07.06.06.0005.01

$$F_\nu(z) = \sin^2\left(\frac{\pi \nu}{2}\right) \sum_{k=0}^{\infty} \frac{\left(\frac{1-\nu}{2}\right)_k \left(\frac{\nu+1}{2}\right)_k}{\left(\frac{1}{2}\right)_k k!} \left(-\frac{z^2}{4}\right)^k + \frac{\nu z}{2} \cos^2\left(\frac{\pi \nu}{2}\right) \sum_{k=0}^{\infty} \frac{\left(1-\frac{\nu}{2}\right)_k \left(\frac{\nu}{2}+1\right)_k}{\left(\frac{3}{2}\right)_k k!} \left(-\frac{z^2}{4}\right)^k /; |z| < 2$$

07.06.06.0006.01

$$F_\nu(z) = \sin^2\left(\frac{\pi \nu}{2}\right) {}_2F_1\left(\frac{1-\nu}{2}, \frac{\nu+1}{2}; \frac{1}{2}; -\frac{z^2}{4}\right) + \frac{\nu z}{2} \cos^2\left(\frac{\pi \nu}{2}\right) {}_2F_1\left(1-\frac{\nu}{2}, \frac{\nu}{2}+1; \frac{3}{2}; -\frac{z^2}{4}\right)$$

07.06.06.0007.01

$$F_\nu(z) \propto \sin^2\left(\frac{\pi \nu}{2}\right) + \frac{\nu}{2} \cos^2\left(\frac{\pi \nu}{2}\right) z\,(1 + O[z]) /; (z \to 0)$$

07.06.06.0008.01

$$F_n(z) = \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n-k-1}{k} z^{n-2k-1} /; n \in \mathbb{N}$$

Expansions at $z = 2i$

07.06.06.0009.01

$$F_\nu(z) \propto \frac{e^{-\frac{i\pi\nu}{2}}}{2}\left(\nu\,(\sin(\pi\,\nu) - 2\,i\cos(\pi\,\nu))\left(1 + \frac{i\left(1-\nu^2\right)}{6}\,(z-2\,i) + \frac{-\nu^4+5\,\nu^2-4}{120}\,(z-2\,i)^2 + \ldots\right) + \frac{i\sin(\pi\,\nu)}{\sqrt{i\,(z-2\,i)}}\left(1 + \frac{i\left(1-4\,\nu^2\right)}{8}\,(z-2\,i) + \frac{-16\,\nu^4+40\,\nu^2-9}{384}\,(z-2\,i)^2 + \ldots\right)\right) /;\ (z \to 2\,i)$$

07.06.06.0039.01

$$F_\nu(z) \propto \frac{e^{-\frac{i\pi\nu}{2}}}{2}\left(\nu\,(\sin(\pi\,\nu) - 2\,i\cos(\pi\,\nu))\left(1 + \frac{i\left(1-\nu^2\right)}{6}\,(z-2\,i) + \frac{-\nu^4+5\,\nu^2-4}{120}\,(z-2\,i)^2 + \ldots\right) + \frac{i\sin(\pi\,\nu)}{\sqrt{i\,(z-2\,i)}}\left(1 + \frac{i\left(1-4\,\nu^2\right)}{8}\,(z-2\,i) + \frac{-16\,\nu^4+40\,\nu^2-9}{384}\,(z-2\,i)^2 + O\!\left((z-2\,i)^3\right)\right)\right)$$

07.06.06.0010.01

$$F_\nu(z) = \frac{i\sin(\pi\,\nu)}{2\sqrt{i\,(z-2\,i)}}\,e^{-\frac{i\pi\nu}{2}}\sum_{k=0}^{\infty}\frac{\left(\frac{i}{4}\right)^k\left(\frac{1}{2}-\nu\right)_k\left(\nu+\frac{1}{2}\right)_k}{k!\left(\frac{1}{2}\right)_k}\,(z-2\,i)^k + \frac{\nu\,(\sin(\pi\,\nu) - 2\,i\cos(\pi\,\nu))}{2}\,e^{-\frac{i\pi\nu}{2}}\sum_{k=0}^{\infty}\frac{\left(\frac{i}{4}\right)^k(1-\nu)_k\,(\nu+1)_k}{k!\left(\frac{3}{2}\right)_k}\,(z-2\,i)^k /;\ |z-2\,i| < 4$$

07.06.06.0011.01

$$F_\nu(z) = \frac{\nu}{2}\,e^{-\frac{i\pi\nu}{2}}\,(\sin(\pi\,\nu) - 2\,i\cos(\pi\,\nu))\,{}_2F_1\!\left(1-\nu,\,\nu+1;\,\frac{3}{2};\,\frac{i\,(z-2\,i)}{4}\right) + \frac{i\sin(\pi\,\nu)}{2\sqrt{i\,(z-2\,i)}}\,e^{-\frac{i\pi\nu}{2}}\,{}_2F_1\!\left(\frac{1}{2}-\nu,\,\frac{1}{2}+\nu;\,\frac{1}{2};\,\frac{i\,(z-2\,i)}{4}\right)$$

07.06.06.0012.01

$$F_\nu(z) \propto \frac{1}{2}\,e^{-\frac{i\pi\nu}{2}}\left(\frac{i\sin(\pi\,\nu)}{\sqrt{i\,(z-2\,i)}}\,(1+O(z-2\,i)) + \nu\,(\sin(\pi\,\nu) - 2\,i\cos(\pi\,\nu))\,(1+O(z-2\,i))\right) /;\ (z \to 2\,i)$$

Expansions at $z = -2\,i$

07.06.06.0013.01

$$F_\nu(z) \propto \frac{1}{2}\,e^{\frac{i\pi\nu}{2}}\left(\nu\,(\sin(\pi\,\nu) + 2\,i\cos(\pi\,\nu))\left(1 + \frac{i\left(\nu^2-1\right)}{6}\,(z+2\,i) + \frac{-\nu^4+5\,\nu^2-4}{120}\,(z+2\,i)^2 + \ldots\right) - \frac{i\sin(\pi\,\nu)}{\sqrt{-i\,(z+2\,i)}}\left(1 + \frac{i\left(4\,\nu^2-1\right)}{8}\,(z+2\,i) + \frac{-16\,\nu^4+40\,\nu^2-9}{384}\,(z+2\,i)^2 + \ldots\right)\right) /;\ (z \to -2\,i)$$

07.06.06.0040.01

$$F_\nu(z) \propto \frac{1}{2}\,e^{\frac{i\pi\nu}{2}}\left(\nu\,(\sin(\pi\,\nu) + 2\,i\cos(\pi\,\nu))\left(1 + \frac{i\left(\nu^2-1\right)}{6}\,(z+2\,i) + \frac{-\nu^4+5\,\nu^2-4}{120}\,(z+2\,i)^2 + \ldots\right) - \frac{i\sin(\pi\,\nu)}{\sqrt{-i\,(z+2\,i)}}\left(1 + \frac{i\left(4\,\nu^2-1\right)}{8}\,(z+2\,i) + \frac{-16\,\nu^4+40\,\nu^2-9}{384}\,(z+2\,i)^2 + O\!\left((z+2\,i)^3\right)\right)\right)$$

