HarmonicNumber

Notations

Traditional name

Harmonic number

Traditional notation

H_z

Mathematica StandardForm notation

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HarmonicNumber[z]
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Primary definition

06.16.02.0001.01

$$H_z = \psi(z+1) + \gamma$$

06.16.02.0002.02

$$H_n = \sum_{k=1}^{n} \frac{1}{k} \;/;\; n \in \mathbb{N}$$

Specific values

Specialized values

06.16.03.0001.01

$$H_{-n} = \tilde{\infty} \;/;\; n \in \mathbb{N}^+$$

06.16.03.0002.01

$$H_{n+\frac{1}{4}} = 4 \sum_{k=0}^{n} \frac{1}{4k+1} - \frac{\pi}{2} - \log(8) \;/;\; n \in \mathbb{N}$$

06.16.03.0003.01

$$H_{\frac{1}{4}-n} = 4 \sum_{k=0}^{n-2} \frac{1}{4k+3} - \frac{\pi}{2} - \log(8) \;/;\; n \in \mathbb{N}^+$$

06.16.03.0004.01

$$H_{n+\frac{1}{3}} = 3 \sum_{k=0}^{n} \frac{1}{3k+1} - \frac{1}{6}\left(9\log(3) + \sqrt{3}\,\pi\right) \;/;\; n \in \mathbb{N}$$

06.16.03.0005.01

$$H_{\frac{1}{3}-n} = 3\sum_{k=0}^{n-2}\frac{1}{3k+2} - \frac{1}{6}\left(9\log(3)+\sqrt{3}\,\pi\right) /; n\in\mathbb{N}^+$$

06.16.03.0006.01

$$H_{n+\frac{1}{2}} = \sum_{k=1}^{n-1}\frac{1}{k} + \sum_{k=n}^{2n-1}\frac{2}{k} + \frac{2}{2n+1} - \log(4) - \gamma /; n\in\mathbb{N}$$

06.16.03.0007.01

$$H_{\frac{1}{2}-n} = \sum_{k=1}^{n-1}\frac{1}{k} + \sum_{k=n}^{2n-1}\frac{2}{k} + \frac{2}{1-2n} - \log(4) /; n\in\mathbb{N}$$

06.16.03.0008.01

$$H_{n+\frac{2}{3}} = 3\sum_{k=0}^{n}\frac{1}{3k+2} + \frac{1}{6}\left(\sqrt{3}\,\pi - 9\log(3)\right) /; n\in\mathbb{N}$$

06.16.03.0009.01

$$H_{\frac{2}{3}-n} = 3\sum_{k=0}^{n-2}\frac{1}{3k+1} + \frac{1}{6}\left(\sqrt{3}\,\pi - \log(19683)\right) /; n\in\mathbb{N}^+$$

06.16.03.0010.01

$$H_{n+\frac{3}{4}} = 4\sum_{k=0}^{n}\frac{1}{4k+3} + \frac{\pi}{2} - \log(8) /; n\in\mathbb{N}$$

06.16.03.0011.01

$$H_{\frac{3}{4}-n} = 4\sum_{k=0}^{n-2}\frac{1}{4k+1} + \frac{\pi}{2} - \log(8) /; n\in\mathbb{N}^+$$

06.16.03.0012.01

$$H_{n+\frac{p}{q}} = q\sum_{k=0}^{n}\frac{1}{p+kq} + 2\sum_{k=1}^{\left\lfloor\frac{q-1}{2}\right\rfloor}\cos\left(\frac{2\pi p k}{q}\right)\log\left(\sin\left(\frac{\pi k}{q}\right)\right) - \frac{\pi}{2}\cot\left(\frac{\pi p}{q}\right) - \log(2q) /; n\in\mathbb{N}\bigwedge p\in\mathbb{N}^+\bigwedge q\in\mathbb{N}^+\bigwedge p<q$$

06.16.03.0013.01

$$H_{\frac{p}{q}-n} = q\sum_{k=0}^{n-2}\frac{1}{q(k+1)-p} + 2\sum_{k=1}^{\left\lfloor\frac{q-1}{2}\right\rfloor}\cos\left(\frac{2\pi p k}{q}\right)\log\left(\sin\left(\frac{\pi k}{q}\right)\right) - \frac{\pi}{2}\cot\left(\frac{\pi p}{q}\right) - \log(2q) /; n\in\mathbb{N}^+\bigwedge p\in\mathbb{N}^+\bigwedge q\in\mathbb{N}^+\bigwedge p<q$$

