HeavisideTheta

Notations

Traditional name

Heaviside step function

Traditional notation

$\theta(x)$

Mathematica StandardForm notation

HeavisideTheta[x]

Primary definition

14.05.02.0001.01

$\theta(x) = 1 \,/;\, x \in \mathbb{R} \land x > 0$

14.05.02.0002.01

$\theta(x) = 0 \,/;\, x \in \mathbb{R} \land x < 0$

14.05.02.0003.01

$$\frac{\partial \theta(x)}{\partial x} = \delta(x)$$

Specific values

Values at fixed points

14.05.03.0001.01

$\theta(1) = 1$

14.05.03.0002.01

$\theta(-1) = 0$

Values at infinities

14.05.03.0003.01

$\theta(\infty) = 1$

14.05.03.0004.01

$\theta(-\infty) = 0$

General characteristics

Domain and analyticity

$\theta(x)$ is a nonanalytical function; it is a piecewise constant function defined for all nonzero real x.

14.05.04.0001.01

$$x \longrightarrow \theta(x) :: \mathbb{R} \longrightarrow \mathbb{Z}$$

Symmetries and periodicities

Parity

14.05.04.0002.01

$$\theta(-x) == 1 - \theta(x) \;/;\; x \neq 0$$

Periodicity

No periodicity

Sets of discontinuity

The function $\theta(x)$ is continuous function in $\mathbb{R} \setminus \{0\}$.

14.05.04.0003.01

$$\mathcal{DS}_x(\theta(x)) == \{0\}$$

14.05.04.0004.01

$$\lim_{\epsilon \to +0} \theta(\epsilon) == 1$$

14.05.04.0005.01

$$\lim_{\epsilon \to +0} \theta(-\epsilon) == 0$$

Series representations

Exponential Fourier series

14.05.06.0001.01

$$\theta(x) == \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{\sin((2k+1)x)}{2k+1} + \frac{1}{2} \;/;\; -\pi < x < \pi$$

Residue representations

14.05.06.0002.01

$$\theta(x) == \operatorname{res}_s\left((1-x)^{-s} \frac{1}{s}\right)(0) \;/;\; 0 < x < 2$$

14.05.06.0003.01

$$\theta(x) == 0 \;/;\; x < 0$$

14.05.06.0004.01

$$\theta(x) == \operatorname{res}_s\left((1+x)^{-s}\,\frac{1}{s}\right)(0)\,/;\,x>0$$

14.05.06.0005.01

$$\theta(x) == 0\,/;\,-2<x<0$$

Integral representations

On the real axis

Of the direct function

14.05.07.0001.01

$$\theta(x) == \frac{1}{2\pi i}\lim_{\varepsilon\to+0}\int_{-\infty}^{\infty}\frac{e^{i t x}}{t-i\varepsilon}\,dt$$

14.05.07.0002.01

$$\theta(x) == \frac{1}{2\pi i}\lim_{\varepsilon\to+0}\int_{-\infty}^{\infty}\frac{e^{-i t x}}{t+i\varepsilon}\,dt$$

14.05.07.0003.01

$$\theta(x) == \frac{1}{\sqrt{\pi}}\lim_{\varepsilon\to+0}\int_{-x}^{\infty}\frac{1}{\varepsilon}\,e^{-\frac{t^2}{\varepsilon^2}}\,dt$$

14.05.07.0004.01

$$\theta(x) == \frac{1}{\pi}\lim_{\varepsilon\to+0}\int_{-\infty}^{x}\frac{1}{t}\sin\left(\frac{t}{\varepsilon}\right)dt$$

Contour integral representations

14.05.07.0005.01

$$\theta(x) == \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\Gamma(s)\,(1-x)^{-s}}{\Gamma(s+1)}\,ds\,/;\;0<\gamma\bigwedge x<2$$

14.05.07.0006.01

$$\theta(x) == \frac{1}{2\pi i}\int_{\mathcal{L}}\frac{\Gamma(s)\,(1-x)^{-s}}{\Gamma(s+1)}\,ds\,/;\,x<2$$