07.06.06.0014.01

$$F_\nu(z) = \frac{\nu\,(2\,i\cos(\pi\,\nu)+\sin(\pi\,\nu))}{2}\,e^{\frac{i\pi\nu}{2}}\sum_{k=0}^{\infty}\frac{\left(-\frac{i}{4}\right)^k (1-\nu)_k\,(\nu+1)_k}{k!\left(\frac{3}{2}\right)_k}(2\,i+z)^k - \frac{i\sin(\pi\,\nu)}{2\sqrt{-i\,(z+2\,i)}}\,e^{\frac{i\pi\nu}{2}}\sum_{k=0}^{\infty}\frac{\left(-\frac{i}{4}\right)^k\left(\frac{1}{2}-\nu\right)_k\left(\nu+\frac{1}{2}\right)_k}{k!\left(\frac{1}{2}\right)_k}(2\,i+z)^k \;/;\; |z+2\,i|<4$$

07.06.06.0015.01

$$F_\nu(z) = \frac{\nu}{2}\,e^{\frac{i\pi\nu}{2}}\,(2\,i\cos(\pi\,\nu)+\sin(\pi\,\nu))\,{}_2F_1\!\left(1-\nu,\,\nu+1;\,\frac{3}{2};\,\frac{-i\,(z+2\,i)}{4}\right) - \frac{i\sin(\pi\,\nu)}{2\sqrt{-i\,(z+2\,i)}}\,e^{\frac{i\pi\nu}{2}}\,{}_2F_1\!\left(\frac{1}{2}-\nu,\,\nu+\frac{1}{2};\,\frac{1}{2};\,\frac{-i\,(z+2\,i)}{4}\right)$$

07.06.06.0016.01

$$F_\nu(z) \propto \frac{1}{2}\,e^{\frac{i\pi\nu}{2}}\left(\nu\,(\sin(\pi\,\nu)+2\,i\cos(\pi\,\nu))\,(1+O(z+2\,i)) - \frac{i\sin(\pi\,\nu)}{\sqrt{-i\,(z+2\,i)}}\,(1+O(z+2\,i))\right) \;/;\; (z\to-2\,i)$$

Expansions at $z == \infty$

07.06.06.0017.01

$$F_\nu(z) = \frac{1}{z}\left(z^2\right)^{-\frac{\nu+1}{2}}\left(z\sin^2\!\left(\frac{\pi\,\nu}{2}\right)-\sqrt{z^2}\,\cos^2\!\left(\frac{\pi\,\nu}{2}\right)\right)\left(1-\frac{2+\nu}{z^2}+\frac{\nu^2+7\,\nu+12}{2\,z^4}-\ldots\right) + \frac{1}{z}\left(z^2\right)^{\frac{\nu-1}{2}}\left(z\sin^2\!\left(\frac{\pi\,\nu}{2}\right)+\sqrt{z^2}\,\cos^2\!\left(\frac{\pi\,\nu}{2}\right)\right)\left(1-\frac{2-\nu}{z^2}+\frac{\nu^2-7\,\nu+12}{2\,z^4}-\ldots\right) \;/;\; \nu\notin\mathbb{Z}\bigwedge(|z|\to\infty)$$

07.06.06.0041.01

$$F_\nu(z) = \frac{1}{z}\left(z^2\right)^{-\frac{\nu+1}{2}}\left(z\sin^2\!\left(\frac{\pi\,\nu}{2}\right)-\sqrt{z^2}\,\cos^2\!\left(\frac{\pi\,\nu}{2}\right)\right)\left(1-\frac{2+\nu}{z^2}+\frac{\nu^2+7\,\nu+12}{2\,z^4}-\ldots\right) + \frac{1}{z}\left(z^2\right)^{\frac{\nu-1}{2}}\left(z\sin^2\!\left(\frac{\pi\,\nu}{2}\right)+\sqrt{z^2}\,\cos^2\!\left(\frac{\pi\,\nu}{2}\right)\right)\left(1-\frac{2-\nu}{z^2}+\frac{\nu^2-7\,\nu+12}{2\,z^4}+O\!\left(\frac{1}{z^6}\right)\right) \;/;\; \nu\notin\mathbb{Z}$$

07.06.06.0018.01

$$F_\nu(z) = \frac{1}{z}\left(z^2\right)^{\frac{\nu-1}{2}}\left(z\sin^2\!\left(\frac{\pi\,\nu}{2}\right)+\sqrt{z^2}\,\cos^2\!\left(\frac{\pi\,\nu}{2}\right)\right)\sum_{k=0}^{\infty}\frac{4^k\left(\frac{1-\nu}{2}\right)_k\left(1-\frac{\nu}{2}\right)_k}{k!\,(1-\nu)_k}\left(-\frac{1}{z^2}\right)^k + \frac{1}{z}\left(z^2\right)^{-\frac{\nu+1}{2}}\left(z\sin^2\!\left(\frac{\pi\,\nu}{2}\right)-\sqrt{z^2}\,\cos^2\!\left(\frac{\pi\,\nu}{2}\right)\right)\sum_{k=0}^{\infty}\frac{4^k\left(\frac{\nu}{2}+1\right)_k\left(\frac{\nu+1}{2}\right)_k}{k!\,(\nu+1)_k}\left(-\frac{1}{z^2}\right)^k \;/;\; |z|>2\bigwedge\nu\notin\mathbb{Z}$$

07.06.06.0019.01

$$F_\nu(z) = \frac{1}{z}\left(z^2\right)^{-\frac{\nu+1}{2}}\left(z\sin^2\!\left(\frac{\pi\,\nu}{2}\right)-\sqrt{z^2}\,\cos^2\!\left(\frac{\pi\,\nu}{2}\right)\right){}_2F_1\!\left(1+\frac{\nu}{2},\,\frac{1+\nu}{2};\,1+\nu;\,-\frac{4}{z^2}\right) + \frac{1}{z}\left(z^2\right)^{\frac{\nu-1}{2}}\left(z\sin^2\!\left(\frac{\pi\,\nu}{2}\right)+\sqrt{z^2}\,\cos^2\!\left(\frac{\pi\,\nu}{2}\right)\right){}_2F_1\!\left(1-\frac{\nu}{2},\,\frac{1-\nu}{2};\,1-\nu;\,-\frac{4}{z^2}\right) \;/;\; i\,z\notin(-2,0)\bigwedge i\,z\notin(0,2)\bigwedge\nu\notin\mathbb{Z}$$

07.06.06.0020.01

$$F_\nu(z) \propto \frac{1}{z}\left(z^2\right)^{\frac{\nu-1}{2}}\left(z\sin^2\left(\frac{\pi\nu}{2}\right)+\sqrt{z^2}\cos^2\left(\frac{\pi\nu}{2}\right)\right)\left(1+O\left(\frac{1}{z^2}\right)\right)+\frac{1}{z}\left(z^2\right)^{-\frac{\nu+1}{2}}\left(z\sin^2\left(\frac{\pi\nu}{2}\right)-\sqrt{z^2}\cos^2\left(\frac{\pi\nu}{2}\right)\right)\left(1+O\left(\frac{1}{z^2}\right)\right) /; (|z|\to\infty) \bigwedge \nu\notin\mathbb{Z}$$