Values at fixed points

06.16.03.0014.01

$$H_{-3} = \tilde{\infty}$$

06.16.03.0015.01

$$H_{-\frac{5}{2}} = \frac{8}{3} - \log(4)$$

06.16.03.0016.01

$$H_{-2} = \tilde{\infty}$$

06.16.03.0017.01

$$H_{-\frac{3}{2}} = 2 - \log(4)$$

06.16.03.0018.01

$$H_{-1} = \tilde{\infty}$$

06.16.03.0019.01

$$H_{-\frac{1}{2}} = -\log(4)$$

06.16.03.0020.01

$$H_0 = 0$$

06.16.03.0021.01

$$H_{\frac{1}{2}} = 2 - \log(4)$$

06.16.03.0022.01

$$H_1 = 1$$

06.16.03.0023.01

$$H_{\frac{3}{2}} = \frac{8}{3} - \log(4)$$

06.16.03.0024.01

$$H_2 = \frac{3}{2}$$

06.16.03.0025.01

$$H_{\frac{5}{2}} = \frac{46}{15} - \log(4)$$

06.16.03.0026.01

$$H_3 = \frac{11}{6}$$

Values at infinities

06.16.03.0027.01

$$H_{\infty} = \infty$$

06.16.03.0028.01

$$H_{-\infty} = \infty$$

06.16.03.0029.01

$$H_{i\infty} = \infty$$

06.16.03.0030.01

$$H_{-i\infty} = \infty$$

06.16.03.0031.01

$$H_{\tilde{\infty}} = \infty$$

General characteristics

Domain and analyticity

H_z is an analytical function of z which is defined over the whole complex z-plane.

06.16.04.0001.01

$$z \longrightarrow H_z :: \mathbb{C} \longrightarrow \mathbb{C}$$

Symmetries and periodicities

Mirror symmetry

06.16.04.0002.01

$H_{\bar{z}} = \overline{H_z}$

Periodicity

No periodicity

Poles and essential singularities

The function H_z has an infinite set of singular points:

a) $z = -k$ /; $k \in \mathbb{N}^+$, are the simple poles with residues -1;

b) $z = \tilde{\infty}$ is the point of convergence of poles, which is similar to considering $\tilde{\infty}$ as an essential singular point.

06.16.04.0003.01

$\mathcal{Sing}_z(H_z) = \{\{\{-k, 1\} /; k \in \mathbb{N}^+\}, \{\tilde{\infty}, \infty\}\}$

06.16.04.0004.01

$\text{res}_z(H_z)(-k) = -1 /; k \in \mathbb{N}^+$

Branch points

The function H_z does not have branch points.

06.16.04.0005.01

$\mathcal{BP}_z(H_z) = \{\}$

Branch cuts

The function H_z does not have branch cuts.

06.16.04.0006.01

$\mathcal{BC}_z(H_z) = \{\}$

Series representations

Generalized power series

Expansions at generic point $z = z_0$

06.16.06.0018.01

$$H_z \propto H_{z_0} + \left(\frac{\pi^2}{6} - H_{z_0}^{(2)}\right)(z - z_0) + \left(H_{z_0}^{(3)} - \zeta(3)\right)(z - z_0)^2 + \ldots /; (z \to z_0)$$

06.16.06.0019.01

$$H_z \propto H_{z_0} + \left(\frac{\pi^2}{6} - H_{z_0}^{(2)}\right)(z - z_0) + \left(H_{z_0}^{(3)} - \zeta(3)\right)(z - z_0)^2 + O\left((z - z_0)^3\right)$$

06.16.06.0020.01

$$H_z = \sum_{k=0}^{\infty} \frac{H^{(k)}(z_0)}{k!} (z - z_0)^k$$

06.16.06.0021.01

$$H_z = H_{z_0} + \sum_{k=1}^{\infty} \frac{(-1)^k k! \, z_0^{-k-1} + \psi^{(k)}(z_0)}{k!} (z - z_0)^k$$

06.16.06.0022.01

$$H_z = H_{z_0} + \sum_{k=1}^{\infty} \frac{(-1)^k k!}{k!} \left(H_z^{(k+1)} - \zeta(k+1)\right) (z - z_0)^k$$

