JacobiCN

Notations

Traditional name

Jacobi elliptic function cn

Traditional notation

$\operatorname{cn}(z \mid m)$

Mathematica StandardForm notation

`JacobiCN[z, m]`

Primary definition

09.26.02.0001.01

$$\operatorname{cn}(z \mid m) == \cos(\operatorname{am}(z \mid m))$$

Specific values

Specialized values

For fixed z

Case $m = 0$

09.26.03.0001.01

$$\operatorname{cn}(z \mid 0) == \cos(z)$$

09.26.03.0002.01

$$\operatorname{cn}\left(z + \frac{\pi}{2} \,\middle|\, 0\right) == -\sin(z)$$

09.26.03.0025.01

$$\operatorname{cn}\left(z + \frac{\pi k}{2} \,\middle|\, 0\right) == \cos\left(z + \frac{\pi k}{2}\right) /; k \in \mathbb{Z}$$

Case $m = 1$

09.26.03.0003.01

$$\operatorname{cn}(z \mid 1) == \operatorname{sech}(z)$$

09.26.03.0026.01

$$\operatorname{cn}\left(z+\frac{\pi i}{2},\,1\right)==-i\operatorname{csch}(z)$$

09.26.03.0027.01

$$\operatorname{cn}\left(z+\frac{i\pi k}{2}\,\middle|\,1\right)==\operatorname{sech}\left(z+\frac{i\pi k}{2}\right)\;/;\;k\in\mathbb{Z}$$

For fixed *m*

Values at quarter-period points in the fundamental period parallelogram

09.26.03.0004.01

$$\operatorname{cn}(0\mid m)==1$$

09.26.03.0005.01

$$\operatorname{cn}(K(m)\mid m)==0$$

09.26.03.0006.01

$$\operatorname{cn}(2K(m)\mid m)==-1$$

09.26.03.0007.01

$$\operatorname{cn}(3K(m)\mid m)==0$$

09.26.03.0008.01

$$\operatorname{cn}(4K(m)\mid m)==1$$

09.26.03.0009.01

$$\operatorname{cn}(i K(1-m)\mid m)==\tilde{\infty}$$

09.26.03.0010.01

$$\operatorname{cn}(2 i K(1-m)\mid m)==-1$$

09.26.03.0011.01

$$\operatorname{cn}(3 i K(1-m)\mid m)==\tilde{\infty}$$

09.26.03.0012.01

$$\operatorname{cn}(4 i K(1-m)\mid m)==1$$

09.26.03.0013.01

$$\operatorname{cn}(K(m)+i K(1-m)\mid m)==-i\,\frac{\sqrt{1-m}}{\sqrt{m}}$$

09.26.03.0014.01

$$\operatorname{cn}(2K(m)+i K(1-m)\mid m)==\tilde{\infty}$$

09.26.03.0015.01

$$\operatorname{cn}(3K(m)+i K(1-m)\mid m)==i\,\frac{\sqrt{1-m}}{\sqrt{m}}$$

09.26.03.0016.01

$$\operatorname{cn}(4K(m)+i K(1-m)\mid m)==\tilde{\infty}$$

09.26.03.0017.01

$$\operatorname{cn}(2rK(m)+(2s+1)\,i K(1-m)\mid m)==\tilde{\infty}\;/;\;\{r,s\}\in\mathbb{Z}$$

09.26.03.0018.01

$$\operatorname{cn}(K(m)+2 i K(1-m)\mid m)==0$$

09.26.03.0019.01
$$\operatorname{cn}(2K(m) + 2 i K(1-m) \mid m) = 1$$

09.26.03.0020.01
$$\operatorname{cn}(3K(m) + 2 i K(1-m) \mid m) = 0$$

09.26.03.0021.01
$$\operatorname{cn}(4K(m) + 2 i K(1-m) \mid m) = -1$$

Values at half-quarter-period points

09.26.03.0022.01
$$\operatorname{cn}\left(\frac{K(m)}{2} \,\middle|\, m\right) = \frac{\sqrt[4]{1-m}}{\sqrt{1+\sqrt{1-m}}}$$

09.26.03.0023.01
$$\operatorname{cn}\left(\frac{i K(1-m)}{2} \,\middle|\, m\right) = \frac{\sqrt{1+\sqrt{m}}}{\sqrt[4]{m}}$$

09.26.03.0024.01
$$\operatorname{cn}\left(\frac{K(m)}{2} + \frac{i K(1-m)}{2} \,\middle|\, m\right) = \frac{\sqrt[4]{1-m}}{\sqrt{2}\,\sqrt[4]{m}}(1-i)$$

General characteristics

Domain and analyticity

$\operatorname{cn}(z \mid m)$ is a meromorphic function of z and m which is defined over $\mathbb{C}^2$.

09.26.04.0001.01
$$(z * m) \longrightarrow \operatorname{cn}(z \mid m) :: (\mathbb{C} \otimes \mathbb{C}) \longrightarrow \mathbb{C}$$

Symmetries and periodicities

Parity

$\operatorname{cn}(z \mid m)$ is an even function with respect to z.

09.26.04.0002.01
$$\operatorname{cn}(-z \mid m) = \operatorname{cn}(z \mid m)$$

Mirror symmetry

09.26.04.0003.01
$$\operatorname{cn}(\bar{z} \mid \bar{m}) = \overline{\operatorname{cn}(z \mid m)}$$

Periodicity

$\operatorname{cn}(z \mid m)$ is a doubly periodic function with respect to z with periods $4 i K(1-m)$ and $4K(m)$.

09.26.04.0004.01
$$\operatorname{cn}(z + 2K(m) \mid m) = -\operatorname{cn}(z \mid m)$$

09.26.04.0005.01

$\operatorname{cn}(z + 4\,K(m) \mid m) = \operatorname{cn}(z \mid m)$

09.26.04.0006.01

$\operatorname{cn}(z + 2\,i\,K(1-m) \mid m) = -\operatorname{cn}(z \mid m)$

09.26.04.0007.01

$\operatorname{cn}(z + 4\,i\,K(1-m) \mid m) = \operatorname{cn}(z \mid m)$

09.26.04.0008.01

$\operatorname{cn}(z + 2\,K(m) + 2\,i\,K(1-m) \mid m) = \operatorname{cn}(z \mid m)$

09.26.04.0009.01

$\operatorname{cn}(z + 2\,i\,s\,K(1-m) + 2\,r\,K(m) \mid m) = (-1)^{r+s} \operatorname{cn}(z \mid m) \;/;\; \{r, s\} \in \mathbb{Z}$

Poles and essential singularities

With respect to *z*

For fixed m, the function cn($z \mid m$) has an infinite set of singular points:

a) $z = 2\,r\,K(m) + (2\,s+1)\,i\,K(1-m)$, $\{r, s\} \in \mathbb{Z}$, are the simple poles with residues $\frac{(-1)^{r+s-1}\,i}{\sqrt{m}}$;

b) $z = \tilde{\infty}$ is an essential singular point.

09.26.04.0010.01

$\mathcal{Sing}_z(\operatorname{cn}(z \mid m)) = \{\{\{(2\,s+1)\,i\,K(1-m) + 2\,r\,K(m), 1\} \;/;\; \{r, s\} \in \mathbb{Z}\}, \{\tilde{\infty}, \infty\}\}$

09.26.04.0011.01

$$\operatorname{res}_z(\operatorname{cn}(z \mid m))((2\,s+1)\,i\,K(1-m) + 2\,r\,K(m)) = \frac{(-1)^{r+s-1}\,i}{\sqrt{m}} \;/;\; \{r, s\} \in \mathbb{Z}$$

Branch points

With respect to *m*

For fixed z, the function cn($z \mid m$) is a meromorphic function in m that has no branch points.

09.26.04.0014.01

$\mathcal{BP}_m(\operatorname{cn}(z \mid m)) = \{\}$

P. Walker

With respect to *z*

For fixed m, the function cn($z \mid m$) does not have branch points.

09.26.04.0012.01

$\mathcal{BP}_z(\operatorname{cn}(z \mid m)) = \{\}$

Branch cuts

With respect to *m*

For fixed z, the function cn($z \mid m$) is a meromorphic function in m that has no branch cuts.

09.26.04.0015.01

$\mathcal{BC}_m(\mathrm{cn}(z \mid m)) = \{\}$

P. Walker

With respect to *z*

For fixed m, the function cn$(z \mid m)$ does not have branch cuts.

09.26.04.0013.01

$\mathcal{BC}_z(\mathrm{cn}(z \mid m)) = \{\}$

Series representations

Generalized power series

Expansions at $z = 0$

09.26.06.0006.01

$$\mathrm{cn}(z \mid m) \propto 1 - \frac{z^2}{2} + \frac{1}{24}(1 + 4m) z^4 + \ldots /; (z \to 0)$$

09.26.06.0001.02

$$\mathrm{cn}(z \mid m) \propto 1 - \frac{z^2}{2} + \frac{1}{24}(1 + 4m) z^4 + \frac{1}{720}\left(-1 - 44m - 16m^2\right) z^6 + \frac{\left(1 + 408m + 912m^2 + 64m^3\right) z^8}{40\,320} + \frac{\left(-1 - 3688m - 30\,768m^2 - 15\,808m^3 - 256m^4\right) z^{10}}{3\,628\,800} + O\left(z^{12}\right)$$

09.26.06.0007.01

$$\mathrm{cn}(z \mid m) = \sum_{k=0}^{\infty} \frac{(-1)^k \mathrm{cn}_k(m) z^{2k}}{(2k)!} /; \mathrm{sn}_0(m) = 1 \bigwedge \mathrm{sn}_n(m) = \sum_{j=0}^{n} \sum_{k=0}^{n} \binom{2n}{2j} \mathrm{cn}_j(m)\, \mathrm{dn}_k(m)\, \delta_{j+k-n} \bigwedge \mathrm{cn}_0(m) = 1 \bigwedge \mathrm{cn}_n(m) = \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} \binom{2n-1}{2j+1} \mathrm{sn}_j(m)\, \mathrm{dn}_k(m)\, \delta_{j+k-n+1} \bigwedge \mathrm{dn}_0(m) = 1 \bigwedge \mathrm{dn}_n(m) = m \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} \binom{2n-1}{2j+1} \mathrm{sn}_j(m)\, \mathrm{cn}_k(m)\, \delta_{j+k-n+1}$$

09.26.06.0008.01

$$\mathrm{cn}(z \mid m) \propto 1 + O\left(z^2\right)$$

Expansions at $z = 2rK(m) + (2s+1)\, i\, K(1-m)$

09.26.06.0009.01

$$\mathrm{cn}(z \mid m) \propto \frac{i\,(-1)^{r+s-1}}{\sqrt{m}} \left(\frac{1}{z - z_0} + \frac{1}{6}(1 - 2m)(z - z_0) + \frac{1}{360}\left(-8m^2 + 8m + 7\right)(z - z_0)^3 + \ldots\right) /; (z \to z_0) \bigwedge z_0 = 2rK(m) + (2s+1)\, i\, K(1-m) \bigwedge r \in \mathbb{Z} \bigwedge s \in \mathbb{Z}$$

09.26.06.0010.01

$$\operatorname{cn}(z \mid m) = \frac{i\,(-1)^{r+s-1}}{\sqrt{m}\,(z-z_0)} \sum_{k=0}^{\infty} \sum_{j=0}^{k} \frac{(j+1)\,(-1)^{k-j}\,\operatorname{dn}_{k-j}(m)}{(2k-2j)!} \sum_{r=0}^{j} \frac{(-1)^r}{r+1} \binom{j}{r} q_{r,j}\,(z-z_0)^{2k-1} \,/;$$

$$z_0 = 2\,r\,K(m) + (2\,s+1)\,i\,K(1-m) \bigwedge r \in \mathbb{Z} \bigwedge s \in \mathbb{Z} \bigwedge q_{j,0} = 1 \bigwedge q_{j,k} = \frac{1}{k} \sum_{i=1}^{k} \frac{(j\,i+i-k)\,(-1)^i\,\operatorname{sn}_i(m)\,q_{j,k-i}}{(2\,i+1)!} \bigwedge$$

$$k \in \mathbb{N}^+ \bigwedge \operatorname{sn}_0(m) = 1 \bigwedge \operatorname{sn}_n(m) = \sum_{j=0}^{n} \sum_{k=0}^{n} \binom{2n}{2j} \operatorname{cn}_j(m)\,\operatorname{dn}_k(m)\,\delta_{j+k-n} \bigwedge \operatorname{cn}_0(m) = 1 \bigwedge$$

$$\operatorname{cn}_n(m) = \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} \binom{2n-1}{2j+1} \operatorname{sn}_j(m)\,\operatorname{dn}_k(m)\,\delta_{j+k-n+1} \bigwedge \operatorname{dn}_0(m) = 1 \bigwedge \operatorname{dn}_n(m) = m \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} \binom{2n-1}{2j+1} \operatorname{sn}_j(m)\,\operatorname{cn}_k(m)\,\delta_{j+k-n+1}$$

09.26.06.0011.01

$$\operatorname{cn}(z \mid m) \propto \frac{i\,(-1)^{r+s-1}}{\sqrt{m}\,(z-z_0)} \left(1 + O\left((z-z_0)^2\right)\right) /;\ z_0 = 2\,r\,K(m) + (2\,s+1)\,i\,K(1-m) \land r \in \mathbb{Z} \land s \in \mathbb{Z}$$

Expansions at *m* == 0

09.26.06.0012.01

$$\operatorname{cn}(z \mid m) \propto \cos(z) + \frac{1}{8} \sin(z)\,(2\,z - \sin(2\,z))\,m + \frac{1}{256} \left(-\left(8\,z^2+9\right) \cos(z) + 8 \cos(3\,z) + \cos(5\,z) + 16\,z \sin(z) + 12\,z \sin(3\,z)\right) m^2 + \ldots /;\ (m \to 0)$$

