HarmonicNumber2

Notations

Traditional name

Generalized harmonic number

Traditional notation

$H_z^{(r)}$

Mathematica StandardForm notation

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HarmonicNumber[z, r]
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Primary definition

06.17.02.0001.01

$$H_z^{(r)} == \zeta(r) - \hat{\zeta}(r, z+1) /; r \neq 1$$

06.17.02.0003.01

$$H_z^{(1)} = \psi(z+1) + \gamma$$

06.17.02.0002.01

$$H_n^{(r)} == \sum_{k=1}^{n} \frac{1}{k^r} /; n \in \mathbb{N}$$

Specific values

Specialized values

For fixed *z*

06.17.03.0017.01

$$H_z^{(-n)} = \frac{(-1)^n}{n+1} B_{n+1} + \frac{1}{n+1} B_{n+1}(z+1) /; n \in \mathbb{N}$$

06.17.03.0001.01

$$H_z^{(-5)} == \frac{1}{12} z^2 (z+1)^2 \left(2 z^2 + 2 z - 1\right)$$

06.17.03.0002.01

$$H_z^{(-4)} == \frac{1}{30} z (z+1) (2 z+1) \left(3 z^2 + 3 z - 1\right)$$

06.17.03.0003.01

$$H_z^{(-3)} == \frac{1}{4} z^2 (z+1)^2$$

06.17.03.0004.01

$$H_z^{(-2)} == \frac{1}{6} z (z+1) (2z+1)$$

06.17.03.0005.01

$$H_z^{(-1)} == \frac{1}{2} z (z+1)$$

06.17.03.0006.01

$$H_z^{(0)} == z$$

06.17.03.0007.01

$$H_z^{(1)} == H_z$$

06.17.03.0008.01

$$H_z^{(r)} == \frac{(-1)^r}{(r-1)!} \left(\psi^{(r-1)}(1) - \psi^{(r-1)}(z+1)\right) /; r \in \mathbb{N}^+$$

06.17.03.0018.01

$$H_z^{(r)} == \frac{(-1)^r}{(r-1)!} \left(\left(\frac{\partial^r \log\Gamma(z+1)}{\partial z^r} /. \{z \to 0\}\right) - \frac{\partial^r \log\Gamma(z+1)}{\partial z^r}\right) /; r \in \mathbb{N}^+$$

06.17.03.0019.01

$$H_z^{(r)} == \frac{(-1)^r}{(r-1)!} \left(\left(\frac{\partial^{r-1} \frac{\frac{\partial\Gamma(z+1)}{\partial z}}{\Gamma(z+1)}}{\partial z^{r-1}} /. \{z \to 0\}\right) - \frac{\partial^{r-1} \frac{\frac{\partial\Gamma(z+1)}{\partial z}}{\Gamma(z+1)}}{\partial z^{r-1}}\right) /; r \in \mathbb{N}^+$$

For fixed r

06.17.03.0009.01

$$H_{-1}^{(r)} == 0$$

06.17.03.0010.01

$$H_0^{(r)} == 0$$

06.17.03.0011.01

$$H_1^{(r)} == 1$$

06.17.03.0012.01

$$H_2^{(r)} == 1 + 2^{-r}$$

06.17.03.0013.01

$$H_3^{(r)} == 1 + 2^{-r} + 3^{-r}$$

06.17.03.0014.01

$$H_4^{(r)} == 1 + 2^{-r} + 3^{-r} + 4^{-r}$$

06.17.03.0015.01

$$H_5^{(r)} == 1 + 2^{-r} + 3^{-r} + 4^{-r} + 5^{-r}$$

06.17.03.0016.01

$$H_n^{(-m)} = \frac{B_{m+1}(n+1) - B_{m+1}}{m+1} \;/;\; m \in \mathbb{N}^+ \bigwedge n \in \mathbb{N}$$

General characteristics

Domain and analyticity

$H_z^{(r)}$ is an analytical function of z and r which is defined in $\mathbb{C}^2$. For fixed z, it is an entire function of r.