06.16.06.0023.01

$$H_z \propto H_{z_0} (1 + O(z - z_0))$$

Expansions at $z == 0$

06.16.06.0001.02

$$H_z \propto \frac{\pi^2 z}{6} - \zeta(3) z^2 + \frac{\pi^4 z^3}{90} - \ldots \;/;\; (z \to 0)$$

06.16.06.0024.01

$$H_z \propto \frac{\pi^2 z}{6} - \zeta(3) z^2 + \frac{\pi^4 z^3}{90} + O\left(z^4\right)$$

06.16.06.0002.02

$$H_z == \sum_{j=0}^{\infty} (-1)^j \zeta(j+2) z^{j+1} \;/;\; |z| < 1$$

06.16.06.0003.01

$$H_z == \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^j z^{j+1}}{(k+1)^{j+2}} \;/;\; |z| < 1$$

06.16.06.0004.02

$$H_z \propto \frac{\pi^2 z}{6} (1 + O(z))$$

Expansions at $z == z_0 \;/;\; z_0 \neq -n$

For the function itself

06.16.06.0005.02

$$H_z \propto H_{z_0} + \zeta(2, z_0 + 1) (z - z_0) - \zeta(3, z_0 + 1) (z - z_0)^2 + \zeta(4, z_0 + 1) (z - z_0)^3 - \ldots \;/;\; (z \to z_0) \land \neg (z_0 \in \mathbb{Z} \land z_0 < 0)$$

06.16.06.0025.01

$$H_z \propto H_{z_0} + \left(\frac{\pi^2}{6} - H_{z_0}^{(2)}\right) (z - z_0) + \left(H_{z_0}^{(3)} - \zeta(3)\right) (z - z_0)^2 + \ldots \;/;\; (z \to z_0) \land \neg (z_0 \in \mathbb{Z} \land z_0 < 0)$$

06.16.06.0026.01

$$H_z \propto H_{z_0} + \zeta(2, z_0 + 1) (z - z_0) - \zeta(3, z_0 + 1) (z - z_0)^2 + \zeta(4, z_0 + 1) (z - z_0)^3 + O\left((z - z_0)^4\right) \;/;\; \neg (z_0 \in \mathbb{Z} \land z_0 < 0)$$

06.16.06.0027.01

$$H_z \propto H_{z_0} + \left(\frac{\pi^2}{6} - H_{z_0}^{(2)}\right)(z - z_0) + \left(H_{z_0}^{(3)} - \zeta(3)\right)(z - z_0)^2 + O\left((z - z_0)^3\right) /; \neg (z_0 \in \mathbb{Z} \land z_0 < 0)$$

06.16.06.0028.01

$$H_z = \sum_{k=0}^{\infty} \frac{H^{(k)}(z_0)}{k!} (z - z_0)^k /; \neg (z_0 \in \mathbb{Z} \land z_0 < 0)$$

06.16.06.0006.02

$$H_z = H_{z_0} + \sum_{j=0}^{\infty} (-1)^j \zeta(j + 2, z_0 + 1) (z - z_0)^{j+1} /; \neg (z_0 \in \mathbb{Z} \land z_0 < 0)$$

06.16.06.0007.02

$$H_z = H_{z_0} + \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^j (z - z_0)^{j+1}}{(k + z_0 + 1)^{j+2}} /; \neg (z_0 \in \mathbb{Z} \land z_0 < 0)$$

06.16.06.0029.01

$$H_z = H_{z_0} + \sum_{k=1}^{\infty} (-1)^k \left(H_{z_0}^{(k+1)} - \zeta(k + 1)\right) (z - z_0)^k /; \neg (z_0 \in \mathbb{Z} \land z_0 < 0)$$

06.16.06.0030.01

$$H_z = H_{z_0} + \sum_{k=1}^{\infty} \left((-1)^k z_0^{-k-1} + \frac{\psi^{(k)}(z_0)}{k!}\right) (z - z_0)^k /; \neg (z_0 \in \mathbb{Z} \land z_0 < 0)$$

06.16.06.0008.02

$$H_z \propto H_{z_0} (1 + O(z - z_0)) /; \neg (z_0 \in \mathbb{Z} \land z_0 < 0)$$

