WOLFRAM RESEARCH functions.wolfram.com
The single most comprehensive and unified source of information about mathematical functions.

InverseJacobiCD

View the online version at
functions.wolfram.com
Download the
PDF File

Notations

Traditional name

Inverse of the Jacobi elliptic function cd

Traditional notation

$\mathrm{cd}^{-1}(z \mid m)$

Mathematica StandardForm notation

```
InverseJacobiCD[z, m]
```

Primary definition

09.37.02.0001.01

$$z = \mathrm{cd}(w \mid m) \;/;\; w = \mathrm{cd}^{-1}(z \mid m)$$

09.37.02.0002.01

$$\mathrm{cd}^{-1}(z \mid m) = \int_z^1 \frac{1}{\sqrt{1-t^2}\,\sqrt{1-m\,t^2}}\,dt \;/;\; -1 < z < 1 \land m < 1$$

Specific values

Specialized values

For fixed *z*

09.37.03.0001.01

$$\mathrm{cd}^{-1}(z \mid 0) = \cos^{-1}(z)$$

09.37.03.0002.01

$$\mathrm{cd}^{-1}\left(z \,\middle|\, \frac{1}{2}\right) = K\left(\frac{1}{2}\right) - F\left(\sin^{-1}(z) \,\middle|\, \frac{1}{2}\right)$$

09.37.03.0003.01

$$\mathrm{cd}^{-1}(z \mid 1) = \tilde{\infty}$$

For fixed *m*

09.37.03.0004.01

$$\mathrm{cd}^{-1}(-1 \mid m) = 2\,K(m)$$

09.37.03.0005.01

$$\mathrm{cd}^{-1}\left(-\frac{1}{2}\,\middle|\,m\right) = F\left(\frac{\pi}{6}\,\middle|\,m\right) + K(m)$$

09.37.03.0006.01

$$\mathrm{cd}^{-1}(0 \mid m) = K(m)$$

09.37.03.0007.01

$$\mathrm{cd}^{-1}\left(\frac{1}{2}\,\middle|\,m\right) = K(m) - F\left(\frac{\pi}{6}\,\middle|\,m\right)$$

09.37.03.0008.01

$$\mathrm{cd}^{-1}(1 \mid m) = 0$$

09.37.03.0009.01

$$\mathrm{cd}^{-1}(i \mid m) = K(m) - F\left(\sin^{-1}(i) \,\middle|\, m\right)$$

09.37.03.0010.01

$$\mathrm{cd}^{-1}(-i \mid m) = K(m) + F\left(\sin^{-1}(i) \,\middle|\, m\right)$$

Values at infinities

09.37.03.0011.01

$$\mathrm{cd}^{-1}(z \mid \infty) = 0$$

09.37.03.0012.01

$$\mathrm{cd}^{-1}(z \mid -\infty) = 0$$

09.37.03.0013.01

$$\mathrm{cd}^{-1}(\infty \mid m) = \frac{1}{\sqrt{m}} K\left(\frac{1}{m}\right)$$

09.37.03.0014.01

$$\mathrm{cd}^{-1}(-\infty \mid m) = 2K(m) - \frac{1}{\sqrt{m}} K\left(\frac{1}{m}\right)$$

09.37.03.0015.01

$$\mathrm{cd}^{-1}(i\infty \mid m) = \frac{i}{\sqrt{-m}} K\left(\frac{1}{m}\right) /;\ 0 < m < 1$$

09.37.03.0016.01

$$\mathrm{cd}^{-1}(-i\infty \mid m) = 2K(m) - \frac{i}{\sqrt{-m}} K\left(\frac{1}{m}\right) /;\ 0 < m < 1$$

General characteristics

Domain and analyticity

$\mathrm{cd}^{-1}(z \mid m)$ is an analytical function of z and m which is defined over $\mathbb{C}^2$.

09.37.04.0001.01

$$(z * m) \longrightarrow \mathrm{cd}^{-1}(z \mid m) :: (\mathbb{C} \otimes \mathbb{C}) \longrightarrow \mathbb{C}$$

Symmetries and periodicities

Mirror symmetry

09.37.04.0002.01

$$\operatorname{cd}^{-1}(\bar{z} \mid \bar{m}) == \overline{\operatorname{cd}^{-1}(z \mid m)}$$

Quasi-reflection symmetry

09.37.04.0003.01

$$\operatorname{cd}^{-1}(-z \mid m) == 2\,K(m) - \operatorname{cd}^{-1}(z \mid m)$$

Poles and essential singularities

With respect to *m*

The function $\operatorname{cd}^{-1}(z \mid m)$ does not have poles and essential singularities with respect to m.

09.37.04.0004.01

$$\mathcal{S}ing_m\left(\operatorname{cd}^{-1}(z \mid m)\right) == \{\}$$

With respect to *z*

The function $\operatorname{cd}^{-1}(z \mid m)$ does not have poles and essential singularities with respect to z.