07.06.06.0042.01

$$F_\nu(z) \propto \begin{cases} e^{i\pi\nu} z^{\nu-1}\cos(\pi\nu) - e^{-i\pi\nu} z^{-\nu-1} & \arg(z) \le -\frac{\pi}{2} \\ z^{\nu-1} - \cos(\pi\nu) z^{-\nu-1} & -\frac{\pi}{2} < \arg(z) \le \frac{\pi}{2} \\ e^{-i\pi\nu}\cos(\pi\nu) z^{\nu-1} - e^{i\pi\nu} z^{-\nu-1} & \text{True} \end{cases} /; (|z|\to\infty) \bigwedge \nu\notin\mathbb{Z}$$

07.06.06.0021.01

$$F_n(z) \propto z^{n-1}\left(1+\frac{n-2}{z^2}+\frac{(n-4)(n-3)}{2z^4}+\ldots\right) /; (|z|\to\infty)\land n\in\mathbb{N}$$

07.06.06.0022.01

$$F_n(z) = \frac{2^{1-n}\sqrt{\pi}\, z^{n-1}}{\Gamma\left(\frac{n}{2}\right)}\sum_{k=0}^{n-1}\frac{(n-k-1)!}{k!\,\Gamma\left(\frac{n+1}{2}-k\right)}\left(1-\frac{n}{2}\right)_k\left(-\frac{4}{z^2}\right)^k /; n\in\mathbb{N}$$

07.06.06.0023.01

$$F_n(z) = z^{n-1}\,{}_2F_1\left(\frac{1-n}{2}, 1-\frac{n}{2}; 1-n; -\frac{4}{z^2}\right) /; n\in\mathbb{N}^+$$

07.06.06.0024.01

$$F_n(z) \propto z^{n-1}\left(1+O\left(\frac{1}{z^2}\right)\right) /; (|z|\to\infty)$$

Asymptotic series expansions

07.06.06.0025.01

$$F_\nu(z) \propto \frac{1}{\sqrt{z^2+4}}\left(\left(\frac{1}{2}\left(z+\sqrt{z^2+4}\right)\right)^\nu - \cos(\nu\pi)\left(\frac{1}{2}\left(z+\sqrt{z^2+4}\right)\right)^{-\nu}\right) /; (|\nu|\to\infty)$$

Residue representations

07.06.06.0026.01

$$F_\nu(z) = \frac{\sin(\pi\nu)}{2\sqrt{\pi}}\left(\frac{\sqrt{z^2}}{z}\cos\left(\frac{\pi\nu}{2}\right)\sum_{j=0}^{\infty}\operatorname{res}_s\left(\frac{\Gamma\left(s+\frac{1}{2}\right)\Gamma\left(\frac{1-\nu}{2}-s\right)\Gamma\left(\frac{\nu+1}{2}-s\right)}{\Gamma(1-s)}\left(\frac{z^2}{4}\right)^{-s}\right)\left(-j-\frac{1}{2}\right)+\right.$$
$$\left.\sin\left(\frac{\pi\nu}{2}\right)\sum_{j=0}^{\infty}\operatorname{res}_s\left(\frac{\Gamma(s)\Gamma\left(\frac{1-\nu}{2}-s\right)\Gamma\left(\frac{\nu+1}{2}-s\right)}{\Gamma\left(\frac{1}{2}-s\right)}\left(\frac{z^2}{4}\right)^{-s}\right)(-j)\right) /; |z|<1 \bigwedge \nu\notin\mathbb{Z}$$

07.06.06.0027.01

$$F_\nu(z) = \frac{\sin(\pi\nu)}{2\sqrt{\pi}}\left(\sum_{j=0}^{\infty}\operatorname{res}_s\left(\frac{\Gamma(s)\Gamma\left(s+\frac{1}{2}\right)\Gamma\left(\frac{1-\nu}{2}-s\right)\Gamma\left(\frac{\nu+1}{2}-s\right)}{\Gamma\left(s+\frac{\nu}{2}\right)\Gamma\left(-s-\frac{\nu}{2}+1\right)}\left(\frac{z}{2}\right)^{-2s}\right)\left(-j-\frac{1}{2}\right)+\right.$$
$$\left.\sum_{j=0}^{\infty}\operatorname{res}_s\left(\frac{\Gamma(s)\Gamma\left(s+\frac{1}{2}\right)\Gamma\left(\frac{1-\nu}{2}-s\right)\Gamma\left(\frac{\nu+1}{2}-s\right)}{\Gamma\left(s+\frac{\nu}{2}\right)\Gamma\left(-s-\frac{\nu}{2}+1\right)}\left(\frac{z}{2}\right)^{-2s}\right)(-j)\right) /; |z|<1 \bigwedge \nu\notin\mathbb{Z}$$

Integral representations

Contour integral representations

07.06.07.0001.01

$$F_\nu(z) = \frac{\sin(\pi\nu)}{2\sqrt{\pi}} \left(\frac{\sqrt{z^2}}{z} \cos\left(\frac{\pi\nu}{2}\right) \frac{1}{2\pi i} \int_{\mathcal{L}} \frac{\Gamma\left(s+\frac{1}{2}\right)\Gamma\left(\frac{1-\nu}{2}-s\right)\Gamma\left(\frac{\nu+1}{2}-s\right)}{\Gamma(1-s)} \left(\frac{z^2}{4}\right)^{-s} ds + \sin\left(\frac{\pi\nu}{2}\right) \frac{1}{2\pi i} \int_{\mathcal{L}} \frac{\Gamma(s)\,\Gamma\left(\frac{1-\nu}{2}-s\right)\Gamma\left(\frac{\nu+1}{2}-s\right)}{\Gamma\left(\frac{1}{2}-s\right)} \left(\frac{z^2}{4}\right)^{-s} ds \right) /;\ \nu \notin \mathbb{Z}$$

07.06.07.0002.01

$$F_\nu(z) = \frac{\sin(\pi\nu)}{2\sqrt{\pi}} \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(s)\,\Gamma\left(s+\frac{1}{2}\right)\Gamma\left(\frac{1-\nu}{2}-s\right)\Gamma\left(\frac{\nu+1}{2}-s\right)}{\Gamma\left(s+\frac{\nu}{2}\right)\Gamma\left(1-\frac{\nu}{2}-s\right)} \left(\frac{z}{2}\right)^{-2s} ds /;\ 0 < \gamma < \frac{1}{2}(1-|\mathrm{Re}(\nu)|) \bigwedge |\arg(z)| < \pi$$

Generating functions

07.06.11.0001.01

$$F_n(z) = \left([t^n]\, \frac{t}{1-z\,t-t^2}\right) /;\ n \in \mathbb{N}$$

Differential equations

Ordinary linear differential equations and wronskians

For the direct function itself

07.06.13.0001.01

$$\left(z^2+4\right) w''(z) + 3\,z\,w'(z) + \left(1-\nu^2\right) w(z) = 0 /;\ w(z) = c_1\,F_\nu(z) + \frac{c_2}{\sqrt[4]{z^2+4}}\, P_{\nu-\frac{1}{2}}^{\frac{1}{2}}\left(\frac{i\,z}{2}\right)$$