09.26.06.0013.01

$$\operatorname{cn}(z \mid m) \propto \cos(z) + \frac{1}{8} \sin(z)\,(2\,z - \sin(2\,z))\,m + \frac{1}{256} \left(-\left(8\,z^2+9\right) \cos(z) + 8 \cos(3\,z) + \cos(5\,z) + 16\,z \sin(z) + 12\,z \sin(3\,z)\right) m^2 + \frac{1}{12\,288} \left(-27 \left(8\,z^2+11\right) \cos(z) - 6\left(36\,z^2-41\right) \cos(3\,z) + 48 \cos(5\,z) + 3 \cos(7\,z) - 8\,z\left(4\,z^2-45\right) \sin(z) + 468\,z \sin(3\,z) + 60\,z \sin(5\,z)\right) m^3 + \frac{1}{196\,608} \left(2\left(16\,z^4 - 1128\,z^2 - 1797\right) \cos(z) + \left(2829 - 3888\,z^2\right) \cos(3\,z) - 30\left(20\,z^2-23\right) \cos(5\,z) + 72 \cos(7\,z) + 3 \cos(9\,z) - 4\,z\left(112\,z^2-843\right) \sin(z) - 72\,z\left(12\,z^2-83\right) \sin(3\,z) + 1260\,z \sin(5\,z) + 84\,z \sin(7\,z)\right) m^4 + \frac{1}{15\,728\,640} \left(40\left(76\,z^4 - 3228\,z^2 - 5751\right) \cos(z) + 15\left(864\,z^4 - 19\,800\,z^2 + 11\,597\right) \cos(3\,z) - 240\left(325\,z^2-204\right) \cos(5\,z) - 30\left(196\,z^2-221\right) \cos(7\,z) + 480 \cos(9\,z) + 15 \cos(11\,z) + 8\,z\left(16\,z^4 - 3660\,z^2 + 22\,185\right) \sin(z) - 360\,z\left(276\,z^2-1087\right) \sin(3\,z) - 200\,z\left(100\,z^2-531\right) \sin(5\,z) + 12\,180\,z \sin(7\,z) + 540\,z \sin(9\,z)\right) m^5 + \frac{1}{754\,974\,720} \left(-8\left(32\,z^6 - 17\,820\,z^4 + 588\,375\,z^2 + 1\,146\,195\right) \cos(z) + 90\left(12\,096\,z^4 - 145\,620\,z^2 + 74\,525\right) \cos(3\,z) + 15\left(20\,000\,z^4 - 301\,800\,z^2 + 139\,557\right) \cos(5\,z) - 720\left(833\,z^2-465\right) \cos(7\,z) - 90\left(324\,z^2-359\right) \cos(9\,z) + 1800 \cos(11\,z) + 45 \cos(13\,z) + 12\,z\left(768\,z^4 - 96\,440\,z^2 + 502\,095\right) \sin(z) + 432\,z\left(216\,z^4 - 12\,270\,z^2 + 36\,485\right) \sin(3\,z) - 300\,z\left(6200\,z^2-16\,761\right) \sin(5\,z) - 840\,z\left(196\,z^2-909\right) \sin(7\,z) + 59\,940\,z \sin(9\,z) + 1980\,z \sin(11\,z)\right) m^6 + \frac{1}{84\,557\,168\,640} \left(-14\left(3712\,z^6 - 1\,058\,640\,z^4 + 29\,817\,540\,z^2 + 62\,553\,375\right) \cos(z) - 63\left(20\,736\,z^6 - 2\,453\,760\,z^4 + 21\,231\,000\,z^2 - 9\,882\,415\right) \cos(3\,z) + 315\left(240\,000\,z^4 - 1\,750\,000\,z^2 + 668\,239\right) \cos(5\,z) + 735\left(10\,976\,z^4 - 133\,560\,z^2 + 51\,573\right) \cos(7\,z) - 5040\left(1701\,z^2-874\right) \cos(9\,z) - 630\,(22\,z-23)\,(22\,z+23) \cos(11\,z) + \right.$$

$$15\,120\cos(13z) + 315\cos(15z) - 4z\left(256z^6 - 292\,992z^4 + 27\,155\,520z^2 - 125\,335\,035\right)\sin(z) +$$
$$6804z\left(3168z^4 - 89\,080z^2 + 222\,225\right)\sin(3z) + 2100z\left(4000z^4 - 135\,400z^2 + 257\,937\right)\sin(5z) -$$
$$8820z\left(5096z^2 - 11\,175\right)\sin(7z) - 22\,680z\left(108z^2 - 461\right)\sin(9z) + 623\,700z\sin(11z) + 16\,380z\sin(13z)\Big)m^7 +$$
$$\frac{1}{1\,352\,914\,698\,240}\Big(\left(\left(512z^8 - 1\,075\,200z^6 + 216\,777\,120z^4 - 5\,453\,932\,680z^2 - 12\,187\,866\,915\right)\cos(z) -\right.$$
$$63\left(787\,968z^6 - 44\,094\,240z^4 + 309\,619\,800z^2 - 134\,140\,385\right)\cos(3z) -$$
$$350\left(80\,000z^6 - 5\,256\,000z^4 + 26\,073\,900z^2 - 8\,709\,921\right)\cos(5z) + 5880\left(60\,368z^4 - 334\,950z^2 + 102\,195\right)\cos(7z) +$$
$$315\left(69\,984z^4 - 745\,848z^2 + 253\,307\right)\cos(9z) - 55\,440\left(275z^2 - 133\right)\cos(11z) - 630\left(676z^2 - 731\right)\cos(13z) +$$
$$17\,640\cos(15z) + 315\cos(17z) - 4z\left(8704z^6 - 4\,839\,072z^4 + 370\,434\,960z^2 - 1\,543\,848\,075\right)\sin(z) -$$
$$432z\left(5184z^6 - 1\,137\,024z^4 + 22\,104\,180z^2 - 48\,770\,155\right)\sin(3z) +$$
$$8400z\left(41\,000z^4 - 643\,900z^2 + 982\,101\right)\sin(5z) + 588z\left(76\,832z^4 - 1\,977\,640z^2 + 2\,906\,355\right)\sin(7z) -$$
$$11\,340z\left(10\,152z^2 - 19\,451\right)\sin(9z) - 9240z\left(484z^2 - 1953\right)\sin(11z) + 868\,140z\sin(13z) + 18\,900z\sin(15z)\Big)m^8 +$$
$$\frac{1}{194\,819\,716\,546\,560}\Big(18\left(9984z^8 - 9\,771\,776z^6 + 1\,580\,310\,480z^4 - 36\,477\,444\,780z^2 - 86\,127\,832\,035\right)\cos(z) +$$
$$162\left(186\,624z^8 - 69\,745\,536z^6 + 2\,615\,185\,440z^4 - 15\,856\,977\,780z^2 + 6\,489\,123\,935\right)\cos(3z) -$$
$$630\left(18\,400\,000z^6 - 542\,880\,000z^4 + 2\,102\,350\,500z^2 - 635\,467\,347\right)\cos(5z) -$$
$$126\left(15\,059\,072z^6 - 719\,147\,520z^4 + 2\,605\,542\,660z^2 - 673\,078\,005\right)\cos(7z) +$$
$$45\,360\left(227\,448z^4 - 1\,059\,399z^2 + 274\,924\right)\cos(9z) + 945\left(468\,512z^4 - 4\,562\,184z^2 + 1\,409\,247\right)\cos(11z) -$$
$$45\,360\left(4901z^2 - 2264\right)\cos(13z) - 28\,350\left(180z^2 - 193\right)\cos(15z) + 181\,440\cos(17z) +$$
$$2835\cos(19z) + 16z\left(128z^8 - 454\,464z^6 + 172\,469\,304z^4 - 11\,542\,260\,240z^2 + 43\,954\,585\,605\right)\sin(z) -$$
$$1944z\left(445\,824z^6 - 44\,597\,952z^4 + 684\,627\,300z^2 - 1\,377\,694\,745\right)\sin(3z) -$$
$$4500z\left(160\,000z^6 - 18\,480\,000z^4 + 191\,895\,480z^2 - 250\,280\,163\right)\sin(5z) +$$
$$5292z\left(3\,764\,768z^4 - 42\,904\,400z^2 + 48\,678\,885\right)\sin(7z) + 20\,412z\left(69\,984z^4 - 1\,517\,400z^2 + 1\,876\,715\right)\sin(9z) -$$
$$41\,580z\left(53\,240z^2 - 92\,781\right)\sin(11z) - 98\,280z\left(676z^2 - 2619\right)\sin(13z) +$$
$$10\,376\,100z\sin(15z) + 192\,780z\sin(17z)\Big)m^9 + \frac{1}{15\,585\,577\,323\,724\,800}$$
$$\Big(-\left(4096z^{10} - 23\,109\,120z^8 + 15\,001\,297\,920z^6 - 2\,076\,748\,480\,800z^4 + 44\,740\,696\,971\,600z^2 + 110\,914\,462\,696\,275\right)$$
$$\cos(z) + 405\left(17\,915\,904z^8 - 2\,965\,310\,208z^6 + 85\,818\,096\,000z^4 - 466\,095\,165\,480z^2 + 181\,992\,844\,775\right)\cos(3z) +$$
$$225\left(40\,000\,000z^8 - 7\,554\,400\,000z^6 + 143\,925\,600\,000z^4 - 467\,581\,413\,600z^2 + 130\,691\,579\,823\right)\cos(5z) -$$
$$945\left(542\,126\,592z^6 - 11\,186\,739\,200z^4 + 30\,686\,626\,320z^2 - 6\,994\,157\,025\right)\cos(7z) -$$
$$17\,010\left(2\,519\,424z^6 - 98\,210\,880z^4 + 289\,458\,900z^2 - 61\,983\,655\right)\cos(9z) +$$
$$113\,400\left(1\,171\,280z^4 - 4\,826\,206z^2 + 1\,109\,661\right)\cos(11z) + 4725\left(913\,952z^4 - 8\,335\,080z^2 + 2\,394\,237\right)\cos(13z) -$$
$$680\,400\left(2475z^2 - 1103\right)\cos(15z) - 28\,350\left(1156z^2 - 1231\right)\cos(17z) + 1\,020\,600\cos(19z) + 14\,175\cos(21z) +$$
$$80z\left(5632z^8 - 9\,014\,976z^6 + 2\,676\,722\,328z^4 - 161\,909\,934\,480z^2 + 567\,983\,549\,385\right)\sin(z) +$$
$$43\,740z\left(4608z^8 - 2\,753\,280z^6 + 179\,852\,064z^4 - 2\,336\,147\,800z^2 + 4\,372\,958\,135\right)\sin(3z) -$$
$$54\,000z\left(3\,400\,000z^6 - 171\,703\,000z^4 + 1\,362\,702\,250z^2 - 1\,580\,291\,223\right)\sin(5z) -$$
$$17\,640z\left(2\,151\,296z^6 - 174\,254\,976z^4 + 1\,268\,918\,700z^2 - 1\,197\,550\,395\right)\sin(7z) +$$
$$918\,540z\left(443\,232z^4 - 4\,107\,600z^2 + 3\,812\,645\right)\sin(9z) +$$
$$41\,580z\left(468\,512z^4 - 9\,026\,600z^2 + 9\,864\,255\right)\sin(11z) - 737\,100z\left(28\,392z^2 - 46\,115\right)\sin(13z) -$$
$$1\,701\,000z\left(300z^2 - 1127\right)\sin(15z) + 66\,509\,100z\sin(17z) + 1\,077\,300z\sin(19z)\Big)m^{10} + O\left(m^{11}\right)$$

09.26.06.0014.01

$$\mathrm{cn}(z \mid m) \propto \cos(z)\,(1 + O(m))$$

Expansions at $m == 1$

09.26.06.0015.01

$$\mathrm{cn}(z \mid m) \propto \mathrm{sech}(z) + \frac{1}{4}(\sinh(z) - z\,\mathrm{sech}(z))\tanh(z)\,(m-1) + \frac{1}{512}\left(8\cosh(2z)\,z^2 - 24z^2 + 44\sinh(2z)\,z + 4\sinh(4z)\,z - 11\cosh(4z) + 11\right)\mathrm{sech}^3(z)\,(m-1)^2 + \ldots /; (m \to 1)$$

09.26.06.0016.01

$$\mathrm{cn}(z \mid m) \propto \mathrm{sech}(z) + \frac{1}{4}(m-1)(\sinh(z) - z\,\mathrm{sech}(z))\tanh(z) + \frac{1}{512}\left(8\cosh(2z)\,z^2 - 24z^2 + 44\sinh(2z)\,z + 4\sinh(4z)\,z - 11\cosh(4z) + 11\right)\mathrm{sech}^3(z)\,(m-1)^2 - \frac{1}{49\,152}\left(-15\left(128z^2 - 55\right)\cosh(z) + 3\left(72z^2 - 139\right)\cosh(3z) - 3\left(8z^2 + 137\right)\cosh(5z) + 3\cosh(7z) - 16z\left(46z^2 - 105\right)\sinh(z) + 16z\left(2z^2 + 117\right)\sinh(3z) + 192z\sinh(5z)\right)\mathrm{sech}^4(z)\,(m-1)^3 - \frac{1}{1\,572\,864}\left(-2\left(1840z^4 - 14\,340z^2 + 5487\right) + \left(2432z^4 + 26\,472z^2 - 5361\right)\cosh(2z) - 2\left(16z^4 + 852z^2 - 5529\right)\cosh(4z) + 3\left(168z^2 + 1787\right)\cosh(6z) - 84\cosh(8z) + 4z\left(3304z^2 - 12\,069\right)\sinh(2z) - 4z\left(184z^2 + 7065\right)\sinh(4z) - 4z\left(8z^2 + 675\right)\sinh(6z) + 36z\sinh(8z)\right)\mathrm{sech}^5(z)\,(m-1)^4 + \frac{1}{251\,658\,240}\left(-40\left(14\,644z^4 - 156\,513z^2 + 47\,007\right)\cosh(z) + 30\left(9712z^4 + 63\,780z^2 + 13\,347\right)\cosh(3z) - 5\left(544z^4 + 10\,008z^2 - 226\,323\right)\cosh(5z) + 10\left(16z^4 + 3720z^2 + 35\,691\right)\cosh(7z) - 30\left(36z^2 + 289\right)\cosh(9z) + 15\cosh(11z) - 8z\left(26\,912z^4 - 134\,240z^2 + 423\,615\right)\sinh(z) + 48z\left(632z^4 + 20\,990z^2 - 111\,945\right)\sinh(3z) - 16z\left(8z^4 + 4410z^2 + 135\,615\right)\sinh(5z) - 20z\left(208z^2 + 8973\right)\sinh(7z) + 5940z\sinh(9z)\right)\mathrm{sech}^6(z)\,(m-1)^5 + \frac{1}{24\,159\,191\,040}\left(-2\left(1\,507\,072z^6 - 16\,371\,600z^4 + 140\,414\,400z^2 - 39\,809\,205\right) + 2\left(1\,349\,504z^6 + 8\,115\,360z^4 - 183\,965\,400z^2 + 31\,171\,455\right)\cosh(2z) - 4\left(46\,208z^6 + 4\,100\,160z^4 + 21\,346\,920z^2 + 16\,308\,855\right)\cosh(4z) + \left(256z^6 + 96\,960z^4 + 308\,160z^2 - 62\,820\,045\right)\cosh(6z) - 30\left(496z^4 + 42\,768z^2 + 479\,367\right)\cosh(8z) + 45\left(2736z^2 + 10\,603\right)\cosh(10z) - 1980\cosh(12z) + 24z\left(670\,704z^4 - 4\,085\,480z^2 + 15\,703\,545\right)\sinh(2z) - 60z\left(43\,712z^4 + 733\,888z^2 - 5\,397\,135\right)\sinh(4z) + 60z\left(224z^4 + 59\,288z^2 + 1\,666\,755\right)\sinh(6z) + 24z\left(16z^4 + 6480z^2 + 283\,305\right)\sinh(8z) - 180z\left(72z^2 + 2269\right)\sinh(10z) + 900z\sinh(12z)\right)\mathrm{sech}^7(z)\,(m-1)^6 + \frac{1}{5\,411\,658\,792\,960}\left(63\left(12\,580\,864z^6 - 155\,912\,640z^4 + 1\,551\,549\,600z^2 - 353\,219\,675\right)\cosh(z) - 140\left(4\,273\,024z^6 - 302\,160z^4 - 340\,916\,742z^2 - 2\,529\,567\right)\cosh(3z) + 70\left(628\,480z^6 + 28\,156\,560z^4 + 127\,545\,876z^2 + 184\,163\,139\right)\cosh(5z) - 35\left(1280z^6 + 369\,120z^4 - 100\,368z^2 - 220\,288\,689\right)\cosh(7z) + 7\left(256z^6 + 18\,720z^4 + 9\,236\,880z^2 + 193\,488\,075\right)\cosh(9z) - 630\left(432z^4 + 34\,372z^2 + 91\,485\right)\cosh(11z) + 2520\left(25z^2 + 157\right)\cosh(13z) - 315\cosh(15z) + 8z\left(33\,244\,544z^6 - 298\,278\,624z^4 + 1\,336\,586\,160z^2 - 4\,800\,015\,045\right)\sinh(z) - 12z\left(5\,176\,064z^6 + 166\,506\,816z^4 - 1\,293\,092\,640z^2 + 5\,942\,460\,615\right)\sinh(3z) + 8z\left(278\,912z^6 + 48\,239\,520z^4 + 558\,161\,940z^2 - 5\,371\,069\,725\right)\sinh(5z) - 4z\left(256z^6 + 581\,952z^4 + 90\,658\,680z^2 + 2\,669\,422\,455\right)\sinh(7z) - 252z\left(384z^4 - 1560z^2 + 2\,230\,115\right)\sinh(9z) + 5040z\left(774z^2 + 11\,159\right)\sinh(11z) - 308\,700z\sinh(13z)\right)\mathrm{sech}^8(z)\,(m-1)^7 -$$