09.37.04.0005.01

$$\mathcal{S}ing_z\left(\operatorname{cd}^{-1}(z \mid m)\right) == \{\}$$

Branch points

With respect to *m*

For fixed z, the function $\operatorname{cd}^{-1}(z \mid m)$ has two branch points: $m = \frac{1}{z^2}$, $m = \tilde{\infty}$.

09.37.04.0006.01

$$\mathcal{BP}_m\left(\operatorname{cd}^{-1}(z \mid m)\right) == \left\{\frac{1}{z^2}, \tilde{\infty}\right\}$$

09.37.04.0007.01

$$\mathcal{R}_m\left(\operatorname{cd}^{-1}(z \mid m), \frac{1}{z^2}\right) == \log$$

09.37.04.0008.01

$$\mathcal{R}_m\left(\operatorname{cd}^{-1}(z \mid m), \tilde{\infty}\right) == \log$$

With respect to *z*

For fixed m, the function $\operatorname{cd}^{-1}(z \mid m)$ has five branch points: $z = \pm 1$, $z == \pm \frac{1}{\sqrt{m}}$, $z = \tilde{\infty}$.

09.37.04.0009.01

$$\mathcal{BP}_z\left(\operatorname{cd}^{-1}(z \mid m)\right) == \left\{1, -1, \frac{1}{\sqrt{m}}, -\frac{1}{\sqrt{m}}, \tilde{\infty}\right\}$$

09.37.04.0010.01

$$\mathcal{R}_z\left(\operatorname{cd}^{-1}(z \mid m), 1\right) == 2$$

09.37.04.0011.01

$$\mathcal{R}_z\left(\text{cd}^{-1}(z \mid m),\, -1\right) == 2$$

09.37.04.0012.01

$$\mathcal{R}_z\left(\text{cd}^{-1}(z \mid m),\, \frac{1}{\sqrt{m}}\right) == 2$$

09.37.04.0013.01

$$\mathcal{R}_z\left(\text{cd}^{-1}(z \mid m),\, -\frac{1}{\sqrt{m}}\right) == 2$$

09.37.04.0014.01

$$\mathcal{R}_z\left(\text{cd}^{-1}(z \mid m),\, \tilde{\infty}\right) == \log$$

Branch cuts

Branch cut locations: complicated

Series representations

Generalized power series

Expansions at generic point $z == z_0$

For the function itself

09.37.06.0007.01

$$\text{cd}^{-1}(z \mid m) \propto \text{cd}^{-1}(z_0 \mid m) - \frac{z - z_0}{\sqrt{1 - z_0^2}\,\sqrt{1 - m z_0^2}} - \frac{z_0\left(-2 m z_0^2 + m + 1\right)}{2\left(1 - z_0^2\right)^{3/2}\left(1 - m z_0^2\right)^{3/2}}(z - z_0)^2 + \ldots /; (z \to z_0)$$

09.37.06.0008.01

$$\text{cd}^{-1}(z \mid m) \propto \text{cd}^{-1}(z_0 \mid m) - \frac{z - z_0}{\sqrt{1 - z_0^2}\,\sqrt{1 - m z_0^2}} - \frac{z_0\left(-2 m z_0^2 + m + 1\right)}{2\left(1 - z_0^2\right)^{3/2}\left(1 - m z_0^2\right)^{3/2}}(z - z_0)^2 + O\left((z - z_0)^3\right)$$

09.37.06.0009.01

$$\text{cd}^{-1}(z \mid m) == \text{cd}^{-1}(z_0 \mid m) - \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{j=0}^{k-1} \frac{(1-k)_{2(k-j)-2}}{(k-j-1)!\,(2 z_0)^{-2j+k-1}} \sum_{s=0}^{j} \binom{j}{s} \left(\frac{1}{2}\right)_s \left(\frac{1}{2}\right)_{j-s} m^{j-s} \left(1 - z_0^2\right)^{-s-\frac{1}{2}} \left(1 - m z_0^2\right)^{s-j-\frac{1}{2}} (z - z_0)^k$$

09.37.06.0010.01

$$\text{cd}^{-1}(z \mid m) == \text{cd}^{-1}(z_0 \mid m) + \frac{\pi}{(m-1)\,\text{nd}\left(\text{cd}^{-1}(z_0 \mid m) \mid m\right) \text{sd}\left(\text{cd}^{-1}(z_0 \mid m) \mid m\right)} \sum_{k=1}^{\infty} \frac{(-2 z_0)^{k-1}}{k} \sum_{j=0}^{k-1} \frac{m^{k-j-1}\left(1 - z_0^2\right)^{-j}\left(1 - m z_0^2\right)^{j-k+1}}{j!\,(k-j-1)!\,\Gamma\left(\frac{1}{2} - j\right)\Gamma\left(j - k + \frac{3}{2}\right)}$$
$$\,_2F_1\left(\frac{1-j}{2}, -\frac{j}{2}; \frac{1}{2} - j; 1 - \frac{1}{z_0^2}\right) \,_2F_1\left(\frac{j-k+2}{2}, \frac{j-k+1}{2}; j - k + \frac{3}{2}; 1 - \frac{1}{m z_0^2}\right)(z - z_0)^k$$