Expansions at $z = -n$

For the function itself

06.16.06.0009.02

$$H_z \propto -\frac{1}{z + n} + H_{n-1} + \left(\frac{\pi^2}{3} - \zeta(2, n)\right)(z + n) - \zeta(3, n) (z + n)^2 + \left(\frac{\pi^4}{45} - \zeta(4, n)\right)(z + n)^3 - \ldots /; (z \to -n) \land n \in \mathbb{N}^+$$

06.16.06.0031.01

$$H_z \propto -\frac{1}{z + n} + H_{n-1} + \left(\frac{\pi^2}{3} - \zeta(2, n)\right)(z + n) - \zeta(3, n) (z + n)^2 + \left(\frac{\pi^4}{45} - \zeta(4, n)\right)(z + n)^3 + O\left((n + z)^4\right) /; n \in \mathbb{N}^+$$

06.16.06.0010.02

$$H_z = -\frac{1}{z + n} + H_{n-1} + \sum_{k=1}^{\infty} \left(\frac{\psi^{(k)}(1)}{k!} + \zeta(k + 1) - \zeta(k + 1, n)\right)(z + n)^k /; n \in \mathbb{N}^+$$

06.16.06.0011.02

$$H_z \propto -\frac{1}{z + n} + H_{n-1} (1 + O(z + n)) /; n \in \mathbb{N}^+$$

Asymptotic series expansions

06.16.06.0032.01

$$H_z \propto \log(z) + \gamma + \frac{1}{2z} - \frac{1}{12 z^2} + \frac{1}{120 z^4} - \frac{1}{252 z^6} + \ldots \;/;\; |\arg(z)| < \pi \bigwedge (|z| \to \infty)$$

06.16.06.0012.01

$$H_z \propto \log(z) + \gamma + \frac{1}{2z} - \sum_{k=1}^{\infty} \frac{B_{2k}}{2k\, z^{2k}} \;/;\; |\arg(z)| < \pi \bigwedge (|z| \to \infty)$$

06.16.06.0033.01

$$H_z \propto \log(z+1) + \gamma - \frac{1}{2(z+1)} + i\pi(i\cot(\pi z) - 1)\left\lfloor \frac{|\arg(z+1)|}{\pi} \right\rfloor - \sum_{k=1}^{\infty} \frac{B_{2k}}{2k(z+1)^{2k}} \;/;\; \neg(z \in \mathbb{Z} \land z < 0) \land (|z| \to \infty)$$

06.16.06.0013.01

$$H_z \propto \log(z) + \gamma + \frac{1}{2z} - \frac{1}{12 z^2}\left(1 + O\left(\frac{1}{z^2}\right)\right) \;/;\; |\arg(z)| < \pi \bigwedge (|z| \to \infty)$$

Residue representations

06.16.06.0014.01

$$H_z = -\frac{1}{\Gamma(-z)} \sum_{j=0}^{\infty} \text{res}_s\left(\frac{\Gamma(s)\,\Gamma(1-s)\,\Gamma(1-s)\,\Gamma(1-z-s)}{\Gamma(2-s)\,\Gamma(2-s)} (-1)^{-s}\right)(-j)$$

Other series representations

06.16.06.0015.01

$$H_z = z\left(\frac{1}{z+1} + \frac{1}{2(z+2)} + \frac{1}{3(z+3)} + \ldots\right)$$

06.16.06.0016.01

$$H_z = z \sum_{k=0}^{\infty} \frac{1}{(k+1)(k+z+1)}$$

06.16.06.0017.01

$$H_n = \frac{1}{B_n}\left(\sum_{k=1}^{n} \frac{1}{k} B_k B_{n-k} - \sum_{k=1}^{n} \frac{1}{k}\binom{n}{k} B_k B_{n-k}\right) /;\; 2n \in \mathbb{N}$$

Integral representations

On the real axis

Of the direct function

06.16.07.0001.01

$$H_z = \int_0^1 \frac{1-t^z}{1-t}\, dt \;/;\; \text{Re}(z) > -1$$

06.16.07.0002.01

$$H_z = \int_0^{\infty} \left(\frac{e^{-t}}{t} - \frac{(t+1)^{-z-1}}{t}\right) dt + \gamma \;/;\; \text{Re}(z) > -1$$

06.16.07.0003.01

$$H_z = \int_0^\infty \frac{e^{-t} - e^{-(z+1)t}}{1 - e^{-t}} \, dt \,/;\, \operatorname{Re}(z) > -1$$