$$\frac{1}{173\,173\,081\,374\,720}\Big(-18\left(66\,489\,088\,z^8-1\,040\,174\,464\,z^6+9\,055\,047\,120\,z^4-81\,910\,066\,140\,z^2+17\,975\,180\,825\right)+ 2\left(636\,233\,728\,z^8+2\,298\,564\,352\,z^6-81\,203\,428\,320\,z^4+1\,096\,786\,328\,400\,z^2-158\,774\,880\,285\right)\cosh(2\,z)- 4\left(42\,446\,336\,z^8+3\,271\,466\,240\,z^6-8\,264\,534\,880\,z^4-212\,940\,349\,650\,z^2-48\,463\,987\,005\right)\cosh(4\,z)+ 63\left(53\,248\,z^8+16\,497\,920\,z^6+509\,192\,160\,z^4+2\,070\,293\,400\,z^2+4\,751\,990\,815\right)\cosh(6\,z)- 2\left(256\,z^8+352\,128\,z^6+227\,349\,360\,z^4+1\,054\,258\,380\,z^2-65\,327\,886\,855\right)\cosh(8\,z)+ 7\left(10\,496\,z^6-10\,388\,640\,z^4-60\,396\,840\,z^2+2\,595\,004\,875\right)\cosh(10\,z)- 1260\left(10\,368\,z^4+358\,974\,z^2+760\,673\right)\cosh(12\,z)+6300\left(540\,z^2+1477\right)\cosh(14\,z)- 18\,900\cosh(16\,z)+84\,z\left(94\,641\,152\,z^6-980\,135\,424\,z^4+4\,862\,912\,920\,z^2-19\,212\,070\,275\right)\sinh(2\,z)- 36\,z\left(64\,872\,960\,z^6+839\,939\,296\,z^4-8\,695\,009\,680\,z^2+46\,600\,990\,065\right)\sinh(4\,z)+ 216\,z\left(402\,560\,z^6+33\,543\,216\,z^4+306\,998\,720\,z^2-3\,625\,985\,335\right)\sinh(6\,z)- 4\,z\left(12\,032\,z^6+6\,636\,000\,z^4+1\,063\,802\,040\,z^2+40\,252\,129\,785\right)\sinh(8\,z)- 8\,z\left(128\,z^6-437\,136\,z^4-55\,986\,000\,z^2+742\,062\,195\right)\sinh(10\,z)+756\,z\left(864\,z^4+140\,320\,z^2+1\,345\,585\right)\sinh(12\,z)- 420\,z\left(1000\,z^2+22\,731\right)\sinh(14\,z)+8820\,z\sinh(16\,z)\Big)\operatorname{sech}^9(z)\,(m-1)^8-\frac{1}{49\,873\,847\,435\,919\,360}$$

$$\Big(693\left(640\,648\,192\,z^8-11\,114\,918\,400\,z^6+103\,544\,164\,320\,z^4-1\,011\,114\,289\,440\,z^2+183\,912\,469\,155\right)\cosh(z)- 216\left(2\,021\,590\,912\,z^8-7\,211\,068\,704\,z^6-88\,869\,508\,840\,z^4+1\,917\,658\,532\,895\,z^2-74\,520\,831\,420\right)\cosh(3\,z)+ 720\left(91\,535\,840\,z^8+3\,068\,581\,152\,z^6-12\,921\,515\,670\,z^4-183\,960\,186\,480\,z^2-90\,768\,299\,517\right)\cosh(5\,z)- 9\left(147\,609\,088\,z^8+21\,017\,455\,872\,z^6+497\,265\,867\,840\,z^4+1\,804\,580\,711\,640\,z^2+6\,300\,177\,480\,765\right)\cosh(7\,z)+ 27\left(5632\,z^8+22\,068\,480\,z^6+5\,277\,466\,880\,z^4+35\,863\,419\,480\,z^2-720\,548\,768\,625\right)\cosh(9\,z)- 36\left(128\,z^8-2\,768\,640\,z^6-772\,770\,600\,z^4-7\,816\,771\,620\,z^2+60\,456\,064\,095\right)\cosh(11\,z)+ 189\left(62\,208\,z^6+18\,051\,200\,z^4+404\,104\,440\,z^2+740\,346\,795\right)\cosh(13\,z)- 945\left(20\,000\,z^4+1\,078\,440\,z^2+1\,867\,113\right)\cosh(15\,z)+5670\left(196\,z^2+1095\right)\cosh(17\,z)-2835\cosh(19\,z)+8\,z \left(17\,769\,803\,264\,z^8-233\,220\,314\,880\,z^6+1\,637\,875\,022\,112\,z^4-7\,171\,584\,281\,820\,z^2+26\,945\,149\,506\,345\right)\sinh(z)- 4\,z\left(11\,114\,481\,664\,z^8+329\,411\,764\,224\,z^6-4\,475\,056\,511\,520\,z^4+25\,403\,813\,288\,280\,z^2-109\,796\,213\,117\,115\right)\sinh(3\,z)+20\,z\left(179\,849\,216\,z^8+26\,385\,661\,440\,z^6+181\,372\,280\,544\,z^4-2\,698\,100\,685\,000\,z^2+16\,371\,991\,876\,905\right) \sinh(5\,z)-8\,z\left(5\,036\,288\,z^8+2\,548\,683\,648\,z^6+146\,556\,644\,976\,z^4+1\,168\,817\,466\,600\,z^2-15\,593\,642\,462\,475\right) \sinh(7\,z)+8\,z\left(256\,z^8+1\,648\,512\,z^6-628\,329\,744\,z^4+30\,940\,675\,920\,z^2+2\,691\,609\,178\,545\right)\sinh(9\,z)+ 36\,z\left(11\,776\,z^6-74\,036\,256\,z^4-3\,957\,899\,400\,z^2+12\,244\,321\,845\right)\sinh(11\,z)- 756\,z\left(412\,128\,z^4+27\,634\,520\,z^2+207\,626\,265\right)\sinh(13\,z)+ 94\,500\,z\left(2360\,z^2+22\,737\right)\sinh(15\,z)-5\,159\,700\,z\sinh(17\,z)\Big)\operatorname{sech}^{10}(z)\,(m-1)^9+$$

$$\frac{1}{7\,979\,815\,589\,747\,097\,600}\Big((m-1)^{10}\Big(-3\,127\,485\,374\,464\,z^{10}+63\,366\,273\,653\,760\,z^8- 724\,973\,096\,010\,240\,z^6+5\,803\,927\,864\,257\,600\,z^4-53\,237\,719\,525\,636\,800\,z^2+ 2160\left(11\,081\,770\,240\,z^8-161\,812\,138\,048\,z^6+1\,225\,340\,001\,032\,z^4-5\,764\,403\,332\,800\,z^2+22\,640\,041\,844\,805\right) \sinh(2\,z)\,z-120\left(83\,424\,409\,856\,z^8+723\,001\,694\,976\,z^6-15\,277\,284\,541\,008\,z^4+102\,021\,535\,808\,640\,z^2- 476\,425\,577\,821\,725\right)\sinh(4\,z)\,z+40\left(21\,784\,779\,008\,z^8+1\,391\,637\,227\,904\,z^6+5\,072\,163\,219\,648\,z^4- 126\,500\,645\,043\,360\,z^2+840\,081\,479\,030\,325\right)\sinh(6\,z)\,z-40\left(246\,754\,304\,z^8+56\,320\,072\,704\,z^6+ 2\,612\,805\,933\,024\,z^4+19\,701\,553\,793\,040\,z^2-269\,430\,695\,986\,725\right)\sinh(8\,z)\,z+ 40\left(15\,104\,z^8-97\,188\,480\,z^6-47\,883\,860\,640\,z^4-504\,590\,189\,040\,z^2+39\,292\,204\,182\,525\right)\sinh(10\,z)\,z+ 20\left(512\,z^8-52\,296\,192\,z^6-25\,141\,106\,592\,z^4-908\,266\,867\,200\,z^2+225\,436\,486\,905\right)\sinh(12\,z)\,z- 540\left(186\,624\,z^6+87\,773\,728\,z^4+3\,699\,596\,040\,z^2+23\,994\,177\,585\right)\sinh(14\,z)\,z+ 18\,900\left(20\,000\,z^4+2\,100\,816\,z^2+12\,561\,231\right)\sinh(16\,z)\,z-18\,900\left(2744\,z^2+53\,127\right)\sinh(18\,z)\,z+$$

$$510\,300\sinh(20\,z)\,z + \big(3\,714\,757\,763\,072\,z^{10} + 1\,042\,292\,920\,320\,z^8 - 577\,678\,143\,006\,720\,z^6 + 7\,378\,698\,967\,466\,400\,z^4 - 84\,637\,606\,349\,986\,200\,z^2 + 10\,608\,569\,972\,968\,275\big)\cosh(2\,z) - 2\,\big(365\,160\,251\,392\,z^{10} + 26\,458\,755\,160\,320\,z^8 - 177\,932\,880\,840\,960\,z^6 - 425\,836\,073\,804\,400\,z^4 + 20\,669\,808\,618\,547\,200\,z^2 + 1\,837\,886\,114\,913\,975\big)\cosh(4\,z) + \big(37\,339\,713\,536\,z^{10} + 9\,216\,831\,375\,360\,z^8 + 191\,012\,478\,531\,840\,z^6 - 1\,067\,674\,612\,687\,200\,z^4 - 11\,070\,187\,941\,216\,600\,z^2 - 9\,039\,440\,287\,308\,375\big)\cosh(6\,z) - 224\,\big(1\,079\,552\,z^{10} + 846\,531\,360\,z^8 + 77\,688\,939\,600\,z^6 + 1\,459\,842\,367\,950\,z^4 + 4\,501\,803\,215\,475\,z^2 + 25\,066\,618\,232\,625\big)\cosh(8\,z) + \big(4096\,z^{10} + 14\,837\,760\,z^8 + 188\,079\,816\,960\,z^6 + 21\,156\,536\,469\,600\,z^4 + 155\,271\,976\,234\,200\,z^2 - 1\,580\,276\,908\,253\,925\big)\cosh(10\,z) - 90\,\big(13\,056\,z^8 - 394\,241\,792\,z^6 - 44\,166\,563\,280\,z^4 - 444\,642\,569\,280\,z^2 + 1\,590\,783\,707\,835\big)\cosh(12\,z) + 945\,\big(3\,608\,064\,z^6 + 405\,685\,600\,z^4 + 7\,052\,105\,160\,z^2 + 11\,795\,057\,685\big)\cosh(14\,z) - 4725\,\big(1\,280\,000\,z^4 + 28\,536\,480\,z^2 + 36\,656\,241\big)\cosh(16\,z) + 56\,700\,\big(6860\,z^2 + 15\,751\big)\cosh(18\,z) - 1\,077\,300\cosh(20\,z) + 9\,434\,038\,449\,457\,125\big)\operatorname{sech}^{11}(z)\big) + O\big((m-1)^{11}\big)$$

09.26.06.0017.01

$$\operatorname{cn}(z\mid m) \propto \operatorname{sech}(z) + O(m-1)$$

q-series

09.26.06.0002.01

$$\operatorname{cn}(z\mid m) = \frac{2\pi}{\sqrt{m}\,K(m)}\sum_{n=0}^{\infty}\frac{q(m)^{n+\frac{1}{2}}}{1+q(m)^{2n+1}}\cos\left((2n+1)\frac{\pi z}{2K(m)}\right)$$

09.26.06.0003.01

$$\log(\operatorname{cn}(2K(m)\,z\mid m)) = \log\left(\sin\left(\pi\left(z+\frac{1}{2}\right)\right)\right) - 4\sum_{r=1}^{\infty}\frac{q(m)^r}{r\,(1+(-1)^r q(m)^r)}\sin^2(r\pi z)$$