06.17.04.0001.01

$$(z * r) \longrightarrow H_z^{(r)} :: (\mathbb{C} \otimes \mathbb{C}) \longrightarrow \mathbb{C}$$

Symmetries and periodicities

Mirror symmetry

06.17.04.0002.01

$$H_{\bar{z}}^{(\bar{r})} = \overline{H_z^{(r)}}$$

Periodicity

No periodicity

Poles and essential singularities

With respect to *r*

For fixed z, the function $H_z^{(r)}$ has only one singular point at $r = \tilde{\infty}$. It is an essential singular point.

06.17.04.0003.01

$$\mathcal{Sing}_r\left(H_z^{(r)}\right) = \{\{\tilde{\infty}, \infty\}\}$$

Branch points

With respect to *r*

For fixed z, the function $H_z^{(r)}$ does not have branch points.

06.17.04.0004.01

$$\mathcal{BP}_r\left(H_z^{(r)}\right) = \{\}$$

Branch cuts

With respect to *r*

For fixed z, the function $H_z^{(r)}$ does not have branch cuts.

06.17.04.0005.01

$$\mathcal{BC}_r\left(H_z^{(r)}\right) = \{\}$$

Series representations

Generalized power series

Expansions at generic point $z == z_0$

06.17.06.0015.01

$$H_z^{(r)} \propto H_{z_0}^{(r)} - r\left(H_z^{(r+1)} - \zeta(r+1)\right)(z-z_0) + \frac{r(r+1)}{2}\left(H_z^{(r+2)} - \zeta(r+2)\right)(z-z_0)^2 + \ldots \;/;\; (z \to z_0)$$

06.17.06.0016.01

$$H_z^{(r)} \propto H_{z_0}^{(r)} - r\left(H_z^{(r+1)} - \zeta(r+1)\right)(z-z_0) + \frac{r(r+1)}{2}\left(H_z^{(r+2)} - \zeta(r+2)\right)(z-z_0)^2 + O\left((z-z_0)^3\right)$$

06.17.06.0017.01

$$H_z^{(r)} = \sum_{k=0}^{\infty} \frac{H^{(k,0)}(z_0, r)}{k!}(z-z_0)^k$$

06.17.06.0018.01

$$H_z^{(r)} = H_{z_0}^{(r)} + \sum_{k=1}^{\infty} \frac{(-1)^k (r)_k}{k!}\left(H_z^{(k+r)} - \zeta(k+r)\right)(z-z_0)^k$$

06.17.06.0019.01

$$H_z^{(r)} \propto H_{z_0}^{(r)}(1 + O(z-z_0))$$

Expansions at $z == 0$

06.17.06.0001.01

$$H_z^{(r)} == r\,\zeta(r+1)\,z - \frac{1}{2}r(r+1)\,\zeta(r+2)\,z^2 + \frac{1}{6}r(r+1)(r+2)\,\zeta(r+3)\,z^3 - \ldots \;/;\; \mathrm{Re}(r) > 1 \land |z| < 1$$

06.17.06.0002.01

$$H_z^{(r)} == \sum_{j=1}^{\infty} \frac{(-1)^{j-1}(r)_j\,\zeta(j+r)\,z^j}{j!} \;/;\; \mathrm{Re}(r) > 1 \land |z| < 1$$

06.17.06.0003.01

$$H_z^{(r)} == \zeta(r) - \sum_{k=1}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^j (r)_j\, z^j}{j!\, k^{j+r}} \;/;\; \mathrm{Re}(r) > 1 \land |z| < 1$$

06.17.06.0004.01

$$H_z^{(r)} \propto r\,\zeta(r+1)\,z\,(1 + O(z)) \;/;\; (z \to 0)$$

Expansions at $z == -m$

06.17.06.0005.01

$$H_z^{(r)} \propto -\frac{1}{(z+m)^r} + (-1)^{r-1} H_m^{(r)} - \frac{(-1)^r}{(r-1)!}\sum_{k=1}^{\infty} \frac{\left(\Gamma(k+r)\,H_m^{(k+r)} + \psi^{(k+r-1)}(1)\right)}{k!}(z+m)^k \;/;\; (z \to -m) \bigwedge m \in \mathbb{N}^+ \bigwedge r \in \mathbb{N}^+$$