07.06.13.0002.02

$$W_z\left(F_\nu(z),\ \frac{1}{\sqrt[4]{z^2+4}}\, P_{\nu-\frac{1}{2}}^{\frac{1}{2}}\left(\frac{i\,z}{2}\right)\right) = -\frac{e^{-\frac{1}{2} i \nu \pi}\left(3+e^{2 i \nu \pi}\right)\nu}{\sqrt{\pi}\,\left(z^2+4\right)^{3/2}}$$

07.06.13.0003.01

$$w''(z) + \left(\frac{3\,g(z)\,g'(z)}{g(z)^2+4} - \frac{g''(z)}{g'(z)}\right) w'(z) + \frac{\left(1-\nu^2\right) g'(z)^2}{g(z)^2+4}\, w(z) = 0 /;\ w(z) = c_1\,F_\nu(g(z)) + c_2\, \frac{1}{\sqrt[4]{g(z)^2+4}}\, P_{\nu-\frac{1}{2}}^{\frac{1}{2}}\left(\frac{i}{2}\,g(z)\right)$$

07.06.13.0004.01

$$W_z\left(F_\nu(g(z)),\ \frac{1}{\sqrt[4]{g(z)^2+4}}\, P_{\nu-\frac{1}{2}}^{\frac{1}{2}}\left(\frac{1}{2}\,i\,g(z)\right)\right) = -\frac{e^{-\frac{1}{2} i \nu \pi}\left(3+e^{2 i \nu \pi}\right)\nu\,g'(z)}{\sqrt{\pi}\,\left(g(z)^2+4\right)^{3/2}}$$

07.06.13.0005.01

$$w''(z)+\left(\frac{3\,g(z)\,g'(z)}{g(z)^2+4}-\frac{2\,h'(z)}{h(z)}-\frac{g''(z)}{g'(z)}\right)w'(z)+\left(\frac{(1-\nu^2)\,g'(z)^2}{g(z)^2+4}-\frac{3\,g(z)\,h'(z)\,g'(z)}{(g(z)^2+4)\,h(z)}+\frac{2\,h'(z)^2}{h(z)^2}+\frac{h'(z)\,g''(z)}{h(z)\,g'(z)}-\frac{h''(z)}{h(z)}\right)w(z)=0\,/;$$

$$w(z)=c_1\,h(z)\,F_\nu(g(z))\;+c_2\,\frac{h(z)}{\sqrt[4]{g(z)^2+4}}\,P_{\nu-\frac{1}{2}}^{\frac{1}{2}}\left(\frac{i}{2}\,g(z)\right)$$

07.06.13.0006.01

$$W_z\left(h(z)\,F_\nu(g(z)),\;\frac{h(z)}{\sqrt[4]{g(z)^2+4}}\,P_{\nu-\frac{1}{2}}^{\frac{1}{2}}\left(\frac{1}{2}\,i\,g(z)\right)\right)=-\frac{e^{-\frac{1}{2}i\nu\pi}\left(3+e^{2i\nu\pi}\right)\nu\,h(z)^2\,g'(z)}{\sqrt{\pi}\,\left(g(z)^2+4\right)^{3/2}}$$

07.06.13.0007.01

$$z^2\left(a^2\,z^{2r}+4\right)w''(z)\;+\left(a^2\,(2\,r-2\,s+1)\,z^{2r}-4\,(r+2\,s-1)\right)z\;w'(z)+\left(a^2\left((r-s)^2-r^2\,\nu^2\right)z^{2r}+4\,s\,(r+s)\right)w(z)=0\,/;$$

$$w(z)=c_1\,z^s\,F_\nu\,(a\,z^r)\;+c_2\,z^s\,\frac{1}{\sqrt[4]{a^2\,z^{2r}+4}}\,P_{\nu-\frac{1}{2}}^{\frac{1}{2}}\left(\frac{1}{2}\,i\,a\,z^r\right)$$

07.06.13.0008.01

$$W_z\left(z^s\,F_\nu(a\,z^r),\;\frac{z^s}{\sqrt[4]{a^2\,z^{2r}+4}}\,P_{\nu-\frac{1}{2}}^{\frac{1}{2}}\left(\frac{1}{2}\,i\,a\,z^r\right)\right)=-\frac{a\,e^{-\frac{1}{2}i\pi\nu}\left(3+e^{2i\pi\nu}\right)r\,z^{r+2s-1}\,\nu}{\sqrt{\pi}\,\left(a^2\,z^{2r}+4\right)^{3/2}}$$

07.06.13.0009.01

$$w''(z)+\frac{-2\,a^2\,(\log(s)-\log(r))\,r^{2z}-4\,(\log(r)+2\log(s))}{a^2\,r^{2z}+4}\,w'(z)+$$

$$\frac{1}{a^2\,r^{2z}+4}\left(4\log(s)\,(\log(r)+\log(s))-a^2\,r^{2z}\left(\left(\nu^2-1\right)\log^2(r)+2\log(s)\log(r)-\log^2(s)\right)\right)w(z)=0\,/;$$

$$w(z)=c_1\,s^z\,F_\nu(a\,r^z)\;+c_2\,\frac{s^z}{\sqrt[4]{a^2\,r^{2z}+4}}\,P_{\nu-\frac{1}{2}}^{\frac{1}{2}}\left(\frac{1}{2}\,i\,a\,r^z\right)$$

07.06.13.0010.01

$$W_z\left(s^z\,F_\nu(a\,r^z),\;\frac{s^z}{\sqrt[4]{a^2\,r^{2z}+4}}\,P_{\nu-\frac{1}{2}}^{\frac{1}{2}}\left(\frac{i\,a\,r^z}{2}\right)\right)=-\frac{a\,e^{-\frac{1}{2}i\pi\nu}\left(3+e^{2i\pi\nu}\right)r^z\,s^{2z}\,\nu\log(r)}{\sqrt{\pi}\,\left(a^2\,r^{2z}+4\right)^{3/2}}$$

Identities

Recurrence identities

Consecutive neighbors

07.06.17.0001.01

$$F_\nu(z)=-z\,F_{\nu+1}(z)+F_{\nu+2}(z)$$

07.06.17.0002.01

$$F_\nu(z)=z\,F_{\nu-1}(z)+F_{\nu-2}(z)$$

Distant neighbors

07.06.17.0003.01

$$F_\nu(z) = (-1)^{\left\lfloor \frac{m}{2} \right\rfloor} (-z)^{m-2\left\lfloor \frac{m}{2} \right\rfloor} \left(-z^2\right)^{\frac{1-m}{2}+\left\lfloor \frac{m}{2} \right\rfloor} U_{\frac{m-1}{2}}\left(-\frac{z^2}{2}-1\right) F_{\nu+m}(z) + (-1)^{\left\lfloor \frac{m-1}{2} \right\rfloor} (-z)^{1-m+2\left\lfloor \frac{m}{2} \right\rfloor} \left(-z^2\right)^{\left\lfloor \frac{m+1}{2} \right\rfloor-\frac{m}{2}} U_{\frac{m}{2}-1}\left(-\frac{z^2}{2}-1\right) F_{\nu+m+1}(z) \,/;\, m \in \mathbb{N}^+$$