Other series representations

09.26.06.0004.01

$$\operatorname{cn}(z\mid m) = \frac{\pi}{2\sqrt{m}\,K(1-m)}\sum_{k=-\infty}^{\infty}(-1)^k\operatorname{sech}\left(\pi\frac{K(m)}{K(1-m)}\left(k+\frac{z}{2K(m)}\right)\right)$$

09.26.06.0005.01

$$\operatorname{cn}(z\mid m) \propto \frac{(-1)^{r+s-1}\,i}{\sqrt{m}\,(z-i\,(2s+1)\,K(1-m)-2r\,K(m))} + O(1)\,/;\,(z\to(2s+1)\,i\,K(1-m)+2r\,K(m))\land\{r,s\}\in\mathbb{Z}$$

Product representations

09.26.08.0001.01

$$\operatorname{cn}(z\mid m) = 2\sqrt[4]{q(m)}\,\frac{\sqrt[4]{1-m}}{\sqrt[4]{m}}\cos\left(\frac{\pi z}{2K(m)}\right)\prod_{n=1}^{\infty}\frac{1+2\,q(m)^{2n}\cos\left(\frac{\pi z}{K(m)}\right)+q(m)^{4n}}{1-2\,q(m)^{2n-1}\cos\left(\frac{\pi z}{K(m)}\right)+q(m)^{4n-2}}$$

Differential equations

Ordinary nonlinear differential equations

With respect to *m*

09.26.13.0003.01

$$m^3 z^2 w(m)^{12} + \left(-6 z^2 m^3 + 3 z^2 m^2 + 1\right) w(m)^{10} + \left(15 z^2 m^3 - 15 z^2 m^2 + 3 z^2 m - 4\right) w(m)^8 + \left(-20 z^2 m^3 + 30 z^2 m^2 - 12 z^2 m + z^2 + 6\right) w(m)^6 + \left(15 z^2 m^3 - 30 z^2 m^2 + 18 z^2 m - 3 z^2 - 4\right) w(m)^4 + \left(-6 z^2 m^3 + 15 z^2 m^2 - 12 z^2 m + 3 z^2 + 1\right) w(m)^2 + \left(64 (m-1)^2 m^4 w(m)^6 - 64 (m-1)^2 m^3 (2 m - 1) w(m)^4 + 16 (m-1)^2 m^2 (2 m - 1)^2 w(m)^2\right) w'(m)^4 + \left(-64 (m-1) m^4 w(m)^7 + 32 (m-1) m^2 \left(6 m^2 - 5 m + 2\right) w(m)^5 - 32 (m-1) m \left(6 m^3 - 10 m^2 + 6 m - 1\right) w(m)^3 + 32 (m-1)^3 m (2 m - 1) w(m)\right) w'(m)^3 + (m-1)^3 z^2 + \left(16 m^2 \left(m^2 - m + 1\right) w(m)^8 - 8 m \left(8 m^3 - 14 m^2 + 11 m - 1\right) w(m)^6 + 16 \left(6 m^4 - 15 m^3 + 14 m^2 - 5 m + 1\right) w(m)^4 - 8 (m-1) \left(8 m^3 - 18 m^2 + 13 m - 4\right) w(m)^2 + 16 (m-1)^4\right) w'(m)^2 + \left(16 (m-1)^2 m^4 w(m)^8 - 32 (m-1)^2 m^3 (2 m - 1) w(m)^6 + 16 (m-1)^2 m^2 \left(6 m^2 - 6 m + 1\right) w(m)^4 - 32 (m-1)^3 m^2 (2 m - 1) w(m)^2 + 16 (m-1)^4 m^2\right) w''(m)^2 + \left(8 m^2 w(m)^9 - 8 \left(4 m^2 - 2 m + 1\right) w(m)^7 + 24 \left(2 m^2 - 2 m + 1\right) w(m)^5 - 8 \left(4 m^2 - 6 m + 3\right) w(m)^3 + 8 (m-1)^2 w(m)\right) w'(m) + \left(8 (m-1) m^2 w(m)^9 - 8 (m-1) m (4 m - 1) w(m)^7 + 24 (m-1) m (2 m - 1) w(m)^5 - 8 (m-1) m (4 m - 3) w(m)^3 + 8 (m-1)^2 m\, w(m) + \left(-64 (m-1)^2 m^4 w(m)^7 + 96 (m-1)^2 m^3 (2 m - 1) w(m)^5 - 32 (m-1)^2 m^2 \left(6 m^2 - 6 m + 1\right) w(m)^3 + 32 (m-1)^3 m^2 (2 m - 1) w(m)\right) w'(m)^2 + \left(32 (m-1) m^4 w(m)^8 - 32 (m-1) m^2 \left(4 m^2 - 3 m + 1\right) w(m)^6 + 32 (m-1) m (2 m - 1) \left(3 m^2 - 3 m + 1\right) w(m)^4 - 32 (m-1)^2 m \left(4 m^2 - 5 m + 2\right) w(m)^2 + 32 (m-1)^4 m\right) w'(m)\right) w''(m) == 0 \;/;\; w(m) == \mathrm{cn}(z \mid m)$$

With respect to *z*

09.26.13.0001.01

$$w''(z) + \left(2 m\, w(z)^2 - 2 m + 1\right) w(z) == 0 \;/;\; w(z) == \mathrm{cn}(z \mid m)$$

09.26.13.0002.01

$$w'(z)^2 == \left(1 - w(z)^2\right) \left(1 - m + m\, w(z)^2\right) \;/;\; w(z) == \mathrm{cn}(z \mid m)$$

Transformations

Transformations and argument simplifications

Argument involving basic arithmetic operations

09.26.16.0001.01

$$\mathrm{cn}(i z \mid m) == \frac{1}{\mathrm{cn}(z \mid 1 - m)}$$

09.26.16.0002.01

$$\mathrm{cn}(z \mid 1 - m) == \frac{1}{\mathrm{cn}(i z \mid m)}$$

09.26.16.0006.01

$$\mathrm{cn}(x + i y \mid m) == (\mathrm{cn}(x \mid m)\, \mathrm{cn}(y \mid 1 - m) - i\, \mathrm{sn}(x \mid m)\, \mathrm{dn}(x \mid m)\, \mathrm{sn}(y \mid 1 - m)\, \mathrm{dn}(y \mid 1 - m)) \Big/ \left(\mathrm{cn}(y \mid 1 - m)^2 + m\, \mathrm{sn}(x \mid m)^2\, \mathrm{sn}(y \mid 1 - m)^2\right) \;/;\; \{x, y\} \in \mathbb{R}$$

09.26.16.0007.01

$$\mathrm{cn}\left(\sqrt{1 - m}\; z \,\middle|\, \frac{m}{m - 1}\right) == \frac{\mathrm{cn}(z \mid m)}{\mathrm{dn}(z \mid m)}$$

09.26.16.0008.01

$$\operatorname{cn}(i\,z \mid 1-m) = \frac{1}{\operatorname{cn}(z \mid m)}$$

09.26.16.0009.01

$$\operatorname{cn}\left(\sqrt{m}\, z \,\middle|\, \frac{1}{m}\right) = \operatorname{dn}(z \mid m)$$

09.26.16.0010.01

$$\operatorname{cn}\left(i\sqrt{m}\, z \,\middle|\, \frac{m-1}{m}\right) = \frac{1}{\operatorname{dn}(z \mid m)}$$

09.26.16.0011.01

$$\operatorname{cn}\left(i\sqrt{1-m}\, z \,\middle|\, \frac{1}{1-m}\right) = \frac{\operatorname{dn}(z \mid m)}{\operatorname{cn}(z \mid m)}$$

Landen's transformation:

09.26.16.0012.01

$$\operatorname{cn}\left(\left(1+\sqrt{1-m}\right) z \,\middle|\, \left(\frac{1-\sqrt{1-m}}{1+\sqrt{1-m}}\right)^2\right) = \frac{1-\left(1+\sqrt{1-m}\right)\operatorname{sn}(z \mid m)^2}{\operatorname{dn}(z \mid m)}$$

Gauss' transformation:

09.26.16.0013.01

$$\operatorname{cn}\left(\left(1+\sqrt{m}\right) z \,\middle|\, \frac{4\sqrt{m}}{\left(1+\sqrt{m}\right)^2}\right) = \frac{\operatorname{cn}(z \mid m)\operatorname{dn}(z \mid m)}{1+\sqrt{m}\,\operatorname{sn}(z \mid m)^2}$$

n th degree transformations:

09.26.16.0014.01

$$\operatorname{cn}\left(\frac{z}{M} \,\middle|\, l\right) = \operatorname{cn}(z \mid m) \prod_{r=1}^{\frac{n-1}{2}} \left(1 - \frac{\operatorname{sn}(z \mid m)^2}{\operatorname{sn}\left(\frac{(2r-1)K(m)}{n} \,\middle|\, m\right)^2}\right) \frac{1}{1 - m\operatorname{sn}\left(\frac{2rK(m)}{n} \,\middle|\, m\right)^2 \operatorname{sn}(z \mid m)^2} \,/;$$

$$\frac{n+1}{2} \in \mathbb{Z}^+ \bigwedge l = m^n \prod_{r=1}^{\frac{n-1}{2}} \operatorname{sn}\left(\frac{(2r-1)K(m)}{n} \,\middle|\, m\right)^8 \bigwedge M = \prod_{r=1}^{\frac{n-1}{2}} \frac{\operatorname{sn}\left(\frac{(2r-1)K(m)}{n} \,\middle|\, m\right)^2}{\operatorname{sn}\left(\frac{2rK(m)}{n} \,\middle|\, m\right)^2}$$

09.26.16.0015.01

$$\operatorname{cn}\left(\frac{z}{M} + \frac{K(m)}{n\,M} \,\middle|\, l\right) = -\frac{\sqrt{1-l}\,\operatorname{sn}(z \mid m)}{M\operatorname{cn}(z \mid m)} \prod_{r=1}^{\frac{n}{2}} \left(1 - \frac{\operatorname{sn}(z \mid m)^2}{\operatorname{sn}\left(\frac{2rK(m)}{n} \,\middle|\, m\right)^2}\right) \frac{1}{1 - m\operatorname{sn}\left(\frac{(2r-1)K(m)}{n} \,\middle|\, m\right)^2 \operatorname{sn}(z \mid m)^2} \,/;$$

$$\frac{n}{2} \in \mathbb{Z}^+ \bigwedge l = m^n \prod_{r=1}^{\frac{n}{2}} \operatorname{sn}\left(\frac{(2r-1)K(m)}{n} \,\middle|\, m\right)^8 \bigwedge M = \prod_{r=1}^{\frac{n}{2}} \frac{\operatorname{sn}\left(\frac{(2r-1)K(m)}{n} \,\middle|\, m\right)^2}{\operatorname{sn}\left(\frac{2rK(m)}{n} \,\middle|\, m\right)^2}$$

Argument involving half-periods

09.26.16.0003.01

$$\text{cn}(z+K(m)\mid m) == -\sqrt{1-m}\ \text{sd}(z\mid m)$$

09.26.16.0145.01

$$\text{cn}(z-K(m)\mid m) = \sqrt{1-m}\ \text{sd}(z\mid m)$$

09.26.16.0146.01

$$\text{cn}(z+3K(m)\mid m) = \sqrt{1-m}\ \text{sd}(z\mid m)$$

09.26.16.0147.01

$$\text{cn}(z+(2r+1)K(m)\mid m) = (-1)^{r-1}\sqrt{1-m}\ \text{sd}(z\mid m)\ /;\ r\in\mathbb{Z}$$

09.26.16.0004.01

$$\text{cn}(z+iK(1-m)\mid m) = -\frac{i\,\text{ds}(z\mid m)}{\sqrt{m}}$$

09.26.16.0148.01

$$\text{cn}(z-iK(1-m)\mid m) = \frac{i\,\text{ds}(z\mid m)}{\sqrt{m}}$$

09.26.16.0149.01

$$\text{cn}(z+3iK(1-m)\mid m) = \frac{i}{\sqrt{m}}\,\text{ds}(z\mid m)\ /;\ s\in\mathbb{Z}$$

09.26.16.0150.01

$$\text{cn}(z+(2s+1)iK(1-m)\mid m) = \frac{(-1)^{s-1}i}{\sqrt{m}}\,\text{ds}(z\mid m)\ /;\ s\in\mathbb{Z}$$

09.26.16.0005.01

$$\text{cn}(z+iK(1-m)+K(m)\mid m) = -\frac{i\sqrt{1-m}\ \text{nc}(z\mid m)}{\sqrt{m}}$$

09.26.16.0151.01

$$\text{cn}(z-iK(1-m)+K(m)\mid m) = \frac{i\sqrt{1-m}\ \text{nc}(z\mid m)}{\sqrt{m}}$$

09.26.16.0152.01

$$\text{cn}(z+iK(1-m)-K(m)\mid m) = \frac{i\sqrt{1-m}\ \text{nc}(z\mid m)}{\sqrt{m}}$$

09.26.16.0153.01

$$\text{cn}(z-iK(1-m)-K(m)\mid m) = -\frac{i\sqrt{1-m}\ \text{nc}(z\mid m)}{\sqrt{m}}$$

09.26.16.0154.01

$$\text{cn}(z+iK(1-m)+3K(m)\mid m) = \frac{i\sqrt{1-m}\ \text{nc}(z\mid m)}{\sqrt{m}}$$

09.26.16.0155.01

$$\text{cn}(z+(4s+1)iK(1-m)+(4r+1)K(m)\mid m) = -\frac{i\sqrt{1-m}\ \text{nc}(z\mid m)}{\sqrt{m}}\ /;\ \{r,s\}\in\mathbb{Z}$$

09.26.16.0156.01

$$\operatorname{cn}(z+(4s+1)\,i\,K(1-m)+(4r-1)\,K(m)\mid m)=\frac{i\sqrt{1-m}\;\operatorname{nc}(z\mid m)}{\sqrt{m}}\;/;\;\{r,s\}\in\mathbb{Z}$$

09.26.16.0157.01

$$\operatorname{cn}(z+(4s-1)\,i\,K(1-m)+(4r+1)\,K(m)\mid m)=\frac{i\sqrt{1-m}\;\operatorname{nc}(z\mid m)}{\sqrt{m}}\;/;\;\{r,s\}\in\mathbb{Z}$$

09.26.16.0158.01

$$\operatorname{cn}(z+(4s-1)\,i\,K(1-m)+(4r-1)\,K(m)\mid m)=-\frac{i\sqrt{1-m}\;\operatorname{nc}(z\mid m)}{\sqrt{m}}\;/;\;\{r,s\}\in\mathbb{Z}$$