06.17.06.0006.01

$$H_z^{(r)} \propto -\frac{1}{(z+m)^r} + (-1)^{r-1} H_m^{(r)}(1 + O(z+m)) \;/;\; (z \to -m) \bigwedge m \in \mathbb{N}^+ \bigwedge r \in \mathbb{N}^+$$

Expansions at $r == 1$

06.17.06.0007.01

$$H_z^{(r)} = H_z + \sum_{k=1}^{\infty} \left(\frac{\log(k+z)}{k+z} - \frac{\log(k)}{k} \right) (r-1) + \frac{1}{2} \sum_{k=1}^{\infty} \left(\frac{\log^2(k+z)}{k+z} - \frac{\log^2(k)}{k} \right) (r-1)^2 + \ldots$$

06.17.06.0008.01

$$H_z^{(r)} = H_z + \sum_{j=1}^{\infty} \frac{1}{j!} \sum_{k=1}^{\infty} \left(\frac{\log^j(k+z)}{k+z} - \frac{\log^j(k)}{k} \right) (r-1)^j$$

06.17.06.0009.01

$$H_z^{(r)} \propto H_z + \sum_{k=1}^{\infty} \left(\frac{\log(k+z)}{k+z} - \frac{\log(k)}{k} \right) (r-1)\,(1 + O(r-1)) \;/;\; (r \to 1)$$

Asymptotic series expansions

06.17.06.0020.01

$$H_z^{(n)} \propto \zeta(n) - \frac{n+2z+1}{2\,(z+1)^n\,(n-1)} + \left\lfloor \frac{|\arg(z+1)|}{\pi} \right\rfloor \frac{(i\cot(\pi z)-1)\,(i\pi)^n\,2^{n-1}}{(n-1)!} \sum_{k=0}^{n-1} \frac{(-1)^k\,k!}{2^k}\,\mathcal{S}_{n-1}^{(k)}\,(i\cot(\pi z)+1)^k - \frac{1}{(n-1)!} \sum_{k=1}^{\infty} \frac{(2k+n-2)!\,B_{2k}}{(2k)!\,(z+1)^{2k+n-1}} \;/;\; (|z| \to \infty) \land n \in \mathbb{Z} \land n > 1$$

06.17.06.0021.01

$$H_z^{(-n)} = \frac{(-1)^n}{n+1} B_{n+1} + \frac{1}{n+1} \sum_{k=0}^{n+1} \binom{n+1}{k} B_{n-k+1}\,(z+1)^k \;/;\; n \in \mathbb{N}$$

06.17.06.0010.01

$$H_z^{(r)} \propto \frac{(-1)^r\,\psi^{(r-1)}(1)}{(r-1)!} + \frac{r-2z-1}{2\,(r-1)\,z^r} - \frac{1}{(r-1)!} \sum_{k=1}^{\infty} \frac{(2k+r-2)!\,B_{2k}}{(2k)!\,z^{2k+r-1}} \;/;\; |\arg(z)| < \pi \bigwedge r-1 \in \mathbb{N}^+ \bigwedge (|z| \to \infty)$$

06.17.06.0022.01

$$H_z^{(2n)} \propto \zeta(2n) - \frac{2n+2z+1}{2\,(z+1)^{2n}\,(2n-1)} + \left\lfloor \frac{|\arg(z+1)|}{\pi} \right\rfloor \frac{(i\pi)^{2n}\,2^{2n-1}\,(i\cot(\pi z)-1)}{(2n-1)!} \sum_{k=0}^{2n-1} \frac{(-1)^k\,k!}{2^k}\,\mathcal{S}_{2n-1}^{(k)}\,(i\cot(\pi z)+1)^k - \frac{1}{(2n-1)!} \sum_{k=1}^{\infty} \frac{(2k+2n-2)!\,B_{2k}}{(2k)!\,(z+1)^{2k+2n-1}} \;/;\; (|z| \to \infty) \bigwedge n \in \mathbb{N}^+$$