14.05.07.0007.01

$$\theta(x) == \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\Gamma(-s)\,(1+x)^{-s}}{\Gamma(1-s)}\,ds\,/;\;\gamma<0\bigwedge x>-2$$

14.05.07.0008.01

$$\theta(x) == \frac{1}{2\pi i}\int_{\mathcal{L}}\frac{\Gamma(-s)\,(1+x)^{-s}}{\Gamma(1-s)}\,ds\,/;\,x>-2$$

Other integral representations

14.05.07.0009.01

$$\theta(x) = \int\delta(x)\,dx$$

Limit representations

14.05.09.0001.01

$$\theta(x) = \lim_{\varepsilon \to +0} \frac{1}{e^{-\frac{x}{\varepsilon}} + 1}$$

14.05.09.0002.01

$$\theta(x) = \lim_{\varepsilon \to +0} e^{e^{-\frac{x}{\varepsilon}}}$$

14.05.09.0003.01

$$\theta(x) = \frac{1}{2} \left(\lim_{\varepsilon \to +0} \tanh\left(\frac{x}{\varepsilon}\right) + 1 \right)$$

14.05.09.0004.01

$$\theta(x) = \frac{1}{2 \pi i} \lim_{\varepsilon \to +0} (\log(i \varepsilon - x) - \log(-x - i \varepsilon))$$

14.05.09.0005.01

$$\theta(x) = \frac{1}{\pi} \lim_{\varepsilon \to +0} \tan^{-1}\left(\frac{x}{\varepsilon}\right) + \frac{1}{2}$$

14.05.09.0006.01

$$\theta(x) = \frac{1}{2} \left(\lim_{\varepsilon \to +0} \operatorname{erf}\left(\frac{x}{\varepsilon}\right) + 1 \right)$$

14.05.09.0007.01

$$\theta(x) = \frac{1}{2} \left(\lim_{\varepsilon \to +0} \operatorname{erfc}\left(-\frac{x}{\varepsilon}\right) \right)$$

14.05.09.0008.01

$$\theta(x) = \frac{1}{\pi} \lim_{\varepsilon \to +0} \operatorname{Si}\left(\frac{\pi x}{\varepsilon}\right) + \frac{1}{2}$$

14.05.09.0009.01

$$e^{i \lambda \pi (\theta(x) - \theta(-x))} = \lim_{\varepsilon \to +0} \left(\frac{\varepsilon + i x}{\varepsilon - i x} \right)^{\lambda}$$

Transformations

Transformations and argument simplifications

Argument involving basic arithmetic operations

14.05.16.0001.01

$$\theta(-x) = 1 - \theta(x) \;/;\; x \neq 0$$

Multiple arguments

14.05.16.0002.01

$$\theta\left(\prod_{k=1}^{n} (x - a_k) \right) = \sum_{k=1}^{n} (-1)^{n-k} \theta(x - a_k) + n - 2 \left\lfloor \frac{n-1}{2} \right\rfloor - 1 \;/;\; x \neq a_k \bigwedge a_k < a_{k+1} \bigwedge a_k \in \mathbb{R} \bigwedge 1 \leq k \leq n$$

Products, sums, and powers of the direct function

Powers of the direct function

14.05.16.0003.01

$\theta(x)^a == \theta(x) \;/;\; a > 0$

Identities

Functional identities

14.05.17.0001.01

$\theta(a\,x + b) = \theta\left(-\frac{b}{a} - x\right)\theta(-a) + \theta\left(\frac{b}{a} + x\right)\theta(a) \;/;\; a \in \mathbb{R} \bigwedge b \in \mathbb{R}$

14.05.17.0002.01

$\theta(x)\,\theta(y) = \theta(x\,y)\,\theta\left(\frac{x+y}{2}\right) \;/;\; y \in \mathbb{R} \bigwedge y \in \mathbb{R}$

Complex characteristics

Real part

14.05.19.0001.01

$\operatorname{Re}(\theta(x)) = \theta(x)$

Imaginary part

14.05.19.0002.01

$\operatorname{Im}(\theta(x)) = 0$

Absolute value

14.05.19.0003.01

$|\theta(x)| = \theta(x)$

Argument

14.05.19.0004.01

$\arg(\theta(x)) = \tan^{-1}(\theta(x),\, 0)$

Conjugate value

14.05.19.0005.01

$\overline{\theta(x)} == \theta(x)$

Differentiation

Low-order differentiation

In a distributional sense for $x \in \mathbb{R}$.