09.26.16.0159.01

$$\operatorname{cn}(z+(2s+1)\,i\,K(1-m)+(2r+1)\,K(m)\mid m)=\frac{(-1)^{r+s-1}\,i\sqrt{1-m}\;\operatorname{nc}(z\mid m)}{\sqrt{m}}\;/;\;\{r,s\}\in\mathbb{Z}$$

Argument involving inverse Jacobi functions

09.26.16.0160.01

$$\operatorname{cn}\left(\operatorname{cd}^{-1}(z\mid m)\mid m\right)^2=\frac{(m-1)\,z^2}{m\,z^2-1}$$

09.26.16.0161.01

$$\operatorname{cn}\left(\operatorname{cs}^{-1}(z\mid m)\mid m\right)^2=\frac{z^2}{z^2+1}$$

09.26.16.0162.01

$$\operatorname{cn}\left(\operatorname{dc}^{-1}(z\mid m)\mid m\right)^2=\frac{1-m}{z^2-m}$$

09.26.16.0163.01

$$\operatorname{cn}\left(\operatorname{dn}^{-1}(z\mid m)\mid m\right)^2=\frac{z^2-1}{m}+1$$

09.26.16.0164.01

$$\operatorname{cn}\left(\operatorname{ds}^{-1}(z\mid m)\mid m\right)^2=1-\frac{1}{m+z^2}$$

09.26.16.0165.01

$$\operatorname{cn}\left(\operatorname{nc}^{-1}(z\mid m)\mid m\right)=\frac{1}{z}$$

09.26.16.0166.01

$$\operatorname{cn}\left(\operatorname{nd}^{-1}(z\mid m)\mid m\right)^2=1+\frac{1-z^2}{m\,z^2}$$

09.26.16.0167.01

$$\operatorname{cn}\left(\operatorname{ns}^{-1}(z\mid m)\mid m\right)^2=1-\frac{1}{z^2}$$

09.26.16.0168.01

$$\operatorname{cn}\left(\operatorname{sc}^{-1}(z\mid m)\mid m\right)^2=\frac{1}{z^2+1}$$

09.26.16.0169.01

$$\operatorname{cn}\left(\operatorname{sd}^{-1}(z \mid m) \mid m\right)^2 = 1 - \frac{z^2}{m z^2 + 1}$$

09.26.16.0170.01

$$\operatorname{cn}\left(\operatorname{sn}^{-1}(z \mid m) \mid m\right)^2 = 1 - z^2$$

09.26.16.0171.01

$$\operatorname{cn}\left(\operatorname{sn}^{-1}(z \mid m) \mid m\right) = \sqrt{1 - z^2}$$

Addition formulas

09.26.16.0016.01

$$\operatorname{cn}(u + v \mid m) = \frac{\operatorname{cn}(u \mid m) \operatorname{cn}(v \mid m) - \operatorname{sn}(u \mid m) \operatorname{dn}(u \mid m) \operatorname{sn}(v \mid m) \operatorname{dn}(v \mid m)}{1 - m \operatorname{sn}(u \mid m)^2 \operatorname{sn}(v \mid m)^2}$$

09.26.16.0017.01

$$\operatorname{cn}(u + v \mid m) + \operatorname{cn}(u - v \mid m) = \frac{2 \operatorname{cn}(u \mid m) \operatorname{cn}(v \mid m)}{1 - m \operatorname{sn}(u \mid m)^2 \operatorname{sn}(v \mid m)^2}$$

09.26.16.0018.01

$$\operatorname{cn}(u + v \mid m) - \operatorname{cn}(u - v \mid m) = -\frac{2 \operatorname{sn}(u \mid m) \operatorname{dn}(u \mid m) \operatorname{sn}(v \mid m) \operatorname{dn}(v \mid m)}{1 - m \operatorname{sn}(u \mid m)^2 \operatorname{sn}(v \mid m)^2}$$

09.26.16.0019.01

$$\operatorname{cn}(u + v \mid m) \operatorname{cn}(u - v \mid m) = \frac{\operatorname{cn}(v \mid m)^2 - \operatorname{dn}(v \mid m)^2 \operatorname{sn}(u \mid m)^2}{1 - m \operatorname{sn}(u \mid m)^2 \operatorname{sn}(v \mid m)^2}$$

09.26.16.0020.01

$$\operatorname{cn}(u - v \mid m) \operatorname{cn}(u + v \mid m) = \frac{\operatorname{cn}(u \mid m)^2 + \operatorname{cn}(v \mid m)^2}{1 - m \operatorname{sn}(u \mid m)^2 \operatorname{sn}(v \mid m)^2} - 1$$

09.26.16.0021.01

$$\operatorname{cn}(u + v \mid m) \operatorname{cn}(u - v \mid m) = 1 - \frac{\operatorname{dn}(v \mid m)^2 \operatorname{sn}(u \mid m)^2 + \operatorname{dn}(u \mid m)^2 \operatorname{sn}(v \mid m)^2}{1 - m \operatorname{sn}(u \mid m)^2 \operatorname{sn}(v \mid m)^2}$$

09.26.16.0022.01

$$(1 + \operatorname{cn}(u + v \mid m)) (1 + \operatorname{cn}(u - v \mid m)) = \frac{(\operatorname{cn}(u \mid m) + \operatorname{cn}(v \mid m))^2}{1 - m \operatorname{sn}(u \mid m)^2 \operatorname{sn}(v \mid m)^2}$$

09.26.16.0023.01

$$(1 + \operatorname{cn}(u + v \mid m)) (1 - \operatorname{cn}(u - v \mid m)) = \frac{(\operatorname{sn}(u \mid m) \operatorname{dn}(v \mid m) - \operatorname{sn}(v \mid m) \operatorname{dn}(u \mid m))^2}{1 - m \operatorname{sn}(u \mid m)^2 \operatorname{sn}(v \mid m)^2}$$

09.26.16.0024.01

$$\operatorname{cn}(u \mid m) \operatorname{cn}(v \mid m) \operatorname{dn}(u + v \mid m) = \operatorname{cn}(u + v \mid m) \operatorname{dn}(v \mid m) \operatorname{dn}(u \mid m) + (1 - m) \operatorname{sn}(u \mid m) \operatorname{sn}(v \mid m)$$

09.26.16.0025.01

$$\operatorname{dn}(v \mid m) \operatorname{cn}(u \mid m) \operatorname{sn}(u + v \mid m) = \operatorname{dn}(u + v \mid m) \operatorname{sn}(v \mid m) + \operatorname{cn}(u + v \mid m) \operatorname{sn}(u \mid m)$$

09.26.16.0026.01

$$\operatorname{cn}(v \mid m) \operatorname{cn}(u \mid m) \operatorname{cn}(u + v \mid m) = \frac{\operatorname{dn}(v \mid m) \operatorname{dn}(u \mid m) \operatorname{dn}(u + v \mid m) - 1}{m} + 1$$

09.26.16.0027.01

$$\operatorname{cn}(v \mid m)\operatorname{cn}(u \mid m)\operatorname{cn}(u+v \mid m) = \frac{\operatorname{dn}(v \mid m)\operatorname{dn}(u \mid m)\operatorname{dn}(u+v \mid m)-1}{m}+1$$

09.26.16.0028.01

$$\operatorname{sn}(v \mid m)\operatorname{sn}(u \mid m)\operatorname{sn}(u+v \mid m) = -\frac{\operatorname{sn}(u+v \mid m)\left(\operatorname{cn}(u+v \mid m)-\operatorname{cn}(v \mid m)\operatorname{cn}(u \mid m)\right)}{\operatorname{dn}(u+v \mid m)}$$

09.26.16.0029.01

$$\operatorname{sn}(v \mid m)\operatorname{sn}(u \mid m)\operatorname{sn}(u+v \mid m) = -\frac{\operatorname{sn}(u+v \mid m)\left(\operatorname{dn}(u+v \mid m)-\operatorname{dn}(v \mid m)\operatorname{dn}(u \mid m)\right)}{m\operatorname{cn}(u+v \mid m)}$$

09.26.16.0030.01

$$\operatorname{sn}(v \mid m)\operatorname{sn}(u \mid m)\operatorname{sn}(u+v \mid m) = -\frac{\operatorname{sn}(u+v \mid m)\left(\operatorname{cn}(u+v \mid m)-\operatorname{cn}(v \mid m)\operatorname{cn}(u \mid m)\right)}{\operatorname{dn}(u+v \mid m)}$$

09.26.16.0031.01

$$\operatorname{sn}(v \mid m)\operatorname{sn}(u \mid m)\operatorname{sn}(u+v \mid m) = -\frac{\operatorname{sn}(u+v \mid m)\left(\operatorname{dn}(u+v \mid m)-\operatorname{dn}(v \mid m)\operatorname{dn}(u \mid m)\right)}{m\operatorname{cn}(u+v \mid m)}$$

09.26.16.0032.01

$$\operatorname{sn}(u+v \mid m)\operatorname{cn}(v \mid m)\operatorname{dn}(u \mid m) = \operatorname{cn}(u+v \mid m)\operatorname{sn}(v \mid m)+\operatorname{dn}(u+v \mid m)\operatorname{sn}(u \mid m)$$

09.26.16.0033.01

$$\operatorname{cn}(u+v \mid m)\operatorname{cn}(v \mid m)\operatorname{dn}(u \mid m) = \operatorname{dn}(u+v \mid m)\operatorname{cn}(u \mid m)\operatorname{dn}(v \mid m)-(1-m)\operatorname{sn}(u+v \mid m)\operatorname{sn}(v \mid m)$$

Half-angle formulas

09.26.16.0034.01

$$\operatorname{cn}\left(\frac{z}{2} \,\middle|\, m\right)^2 = \frac{\operatorname{cn}(z \mid m)+\operatorname{dn}(z \mid m)}{1+\operatorname{dn}(z \mid m)}$$

Multiple arguments

Double angle formulas

09.26.16.0035.01

$$\operatorname{cn}(2z \mid m) = \frac{\operatorname{cn}(z \mid m)^2-\operatorname{sn}(z \mid m)^2\operatorname{dn}(z \mid m)^2}{1-m\operatorname{sn}(z \mid m)^4}$$

09.26.16.0036.01

$$\operatorname{cn}(2z \mid m) = \frac{\operatorname{cn}(z \mid m)^2-\operatorname{sn}(z \mid m)^2\operatorname{dn}(z \mid m)^2}{\operatorname{cn}(z \mid m)^2+\operatorname{dn}(z \mid m)^2\operatorname{sn}(z \mid m)^2}$$

09.26.16.0037.01

$$\frac{1-\operatorname{cn}(2z \mid m)}{1+\operatorname{cn}(2z \mid m)} = \frac{\operatorname{sn}(z \mid m)^2\operatorname{dn}(z \mid m)^2}{\operatorname{cn}(z \mid m)^2}$$

Multiple angle formulas

09.26.16.0038.01

$$\operatorname{cn}(n z \mid m) = \left(\frac{m}{1-m}\right)^{\frac{n^2-1}{4}} \prod_{\mu,\nu=0}^{n-1} \operatorname{cn}\left(z+\frac{4K(m)(\mu+\nu\tau)}{n} \,\middle|\, m\right) /;\ \frac{n+1}{2} \in \mathbb{Z}^+$$

09.26.16.0039.01

$$n\,\mathrm{cn}(n\,z\mid m) = (-1)^{\frac{1-n}{2}} \sum_{r,s=0}^{n-1} \mathrm{cn}\left(z + \frac{4\,(K(m)\,r + K(m)\,s\,\tau)}{n} \,\middle|\, m\right) /;\ \frac{n+1}{2} \in \mathbb{Z}^+$$

09.26.16.0040.01

$$\mathrm{cn}\left(\frac{2\,n}{\pi}\,K\left(\lambda\left(\frac{n}{\pi\,i}\log(q(m))\right)\right)x \,\middle|\, \lambda\left(\frac{n}{\pi\,i}\log(q(m))\right)\right) = (-1)^{\frac{n-1}{2}} \frac{\sqrt[4]{q(m)^n}}{q(m)^{n/4}} \frac{m^{n/4}}{\sqrt[4]{\lambda\left(\frac{n}{\pi\,i}\log(q(m))\right)}} \frac{\sqrt[4]{1-\lambda\left(\frac{n}{\pi\,i}\log(q(m))\right)}}{(1-m)^{n/4}} \prod_{r=0}^{n-1} \mathrm{cn}\left(\frac{2\,K(m)}{\pi}\left(\frac{\pi\,r}{n} + x\right) \,\middle|\, m\right) /;\ \frac{n+1}{2} \in \mathbb{Z}^+$$

Products of a single Jacobi function

09.26.16.0048.01

$$m^{\frac{p-1}{2}} \prod_{k=0}^{p-1} \mathrm{sn}\left(z + \frac{4\,k\,K(m)}{p} \,\middle|\, m\right) = (-1)^{\frac{p-1}{2}} \left(\prod_{k=1}^{\frac{p-1}{2}} \mathrm{ns}\left(\frac{4\,k\,K(m)}{p} \,\middle|\, m\right)^2\right) \sum_{k=0}^{p-1} \mathrm{sn}\left(z + \frac{4\,k\,K(m)}{p} \,\middle|\, m\right) /;\ \frac{p-1}{2} \in \mathbb{N}$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.26.16.0049.01

$$m^{\frac{p-1}{2}} \prod_{k=0}^{p-1} \mathrm{cn}\left(z + \frac{4\,k\,K(m)}{p} \,\middle|\, m\right) = \left(\prod_{k=1}^{\frac{p-1}{2}} \mathrm{ds}\left(\frac{4\,k\,K(m)}{p} \,\middle|\, m\right)^2\right) \sum_{k=0}^{p-1} \mathrm{cn}\left(z + \frac{4\,k\,K(m)}{p} \,\middle|\, m\right) /;\ \frac{p-1}{2} \in \mathbb{N} \bigwedge r \in \mathbb{N}^+ \bigwedge r < p$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.26.16.0050.01

$$m^{p/2} \prod_{k=0}^{p-1} \mathrm{sn}\left(z + \frac{2\,k\,K(m)}{p} \,\middle|\, m\right) = \left(\prod_{k=1}^{p/2-1} \mathrm{ns}\left(\frac{2\,k\,K(m)}{p} \,\middle|\, m\right)^2\right) \sum_{k=0}^{p-1} (-1)^k\, \mathrm{Z}\left(\mathrm{am}\left(z + \frac{2\,k\,K(m)}{p} \,\middle|\, m\right) \,\middle|\, m\right) /;$$
$$\frac{p}{2} \in \mathbb{N}^+ \bigwedge r \in \mathbb{N}^+ \bigwedge r < p \bigwedge \gcd(p, r) = 1 \bigwedge 1 - m > 0$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.26.16.0051.01

$$m^{p/2} \prod_{k=0}^{p-1} \mathrm{cn}\left(z + \frac{2\,k\,K(m)}{p} \,\middle|\, m\right) = \sqrt{1-m}\ (-1)^{p/2} \left(\prod_{k=1}^{p/2-1} \mathrm{ds}\left(\frac{2\,k\,K(m)}{p} \,\middle|\, m\right)^2\right) \sum_{k=0}^{p-1} (-1)^k\, \mathrm{Z}\left(\mathrm{am}\left(z + \frac{2\,k\,K(m)}{p} \,\middle|\, m\right) \,\middle|\, m\right) /;$$
$$\frac{p}{2} \in \mathbb{N}^+ \bigwedge r \in \mathbb{N}^+ \bigwedge r < p \bigwedge \gcd(p, r) = 1 \bigwedge 1 - m > 0$$