09.37.06.0011.01

$$\text{cd}^{-1}(z \mid m) \propto \text{cd}^{-1}(z_0 \mid m)\,(1 + O(z - z_0))$$

Expansions at $z == 0$

09.37.06.0001.02

$$\text{cd}^{-1}(z \mid m) \propto K(m) - z - \frac{m+1}{6} z^3 - \frac{3+2m+3m^2}{40} z^5 - \ldots /; (z \to 0)$$

09.37.06.0002.01

$$\text{cd}^{-1}(z \mid m) == K(m) - \sum_{k=0}^{\infty} \frac{m^k \left(\frac{1}{2}\right)_k}{(2k+1)\,k!} \, {}_2F_1\!\left(\frac{1}{2}, -k; \frac{1}{2} - k; \frac{1}{m}\right) z^{2k+1} /; |z| < 1$$

09.37.06.0012.01

$$\text{cd}^{-1}(z \mid m) \propto K(m)\,(1 + O(z))$$

Expansions at generic point $m == m_0$

For the function itself

09.37.06.0013.01

$$\text{cd}^{-1}(z \mid m) \propto \text{cd}^{-1}(z \mid m_0) + \frac{1}{6}\left(-z^3 F_1\!\left(\frac{3}{2}; \frac{1}{2}, \frac{3}{2}; \frac{5}{2}; z^2, z^2 m_0\right) - \frac{3K(m_0)}{m_0} + \frac{3E(m_0)}{m_0 - m_0^2}\right)(m - m_0) + \frac{1}{2}\left(\frac{-2E(m_0) + 2K(m_0) + m_0\,(4E(m_0) - 5K(m_0) + 3K(m_0)\,m_0)}{4\,(m_0-1)^2\,m_0^2} - \frac{3}{20} z^5 F_1\!\left(\frac{5}{2}; \frac{1}{2}, \frac{5}{2}; \frac{7}{2}; z^2, z^2 m_0\right)\right)(m - m_0)^2 + \ldots /; (m \to m_0)$$

09.37.06.0014.01

$$\text{cd}^{-1}(z \mid m) \propto \text{cd}^{-1}(z \mid m_0) + \frac{1}{6}\left(-z^3 F_1\!\left(\frac{3}{2}; \frac{1}{2}, \frac{3}{2}; \frac{5}{2}; z^2, z^2 m_0\right) - \frac{3K(m_0)}{m_0} + \frac{3E(m_0)}{m_0 - m_0^2}\right)(m - m_0) + \frac{1}{2}\left(\frac{-2E(m_0) + 2K(m_0) + m_0\,(4E(m_0) - 5K(m_0) + 3K(m_0)\,m_0)}{4\,(m_0-1)^2\,m_0^2} - \frac{3}{20} z^5 F_1\!\left(\frac{5}{2}; \frac{1}{2}, \frac{5}{2}; \frac{7}{2}; z^2, z^2 m_0\right)\right)(m - m_0)^2 + O\!\left((m - m_0)^3\right)$$

09.37.06.0015.01

$$\text{cd}^{-1}(z \mid m) == \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{1}{2}\pi\, m_0^{-k}\, {}_2\tilde{F}_1\!\left(\frac{1}{2}, \frac{1}{2}; 1-k; m_0\right) - \frac{(-1)^k \sqrt{\pi}\, z^{2k+1}}{(2k+1)\,\Gamma\!\left(\frac{1}{2} - k\right)} F_1\!\left(k + \frac{1}{2}; \frac{1}{2}, k + \frac{1}{2}; k + \frac{3}{2}; z^2, m_0 z^2\right)\right)(m - m_0)^k$$

09.37.06.0016.01

$$\text{cd}^{-1}(z \mid m) \propto \text{cd}^{-1}(z \mid m_0)\,(1 + O(m - m_0))$$

Expansions at $m == 0$

09.37.06.0003.02

$$\text{cd}^{-1}(z \mid m) \propto \cos^{-1}(z) + \frac{1}{4}\left(\sqrt{1-z^2}\; z + \cos^{-1}(z)\right) m + \frac{3}{64}\left(z\sqrt{1-z^2}\;\left(2z^2 + 3\right) + 3\cos^{-1}(z)\right) m^2 + \ldots /; (m \to 0)$$