07.06.17.0004.01

$$F_\nu(z) = (-1)^{\left\lfloor \frac{m}{2} \right\rfloor} z^{m-2\left\lfloor \frac{m}{2} \right\rfloor} \left(-z^2\right)^{\frac{1-m}{2}+\left\lfloor \frac{m}{2} \right\rfloor} U_{\frac{m-1}{2}}\left(-\frac{z^2}{2}-1\right) F_{\nu-m}(z) + (-1)^{\left\lfloor \frac{m-1}{2} \right\rfloor} z^{1-m+2\left\lfloor \frac{m}{2} \right\rfloor} \left(-z^2\right)^{\left\lfloor \frac{m+1}{2} \right\rfloor-\frac{m}{2}} U_{\frac{m}{2}-1}\left(-\frac{z^2}{2}-1\right) F_{\nu-m-1}(z) \,/;\, m \in \mathbb{N}^+$$

Functional identities

Relations of special kind

07.06.17.0005.01

$$F_{\nu+1}(z)\, F_{\nu-1}(z) - F_\nu(z)^2 = \cos(\nu\,\pi)$$

Differentiation

Low-order differentiation

With respect to ν

07.06.20.0001.01

$$\frac{\partial F_\nu(z)}{\partial \nu} = \left(2 \sinh^{-1}\left(\frac{z}{2}\right) \cot(\pi\,\nu) + \pi\right) \csc(\pi\,\nu)\, F_{-\nu}(z) + \left(\pi \cot(\pi\,\nu) + \sinh^{-1}\left(\frac{z}{2}\right) \left(\cot^2(\pi\,\nu) + \csc^2(\pi\,\nu)\right)\right) F_\nu(z)$$

07.06.20.0002.01

$$\frac{\partial^2 F_\nu(z)}{\partial \nu^2} = \frac{1}{2} \csc^2(\pi\,\nu) \left(2\, F_\nu(z) \left(\pi \cos(\pi\,\nu) - \sinh^{-1}\left(\frac{z}{2}\right) \sin(\pi\,\nu)\right)^2 + 2\,\pi\, F_{-\nu}(z) \left(\pi \cos(\pi\,\nu) - 2 \sinh^{-1}\left(\frac{z}{2}\right) \sin(\pi\,\nu)\right)\right)$$

With respect to z

07.06.20.0003.01

$$\frac{\partial F_\nu(z)}{\partial z} = \frac{2\,\nu\, F_{\nu-1}(z) + z\,(\nu-1)\, F_\nu(z)}{z^2+4}$$

07.06.20.0004.01

$$\frac{\partial^2 F_\nu(z)}{\partial z^2} = \frac{4\,(\nu-1)\,\nu\, F_{\nu-2}(z) + 2\,z\,\nu\,(2\,\nu-5)\, F_{\nu-1}(z) + \left((\nu-2)\, z^2 + 4\right) (\nu-1)\, F_\nu(z)}{\left(z^2+4\right)^2}$$

Symbolic differentiation

With respect to ν

07.06.20.0009.01

$$\frac{\partial^n F_\nu(z)}{\partial \nu^n} = \sinh^{-1}\left(\frac{z}{2}\right)^n F_\nu(z) - \frac{2^{\nu-1}\left(z+\sqrt{z^2+4}\right)^{-\nu}}{\sqrt{z^2+4}} \left(-\left(e^{i\pi\nu}+e^{-i\pi\nu}\right)\sinh^{-1}\left(\frac{z}{2}\right)^n + (-1)^n e^{i\pi\nu}\left(-i\pi+\sinh^{-1}\left(\frac{z}{2}\right)\right)^n + (-1)^n e^{-i\pi\nu}\left(i\pi+\sinh^{-1}\left(\frac{z}{2}\right)\right)^n\right) /; n \in \mathbb{N}$$

07.06.20.0005.02

$$\frac{\partial^n F_\nu(z)}{\partial \nu^n} = \frac{1}{\sqrt{z^2+4}}\left(\left(\frac{1}{2}\left(z+\sqrt{z^2+4}\right)\right)^\nu \log^n\left(\frac{1}{2}\left(z+\sqrt{z^2+4}\right)\right) - \frac{(-1)^n}{2}\left(\frac{1}{2}\left(z+\sqrt{z^2+4}\right)\right)^{-\nu}\left(e^{-i\pi\nu}\left(\log\left(\frac{1}{2}\left(z+\sqrt{z^2+4}\right)\right)+i\pi\right)^n + e^{i\pi\nu}\left(\log\left(\frac{1}{2}\left(z+\sqrt{z^2+4}\right)\right)-i\pi\right)^n\right)\right) /; n \in \mathbb{N}$$

With respect to z

07.06.20.0006.02

$$\frac{\partial^m F_\nu(z)}{\partial z^m} = 2^{m-2}\sqrt{\pi}\,\nu\,z^{1-m}\cos^2\left(\frac{\pi\nu}{2}\right) {}_3\tilde{F}_2\left(1, 1-\frac{\nu}{2}, \frac{\nu}{2}+1; 1-\frac{m}{2}, \frac{3-m}{2}; -\frac{z^2}{4}\right) + 2^m\sqrt{\pi}\sin^2\left(\frac{\pi\nu}{2}\right) z^{-m} {}_3\tilde{F}_2\left(1, \frac{1-\nu}{2}, \frac{\nu+1}{2}; \frac{1-m}{2}, 1-\frac{m}{2}; -\frac{z^2}{4}\right) /; m \in \mathbb{N}$$

Fractional integro-differentiation

With respect to ν

07.06.20.0007.01

$$\frac{\partial^\alpha F_\nu(z)}{\partial \nu^\alpha} = \frac{1}{2\sqrt{z^2+4}}\left(2 w^\nu \nu^{-\alpha}(1-Q(-\alpha, \nu\log(w)))(\nu\log(w))^\alpha - e^{-i\pi\nu} w^{-\nu}\nu^{-\alpha}(1-Q(-\alpha, \nu(-i\pi-\log(w))))(\nu(-i\pi-\log(w)))^\alpha - e^{i\pi\nu} w^{-\nu}\nu^{-\alpha}(1-Q(-\alpha, \nu(i\pi-\log(w))))(\nu(i\pi-\log(w)))^\alpha\right) /; w = \frac{1}{2}\left(z+\sqrt{z^2+4}\right)$$