14.05.20.0001.01

$$\frac{\partial \theta(x)}{\partial x} = \delta(x)$$

Integration

Indefinite integration

Involving only one direct function

14.05.21.0001.01

$$\int \theta(x)\, dx = x\, \theta(x)$$

Involving one direct function and elementary functions

Involving power function

14.05.21.0002.01

$$\int x^{\alpha-1}\, \theta(x)\, dx = \frac{x^{\alpha}}{\alpha}\, \theta(x)$$

14.05.21.0003.01

$$\int \frac{\theta(x)}{x}\, dx = \log(x)\, \theta(x)$$

Definite integration

14.05.21.0004.01

$$\int_{-\infty}^{\infty} \theta(t)\, \theta(x-t)\, dt = x\, \theta(x)$$

Integral transforms

Fourier exp transforms

14.05.22.0001.01

$$\mathcal{F}_t[\theta(t)]\,(x) = \frac{i}{\sqrt{2\pi}\; x} + \sqrt{\frac{\pi}{2}}\; \delta(x)$$

Inverse Fourier exp transforms

14.05.22.0002.01

$$\mathcal{F}_t^{-1}[\theta(t)]\,(x) = -\frac{i}{\sqrt{2\pi}\; x} + \sqrt{\frac{\pi}{2}}\; \delta(x)$$

Fourier cos transforms

14.05.22.0003.01

$$\mathcal{F}c_t[\theta(t)]\,(x) == \sqrt{\frac{\pi}{2}}\; \delta(x)$$

Fourier sin transforms

14.05.22.0004.01

$$\mathcal{F}s_t[\theta(t)](z) = \sqrt{\frac{2}{\pi}}\,\frac{1}{z}$$

Laplace transforms

14.05.22.0005.01

$$\mathcal{L}_t[\theta(t)](z) = \frac{1}{z}$$

Representations through more general functions

Through Meijer *G*

Classical cases for the direct function itself

14.05.26.0001.01

$$\theta(x) = G_{1,1}^{1,0}\left(1 - x \,\middle|\, \begin{matrix} 1 \\ 0 \end{matrix}\right) /;\ x < 2$$

14.05.26.0002.01

$$\theta(x) = G_{1,1}^{0,1}\left(x + 1 \,\middle|\, \begin{matrix} 1 \\ 0 \end{matrix}\right) /;\ x > -2$$

14.05.26.0003.01

$$\theta(1 - |z|) = G_{1,1}^{1,0}\left(z \,\middle|\, \begin{matrix} 1 \\ 0 \end{matrix}\right)$$

14.05.26.0004.01

$$\theta(|z| - 1) = G_{1,1}^{0,1}\left(z \,\middle|\, \begin{matrix} 1 \\ 0 \end{matrix}\right)$$

Representations through equivalent functions

14.05.27.0001.01

$$\theta(x) = \theta(x_1, x_2, \ldots, x_n) /;\ x_1 = x \bigwedge n = 1$$

14.05.27.0002.01

$$\theta(x) = \frac{1}{2}(\operatorname{sgn}(x) + 1) /;\ x \in \mathbb{R} \land x \neq 0$$

14.05.27.0003.01

$$\theta(x) = \theta(x_1, x_2, \ldots, x_n) /;\ x_1 == x \neq 0 \bigwedge n = 1$$

14.05.27.0004.01

$$\theta(x) = \theta(x) /;\ x \neq 0$$

Heaviside theta function $\theta(x)$ represents a generalized function,and unit step function $\theta(x)$ represents a piecewise function. They coincide almost everywhere: $\theta(x) = \theta(x) /;\ x \neq 0$.

History

– O. Heaviside (1881)

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