09.37.06.0004.01

$$\text{cd}^{-1}(z \mid m) == \sum_{k=0}^{\infty}\left(\frac{\pi \left(\frac{1}{2}\right)_k \left(\frac{1}{2}\right)_k}{2\,k!^2} - \frac{\left(\frac{1}{2}\right)_k z^{2k+1}}{(2k+1)\,k!}\, {}_2F_1\!\left(\frac{1}{2}, k + \frac{1}{2}; k + \frac{3}{2}; z^2\right)\right) m^k$$

09.37.06.0017.01

$$\operatorname{cd}^{-1}(z \mid m) = \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k^2}{(k!)^2} \left(\cos^{-1}(z) + \frac{\sqrt{1-z^2}}{2\,z} \sum_{j=1}^{k} \frac{(j-1)!\, z^{2j}}{\left(\frac{1}{2}\right)_j} \right) m^k \;/;\; |m| < 1$$

09.37.06.0005.01

$$\operatorname{cd}^{-1}(z \mid m) = K(m) - \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_j \left(\frac{1}{2}\right)_k m^k z^{2j+2k+1}}{(2\,j + 2\,k + 1)\, j!\, k!}$$

09.37.06.0006.01

$$\operatorname{cd}^{-1}(z \mid m) = K(m) - z\, F_{1\times 0\times 0}^{1\times 1\times 1}\left(\begin{matrix} \frac{1}{2};\ \frac{1}{2};\ \frac{1}{2}; \\ \frac{3}{2};;; \end{matrix}\ m\, z^2,\, z^2 \right)$$

09.37.06.0018.01

$$\operatorname{cd}^{-1}(z \mid m) \propto \cos^{-1}(z)\,(1 + O(m))$$

Integral representations

On the real axis

Of the direct function

09.37.07.0001.01

$$\operatorname{cd}^{-1}(z \mid m) = \int_z^1 \frac{1}{\sqrt{1-t^2}\,\sqrt{1-m\,t^2}}\, dt \;/;\; -1 < z < 1 \land m < 1$$

09.37.07.0002.01

$$\operatorname{cd}^{-1}(z \mid m) = \frac{\sqrt{1-m\,z^2}\ \operatorname{sn}\left(\operatorname{cd}^{-1}(z \mid m) \mid m\right)}{\sqrt{1-z^2}} \int_z^1 \frac{1}{\sqrt{1-t^2}\,\sqrt{1-m\,t^2}}\, dt \;/;$$
$$\neg\, \exists_{\tau,\{\tau\in\mathbb{R},0<\tau<1\}} \left(\operatorname{Im}\left(1 - ((z-1)\,\tau + 1)^2\right) = 0 \bigwedge 1 - ((z-1)\,\tau + 1)^2 < 0 \bigwedge \operatorname{Im}\left(1 - m\,((z-1)\,\tau + 1)^2\right) = 0 \bigwedge 1 - m\,((z-1)\,\tau + 1)^2 < 0 \right)$$

09.37.07.0003.01

$$\operatorname{cd}^{-1}(z \mid m) = \operatorname{cd}^{-1}(z_0 \mid m) - \frac{\sqrt{1-m\,z^2}\ \operatorname{sn}\left(\operatorname{cd}^{-1}(z \mid m) \mid m\right)}{\sqrt{1-z^2}} \int_{z_0}^{z} \frac{1}{\sqrt{1-t^2}\,\sqrt{1-m\,t^2}}\, dt \;/;$$
$$\neg\, \exists_{\tau,\{\tau\in\mathbb{R},0<\tau<1\}} \left(\operatorname{Im}\left(1 - (\tau\,(z-z_0) + z_0)^2\right) = 0 \bigwedge 1 - (\tau\,(z-z_0) + z_0)^2 < 0 \bigwedge \operatorname{Im}\left(1 - m\,(\tau\,(z-z_0) + z_0)^2\right) = 0 \bigwedge 1 - m\,(\tau\,(z-z_0) + z_0)^2 < 0 \right)$$

Differential equations

Ordinary nonlinear differential equations

09.37.13.0001.01

$$w''(z) + \left(2\,m\,z^2 - m - 1\right) z\, w'(z)^3 = 0 \;/;\; w(z) = \operatorname{cd}^{-1}(z \mid m)$$

Transformations

Transformations and argument simplifications

Argument involving basic arithmetic operations

09.37.16.0001.01

$$\operatorname{cd}^{-1}(-z \mid m) = 2\,K(m) - \operatorname{cd}^{-1}(z \mid m)$$

Identities

Functional identities

09.37.17.0001.01

$$\left(\left(z_2^2-1\right) m\, z_1^2 - m\, z_2^2 + 1\right) \operatorname{cd}(w(z_1)+w(z_2) \mid m)^2 + 2\,(m-1)\, z_1\, z_2 \operatorname{cd}(w(z_1)+w(z_2) \mid m) + z_2^2 + z_1^2 \left(1 - m\, z_2^2\right) = 1 \;/;\; w(z) = \operatorname{cd}^{-1}(z \mid m)$$