06.16.07.0004.01

$$H_n = \frac{1}{2} \int_{-1}^{1} \frac{1 - P_n(t)}{1 - t} \, dt \,/;\, n \in \mathbb{N}$$

Contour integral representations

06.16.07.0005.01

$$H_z = -\frac{1}{2 \pi i \, \Gamma(-z)} \int_{\mathcal{L}} \frac{\Gamma(s) \, \Gamma(1-s) \, \Gamma(1-s) \, \Gamma(1-z-s)}{\Gamma(2-s) \, \Gamma(2-s)} (-1)^{-s} \, ds$$

06.16.07.0006.01

$$H_z = -\frac{1}{2 \pi i \, \Gamma(-z)} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{\Gamma(s) \, \Gamma(1-s) \, \Gamma(1-s) \, \Gamma(1-z-s)}{\Gamma(2-s) \, \Gamma(2-s)} (-1)^{-s} \, ds \,/;\, 0 < \gamma < 1$$

Limit representations

06.16.09.0001.01

$$H_z = \lim_{n \to \infty} \left(\log(n) - \sum_{k=0}^{n} \frac{1}{k + z + 1} \right) + \gamma$$

06.16.09.0002.01

$$H_z = \gamma - \lim_{s \to 1} \left(\zeta(s, z+1) - \frac{1}{s-1} \right)$$

Generating functions

06.16.11.0001.01

$$H_n = -\left([t^n] \, \frac{\log(1-t)}{1-t} \right) /;\, n \in \mathbb{N}$$

Transformations

Transformations and argument simplifications

Argument involving basic arithmetic operations

06.16.16.0001.01

$$H_{-z-1} = \pi \cot(\pi z) + H_z$$

06.16.16.0002.01

$$H_{1-z} = \pi \cot(\pi z) + H_z + \frac{1}{1-z} - \frac{1}{z}$$

06.16.16.0003.01

$$H_{-z} = H_z - \frac{1}{z} + \pi \cot(\pi z)$$

06.16.16.0004.01

$$H_{z+1} == H_z + \frac{1}{z+1}$$

06.16.16.0005.01

$$H_{z-1} == H_z - \frac{1}{z}$$

06.16.16.0006.01

$$H_{z+n} == H_z + \sum_{k=1}^{n} \frac{1}{k+z} /; n \in \mathbb{N}$$

06.16.16.0007.01

$$H_{z-n} == H_z - \sum_{k=0}^{n-1} \frac{1}{z-k} /; n \in \mathbb{N}$$

Multiple arguments

06.16.16.0008.01

$$H_{2z} == \frac{1}{2}\left(H_{z-\frac{1}{2}} + H_z\right) + \log(2)$$

06.16.16.0010.01

$$H_{3z} == \frac{1}{3}\left(H_z + H_{z-\frac{1}{3}} + H_{z-\frac{2}{3}}\right) + \log(3)$$

06.16.16.0009.01

$$H_{mz} == \frac{1}{m} \sum_{k=0}^{m-1} H_{z-\frac{k}{m}} + \log(m) /; m \in \mathbb{N}^+$$

Products, sums, and powers of the direct function

Sums of the direct function

06.16.16.0011.01

$$H_z + H_{z+\frac{1}{2}} == 2 H_{2z+1} - 2\log(2)$$

Identities

Recurrence identities

Consecutive neighbors

06.16.17.0001.01

$$H_z == H_{z+1} - \frac{1}{z+1}$$

06.16.17.0003.01

$$H_z == H_{z-1} + \frac{1}{z}$$

Distant neighbors

06.16.17.0004.01

$$H_z = H_{z+n} - \sum_{k=1}^{n} \frac{1}{z+k} \text{ /; } n \in \mathbb{N}$$

06.16.17.0005.01

$$H_z = H_{z-n} + \sum_{k=0}^{n-1} \frac{1}{z-k} \text{ /; } n \in \mathbb{N}$$

Functional identities

Relations of special kind

06.16.17.0006.01

$$H_{-z} = H_z - \frac{1}{z} + \pi \cot(\pi z)$$

Complex characteristics

Real part

06.16.19.0001.01

$$\operatorname{Re}(H_{x+i y}) = \operatorname{RootSum}\left[(\#1+1)\left((x+1)^2 + 2\,\#1\,(x+1) + y^2 + \#1^2\right)\&, -\frac{\psi(-\#1)\left((x+1)^2 + (\#1-1)(x+1) + y^2 - \#1\right)}{(x+1)^2 + (4\,\#1+2)(x+1) + y^2 + \#1\,(3\,\#1+2)}\&\right]$$