With respect to z

07.06.20.0008.01

$$\frac{\partial^\alpha F_\nu(z)}{\partial z^\alpha} = 2^{\alpha-2}\sqrt{\pi}\,z^{-\alpha}\left(z\,\nu\cos^2\left(\frac{\pi\nu}{2}\right){}_3\tilde{F}_2\left(1, 1-\frac{\nu}{2}, \frac{\nu}{2}+1; 1-\frac{\alpha}{2}, \frac{3-\alpha}{2}; -\frac{z^2}{4}\right) + 4\sin^2\left(\frac{\pi\nu}{2}\right){}_3\tilde{F}_2\left(1, \frac{1-\vee}{2}, \frac{1+\nu}{2}; \frac{1-\alpha}{2}, 1-\frac{\alpha}{2}; -\frac{z^2}{4}\right)\right)$$

Integration

Indefinite integration

Involving only one direct function

07.06.21.0001.01

$$\int F_\nu(a\,z)\,dz = \frac{2\,F_{\nu+1}(a\,z) - a\,z\,F_\nu(a\,z)}{a\,\nu}$$

07.06.21.0002.01

$$\int F_\nu(z)\,dz = \frac{2\,F_{\nu+1}(z) - z\,F_\nu(z)}{\nu}$$

Involving one direct function and elementary functions

Involving power function

07.06.21.0003.01

$$\int z^{\alpha-1}\,F_\nu(z)\,dz = -\Bigg(2^{\alpha-\nu-1}\,z^\alpha\left(z+\sqrt{z^2+4}\right)^{1-\nu}\left(-z\left(z+\sqrt{z^2+4}\right)\right)^{-\alpha}\left((\alpha+\nu-1)\,{}_2F_1\!\left(\frac{1}{2}(-\alpha+\nu+1),\,1-\alpha;\,\frac{1}{2}(-\alpha+\nu+3);\,\frac{1}{4}\left(z+\sqrt{z^2+4}\right)^2\right)\left(z+\sqrt{z^2+4}\right)^{2\nu}+4^\nu(-\alpha+\nu+1)\cos(\pi\,\nu)\,{}_2F_1\!\left(\frac{1}{2}(-\alpha-\nu+1),\,1-\alpha;\,\frac{1}{2}(-\alpha-\nu+3);\,\frac{1}{4}\left(z+\sqrt{z^2+4}\right)^2\right)\right)\Bigg)\Big/\left((-\alpha+\nu+1)(\alpha+\nu-1)\right)$$

Involving only one direct function with respect to ν

07.06.21.0004.01

$$\int F_\nu(z)\,d\nu = \frac{2^{-\nu}}{\sqrt{z^2+4}}\left(\left(4^\nu\left(\cos(\pi\,\nu)\log\left(\frac{1}{2}\left(z+\sqrt{z^2+4}\right)\right)-\pi\sin(\pi\,\nu)\right)\left(z+\sqrt{z^2+4}\right)^{-\nu}\right)\Big/\left(\log^2\left(\frac{1}{2}\left(z+\sqrt{z^2+4}\right)\right)+\pi^2\right)+\frac{\left(z+\sqrt{z^2+4}\right)^\nu}{\log\left(\frac{1}{2}\left(z+\sqrt{z^2+4}\right)\right)}\right)$$

Involving one direct function and elementary functions with respect to ν

Involving power function

07.06.21.0005.01

$$\int \nu^{\alpha-1}\,F_\nu(z)\,d\nu = \frac{1}{2\sqrt{z^2+4}}\Bigg(\nu^\alpha\left(-2\,\Gamma\left(\alpha,\,-\nu\log\left(\frac{1}{2}\left(z+\sqrt{z^2+4}\right)\right)\right)\right)\left(-\nu\log\left(\frac{1}{2}\left(z+\sqrt{z^2+4}\right)\right)\right)^{-\alpha}+\Gamma\left(\alpha,\,\nu\left(-i\,\pi+\log\left(\frac{1}{2}\left(z+\sqrt{z^2+4}\right)\right)\right)\right)\left(\nu\left(-i\,\pi+\log\left(\frac{1}{2}\left(z+\sqrt{z^2+4}\right)\right)\right)\right)^{-\alpha}+\Gamma\left(\alpha,\,\nu\left(i\,\pi+\log\left(\frac{1}{2}\left(z+\sqrt{z^2+4}\right)\right)\right)\right)\left(\nu\left(i\,\pi+\log\left(\frac{1}{2}\left(z+\sqrt{z^2+4}\right)\right)\right)\right)^{-\alpha}\Bigg)$$

Integral transforms

Laplace transforms

07.06.22.0001.01

$$\mathcal{L}_t[F_t(z)](w) = \frac{1}{2\sqrt{z^2+4}}\left(\frac{2}{w-\log\left(\frac{1}{2}\left(z+\sqrt{z^2+4}\right)\right)} - \frac{1}{w+\pi i+\log\left(\frac{1}{2}\left(z+\sqrt{z^2+4}\right)\right)} - \frac{1}{w-\pi i+\log\left(\frac{1}{2}\left(z+\sqrt{z^2+4}\right)\right)}\right) /; \operatorname{Re}(w) > \left|\log\left(\frac{1}{2}\left(z+\sqrt{z^2+4}\right)\right)\right|$$

Representations through more general functions

Through hypergeometric functions

Involving ${}_2F_1$

07.06.26.0001.01

$$F_\nu(z) = \sin^2\left(\frac{\pi\nu}{2}\right) {}_2F_1\left(\frac{1-\nu}{2}, \frac{\nu+1}{2}; \frac{1}{2}; -\frac{z^2}{4}\right) + \frac{\nu z}{2}\cos^2\left(\frac{\pi\nu}{2}\right) {}_2F_1\left(1-\frac{\nu}{2}, 1+\frac{\nu}{2}; \frac{3}{2}; -\frac{z^2}{4}\right)$$

07.06.26.0002.01

$$F_\nu(z) = (1-\theta(-\nu)\,\delta(\sin(\nu\pi)))\, z^{\nu-1}\, {}_2F_1\left(\frac{1-\nu}{2}, 1-\frac{\nu}{2}; 1-\nu; -\frac{4}{z^2}\right) - (1-\theta(\nu)\,\delta(\sin(\nu\pi)))\cos(\nu\pi)\, z^{-\nu-1}\, {}_2F_1\left(\frac{\nu+1}{2}, \frac{\nu}{2}+1; \nu+1; -\frac{4}{z^2}\right) /; \operatorname{Re}(z) > 0$$

07.06.26.0003.01

$$F_\nu(z) = \frac{1}{z}\left(z^2\right)^{-\frac{\nu+1}{2}}\left(z\sin^2\left(\frac{\pi\nu}{2}\right) - \sqrt{z^2}\cos^2\left(\frac{\pi\nu}{2}\right)\right) {}_2F_1\left(1+\frac{\nu}{2}, \frac{1+\nu}{2}; 1+\nu; -\frac{4}{z^2}\right) + \frac{1}{z}\left(z^2\right)^{\frac{\nu-1}{2}}\left(z\sin^2\left(\frac{\pi\nu}{2}\right) + \sqrt{z^2}\cos^2\left(\frac{\pi\nu}{2}\right)\right) {}_2F_1\left(1-\frac{\nu}{2}, \frac{1-\nu}{2}; 1-\nu; -\frac{4}{z^2}\right) /; \nu \notin \mathbb{Z}$$