Differentiation

Low-order differentiation

With respect to *z*

09.37.20.0001.02

$$\frac{\partial \operatorname{cd}^{-1}(z \mid m)}{\partial z} = \frac{\operatorname{sn}\left(\operatorname{cd}^{-1}(z \mid m) \,\middle|\, m\right)}{z^2-1}$$

09.37.20.0002.01

$$\frac{\partial \operatorname{cd}^{-1}(z \mid m)}{\partial z} = -\frac{1}{\sqrt{1-z^2}\,\sqrt{1-m\,z^2}} \;/;\; -1 < z < 1 \land m < 1$$

09.37.20.0003.02

$$\frac{\partial^2 \operatorname{cd}^{-1}(z \mid m)}{\partial z^2} = \frac{z\left(-2\,m\,z^2 + m + 1\right) \operatorname{sn}\left(\operatorname{cd}^{-1}(z \mid m) \,\middle|\, m\right)}{\left(z^2-1\right)^2 \left(m\,z^2-1\right)}$$

09.37.20.0011.01

$$\frac{\partial^2 \operatorname{cd}^{-1}(z \mid m)}{\partial z^2} = -\frac{\sqrt{1-m\,z^2}\,\operatorname{sn}\left(\operatorname{cd}^{-1}(z \mid m) \,\middle|\, m\right)}{\sqrt{1-z^2}} \, \frac{\partial \frac{1}{\sqrt{1-z^2}\,\sqrt{1-m\,z^2}}}{\partial z}$$

With respect to *m*

09.37.20.0004.01

$$\frac{\partial \operatorname{cd}^{-1}(z \mid m)}{\partial m} = \frac{E\left(\operatorname{am}\left(\operatorname{cd}^{-1}(z \mid m) \,\middle|\, m\right) \,\middle|\, m\right) + (m-1) \operatorname{cd}^{-1}(z \mid m)}{2\,(1-m)\,m}$$

09.37.20.0005.01

$$\frac{\partial \operatorname{cd}^{-1}(z \mid m)}{\partial m} = \frac{1}{2\,(1-m)\,m} \left(\frac{m\,\sqrt{1-z^2}\;z}{\sqrt{1-m\,z^2}} + E(m) - E\left(\sin^{-1}(z) \,\middle|\, m\right) + (m-1) \operatorname{cd}^{-1}(z \mid m)\right) /;\; -1 < z < 1 \land m < 1$$

09.37.20.0006.02

$$\frac{\partial^2 \mathrm{cd}^{-1}(z\mid m)}{\partial m^2} == \frac{1}{4(m-1)^2 m^2}\left(m z^2 \,\mathrm{sc}\left(\mathrm{cd}^{-1}(z\mid m)\mid m\right)\left(\frac{m-1}{m z^2-1}\right)^{3/2} + (4m-2)\, E\left(\mathrm{am}\left(\mathrm{cd}^{-1}(z\mid m)\mid m\right)\mid m\right) + (m-1)\, F\left(\mathrm{am}\left(\mathrm{cd}^{-1}(z\mid m)\mid m\right)\mid m\right) + 3(m-1)^2 \mathrm{cd}^{-1}(z\mid m)\right)$$

09.37.20.0012.01

$$\frac{\partial^3 \mathrm{cd}^{-1}(z\mid m)}{\partial m^3} = \frac{1}{4(m-1)^3 m^3 \left(2 m z^2-2\right)}\left(-m z^2 \left(m\left((11m-7)z^2-8\right)+4\right)\mathrm{sc}\left(\mathrm{cd}^{-1}(z\mid m)\mid m\right)\left(\frac{m-1}{m z^2-1}\right)^{3/2} + \left(m z^2-1\right)\left((-23(m-1)m-8)\, E\left(\mathrm{am}\left(\mathrm{cd}^{-1}(z\mid m)\mid m\right)\mid m\right) - (m-1)(11m-7)\, F\left(\mathrm{am}\left(\mathrm{cd}^{-1}(z\mid m)\mid m\right)\mid m\right)\right) - 15(m-1)^3\left(m z^2-1\right)\mathrm{cd}^{-1}(z\mid m)\right)$$

Symbolic differentiation

With respect to *z*

09.37.20.0013.01

$$\frac{\partial^n \mathrm{cd}^{-1}(z\mid m)}{\partial z^n} = \mathrm{cd}^{-1}(z\mid m)\,\delta_n + \frac{\mathrm{sn}\left(\mathrm{cd}^{-1}(z\mid m)\mid m\right)}{z^2-1}\sum_{j=0}^{n-1}\frac{(1-n)_{2(n-j)-2}}{(n-j-1)!\,(2z)^{n-2j-1}}\sum_{k=0}^{j}\binom{j}{k}\left(\frac{1}{2}\right)_k\left(\frac{1}{2}\right)_{j-k} m^{j-k}\left(1-z^2\right)^{-k}\left(1-m z^2\right)^{k-j} /;\, n\in\mathbb{N}$$