06.16.19.0002.01

$$\operatorname{Re}(H_{x+i y}) = \frac{1}{2}\left(H_{x+i y} + H_{x-i y}\right)$$

Imaginary part

06.16.19.0003.01

$$\operatorname{Im}(H_{x+i y}) = \frac{i}{2}\left(H_{x-i y} - H_{x+i y}\right)$$

Differentiation

Low-order differentiation

06.16.20.0001.01

$$\frac{\partial H_z}{\partial z} = \frac{\pi^2}{6} - H_z^{(2)}$$

06.16.20.0002.01

$$\frac{\partial^2 H_z}{\partial z^2} = 2\left(H_z^{(3)} - \zeta(3)\right)$$

Symbolic differentiation

06.16.20.0003.02

$$\frac{\partial^n H_z}{\partial z^n} = \delta_n\,\gamma + \psi^{(n)}(z) + (-1)^n\, n!\, z^{-n-1} \text{ /; } n \in \mathbb{N}$$

06.16.20.0004.01

$$\frac{\partial^n H_z}{\partial z^n} == (-1)^n\, n!\left(H_z^{(n+1)} - \zeta(n+1)\right) /;\, n \in \mathbb{N}^+$$

Fractional integro-differentiation

06.16.20.0005.01

$$\frac{\partial^\alpha H_z}{\partial z^\alpha} == z^{1-\alpha} \sum_{k=1}^{\infty} \frac{1}{k^2}\, {}_2\tilde{F}_1\left(1,\, 2;\, 2-\alpha;\, -\frac{z}{k}\right)$$

Integration

Indefinite integration

Involving only one direct function

06.16.21.0001.01

$$\int H_z\, dz == \gamma\, z + \log\Gamma(z+1)$$

Involving one direct function and elementary functions

Involving power function

06.16.21.0002.01

$$\int z^{\alpha-1} H_z\, dz == \frac{z^{\alpha+1}}{\alpha+1} \sum_{k=0}^{\infty} \frac{1}{(k+1)^2}\, {}_3F_2\left(1,\, 2,\, \alpha+1;\, 2,\, \alpha+2;\, -\frac{z}{k+1}\right)$$

Summation

Finite summation

06.16.23.0001.01

$$\sum_{k=1}^{n} H_k == (n+1)\,(H_{n+1} - 1) /;\, n \in \mathbb{N}$$

06.16.23.0002.01

$$\sum_{k=1}^{q} H_{\frac{k}{q}}\, e^{\frac{2\pi p k i}{q}} == (-1)^p\, e^{\frac{i p \pi}{q}} \csc\left(\frac{p\,\pi}{q}\right) \sin(p\,\pi)\,\gamma - q\, \mathrm{B}_{e^{\frac{2 i p \pi}{q}}}(q+1,\, 0) /;\, p \in \mathbb{N}^+ \bigwedge q \in \mathbb{N}^+ \bigwedge p < q$$

Infinite summation

06.16.23.0003.01

$$\sum_{k=1}^{\infty} \frac{H_k}{k^2\, 2^k} = \zeta(3) - \frac{1}{12}\, \pi^2 \log(2)$$

G.Huvent (2006)

06.16.23.0004.01

$$\sum_{k=1}^{\infty} \frac{H_k^2}{k\, 2^k} = \frac{7\,\zeta(3)}{8}$$

G.Huvent (2006)

06.16.23.0005.01

$$\sum_{k=1}^{\infty} \frac{H_k^3}{2^k} = \frac{\log^3(2)}{3} + \frac{1}{3}\,\pi^2 \log(2) + \zeta(3)$$

G.Huvent (2006)

06.16.23.0006.01

$$\sum_{k=1}^{\infty} \frac{H_{2\,k}\, H_{2\,k+1}}{(2\,k+1)^2} = \frac{\pi^4}{64}$$

G.Huvent (2006)

06.16.23.0007.01

$$\sum_{k=1}^{\infty} \frac{H_{2\,k}\, H_{2\,k-1}}{k^2} = \frac{17\,\pi^4}{240}$$

G.Huvent (2006)

06.16.23.0008.01

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}\, H_k\, H_{k+1}}{(k+1)^2} = \frac{\pi^4}{480}$$