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Sums over products of two Jacobi functions

09.26.16.0041.01

$$\operatorname{cn}(z \mid m) \operatorname{cn}\left(z+\frac{4 K(m)}{3} \,\middle|\, m\right)+\operatorname{cn}\left(z+\frac{4 K(m)}{3} \,\middle|\, m\right) \operatorname{cn}\left(z+\frac{8 K(m)}{3} \,\middle|\, m\right)+\operatorname{cn}\left(z+\frac{8 K(m)}{3} \,\middle|\, m\right) \operatorname{cn}(z \mid m)= -\operatorname{dn}\left(\frac{2 K(m)}{3} \,\middle|\, m\right)\left(\operatorname{dn}\left(\frac{2 K(m)}{3} \,\middle|\, m\right)+2\right) \Big/ \left(\operatorname{dn}\left(\frac{2 K(m)}{3} \,\middle|\, m\right)+1\right)^2$$

Khare/Sukhatme_2002

Khare/Sukhatme_JMP_2002

09.26.16.0042.01

$$\sum_{k=0}^{p-1} \operatorname{cn}\left(z+\frac{4 k K(m)}{p} \,\middle|\, m\right) \operatorname{cn}\left(z+\frac{4(k+1) K(m)}{p} \,\middle|\, m\right)=\sum_{k=0}^{p-1} \operatorname{cn}\left(\frac{4 k K(m)}{p} \,\middle|\, m\right) \operatorname{cn}\left(\frac{4(k+1) K(m)}{p} \,\middle|\, m\right) /; \frac{p-1}{2} \in \mathbb{N}^+$$

Khare/Sukhatme_2002

Khare/Sukhatme_JMP_2002

09.26.16.0043.01

$$\sum_{k=0}^{p-1} \operatorname{cn}\left(z+\frac{4 k K(m)}{p} \,\middle|\, m\right) \operatorname{cn}\left(z+\frac{4(k+n) K(m)}{p} \,\middle|\, m\right)=\sum_{k=0}^{p-1} \operatorname{cn}\left(\frac{4 k K(m)}{p} \,\middle|\, m\right) \operatorname{cn}\left(\frac{4(k+n) K(m)}{p} \,\middle|\, m\right) /; \frac{p-1}{2} \in \mathbb{N}^+ \bigwedge n \in \mathbb{Z} \bigwedge 1 \leq n \leq \frac{p+1}{2}$$

Khare/Sukhatme_2002

Khare/Sukhatme_JMP_2002

09.26.16.0052.01

$$\sum_{k=0}^{p-1} \operatorname{sn}\left(z+\frac{2 k K(m)}{p} \,\middle|\, m\right)\left(\operatorname{cn}\left(z+\frac{2 K(m)(k-r)}{p} \,\middle|\, m\right)+\operatorname{cn}\left(z+\frac{2 K(m)(k+r)}{p} \,\middle|\, m\right)\right)=0 /; p \in \mathbb{N}^+ \wedge r \in \mathbb{N}^+ \wedge r<p-1$$

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09.26.16.0053.01

$$\sum_{k=0}^{p-1} \operatorname{dn}\left(z+\frac{2 K(m) k}{p} \,\middle|\, m\right) \operatorname{dn}\left(z+\frac{2 K(m)(k+r)}{p} \,\middle|\, m\right)=p \operatorname{dn}\left(\frac{2 r K(m)}{p} \,\middle|\, m\right)\left(1-\frac{\left|\operatorname{cn}\left(\frac{2 r K(m)}{p} \,\middle|\, m\right)\right| Z\left(\sin^{-1}\left(\operatorname{sn}\left(\frac{2 K(m)}{p} \,\middle|\, m\right)\right) \,\middle|\, m\right)}{\left|\operatorname{sn}\left(\frac{2 r K(m)}{p} \,\middle|\, m\right)\right| \operatorname{dn}\left(\frac{2 r K(m)}{p} \,\middle|\, m\right)}\right) /; p \in \mathbb{N}^+ \wedge r \in \mathbb{N}^+ \wedge r<p-1 \wedge m \in \mathbb{R} \wedge m<1$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.26.16.0054.01

$$\sum_{k=0}^{p-1} \operatorname{sn}\left(z+\frac{2 K(m) k}{p} \,\middle|\, m\right) \operatorname{sn}\left(z+\frac{2 K(m)(k+r)}{p} \,\middle|\, m\right)=\frac{p \operatorname{cn}\left(\frac{2 r K(m)}{p} \,\middle|\, m\right) Z\left(\sin^{-1}\left(\operatorname{sn}\left(\frac{2 r K(m)}{p} \,\middle|\, m\right)\right) \,\middle|\, m\right)}{m\left|\operatorname{sn}\left(\frac{2 r K(m)}{p} \,\middle|\, m\right)\right|\left|\operatorname{cn}\left(\frac{2 r K(m)}{p} \,\middle|\, m\right)\right|} /; p-2 \in \mathbb{N} \wedge r \in \mathbb{N}^+ \wedge r<p-1 \wedge m \in \mathbb{R} \wedge m<1$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.26.16.0055.01

$$\sum_{k=0}^{p-1} \operatorname{cn}\left(z+\frac{2K(m)k}{p} \,\middle|\, m\right) \operatorname{cn}\left(z+\frac{2K(m)(k+r)}{p} \,\middle|\, m\right) = p \operatorname{cn}\left(\frac{2rK(m)}{p} \,\middle|\, m\right)\left(1-\frac{\operatorname{dn}\left(\frac{2rK(m)}{p} \,\middle|\, m\right) Z\left(\sin^{-1}\left(\operatorname{sn}\left(\frac{2rK(m)}{p} \,\middle|\, m\right)\right) \,\middle|\, m\right)}{m \left|\operatorname{sn}\left(\frac{2rK(m)}{p} \,\middle|\, m\right)\right| \left|\operatorname{cn}\left(\frac{2rK(m)}{p} \,\middle|\, m\right)\right|}\right) /;$$

$$p-2 \in \mathbb{N} \wedge r \in \mathbb{N}^+ \wedge r < p-1 \wedge m \in \mathbb{R} \wedge m < 1$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.26.16.0056.01

$$\sum_{k=0}^{p-1} \operatorname{cn}\left(z+\frac{4kK(m)}{p} \,\middle|\, m\right)\left(\operatorname{dn}\left(z+\frac{4(k-r)K(m)}{p} \,\middle|\, m\right)+\operatorname{dn}\left(z+\frac{4(k+r)K(m)}{p} \,\middle|\, m\right)\right) = 0 /; p \in \mathbb{N}^+ \wedge r \in \mathbb{N}^+ \wedge r < p-1$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.26.16.0057.01

$$\sum_{k=0}^{p-1} \operatorname{dn}\left(z+\frac{4kK(m)}{p} \,\middle|\, m\right)\left(\operatorname{sn}\left(z+\frac{4(k-r)K(m)}{p} \,\middle|\, m\right)+\operatorname{sn}\left(z+\frac{4(k+r)K(m)}{p} \,\middle|\, m\right)\right) = 0 /; p \in \mathbb{N}^+ \wedge r \in \mathbb{N}^+ \wedge r < p$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.26.16.0058.01

$$\sum_{k=0}^{p-1} (-1)^k \operatorname{sn}\left(z+\frac{2kK(m)}{p} \,\middle|\, m\right)\left(\operatorname{cn}\left(z+\frac{2(k-r)K(m)}{p} \,\middle|\, m\right)+\operatorname{cn}\left(z+\frac{2(k+r)K(m)}{p} \,\middle|\, m\right)\right) = 0 /;$$

$$\frac{p}{2} \in \mathbb{N}^+ \bigwedge r \in \mathbb{N}^+ \bigwedge \gcd(p, r) = 1$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.26.16.0059.01

$$\sum_{k=0}^{p-1} (-1)^k \operatorname{dn}\left(z+\frac{2kK(m)}{p} \,\middle|\, m\right) \operatorname{dn}\left(z+\frac{2(k+r)K(m)}{p} \,\middle|\, m\right) = -2 \operatorname{cs}\left(\frac{2rK(m)}{p} \,\middle|\, m\right) \sum_{k=0}^{p-1} (-1)^k Z\left(\operatorname{am}\left(z+\frac{2kK(m)}{p} \,\middle|\, m\right) \,\middle|\, m\right) /;$$

$$\frac{p}{2} \in \mathbb{N}^+ \bigwedge r \in \mathbb{N}^+ \bigwedge r < p \bigwedge \gcd(p, r) = 1 \bigwedge 1-m > 0$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.26.16.0060.01

$$\sum_{k=0}^{p-1}(-1)^k \operatorname{sn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{2\,(k+r)\,K(m)}{p}\,\middle|\,m\right)=\frac{2}{m}\operatorname{ns}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\sum_{k=0}^{p-1}(-1)^k Z\left(\operatorname{am}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right)\middle|\,m\right)\,/;\,\frac{p}{2}\in\mathbb{N}^+\bigwedge r\in\mathbb{N}^+\bigwedge r<p\bigwedge\gcd(p,r)=1\bigwedge 1-m>0$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.26.16.0061.01

$$\sum_{k=0}^{p-1}(-1)^k \operatorname{cn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right)\operatorname{cn}\left(z+\frac{2\,(k+r)\,K(m)}{p}\,\middle|\,m\right)=-\frac{2}{m}\operatorname{ds}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\sum_{k=0}^{p-1}(-1)^k Z\left(\operatorname{am}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right)\middle|\,m\right)\,/;\,\frac{p}{2}\in\mathbb{N}^+\bigwedge r\in\mathbb{N}^+\bigwedge r<p\bigwedge\gcd(p,r)=1\bigwedge 1-m>0$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

Sums over products of three Jacobi functions

09.26.16.0044.01

$$\operatorname{cn}(z\mid m)\operatorname{cn}\left(z+\frac{4\,K(m)}{3}\,\middle|\,m\right)\operatorname{cn}\left(z+\frac{8\,K(m)}{3}\,\middle|\,m\right)=\frac{\operatorname{dn}\left(\frac{2\,K(m)}{3}\,\middle|\,m\right)^2}{1-\operatorname{dn}\left(\frac{2\,K(m)}{3}\,\middle|\,m\right)^2}\left(\operatorname{cn}(z\mid m)+\operatorname{cn}\left(z+\frac{4\,K(m)}{3}\,\middle|\,m\right)+\operatorname{cn}\left(z+\frac{8\,K(m)}{3}\,\middle|\,m\right)\right)$$

Khare/Sukhatme_2002

Khare/Sukhatme_JMP_2002

09.26.16.0045.01

$$\operatorname{cn}(z\mid m)^2\left(\operatorname{cn}\left(z+\frac{4\,K(m)}{3}\,\middle|\,m\right)+\operatorname{cn}\left(z+\frac{8\,K(m)}{3}\,\middle|\,m\right)\right)+\operatorname{cn}\left(z+\frac{4\,K(m)}{3}\,\middle|\,m\right)^2\left(\operatorname{cn}\left(z+\frac{8\,K(m)}{3}\,\middle|\,m\right)+\operatorname{cn}(z\mid m)\right)+\operatorname{cn}\left(z+\frac{8\,K(m)}{3}\,\middle|\,m\right)^2\left(\operatorname{cn}(z\mid m)+\operatorname{cn}\left(z+\frac{4\,K(m)}{3}\,\middle|\,m\right)\right)=$$
$$-2\operatorname{dn}\left(\frac{2\,K(m)}{3}\,\middle|\,m\right)\left(\operatorname{dn}\left(\frac{2\,K(m)}{3}\,\middle|\,m\right)^2+\operatorname{dn}\left(\frac{2\,K(m)}{3}\,\middle|\,m\right)+1\right)\Big/\left(\left(1+\operatorname{dn}\left(\frac{2\,K(m)}{3}\,\middle|\,m\right)\right)\left(1-\operatorname{dn}\left(\frac{2\,K(m)}{3}\,\middle|\,m\right)^2\right)\right)\left(\operatorname{cn}(z\mid m)+\operatorname{cn}\left(z+\frac{4\,K(m)}{3}\,\middle|\,m\right)+\operatorname{cn}\left(z+\frac{8\,K(m)}{3}\,\middle|\,m\right)\right)$$

Khare/Sukhatme_2002

Khare/Sukhatme_JMP_2002

09.26.16.0046.01

$$\frac{\sum_{k=0}^{p-1} \operatorname{cn}\left(z+\frac{4 k K(m)}{p} \middle| m\right) \operatorname{cn}\left(z+\frac{4 K(m)(k+n_1)}{p} \middle| m\right) \operatorname{cn}\left(z+\frac{4 K(m)(k+n_2)}{p} \middle| m\right)}{\sum_{k=0}^{p-1} \operatorname{cn}\left(z+\frac{4 k K(m)}{p} \middle| m\right)} = \frac{\sum_{k=0}^{p-1} \operatorname{cn}\left(\frac{4 k K(m)}{p} \middle| m\right) \operatorname{cn}\left(\frac{4 (k+n_1) K(m)}{p} \middle| m\right) \operatorname{cn}\left(\frac{4 (k+n_2) K(m)}{p} \middle| m\right)}{\sum_{k=0}^{p-1} \operatorname{cn}\left(\frac{4 k K(m)}{p} \middle| m\right)} /; \frac{p-1}{2} \in \mathbb{N}^+ \bigwedge n_1 \in \mathbb{Z} \bigwedge n_2 \in \mathbb{Z} \bigwedge 1 \leq n_1 < n_2 < p$$