09.37.20.0014.01

$$\frac{\partial^n \mathrm{cd}^{-1}(z\mid m)}{\partial z^n} = \mathrm{cd}^{-1}(z\mid m)\,\delta_n + \frac{\mathrm{sn}\left(\mathrm{cd}^{-1}(z\mid m)\mid m\right)}{z^2-1}\sum_{j=0}^{n-1}\frac{2^{2j-n+1} m^j z^{2j-n+1}\left(1-m z^2\right)^{-j}\left(\frac{1}{2}\right)_j (1-n)_{2(n-j)-2}}{(n-j-1)!}\,{}_2F_1\left(\frac{1}{2}, -j;\, \frac{1}{2}-j;\, \frac{1-m z^2}{m\left(1-z^2\right)}\right) /;\, n\in\mathbb{N}$$

09.37.20.0015.01

$$\frac{\partial^n \mathrm{cd}^{-1}(z\mid m)}{\partial z^n} = \mathrm{cd}^{-1}(z\mid m)\,\delta_n - \frac{\sqrt{1-m z^2}\,\mathrm{sn}\left(\mathrm{cd}^{-1}(z\mid m)\mid m\right)}{\sqrt{1-z^2}}\,\frac{\partial^{n-1}\frac{1}{\sqrt{1-z^2}\sqrt{1-m z^2}}}{\partial z^{n-1}} /;\, n\in\mathbb{N}^+$$

09.37.20.0007.01

$$\frac{\partial^n \mathrm{cd}^{-1}(z\mid m)}{\partial z^n} == \frac{2^{n-1}\pi(-z)^{n-1}(n-1)!\,\mathrm{sn}\left(\mathrm{cd}^{-1}(z\mid m)\mid m\right)}{z^2-1}\sum_{j=0}^{n-1}\frac{m^{n-j-1}\left(1-z^2\right)^{-j}\left(1-m z^2\right)^{j-n+1}}{j!\,(n-j-1)!\,\Gamma\left(\frac{1}{2}-j\right)\Gamma\left(j-n+\frac{3}{2}\right)}\,{}_2F_1\left(\frac{1-j}{2}, -\frac{j}{2};\, \frac{1}{2}-j;\, 1-\frac{1}{z^2}\right){}_2F_1\left(\frac{j-n+2}{2}, \frac{j-n+1}{2};\, j\,\frac{3}{2}-n;\, 1-\frac{1}{m z^2}\right) /;\, n\in\mathbb{N}^+$$

With respect to *m*

09.37.20.0008.02

$$\frac{\partial^n \operatorname{cd}^{-1}(z \mid m)}{\partial m^n} = \frac{\sqrt{1-m z^2}\ \operatorname{cd}\left(\operatorname{sn}^{-1}(z \mid m) \mid m\right)}{\sqrt{1-z^2}} \left(\frac{\pi \left(\frac{1}{2}\right)_n^2}{2\, n!}\, {}_2F_1\left(n+\frac{1}{2}, n+\frac{1}{2}; n+1; m\right) - \frac{z^{2n+1} \left(\frac{1}{2}\right)_n}{2n+1}\, F_1\left(n+\frac{1}{2}; \frac{1}{2}, n+\frac{1}{2}; n+\frac{3}{2}; z^2, m z^2\right)\right) /; n \in \mathbb{N}$$

09.37.20.0016.01

$$\frac{\partial^n \operatorname{cd}^{-1}(z \mid m)}{\partial m^n} = \frac{\sqrt{1-m z^2}\ \operatorname{cd}\left(\operatorname{sn}^{-1}(z \mid m) \mid m\right)}{\sqrt{1-z^2}} \frac{\partial^n \left(K(m) - F\left(\sin^{-1}(z) \mid m\right)\right)}{\partial m^n} /; n \in \mathbb{N}$$

Fractional integro-differentiation

With respect to *z*

09.37.20.0009.01

$$\frac{\partial^\alpha \operatorname{cd}^{-1}(z \mid m)}{\partial z^\alpha} = \frac{K(m)\, z^{-\alpha}}{\Gamma(1-\alpha)} - z^{1-\alpha} \sqrt{\pi}\ \tilde{F}_{2\times 0\times 0}^{2\times 1\times 1}\left(\begin{matrix} \frac{1}{2}, 1; \frac{1}{2}; \frac{1}{2}; \\ \frac{3-\alpha}{2}, 1-\frac{\alpha}{2};;; \end{matrix} z^2, m z^2\right) /; -1 < z < 1 \land m < 1$$