G.Huvent (2006)

06.16.23.0009.01

$$\sum_{k=1}^{\infty} \frac{H_k}{k\, 2^{k/2}} \sin\left(\frac{\pi\, k}{4}\right) = C$$

G.Huvent (2006)

06.16.23.0010.01

$$\sum_{k=1}^{\infty} \frac{H_k}{k^2} \sin\left(\frac{\pi\, k}{3}\right) = \frac{11\,\pi^3}{324}$$

G.Huvent (2006)

06.16.23.0011.01

$$\sum_{k=1}^{\infty} \frac{H_k}{k^3} \cos\left(\frac{\pi\, k}{3}\right) = \frac{17\,\pi^4}{4860}$$

G.Huvent (2006)

06.16.23.0012.01

$$\sum_{k=1}^{\infty} \frac{H_k^2}{k} \sin\left(\frac{\pi k}{3}\right) = \frac{\pi^3}{36}$$

G.Huvent (2006)

06.16.23.0013.01

$$\sum_{k=1}^{\infty} \frac{H_k}{k^3} \cos\left(\frac{\pi k}{3}\right) = \frac{17 \pi^4}{4860}$$

G.Huvent (2006)

06.16.23.0014.01

$$\sum_{k=1}^{\infty} \frac{H_k}{k} \left(\frac{2^k}{3^k} - \frac{2^{3k-1}}{3^{2k+1}}\right) = \frac{\pi^2}{12}$$

G.Huvent (2006)

Representations through more general functions

Through hypergeometric functions

Involving $_pF_q$

06.16.26.0001.01

$$H_z = z \, _3F_2(1, 1, 1-z; 2, 2; 1)$$

Through Meijer G

Classical cases for the direct function itself

06.16.26.0002.01

$$H_z = -\frac{1}{\Gamma(-z)} G_{3,3}^{1,3}\left(-1 \,\middle|\, \begin{matrix} 0, 0, z \\ 0, -1, -1 \end{matrix}\right)$$

Through other functions

Involving some hypergeometric-type functions

06.16.26.0003.01

$$H_n = H_n^{(1)}$$

Representations through equivalent functions

With related functions

06.16.27.0001.01

$$H_z = \frac{1}{\Gamma(z+1)} \frac{\partial \Gamma(z+1)}{\partial z} + \gamma$$

06.16.27.0002.01

$$H_z = \frac{\partial \log\Gamma(z+1)}{\partial z} + \gamma$$

06.16.27.0003.01

$$H_z = \psi(z+1) + \gamma$$

Theorems

The average number of comparisions done within the Quicksort algorithm for sorting

The average number of comparisons done within the Quicksort algorithm for sorting a list of n randomly ordered numbers is $2(n+1)(H_{n+1}-1)$.

The stacking of books (or coins) of equal lengths (diameters)

n books (or coins) of equal length (diameter) l can be stacked on top of each other without falling in such a way that the nth book hangs over by $(H_n - 1)\, l/2$.

The piecewise linear potential in the time independent Schrödinger equation

The piecewise linear potential $V(x) = -\sum_{k=1}^{\infty} \pi^2 k^2 (\theta(x - H_{k-1})\,\theta(H_k - x) + \theta(-x - H_{k-1})\,\theta(H_k + x))$ in the time–independent Schrödinger equation $-\psi''(x) + V(x)\,\psi(x) = E\,\psi(x),\ -\infty < x < \infty$ has a bound state at $E = 0$ in the continuous spectrum. Its wave function is

$$\psi(x) = \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \left(\left(\theta(x - H_{k-1})\,\theta(H_k - x)\ \sin(k\,\pi\,(x - H_k)) + \theta(x + H_k)\,\theta(-H_{k-1} - x)\ \sin(k\,\pi\,(x + H_k))\right)\right).$$

History

– Pythagoras of Samos (569-475 B.C) discovered the harmonic series

– Richard Suiseth (14th century)and Nicole d'Oresme (1350) studied the harmonic series and found its divergence

– Pietro Mengoli (1647) proved divergence of the harmonic series and that the harmonic series with alternating signs converges to $\log(2)$

– Nicolaus Mercator (1668) studied the series corresponding to the series of $\log(1+z)$

– Jacob Bernoulli (1689) again proved divirgence of harmonic series

– G. W. Leibniz (1673)

– L. Euler (1740)

– Jacob Bernoulli (1744)

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