Khare/Sukhatme_2002

Khare/Sukhatme_JMP_2002

09.26.16.0062.01

$$\sum_{k=0}^{p-1} \operatorname{dn}\left(z+\frac{2 K(m) k}{p} \middle| m\right)^2 \left(\operatorname{dn}\left(z+\frac{2 K(m)(k-r)}{p} \middle| m\right)+\operatorname{dn}\left(z+\frac{2 K(m)(k+r)}{p} \middle| m\right)\right) = 2\left(\operatorname{ds}\left(\frac{2 r K(m)}{p} \middle| m\right) \operatorname{ns}\left(\frac{2 r K(m)}{p} \middle| m\right)-\operatorname{cs}\left(\frac{2 r K(m)}{p} \middle| m\right)^2\right) \sum_{k=0}^{p-1} \operatorname{dn}\left(z+\frac{2 K(m) k}{p} \middle| m\right) /; p \in \mathbb{N}^+ \wedge r \in \mathbb{N}^+ \wedge r < p$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.26.16.0063.01

$$\sum_{k=0}^{p-1} \operatorname{cn}\left(z+\frac{2 K(m) k}{p} \middle| m\right) \left(\operatorname{cn}\left(z+\frac{2 K(m)(k-r)}{p} \middle| m\right) \operatorname{dn}\left(z+\frac{2 K(m)(k-r)}{p} \middle| m\right)+\operatorname{cn}\left(z+\frac{2 K(m)(k+r)}{p} \middle| m\right) \operatorname{dn}\left(z+\frac{2 K(m)(k+r)}{p} \middle| m\right)\right) = -\frac{2}{m} \operatorname{cs}\left(\frac{2 r K(m)}{p} \middle| m\right) \left(\operatorname{ds}\left(\frac{2 r K(m)}{p} \middle| m\right)-\operatorname{ns}\left(\frac{2 r K(m)}{p} \middle| m\right)\right) \sum_{k=0}^{p-1} \operatorname{dn}\left(z+\frac{2 K(m) k}{p} \middle| m\right) /; p \in \mathbb{N}^+ \wedge r \in \mathbb{N}^+ \wedge r < p$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.26.16.0064.01

$$\sum_{k=0}^{p-1} \operatorname{sn}\left(z+\frac{2 k K(m)}{p} \middle| m\right) \left(\operatorname{dn}\left(z+\frac{2 K(m)(k-r)}{p} \middle| m\right) \operatorname{sn}\left(z+\frac{2 K(m)(k-r)}{p} \middle| m\right)+\operatorname{dn}\left(z+\frac{2 K(m)(k+r)}{p} \middle| m\right) \operatorname{sn}\left(z+\frac{2 K(m)(k+r)}{p} \middle| m\right)\right) = -\frac{2}{m} \operatorname{cs}\left(\frac{2 r K(m)}{p} \middle| m\right) \left(\operatorname{ds}\left(\frac{2 r K(m)}{p} \middle| m\right)-\operatorname{ns}\left(\frac{2 r K(m)}{p} \middle| m\right)\right) \sum_{k=0}^{p-1} \operatorname{dn}\left(z+\frac{2 K(m) k}{p} \middle| m\right) /; p \in \mathbb{N}^+ \wedge r \in \mathbb{N}^+ \wedge r < p$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.26.16.0065.01

$$\sum_{k=0}^{p-1} \operatorname{dn}\left(z+\frac{2K(m)k}{p}\,\middle|\,m\right) \left(\operatorname{dn}\left(z+\frac{2K(m)(k-r)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{2K(m)(k-s)}{p}\,\middle|\,m\right)+\operatorname{dn}\left(z+\frac{2K(m)(k+r)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{2K(m)(k+s)}{p}\,\middle|\,m\right)\right) = -2\left(\operatorname{cs}\left(\frac{2(r-s)K(m)}{p}\,\middle|\,m\right)\left(\operatorname{cs}\left(\frac{2rK(m)}{p}\,\middle|\,m\right)-\operatorname{cs}\left(\frac{2sK(m)}{p}\,\middle|\,m\right)\right)+\operatorname{cs}\left(\frac{2rK(m)}{p}\,\middle|\,m\right)\operatorname{cs}\left(\frac{2sK(m)}{p}\,\middle|\,m\right)\right) \sum_{k=0}^{p-1}\operatorname{dn}\left(z+\frac{2K(m)k}{p}\,\middle|\,m\right) /;\, p\in\mathbb{N}^+\land r\in\mathbb{N}^+\land r<p\land s\in\mathbb{N}^+\land s<r$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.26.16.0066.01

$$\sum_{k=0}^{p-1} \operatorname{dn}\left(z+\frac{2K(m)k}{p}\,\middle|\,m\right) \left(\operatorname{cn}\left(z+\frac{2K(m)(k-r)}{p}\,\middle|\,m\right)\operatorname{cn}\left(z+\frac{2K(m)(k-s)}{p}\,\middle|\,m\right)+\operatorname{cn}\left(z+\frac{2K(m)(k+r)}{p}\,\middle|\,m\right)\operatorname{cn}\left(z+\frac{2K(m)(k+s)}{p}\,\middle|\,m\right)\right) = -\frac{2}{m}\left(\operatorname{ds}\left(\frac{2rK(m)}{p}\,\middle|\,m\right)\operatorname{ds}\left(\frac{2sK(m)}{p}\,\middle|\,m\right)+\operatorname{ds}\left(\frac{2(r-s)K(m)}{p}\,\middle|\,m\right)\left(\operatorname{cs}\left(\frac{2rK(m)}{p}\,\middle|\,m\right)-\operatorname{cs}\left(\frac{2sK(m)}{p}\,\middle|\,m\right)\right)\right) \sum_{k=0}^{p-1}\operatorname{dn}\left(z+\frac{2K(m)k}{p}\,\middle|\,m\right) /;\, p\in\mathbb{N}^+\land r\in\mathbb{N}^+\land r<p\land s\in\mathbb{N}^+\land s<r$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.26.16.0067.01

$$\sum_{k=0}^{p-1} \operatorname{dn}\left(z+\frac{2K(m)k}{p}\,\middle|\,m\right) \left(\operatorname{sn}\left(z+\frac{2K(m)(k-r)}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{2K(m)(k-s)}{p}\,\middle|\,m\right)+\operatorname{sn}\left(z+\frac{2K(m)(k+r)}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{2K(m)(k+s)}{p}\,\middle|\,m\right)\right) = \frac{2}{m}\left(\operatorname{ns}\left(\frac{2rK(m)}{p}\,\middle|\,m\right)\operatorname{ns}\left(\frac{2sK(m)}{p}\,\middle|\,m\right)+\operatorname{ns}\left(\frac{2(r-s)K(m)}{p}\,\middle|\,m\right)\left(\operatorname{cs}\left(\frac{2rK(m)}{p}\,\middle|\,m\right)-\operatorname{cs}\left(\frac{2sK(m)}{p}\,\middle|\,m\right)\right)\right) \sum_{k=0}^{p-1}\operatorname{dn}\left(z+\frac{2K(m)k}{p}\,\middle|\,m\right) /;\, p\in\mathbb{N}^+\land r\in\mathbb{N}^+\land r<p\land s\in\mathbb{N}^+\land s<r$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.26.16.0068.01

$$\sum_{k=0}^{p-1} \operatorname{cn}\left(z+\frac{2K(m)k}{p} \,\middle|\, m\right) \left(\operatorname{cn}\left(z+\frac{2K(m)(k-r)}{p} \,\middle|\, m\right) \operatorname{dn}\left(z+\frac{2K(m)(k-s)}{p} \,\middle|\, m\right) + \operatorname{cn}\left(z+\frac{2K(m)(k+r)}{p} \,\middle|\, m\right) \operatorname{dn}\left(z+\frac{2K(m)(k+s)}{p} \,\middle|\, m\right)\right) == -\frac{2}{m} \left(\left(\operatorname{cs}\left(\frac{2sK(m)}{p} \,\middle|\, m\right) + \operatorname{cs}\left(\frac{2(r-s)K(m)}{p} \,\middle|\, m\right)\right) \operatorname{ds}\left(\frac{2rK(m)}{p} \,\middle|\, m\right) - \operatorname{ds}\left(\frac{2(r-s)K(m)}{p} \,\middle|\, m\right) \operatorname{ds}\left(\frac{2sK(m)}{p} \,\middle|\, m\right)\right) \sum_{k=0}^{p-1} \operatorname{dn}\left(z+\frac{2K(m)k}{p} \,\middle|\, m\right) /;\ p \in \mathbb{N}^+ \wedge r \in \mathbb{N}^+ \wedge r < p \wedge s \in \mathbb{N}^+ \wedge s < r$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.26.16.0069.01

$$\sum_{k=0}^{p-1} \operatorname{sn}\left(z+\frac{2K(m)k}{p} \,\middle|\, m\right) \left(\operatorname{dn}\left(z+\frac{2K(m)(k-s)}{p} \,\middle|\, m\right) \operatorname{sn}\left(z+\frac{2K(m)(k-r)}{p} \,\middle|\, m\right) + \operatorname{dn}\left(z+\frac{2K(m)(k+s)}{p} \,\middle|\, m\right) \operatorname{sn}\left(z+\frac{2K(m)(k+r)}{p} \,\middle|\, m\right)\right) == \frac{2}{m} \left(\left(\operatorname{cs}\left(\frac{2sK(m)}{p} \,\middle|\, m\right) + \operatorname{cs}\left(\frac{2(r-s)K(m)}{p} \,\middle|\, m\right)\right) \operatorname{ns}\left(\frac{2rK(m)}{p} \,\middle|\, m\right) - \operatorname{ns}\left(\frac{2(r-s)K(m)}{p} \,\middle|\, m\right) \operatorname{ns}\left(\frac{2sK(m)}{p} \,\middle|\, m\right)\right) \sum_{k=0}^{p-1} \operatorname{dn}\left(z+\frac{2K(m)k}{p} \,\middle|\, m\right) /;\ p \in \mathbb{N}^+ \wedge r \in \mathbb{N}^+ \wedge r < p \wedge s \in \mathbb{N}^+ \wedge s < r$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.26.16.0070.01

$$\sum_{k=0}^{p-1} \operatorname{cn}\left(z+\frac{2K(m)k}{p} \,\middle|\, m\right) \operatorname{dn}\left(z+\frac{2K(m)k}{p} \,\middle|\, m\right) \left(\operatorname{sn}\left(z+\frac{2K(m)(k-r)}{p} \,\middle|\, m\right) + \operatorname{sn}\left(z+\frac{2K(m)(k+r)}{p} \,\middle|\, m\right)\right) == 0 /;\ p \in \mathbb{N}^+ \wedge r \in \mathbb{N}^+ \wedge r < p-1$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.26.16.0071.01

$$\sum_{k=0}^{p-1} \operatorname{dn}\left(z+\frac{2K(m)k}{p} \,\middle|\, m\right) \operatorname{sn}\left(z+\frac{2K(m)k}{p} \,\middle|\, m\right) \left(\operatorname{cn}\left(z+\frac{2K(m)(k-r)}{p} \,\middle|\, m\right) + \operatorname{cn}\left(z+\frac{2K(m)(k+r)}{p} \,\middle|\, m\right)\right) == 0 /;\ p \in \mathbb{N}^+ \wedge r \in \mathbb{N}^+ \wedge r < p-1$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.26.16.0072.01

$$\sum_{k=0}^{p-1} \operatorname{cn}\left(z+\frac{2K(m)k}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{2K(m)k}{p}\,\middle|\,m\right)\left(\operatorname{dn}\left(z+\frac{2K(m)(k-r)}{p}\,\middle|\,m\right)+\operatorname{dn}\left(z+\frac{2K(m)(k+r)}{p}\,\middle|\,m\right)\right)=0\ /;\ p\in\mathbb{N}^+\land r\in\mathbb{N}^+\land r<p-1$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.26.16.0073.01

$$\sum_{k=0}^{p-1} \operatorname{cn}\left(z+\frac{2K(m)k}{p}\,\middle|\,m\right)\left(\operatorname{dn}\left(z+\frac{2K(m)(k-s)}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{2K(m)(k-r)}{p}\,\middle|\,m\right)+\operatorname{dn}\left(z+\frac{2K(m)(k+s)}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{2K(m)(k+r)}{p}\,\middle|\,m\right)\right)=0\ /;\ p\in\mathbb{N}^+\land r\in\mathbb{N}^+\land r<p-1\land s\in\mathbb{N}^+\land s<p-1$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.26.16.0074.01

$$\sum_{k=0}^{p-1} \operatorname{sn}\left(z+\frac{4kK(m)}{p}\,\middle|\,m\right)^2\left(\operatorname{sn}\left(z+\frac{4(k-r)K(m)}{p}\,\middle|\,m\right)+\operatorname{sn}\left(z+\frac{4(k+r)K(m)}{p}\,\middle|\,m\right)\right)=\frac{2}{m}\left(\operatorname{ns}\left(\frac{4rK(m)}{p}\,\middle|\,m\right)^2-\operatorname{ds}\left(\frac{4rK(m)}{p}\,\middle|\,m\right)\operatorname{cs}\left(\frac{4rK(m)}{p}\,\middle|\,m\right)\right)\sum_{k=0}^{p-1}\operatorname{sn}\left(z+\frac{4kK(m)}{p}\,\middle|\,m\right)\ /;\ p\in\mathbb{N}^+\land r\in\mathbb{N}^+\land r<p-1$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.26.16.0075.01

$$\sum_{k=0}^{p-1} \operatorname{cn}\left(z+\frac{4kK(m)}{p}\,\middle|\,m\right)\left(\operatorname{cn}\left(z+\frac{4(k-r)K(m)}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{4(k-r)K(m)}{p}\,\middle|\,m\right)+\operatorname{cn}\left(z+\frac{4(k+r)K(m)}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{4(k+r)K(m)}{p}\,\middle|\,m\right)\right)=\frac{2}{m}\operatorname{ns}\left(\frac{4rK(m)}{p}\,\middle|\,m\right)\left(\operatorname{cs}\left(\frac{4rK(m)}{p}\,\middle|\,m\right)-\operatorname{ds}\left(\frac{4rK(m)}{p}\,\middle|\,m\right)\right)\sum_{k=0}^{p-1}\operatorname{sn}\left(z+\frac{4kK(m)}{p}\,\middle|\,m\right)\ /;\ p\in\mathbb{N}^+\land r\in\mathbb{N}^+\land r<p-1$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.26.16.0076.01