With respect to *m*

09.37.20.0010.01

$$\frac{\partial^\alpha \operatorname{cd}^{-1}(z \mid m)}{\partial m^\alpha} = \frac{\pi\, m^{-\alpha}}{2}\, {}_2\tilde{F}_1\left(\frac{1}{2}, \frac{1}{2}; 1-\alpha; m\right) - \frac{m^{-\alpha} \sqrt{\pi}\ z}{2}\, \tilde{F}_{1\times 0\times 1}^{1\times 1\times 2}\left(\begin{matrix} \frac{1}{2}; \frac{1}{2}; \frac{1}{2}, 1; \\ \frac{3}{2};; 1-\alpha; \end{matrix} z^2, m z^2\right) /; -1 < z < 1 \land m < 1$$

Integration

Indefinite integration

Involving only one direct function

09.37.21.0001.01

$$\int \operatorname{cd}^{-1}(z \mid m)\, dz = z \operatorname{cd}^{-1}(z \mid m) + \frac{\log\left(\sqrt{m}\ \operatorname{sd}\left(\operatorname{cd}^{-1}(z \mid m) \mid m\right) - \operatorname{nd}\left(\operatorname{cd}^{-1}(z \mid m) \mid m\right)\right)}{\sqrt{m}}$$

Involving only one direct function with respect to *m*

09.37.21.0002.01

$$\int \operatorname{cd}^{-1}(z \mid m)\, dm = 2\left(E(m) - \frac{1}{z}\left(z\, E\left(\sin^{-1}(z) \mid m\right) + (m-1)\, z\, F\left(\sin^{-1}(z) \mid m\right) + \sqrt{1-z^2}\ \sqrt{1-m z^2}\right) + (m-1)\, K(m)\right) /; -1 < z < 1 \land m < 1$$

Representations through more general functions

Through hypergeometric functions of two variables

09.37.26.0001.01

$$\operatorname{cd}^{-1}(z \mid m) == K(m) - z\, F_{1\times 0\times 0}^{1\times 1\times 1}\left(\begin{matrix} \frac{1}{2};\ \frac{1}{2};\ \frac{1}{2}; \\ \frac{3}{2};;; \end{matrix}\ m\, z^2,\, z^2\right)$$

Through other functions

Involving some hypergeometric-type functions

09.37.26.0002.01

$$\operatorname{cd}^{-1}(z \mid m) == K(m) - z\, F_1\left(\frac{1}{2}; \frac{1}{2}, \frac{1}{2}; \frac{3}{2}; z^2, m\, z^2\right) /; -1 < z < 1 \land m < 1$$

Representations through equivalent functions

With inverse function

09.37.27.0001.01

$$\operatorname{cd}\left(\operatorname{cd}^{-1}(z \mid m) \,\middle|\, m\right) == z$$

With related functions

Involving cn^{-1}

09.37.27.0002.01

$$\operatorname{cd}^{-1}(z \mid m) == K(m) - \operatorname{cn}^{-1}\left(\sqrt{1 - z^2} \,\middle|\, m\right) /; 0 < z < 1 \land m < 1$$

Involving cs^{-1}

09.37.27.0003.01

$$\operatorname{cd}^{-1}(z \mid m) == \frac{1}{\sqrt{m}} \left(K\left(\frac{1}{m}\right) + i\, \operatorname{cs}^{-1}\left(-i\, z \,\middle|\, 1 - \frac{1}{m}\right)\right) /; 0 < z < 1 \land 0 < m < 1$$

Involving dc^{-1}

09.37.27.0004.01

$$\operatorname{cd}^{-1}(z \mid m) == \operatorname{dc}^{-1}\left(\frac{1}{z} \,\middle|\, m\right) /; z < 0 \land m < 0 \lor z < 1 \land m < 1$$

Involving dn^{-1}

09.37.27.0005.01

$$\operatorname{cd}^{-1}(z \mid m) == -\frac{i}{\sqrt{m}} \operatorname{dn}^{-1}\left(z \,\middle|\, \frac{m-1}{m}\right) /; -1 < z < 1 \land m > 1$$

Involving ds^{-1}

09.37.27.0006.01

$$\operatorname{cd}^{-1}(z \mid m) == K(m) - \frac{1}{\sqrt{1-m}} \operatorname{ds}^{-1}\left(\frac{1}{\sqrt{1-m}\ z} \,\middle|\, \frac{m}{m-1}\right) /; z > 0 \land m \in \mathbb{R}$$

Involving nc^{-1}

09.37.27.0007.01

$$\text{cd}^{-1}(z \mid m) = -\frac{i}{\sqrt{1-m}}\,\text{nc}^{-1}\left(z \,\middle|\, \frac{1}{1-m}\right) /; -1 < z < 1 \land m < 1$$