07.06.26.0004.01

$$F_\nu(z) = \frac{1}{2}\left(\nu\left(\sqrt{-z^2}\sin^3\left(\frac{\pi\nu}{2}\right) - z\cos^3\left(\frac{\pi\nu}{2}\right)\right) {}_2F_1\left(1-\frac{\nu}{2}, \frac{\nu}{2}+1; \frac{3}{2}; \frac{z^2}{4}+1\right) + \frac{\sin(\pi\nu)}{\sqrt{z^2+4}}\left(z\cos\left(\frac{\pi\nu}{2}\right) + \sqrt{-z^2}\sin\left(\frac{\pi\nu}{2}\right)\right) {}_2F_1\left(\frac{1+\nu}{2}, \frac{1-\nu}{2}; \frac{1}{2}; \frac{z^2}{4}+1\right)\right)$$

07.06.26.0005.01

$$F_\nu(z) = \frac{\nu}{2} e^{-\frac{i\pi\nu}{2}}(\sin(\pi\nu) - 2i\cos(\pi\nu))\, {}_2F_1\left(1-\nu, 1+\nu; \frac{3}{2}; \frac{2+iz}{4}\right) + \frac{i\sin(\pi\nu)}{2\sqrt{2+iz}} e^{-\frac{i\pi\nu}{2}}\, {}_2F_1\left(\frac{1}{2}-\nu, \frac{1}{2}+\nu; \frac{1}{2}; \frac{2+iz}{4}\right)$$

07.06.26.0006.01

$$F_\nu(z) = \frac{\nu}{2}(2i\cos(\pi\nu) + \sin(\pi\nu))\, e^{\frac{i\pi\nu}{2}}\, {}_2F_1\left(1-\nu, 1+\nu; \frac{3}{2}; \frac{2-iz}{4}\right) - \frac{i\sin(\pi\nu)}{2\sqrt{2-iz}} e^{\frac{i\pi\nu}{2}}\, {}_2F_1\left(\frac{1}{2}-\nu, \nu+\frac{1}{2}; \frac{1}{2}; \frac{2-iz}{4}\right)$$

Involving ${}_pF_q$

07.06.26.0007.01

$$F_\nu(z) = \frac{1}{2\sqrt{z^2+4}} \left(2\,{}_0F_0\left(;;\nu\log\left(\frac{1}{2}\left(z+\sqrt{z^2+4}\right)\right)\right) - {}_0F_0\left(;;\nu\left(-i\pi-\log\left(\frac{1}{2}\left(z+\sqrt{z^2+4}\right)\right)\right)\right) - {}_0F_0\left(;;\nu\left(i\pi-\log\left(\frac{1}{2}\left(z+\sqrt{z^2+4}\right)\right)\right)\right)\right)$$

Through Meijer *G*

Classical cases for the direct function itself

07.06.26.0008.01

$$F_\nu(z) = \frac{\sin(\pi\nu)}{2\sqrt{\pi}} G_{3,3}^{2,2}\left(\frac{z^2}{4} \,\middle|\, \begin{matrix} \frac{\nu+1}{2}, \frac{1-\nu}{2}, \frac{\nu}{2} \\ 0, \frac{1}{2}, \frac{\nu}{2} \end{matrix}\right) /; \operatorname{Re}(z) > 0 \land \nu \notin \mathbb{Z}$$

07.06.26.0009.01

$$F_\nu(z) = \frac{\sin(\pi\nu)}{2\sqrt{\pi}} \left(\frac{\sqrt{z^2}}{z} \cos\left(\frac{\pi\nu}{2}\right) G_{2,2}^{1,2}\left(\frac{z^2}{4} \,\middle|\, \begin{matrix} \frac{\nu+1}{2}, \frac{1-\nu}{2} \\ \frac{1}{2}, 0 \end{matrix}\right) + \sin\left(\frac{\pi\nu}{2}\right) G_{2,2}^{1,2}\left(\frac{z^2}{4} \,\middle|\, \begin{matrix} \frac{\nu+1}{2}, \frac{1-\nu}{2} \\ 0, \frac{1}{2} \end{matrix}\right)\right) /; \nu \notin \mathbb{Z}$$

07.06.26.0010.01

$$F_\nu(z) = \frac{z^{\nu-1}}{2^\nu\sqrt{\pi}} G_{2,2}^{1,2}\left(\frac{4}{z^2} \,\middle|\, \begin{matrix} \frac{\nu+1}{2}, \frac{\nu}{2} \\ 0, \nu \end{matrix}\right) - \frac{\cos(\nu\pi)\, z^{-\nu-1}}{2^{-\nu}\sqrt{\pi}} G_{2,2}^{1,2}\left(\frac{4}{z^2} \,\middle|\, \begin{matrix} \frac{1-\nu}{2}, -\frac{\nu}{2} \\ 0, -\nu \end{matrix}\right) /; \operatorname{Re}(z) > 0 \land \nu \notin \mathbb{Z}$$

07.06.26.0011.01

$$F_\nu(2i(1+2z)) = \frac{(\sin(\pi\nu) - 2i\cos(\pi\nu))\sin(\pi\nu)}{4\sqrt{\pi}} e^{-\frac{i\pi\nu}{2}} G_{2,2}^{1,2}\left(z \,\middle|\, \begin{matrix} \nu, -\nu \\ 0, -\frac{1}{2} \end{matrix}\right) + \frac{i\sin(2\pi\nu)}{8\sqrt{\pi}\sqrt{-z}} e^{-\frac{i\pi\nu}{2}} G_{2,2}^{1,2}\left(z \,\middle|\, \begin{matrix} \nu+\frac{1}{2}, \frac{1}{2}-\nu \\ 0, \frac{1}{2} \end{matrix}\right) /; \nu \notin \mathbb{Z}$$

07.06.26.0012.01

$$F_\nu(-2i(2z+1)) = \frac{\sin(\pi\nu)}{4\sqrt{\pi}} e^{\frac{i\pi\nu}{2}} \left((2i\cos(\pi\nu) + \sin(\pi\nu))\, G_{2,2}^{1,2}\left(z \,\middle|\, \begin{matrix} \nu, -\nu \\ 0, -\frac{1}{2} \end{matrix}\right) - \frac{i\cos(\pi\nu)}{\sqrt{-z}} G_{2,2}^{1,2}\left(z \,\middle|\, \begin{matrix} \nu+\frac{1}{2}, \frac{1}{2}-\nu \\ 0, \frac{1}{2} \end{matrix}\right)\right) /; \nu \notin \mathbb{Z}$$

07.06.26.0013.01

$$F_\nu\left(\sqrt{z}\right) = \frac{\sin(\pi\nu)}{2\sqrt{\pi}} G_{3,3}^{2,2}\left(\frac{z}{4} \,\middle|\, \begin{matrix} \frac{\nu+1}{2}, \frac{1-\nu}{2}, \frac{\nu}{2} \\ 0, \frac{1}{2}, \frac{\nu}{2} \end{matrix}\right) /; z \notin (-\infty, 0) \land \nu \notin \mathbb{Z}$$