06.17.06.0011.01

$$H_z^{(r)} \propto \frac{(-1)^r\,\psi^{(r-1)}(1)}{(r-1)!} - \frac{z^{1-r}}{r-1} \left(1 + O\!\left(\frac{1}{z} \right) \right) \;/;\; |\arg(z)| < \pi \bigwedge r-1 \in \mathbb{N}^+ \bigwedge (|z| \to \infty)$$

Residue representations

06.17.06.0012.01

$$H_z^{(r)} = \sum_{j=0}^{\infty} \operatorname{res}_s \left(\frac{\Gamma(s)\,\Gamma(1-s)\,\Gamma(1-s)^r\,(-1)^{-s}}{\Gamma(2-s)^r} \right)(-j) - \sum_{j=0}^{\infty} \operatorname{res}_s \left(\frac{\Gamma(s)\,\Gamma(1-s)\,\Gamma(1+z-s)^r\,(-1)^{-s}}{\Gamma(2+z-s)^r} \right)(-j) \;/;\; r-1 \in \mathbb{N}^+$$

Other series representations

06.17.06.0013.01

$$H_z^{(r)} = 1 + 2^{-r} + 3^{-r} + \ldots - (1+z)^{-r} - (2+z)^{-r} - (3+z)^{-r} - \ldots \;/;\; \operatorname{Re}(r) > 1$$

06.17.06.0014.01

$$H_z^{(r)} = \sum_{k=1}^{\infty}\left(\frac{1}{k^r} - \frac{1}{(k+z)^r}\right) /; \operatorname{Re}(r) > 1$$

Integral representations

On the real axis

Of the direct function

06.17.07.0001.01

$$H_z^{(r)} = \frac{1}{(r-1)!}\left((-1)^r \psi^{(r-1)}(1) - \int_0^{\infty} \frac{t^{r-1} e^{-t(z+1)}}{1-e^{-t}} \, dt\right) /; r-1 \in \mathbb{N}^+ \land \operatorname{Re}(z) > -1$$

06.17.07.0002.01

$$H_z^{(r)} = \frac{(-1)^{r-1}}{(r-1)!} \int_0^1 \frac{(t^z - 1)\log^{r-1}(t)}{t-1} \, dt /; \operatorname{Re}(z) > -1 \land r \in \mathbb{N}^+$$

Contour integral representations

06.17.07.0003.01

$$H_z^{(r)} = \frac{1}{2\pi i} \int_{\mathcal{L}} \frac{\Gamma(s)\,\Gamma(1-s)\,\Gamma(1-s)^r\,(-1)^{-s}}{\Gamma(2-s)^r} \, ds - \frac{1}{2\pi i} \int_{\mathcal{L}} \frac{\Gamma(s)\,\Gamma(1-s)\,\Gamma(1+z-s)^r\,(-1)^{-s}}{\Gamma(2+z-s)^r} \, ds /; r-1 \in \mathbb{N}^+$$

06.17.07.0004.01

$$H_z^{(r)} = \frac{1}{2\pi i} \int_{\mathcal{L}} \frac{\Gamma(s)\,\Gamma(1-s)\,\Gamma(1-s)^r\,(-1)^{-s}}{\Gamma(2-s)^r} \, ds - \frac{1}{2\pi i} \int_{\mathcal{L}} \frac{\Gamma(s)\,\Gamma(1-s)\,\Gamma(1+z-s)^r\,(-1)^{-s}}{\Gamma(2+z-s)^r} \, ds /; 0 < \gamma < 1 \land r-1 \in \mathbb{N}^+$$

Generating functions

06.17.11.0001.01

$$H_n^{(r)} = -\left([t^n] \frac{\operatorname{Li}_r(t)}{1-t}\right) /; n \in \mathbb{N}$$

Transformations

Transformations and argument simplifications

Argument involving basic arithmetic operations

06.17.16.0001.01

$$H_{1-z}^{(r)} = (-1)^{r-1} H_z^{(r)} + (-1)^r z^{-r} + \frac{(-1)^{\left\lfloor \frac{r}{2} \right\rfloor - 1} (2\pi)^r}{r!} B_r - \pi\,\delta_{r,1} + \frac{1}{(1-z)^r} + \frac{\pi}{(r-1)!} \frac{\partial^{r-1} \cot(z\pi)}{\partial z^{r-1}} /; r \in \mathbb{N}^+$$