Involving nd^{-1}

09.37.27.0008.01

$$\text{cd}^{-1}(z \mid m) = -i\,\text{nd}^{-1}(z \mid 1-m) /; z \in \mathbb{R} \land m < 0$$

Involving ns^{-1}

09.37.27.0009.01

$$\text{cd}^{-1}(z \mid m) = \frac{1}{\sqrt{m}}\left(\text{ns}^{-1}\left(z \,\middle|\, \frac{1}{m}\right) - K\left(\frac{1}{m}\right)\right) /; -1 < z < 1 \land m < 0$$

Involving sc^{-1}

09.37.27.0010.01

$$\text{cd}^{-1}(z \mid m) = K(m) - i\,\text{sc}^{-1}(-i\,z \mid 1-m) /; z \in \mathbb{R} \land m \in \mathbb{R}$$

Involving sd^{-1}

09.37.27.0011.01

$$\text{cd}^{-1}(z \mid m) = K(m) - \frac{1}{\sqrt{1-m}}\,\text{sd}^{-1}\left(z\sqrt{1-m} \,\middle|\, \frac{m}{m-1}\right) /; -1 < z < 1 \land m < 1$$

Involving sn^{-1}

09.37.27.0012.01

$$\text{cd}^{-1}(z \mid m) = K(m) - \text{sn}^{-1}(z \mid m) /; z \in \mathbb{R} \land m \in \mathbb{R}$$

Involving elliptic integrals

09.37.27.0013.01

$$\text{cd}^{-1}(z \mid m) = K(m) - F\left(\sin^{-1}(z) \,\middle|\, m\right) /; m \notin (1, \infty)$$

09.37.27.0015.01

$$\text{cd}^{-1}(z \mid m) = \frac{\sqrt{1-m\,z^2}\;\text{sn}\left(\text{cd}^{-1}(z \mid m) \,\middle|\, m\right)}{\sqrt{1-z^2}}\left(K(m) - F\left(\sin^{-1}(z) \,\middle|\, m\right)\right) /;$$
$$\neg\,\exists_{\tau,\{\tau\in\mathbb{R},0<\tau<1\}}\left(\text{Im}\left(1-((z-1)\,\tau+1)^2\right) = 0 \bigwedge 1-((z-1)\,\tau+1)^2 < 0 \bigwedge \text{Im}\left(1-m\,((z-1)\,\tau+1)^2\right) = 0 \bigwedge 1-m\,((z-1)\,\tau+1)^2 < 0\right)$$

09.37.27.0016.01

$$\text{cd}^{-1}(z \mid m) = \text{cd}^{-1}(z_0 \mid m) - \frac{\sqrt{1-m\,z^2}\;\text{sn}\left(\text{cd}^{-1}(z \mid m) \,\middle|\, m\right)}{\sqrt{1-z^2}}\left(F\left(\sin^{-1}(z) \,\middle|\, m\right) - F\left(\sin^{-1}(z_0) \,\middle|\, m\right)\right) /;$$
$$\neg\,\exists_{\tau,\{\tau\in\mathbb{R},0<\tau<1\}}\left(\text{Im}\left(1-(\tau\,(z-z_0)+z_0)^2\right) = 0 \bigwedge 1-(\tau\,(z-z_0)+z_0)^2 < 0 \bigwedge \text{Im}\left(1-m\,(\tau\,(z-z_0)+z_0)^2\right) = 0 \bigwedge 1-m\,(\tau\,(z-z_0)+z_0)^2 < 0\right)$$

Involving other related functions

09.37.27.0014.01

$$\operatorname{cd}^{-1}(z \mid m) == K(m) + \frac{\sqrt{z_2^2}}{z_2} \operatorname{elog}(z_1, z_2; a, b) /;$$
$$\{a, b, z_1\} == \left\{-m - 1, m, \frac{1}{z^2}\right\} \bigwedge z_1^3 + a z_1^2 + b z_1 - z_2^2 == 0 \bigwedge (0 < z < 1 \land m \in \mathbb{R}) \bigvee z < -1 \land m > 1$$

History

– N. H. Abel (1826)
– A. G. Greenhill (1892)
– L. M. Milne–Thompson (1948)

Copyright

This document was downloaded from functions.wolfram.com, a comprehensive online compendium of formulas involving the special functions of mathematics. For a key to the notations used here, see http://functions.wolfram.com/Notations/.

Please cite this document by referring to the functions.wolfram.com page from which it was downloaded, for example:

http://functions.wolfram.com/Constants/E/

To refer to a particular formula, cite functions.wolfram.com followed by the citation number.

e.g.: http://functions.wolfram.com/01.03.03.0001.01

This document is currently in a preliminary form. If you have comments or suggestions, please email comments@functions.wolfram.com.

© 2001-2008, Wolfram Research, Inc.