Generalized cases for the direct function itself

07.06.26.0014.01

$$F_\nu(z) = \frac{\sin(\pi\nu)}{2\sqrt{\pi}} G_{3,3}^{2,2}\left(\frac{z}{2}, \frac{1}{2} \,\middle|\, \begin{matrix} \frac{1+\nu}{2}, \frac{1-\nu}{2}, \frac{\nu}{2} \\ 0, \frac{1}{2}, \frac{\nu}{2} \end{matrix}\right) /; \nu \notin \mathbb{Z}$$

Through other functions

Involving some hypergeometric-type functions

07.06.26.0015.01

$$F_n(z) = i^{n-1} U_{n-1}\left(-\frac{iz}{2}\right) /; n \in \mathbb{N}$$

07.06.26.0016.01

$$F_\nu(z) = \frac{i\, e^{-\frac{i\pi\nu}{2}}}{2\sqrt{z^2+4}}\left(2\sin(\pi\nu)\, T_\nu\left(-\frac{i z}{2}\right) - \left(\cos(\pi\nu) + e^{i\pi\nu}\right)\sqrt{z^2+4}\; U_{\nu-1}\left(-\frac{i z}{2}\right)\right)$$

07.06.26.0017.01

$$F_\nu(z) = -\frac{i}{2\sqrt{\pi}\,\sqrt[4]{z^2+4}}\, e^{-\frac{i\pi\nu}{2}}\left(\left(3 + e^{2 i\pi\nu}\right) Q_{\nu-\frac{1}{2}}^{\frac{1}{2}}\left(\frac{i z}{2}\right) + e^{i\pi\nu}\,\pi\sin(\pi\nu)\, P_{\nu-\frac{1}{2}}^{\frac{1}{2}}\left(\frac{i z}{2}\right)\right) /;\ \nu \notin \mathbb{Z}$$

07.06.26.0018.01

$$F_\nu(z) = \frac{e^{-\frac{i\pi\nu}{2}}}{2\sqrt[4]{z^2+4}}\left(\frac{2\,(2\, i\cos(\pi\nu) - \sin(\pi\nu))}{\sqrt{\pi}}\, Q_{\nu-\frac{1}{2}}^{\frac{1}{2}}\left(-\frac{i z}{2}\right) + i\sqrt{\pi}\,\sin(\pi\nu)\, P_{\nu-\frac{1}{2}}^{\frac{1}{2}}\left(-\frac{i z}{2}\right)\right) /;\ \nu \notin \mathbb{Z}$$

Representations through equivalent functions

With elementary functions

07.06.27.0001.01

$$F_\nu(z) = \frac{2\, e^{\nu\log(w)} - e^{(\pi i - \log(w))\,\nu} - e^{-(\pi i + \log(w))\,\nu}}{2\sqrt{z^2+4}} /;\ w = \frac{1}{2}\left(z + \sqrt{z^2+4}\right)$$

07.06.27.0002.01

$$F_\nu(z) = \frac{i}{\sqrt{z^2+4}}\, e^{-\frac{i\pi\nu}{2}}\left(\sin(\pi\nu)\exp\left(-i\,\nu\cos^{-1}\left(-\frac{i z}{2}\right)\right) - 2\cos(\pi\nu)\sin\left(\nu\cos^{-1}\left(-\frac{i z}{2}\right)\right)\right)$$

07.06.27.0003.01

$$F_\nu(z) = \frac{\exp\left(2\,\nu\sinh^{-1}\left(\frac{z}{2}\right)\right) - \cos(\nu\pi)}{\exp\left(\nu\sinh^{-1}\left(\frac{z}{2}\right)\right)\sqrt{z^2+4}}$$

07.06.27.0004.01

$$F_\nu(z) = \frac{1}{\sqrt{z^2+4}}\left((1-\cos(\pi\nu))\cosh\left(\nu\sinh^{-1}\left(\frac{z}{2}\right)\right) + (1+\cos(\pi\nu))\sinh\left(\nu\sinh^{-1}\left(\frac{z}{2}\right)\right)\right)$$

07.06.27.0005.01

$$F_\nu(z) = \frac{1}{\sqrt{z^2+4}}\left(2\sin\left(\frac{\pi\nu}{2}\right)\sin\left(\nu\sin^{-1}\left(\frac{\sqrt{z^2+4}}{2}\right)\right) + \left(\cos(\pi\nu) + \frac{\sqrt{-z^2}\,\sin(\pi\nu)}{z} + 1\right)\sinh\left(\nu\sinh^{-1}\left(\frac{z}{2}\right)\right)\right)$$

07.06.27.0006.01

$$F_\nu(z) = \frac{1}{\sqrt{z^2+4}}\left(\frac{\sin(\pi\nu)}{\sqrt{z}}\left(\sqrt{z}\,\sin\left(\frac{\pi\nu}{2}\right) - \sqrt{-z}\,\cos\left(\frac{\pi\nu}{2}\right)\right)\cos\left(\nu\sin^{-1}\left(\frac{\sqrt{z^2+4}}{2}\right)\right) + \right.$$
$$\left.\frac{2}{\sqrt{-z}}\left(\sqrt{-z}\,\sin^3\left(\frac{\pi\nu}{2}\right) - \sqrt{z}\,\cos^3\left(\frac{\pi\nu}{2}\right)\right)\sin\left(\nu\sin^{-1}\left(\frac{\sqrt{z^2+4}}{2}\right)\right)\right)$$

07.06.27.0007.01

$$F_\nu(z) = \frac{i \sin(\pi \nu)}{\sqrt{z^2+4}} e^{-\frac{i \pi \nu}{2}} \cos\left(\frac{\nu}{2}\left(2 i \sinh^{-1}\left(\frac{z}{2}\right)+\pi\right)\right) + \frac{\sqrt[4]{-1}\, (2 \cos(\pi \nu) + i \sin(\pi \nu))}{\sqrt{2 - i z}\, \sqrt{z - 2 i}} e^{-\frac{i \pi \nu}{2}} \sin\left(2 \nu \csc^{-1}\left(\frac{2 (-1)^{3/4}}{\sqrt{z - 2 i}}\right)\right)$$

07.06.27.0008.01

$$F_\nu(z) = \frac{1}{\sqrt{z^2+4}} e^{\frac{i \pi \nu}{2}} \left((2 i \cos(\pi \nu) + \sin(\pi \nu)) \sin\left(2 \nu \sin^{-1}\left(\frac{1}{2} \sqrt{2 - i z}\right)\right) - i \sin(\pi \nu) \cos\left(2 \nu \sin^{-1}\left(\frac{1}{2} \sqrt{2 - i z}\right)\right)\right)$$

07.06.27.0009.01

$$F_\nu(z) = \frac{1}{\sqrt{z^2+4}} e^{-\frac{i \pi \nu}{2}} \left(i \sin(\pi \nu) \cos\left(2 \nu \sin^{-1}\left(\frac{1}{2} \sqrt{2 + i z}\right)\right) - (2 i \cos(\pi \nu) - \sin(\pi \nu)) \sin\left(2 \nu \sin^{-1}\left(\frac{1}{2} \sqrt{2 + i z}\right)\right)\right)$$

History

– M. Bicknell (1970)

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