06.17.16.0002.01

$$H_{-z}^{(r)} = (-1)^{r-1} H_{z-1}^{(r)} + \frac{(-1)^{\left\lfloor \frac{r}{2} \right\rfloor - 1} (2\pi)^r}{r!} B_r - \pi\,\delta_{r,1} + \frac{\pi}{(r-1)!} \frac{\partial^{r-1} \cot(z\pi)}{\partial z^{r-1}} /; r \in \mathbb{N}^+$$

06.17.16.0003.01

$$H_{-z}^{(r)} == (-1)^{r-1} H_z^{(r)} + (-1)^r z^{-r} + \frac{(-1)^{\lfloor \frac{r}{2} \rfloor -1} (2\pi)^r}{r!} B_r - \pi \delta_{r,1} + \frac{\pi}{(r-1)!} \frac{\partial^{r-1} \cot(z\pi)}{\partial z^{r-1}} /; r \in \mathbb{N}^+$$

06.17.16.0004.01

$$H_{z+1}^{(r)} == H_z^{(r)} + \frac{1}{(z+1)^r}$$

06.17.16.0005.01

$$H_{z-1}^{(r)} == H_z^{(r)} - \frac{1}{z^r}$$

06.17.16.0006.01

$$H_{z+n}^{(r)} == H_z^{(r)} + \sum_{k=1}^{n} \frac{1}{(k+z)^r} /; n \in \mathbb{N}$$

06.17.16.0007.01

$$H_{z-n}^{(r)} == H_z^{(r)} - \sum_{k=0}^{n-1} \frac{1}{(z-k)^r} /; n \in \mathbb{N}$$

Multiple arguments

06.17.16.0008.02

$$H_{2z}^{(r)} == 2^{-r} \left(H_{z-\frac{1}{2}}^{(r)} + H_z^{(r)} \right) - \left(2^{1-r} - 1\right) \zeta(r) /; \operatorname{Re}(z) > 0$$

06.17.16.0009.02

$$H_{mz}^{(r)} == m^{-r} \sum_{k=0}^{m-1} H_{z-\frac{k}{m}}^{(r)} + \left(1 - m^{1-r}\right) \zeta(r) /; \operatorname{Re}(z) > 0 \bigwedge m \in \mathbb{N}^+$$

Products, sums, and powers of the direct function

Sums of the direct function

06.17.16.0010.01

$$H_z^{(r)} + H_{z+\frac{1}{2}}^{(r)} == 2^r H_{2z+1}^{(r)} + (2 - 2^r) \zeta(r) /; \operatorname{Re}(z) > -\frac{1}{2}$$

Identities

Recurrence identities

Consecutive neighbors

06.17.17.0001.01

$$H_z^{(r)} == H_{z+1}^{(r)} - \frac{1}{(z+1)^r}$$

06.17.17.0002.01

$$H_z^{(r)} == H_{z-1}^{(r)} + \frac{1}{z^r}$$

Distant neighbors

06.17.17.0003.01

$$H_z^{(r)} = H_{z+n}^{(r)} - \sum_{k=1}^{n} \frac{1}{(z+k)^r} /; n \in \mathbb{N}$$

06.17.17.0004.01

$$H_z^{(r)} = H_{z-n}^{(r)} + \sum_{k=0}^{n-1} \frac{1}{(z-k)^r} /; n \in \mathbb{N}$$

Functional identities

Relations of special kind

06.17.17.0005.01

$$H_{-z}^{(r)} = (-1)^{r-1} H_z^{(r)} + (-1)^r z^{-r} + \frac{(-1)^{\left\lfloor \frac{r}{2} \right\rfloor - 1} (2\pi)^r}{r!} B_r - \pi \delta_{r,1} + \frac{\pi}{(r-1)!} \frac{\partial^{r-1} \cot(z\pi)}{\partial z^{r-1}} /; r \in \mathbb{N}^+$$

Differentiation

Low-order differentiation

With respect to *z*

06.17.20.0001.01

$$\frac{\partial H_z^{(r)}}{\partial z} = r\left(\zeta(r+1) - H_z^{(r+1)}\right)$$

06.17.20.0002.01

$$\frac{\partial^2 H_z^{(r)}}{\partial z^2} = r(r+1)\left(H_z^{(r+2)} - \zeta(r+2)\right)$$

With respect to *r*

06.17.20.0003.01

$$\frac{\partial H_z^{(r)}}{\partial r} = \sum_{k=1}^{\infty} \frac{\log(k+z)}{(k+z)^r} - \sum_{k=2}^{\infty} \frac{\log(k)}{k^r} /; \mathrm{Re}(r) > 1$$

06.17.20.0004.01

$$\frac{\partial H_n^{(r)}}{\partial r} = -\sum_{k=2}^{n} \frac{\log(k)}{k^r} /; n \in \mathbb{N}$$

06.17.20.0005.01

$$\frac{\partial^2 H_z^{(r)}}{\partial r^2} = \sum_{k=2}^{\infty} \frac{\log^2(k)}{k^r} - \sum_{k=1}^{\infty} \frac{\log^2(k+z)}{(k+z)^r} /; \mathrm{Re}(r) > 1$$

Symbolic differentiation

With respect to *z*

06.17.20.0006.02

$$\frac{\partial^n H_z^{(r)}}{\partial z^n} = \delta_n \, \zeta(r) + (-1)^n \, (r)_n \left(H_z^{(n+r)} - \zeta(n+r)\right) /; n \in \mathbb{N}$$

With respect to *r*

06.17.20.0007.02

$$\frac{\partial^n H_z^{(r)}}{\partial r^n} = \delta_n + (-1)^n \left(\sum_{k=2}^{\infty} \frac{\log^n(k)}{k^r} - \sum_{k=1}^{\infty} \frac{\log^n(k+z)}{(k+z)^r} \right) /; \operatorname{Re}(r) > 1 \land n \in \mathbb{N}$$

06.17.20.0008.02

$$\frac{\partial^n H_m^{(r)}}{\partial r^n} = (-1)^n \, \delta_n + (-1)^n \sum_{k=2}^{m} \frac{\log^n(k)}{k^r} /; m \in \mathbb{N}^+ \bigwedge n \in \mathbb{N}$$

Fractional integro-differentiation

With respect to *z*

06.17.20.0009.01

$$\frac{\partial^\alpha H_z^{(r)}}{\partial z^\alpha} = \frac{\zeta(r) \, z^{-\alpha}}{\Gamma(1-\alpha)} - z^{-\alpha} \sum_{k=1}^{\infty} k^{-r} \, {}_2\tilde{F}_1\left(1, r; 1-\alpha; -\frac{z}{k}\right)$$

Integration

Indefinite integration

Involving only one direct function

06.17.21.0001.01

$$\int H_z^{(r)} \, dz = z \, \zeta(r) + \frac{\zeta(r-1) - H_z^{(r-1)}}{r-1}$$

Involving one direct function and elementary functions

Involving power function

06.17.21.0002.01

$$\int z^{\alpha-1} \, H_z^{(r)} \, dz = \frac{(-1)^r \, z^\alpha \, \psi^{(r-1)}(1)}{\alpha \, (r-1)!} - \frac{z^\alpha}{\alpha} \sum_{k=0}^{\infty} \frac{1}{(k+1)^r} \, {}_2F_1\left(\alpha, r; \alpha+1; -\frac{z}{k+1}\right) /; r \in \mathbb{N}^+$$

Involving one direct function with respect to *r*

06.17.21.0003.01

$$\int H_z^{(r)} \, dr = r - \sum_{k=2}^{\infty} \frac{k^{-r}}{\log(k)} + \sum_{k=1}^{\infty} \frac{(k+z)^{-r}}{\log(k+z)} /; \operatorname{Re}(r) > 1 \land \operatorname{Re}(z) > -1$$

Integral transforms

Laplace transforms

06.17.22.0001.01

$$\mathcal{L}_t\left[H_t^{(r)}\right](z) = \frac{\zeta(r)}{z} - z^{r-1} \sum_{k=1}^{\infty} e^{k z}\, \Gamma(1-r, k z) /; \operatorname{Re}(z) > 0 \bigwedge r \in \mathbb{N}^+$$

Summation

Finite summation

06.17.23.0001.01

$$\sum_{k=1}^{n} H_k^{(r)} = (n+1) H_n^{(r)} - H_n^{(r-1)} /; n \in \mathbb{N}$$

Representations through more general functions

Through hypergeometric functions

Involving ${}_pF_q$

06.17.26.0001.01

$$H_z^{(2)} = 2 z\, {}_4F_3(1, 1, 1, 1-z; 2, 2, 2; 1) - z^2\, {}_3F_2(1, 1, 1-z; 2, 2; 1)^2$$

06.17.26.0002.01

$$H_z^{(3)} = z^3\, {}_3F_2(1, 1, 1-z; 2, 2; 1)^3 - 3 z^2\, {}_4F_3(1, 1, 1, 1-z; 2, 2, 2; 1)\, {}_3F_2(1, 1, 1-z; 2, 2; 1) + 3 z\, {}_5F_4(1, 1, 1, 1, 1-z; 2, 2, 2, 2; 1)$$

06.17.26.0003.01

$$H_z^{(r)} = {}_{r+1}F_r(1, a_1, a_2, \ldots, a_r; a_1+1, a_2+1, \ldots, a_r+1; 1) - (z+1)^{-r}\, {}_{r+1}F_r(1, b_1, b_2, \ldots, b_r; b_1+1, b_2+1, \ldots, b_r+1; 1) /; a_1 = a_2 = \ldots = a_r = 1 \bigwedge b_1 = b_2 = \ldots = b_r = z+1 \bigwedge r-1 \in \mathbb{N}^+$$

Through Meijer G

Classical cases for the direct function itself

06.17.26.0004.01

$$H_z^{(r)} = G_{r+1,r+1}^{1,r+1}\left(-1 \,\middle|\, \begin{matrix} 0, 1-a_1, \ldots, 1-a_r \\ 0, -a_1, \ldots, -a_r \end{matrix}\right) - G_{r+1,r+1}^{1,r+1}\left(-1 \,\middle|\, \begin{matrix} 0, 1-b_1, \ldots, 1-b_r \\ 0, -b_1, \ldots, -b_r \end{matrix}\right) /; a_1 = a_2 = \ldots = a_r = 1 \bigwedge b_1 = b_2 = \ldots = b_r = z+1 \bigwedge r-1 \in \mathbb{N}^+$$

Representations through equivalent functions

With related functions

06.17.27.0001.01

$$H_z^{(r)} = \zeta(r) - \hat{\zeta}(r, z+1) /; r \neq 1$$

06.17.27.0004.01

$$H_z^{(n)} = -\frac{(-1)^n \psi^{(n-1)}(z + \lfloor -\operatorname{Re}(z) \rfloor\, \theta(\lfloor -\operatorname{Re}(z) \rfloor) + 1)}{(n-1)!} - \sum_{k=0}^{\lfloor -\operatorname{Re}(z) \rfloor - 1} \frac{1}{(k+z+1)^n} + \zeta(n) /; n \in \mathbb{Z} \land n > 1$$

06.17.27.0005.01

$$H_z^{(-n)} = \frac{(-1)^n B_{n+1} + B_{n+1}(z+1)}{n+1} \;/;\; n \in \mathbb{N}$$

06.17.27.0002.01

$$H_m^{(-n)} = \frac{B_{n+1}(m+1) - B_{n+1}}{n+1} \;/;\; n \in \mathbb{N}^+ \bigwedge m \in \mathbb{N}$$

06.17.27.0003.01

$$H_z^{(r)} == \frac{(-1)^r}{(r-1)!} \left(\psi^{(r-1)}(1) - \psi^{(r-1)}(z+1)\right) /;\; r \in \mathbb{N}^+ \bigwedge -\frac{\pi}{2} < \arg(z+1) \leq \frac{\pi}{2}$$

History

– L. Euler (1740)

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