$$\sum_{k=0}^{p-1}\operatorname{dn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right)\left(\operatorname{dn}\left(z+\frac{4\,(k-r)\,K(m)}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{4\,(k-r)\,K(m)}{p}\,\middle|\,m\right)+\operatorname{dn}\left(z+\frac{4\,(k+r)\,K(m)}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{4\,(k+r)\,K(m)}{p}\,\middle|\,m\right)\right)=2\operatorname{ns}\left(\frac{8\,r\,K(m)}{p}\,\middle|\,m\right)\left(\operatorname{ds}\left(\frac{8\,r\,K(m)}{p}\,\middle|\,m\right)-\operatorname{cs}\left(\frac{8\,r\,K(m)}{p}\,\middle|\,m\right)\right)\sum_{k=0}^{p-1}\operatorname{sn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right)\;/;\;p\in\mathbb{N}^+\land r\in\mathbb{N}^+\land r<p-1$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.26.16.0077.01

$$\sum_{k=0}^{p-1}\operatorname{sn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right)\left(\operatorname{cn}\left(z+\frac{4\,(k-r)\,K(m)}{p}\,\middle|\,m\right)\operatorname{cn}\left(z+\frac{4\,(k-s)\,K(m)}{p}\,\middle|\,m\right)+\operatorname{cn}\left(z+\frac{4\,(k+r)\,K(m)}{p}\,\middle|\,m\right)\operatorname{cn}\left(z+\frac{4\,(k+s)\,K(m)}{p}\,\middle|\,m\right)\right)=-\frac{2}{m}\left(\operatorname{ds}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right)\operatorname{ds}\left(\frac{4\,s\,K(m)}{p}\,\middle|\,m\right)+\operatorname{ds}\left(\frac{4\,(r-s)\,K(m)}{p}\,\middle|\,m\right)\left(\operatorname{ns}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right)-\operatorname{ns}\left(\frac{4\,s\,K(m)}{p}\,\middle|\,m\right)\right)\right)\sum_{k=0}^{p-1}\operatorname{sn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right)\;/;\;p\in\mathbb{N}^+\land r\in\mathbb{N}^+\land r<p\land s\in\mathbb{N}^+\land s<r$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.26.16.0078.01

$$\sum_{k=0}^{p-1}\operatorname{sn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right)\left(\operatorname{dn}\left(z+\frac{4\,(k-r)\,K(m)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{4\,(k-s)\,K(m)}{p}\,\middle|\,m\right)+\operatorname{dn}\left(z+\frac{4\,(k+r)\,K(m)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{4\,(k+s)\,K(m)}{p}\,\middle|\,m\right)\right)=-2\left(\operatorname{cs}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right)\operatorname{cs}\left(\frac{4\,s\,K(m)}{p}\,\middle|\,m\right)+\operatorname{cs}\left(\frac{4\,(r-s)\,K(m)}{p}\,\middle|\,m\right)\left(\operatorname{ns}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right)-\operatorname{ns}\left(\frac{4\,s\,K(m)}{p}\,\middle|\,m\right)\right)\right)\sum_{k=0}^{p-1}\operatorname{sn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right)\;/;\;p\in\mathbb{N}^+\land r\in\mathbb{N}^+\land r<p\land s\in\mathbb{N}^+\land s<r$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.26.16.0079.01

$$\sum_{k=0}^{p-1} \mathrm{dn}\left(z + \frac{4\,k\,K(m)}{p} \,\middle|\, m\right) \left(\mathrm{cn}\left(z + \frac{4\,(k-r)\,K(m)}{p} \,\middle|\, m\right) \mathrm{dn}\left(z + \frac{4\,(k-r)\,K(m)}{p} \,\middle|\, m\right) + \mathrm{cn}\left(z + \frac{4\,(k+r)\,K(m)}{p} \,\middle|\, m\right) \mathrm{dn}\left(z + \frac{4\,(k+r)\,K(m)}{p} \,\middle|\, m\right)\right) = -2\,\mathrm{ds}\left(\frac{4\,r\,K(m)}{p} \,\middle|\, m\right) \left(\mathrm{cs}\left(\frac{4\,r\,K(m)}{p} \,\middle|\, m\right) - \mathrm{ns}\left(\frac{4\,r\,K(m)}{p} \,\middle|\, m\right)\right) \sum_{k=0}^{p-1} \mathrm{cn}\left(z + \frac{4\,k\,K(m)}{p} \,\middle|\, m\right) /;\; p \in \mathbb{N}^+ \bigwedge r \in \mathbb{N}^+ \bigwedge r < p$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.26.16.0080.01

$$\sum_{k=0}^{p-1} \mathrm{sn}\left(z + \frac{4\,k\,K(m)}{p} \,\middle|\, m\right) \left(\mathrm{cn}\left(z + \frac{4\,(k-r)\,K(m)}{p} \,\middle|\, m\right) \mathrm{sn}\left(z + \frac{4\,(k-r)\,K(m)}{p} \,\middle|\, m\right) + \mathrm{cn}\left(z + \frac{4\,(k+r)\,K(m)}{p} \,\middle|\, m\right) \mathrm{sn}\left(z + \frac{4\,(k+r)\,K(m)}{p} \,\middle|\, m\right)\right) = -\frac{2}{m}\,\mathrm{ds}\left(\frac{4\,r\,K(m)}{p} \,\middle|\, m\right) \left(\mathrm{cs}\left(\frac{4\,r\,K(m)}{p} \,\middle|\, m\right) - \mathrm{ns}\left(\frac{4\,r\,K(m)}{p} \,\middle|\, m\right)\right) \sum_{k=0}^{p-1} \mathrm{cn}\left(z + \frac{4\,k\,K(m)}{p} \,\middle|\, m\right) /;\; p \in \mathbb{N}^+ \bigwedge r \in \mathbb{N}^+ \bigwedge r < p$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.26.16.0081.01

$$\sum_{k=0}^{p-1} \mathrm{cn}\left(z + \frac{4\,k\,K(m)}{p} \,\middle|\, m\right) \left(\mathrm{sn}\left(z + \frac{4\,(k-r)\,K(m)}{p} \,\middle|\, m\right) \mathrm{sn}\left(z + \frac{4\,(k-s)\,K(m)}{p} \,\middle|\, m\right) + \mathrm{sn}\left(z + \frac{4\,(k+r)\,K(m)}{p} \,\middle|\, m\right) \mathrm{sn}\left(z + \frac{4\,(k+s)\,K(m)}{p} \,\middle|\, m\right)\right) = \frac{2}{m} \left(\left(\mathrm{ds}\left(\frac{4\,r\,K(m)}{p} \,\middle|\, m\right) - \mathrm{ds}\left(\frac{4\,s\,K(m)}{p} \,\middle|\, m\right)\right) \mathrm{ns}\left(\frac{4\,(r-s)\,K(m)}{p} \,\middle|\, m\right) + \mathrm{ns}\left(\frac{4\,r\,K(m)}{p} \,\middle|\, m\right) \mathrm{ns}\left(\frac{4\,s\,K(m)}{p} \,\middle|\, m\right)\right) \sum_{k=0}^{p-1} \mathrm{cn}\left(z + \frac{4\,k\,K(m)}{p} \,\middle|\, m\right) /;\; p \in \mathbb{N}^+ \bigwedge r \in \mathbb{N}^+ \bigwedge r < p \bigwedge s \in \mathbb{N}^+ \bigwedge s < r$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.26.16.0082.01

$$\sum_{k=0}^{p-1} \mathrm{cn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right) \left(\mathrm{dn}\left(z+\frac{4\,(k-r)\,K(m)}{p}\,\middle|\,m\right) \mathrm{dn}\left(z+\frac{4\,(k-s)\,K(m)}{p}\,\middle|\,m\right)+\mathrm{dn}\left(z+\frac{4\,(k+r)\,K(m)}{p}\,\middle|\,m\right) \mathrm{dn}\left(z+\frac{4\,(k+s)\,K(m)}{p}\,\middle|\,m\right)\right) = -2\left(\mathrm{cs}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right) \mathrm{cs}\left(\frac{4\,s\,K(m)}{p}\,\middle|\,m\right)+\mathrm{cs}\left(\frac{4\,(r-s)\,K(m)}{p}\,\middle|\,m\right)\left(\mathrm{ds}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right)-\mathrm{ds}\left(\frac{4\,s\,K(m)}{p}\,\middle|\,m\right)\right)\right) \sum_{k=0}^{p-1} \mathrm{cn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right) /;\; p\in\mathbb{N}^+ \land r\in\mathbb{N}^+ \land r<p \land s\in\mathbb{N}^+ \land s<r$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.26.16.0083.01

$$\sum_{k=0}^{p-1} (-1)^k\, \mathrm{dn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right) \mathrm{dn}\left(z+\frac{2\,(k+r)\,K(m)}{p}\,\middle|\,m\right) \mathrm{dn}\left(z+\frac{2\,(k+2\,r)\,K(m)}{p}\,\middle|\,m\right) = -\left(\mathrm{cs}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)^2+2\,\mathrm{cs}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right) \mathrm{cs}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right)\right) \sum_{k=0}^{p-1} (-1)^k\, \mathrm{dn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right) /; \frac{p}{2}\in\mathbb{N}^+ \bigwedge r\in\mathbb{N}^+ \bigwedge \gcd(p,\,r)=1$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.26.16.0084.01

$$\sum_{k=0}^{p-1} (-1)^k\, \mathrm{dn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right)^2 \left(\mathrm{dn}\left(z+\frac{2\,(k-r)\,K(m)}{p}\,\middle|\,m\right)+\mathrm{dn}\left(z+\frac{2\,(k+r)\,K(m)}{p}\,\middle|\,m\right)\right) = 2\left(\mathrm{cs}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)^2+\mathrm{ds}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right) \mathrm{ns}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\right) \sum_{k=0}^{p-1} (-1)^k\, \mathrm{dn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right) /; \frac{p}{2}\in\mathbb{N}^+ \bigwedge r\in\mathbb{N}^+ \bigwedge \gcd(p,\,r)=1$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.26.16.0085.01

$$\sum_{k=0}^{p-1}(-1)^k \operatorname{cn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right)\left(\operatorname{cn}\left(z+\frac{2\,(k-r)\,K(m)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{2\,(k-r)\,K(m)}{p}\,\middle|\,m\right)+\operatorname{cn}\left(z+\frac{2\,(k+r)\,K(m)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{2\,(k+r)\,K(m)}{p}\,\middle|\,m\right)\right) = -\frac{2}{m}\operatorname{cs}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\left(\operatorname{ds}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)+\operatorname{ns}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\right)\sum_{k=0}^{p-1}(-1)^k \operatorname{dn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right)\,/;\ \frac{p}{2}\in\mathbb{N}^+\bigwedge r\in\mathbb{N}^+\bigwedge \gcd(p,\,r)=1$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.26.16.0086.01

$$\sum_{k=0}^{p-1}(-1)^k \operatorname{sn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right)\left(\operatorname{dn}\left(z+\frac{2\,(k-r)\,K(m)}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{2\,(k-r)\,K(m)}{p}\,\middle|\,m\right)+\operatorname{dn}\left(z+\frac{2\,(k+r)\,K(m)}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{2\,(k+r)\,K(m)}{p}\,\middle|\,m\right)\right) = \frac{2}{m}\operatorname{cs}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\left(\operatorname{ds}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)+\operatorname{ns}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\right)\sum_{k=0}^{p-1}(-1)^k \operatorname{dn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right)\,/;\ \frac{p}{2}\in\mathbb{N}^+\bigwedge r\in\mathbb{N}^+\bigwedge \gcd(p,\,r)=1$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

Sums over products of four Jacobi functions

09.26.16.0087.01

$$\sum_{k=0}^{p-1}\operatorname{dn}\left(z+\frac{2\,K(m)\,k}{p}\,\middle|\,m\right)^2\left(\operatorname{cn}\left(z+\frac{2\,K(m)\,(k-r)}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{2\,K(m)\,(k-r)}{p}\,\middle|\,m\right)+\operatorname{cn}\left(z+\frac{2\,K(m)\,(k+r)}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{2\,K(m)\,(k+r)}{p}\,\middle|\,m\right)\right) = -2\left(\operatorname{cs}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)^2+\operatorname{ds}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\operatorname{ns}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\right)\sum_{k=0}^{p-1}\operatorname{cn}\left(z+\frac{2\,K(m)\,k}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{2\,K(m)\,k}{p}\,\middle|\,m\right)\,/;\ p\in\mathbb{N}^+\bigwedge r\in\mathbb{N}^+\bigwedge r<p$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.26.16.0088.01

$$\sum_{k=0}^{p-1} \operatorname{sn}\left(z+\frac{2K(m)k}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{2K(m)k}{p}\,\middle|\,m\right) \left(\operatorname{cn}\left(z+\frac{2K(m)(k-r)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{2K(m)(k-r)}{p}\,\middle|\,m\right)+\operatorname{cn}\left(z+\frac{2K(m)(k+r)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{2K(m)(k+r)}{p}\,\middle|\,m\right)\right) = -2\operatorname{cs}\left(\frac{2rK(m)}{p}\,\middle|\,m\right)\left(\operatorname{ds}\left(\frac{2rK(m)}{p}\,\middle|\,m\right)+\operatorname{ns}\left(\frac{2rK(m)}{p}\,\middle|\,m\right)\right)\sum_{k=0}^{p-1}\operatorname{cn}\left(z+\frac{2K(m)k}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{2K(m)k}{p}\,\middle|\,m\right) /;\; p\in\mathbb{N}^+ \land r\in\mathbb{N}^+ \land r<p$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.26.16.0089.01

$$\sum_{k=0}^{p-1} \operatorname{sn}\left(z+\frac{2K(m)k}{p}\,\middle|\,m\right)\operatorname{cn}\left(z+\frac{2K(m)k}{p}\,\middle|\,m\right) \left(\operatorname{dn}\left(z+\frac{2K(m)(k-r)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{2K(m)(k-s)}{p}\,\middle|\,m\right)+\operatorname{dn}\left(z+\frac{2K(m)(k+r)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{2K(m)(k+s)}{p}\,\middle|\,m\right)\right) = -2\operatorname{cs}\left(\frac{2rK(m)}{p}\,\middle|\,m\right)\operatorname{cs}\left(\frac{2sK(m)}{p}\,\middle|\,m\right)\sum_{k=0}^{p-1}\operatorname{cn}\left(z+\frac{2K(m)k}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{2K(m)k}{p}\,\middle|\,m\right) /;\; p\in\mathbb{N}^+ \land r\in\mathbb{N}^+ \land r<p \land s\in\mathbb{N}^+ \land s<r$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.26.16.0090.01

$$\sum_{k=0}^{p-1} \operatorname{sn}\left(z+\frac{2K(m)k}{p}\,\middle|\,m\right)\operatorname{cn}\left(z+\frac{2K(m)k}{p}\,\middle|\,m\right) \left(\operatorname{cn}\left(z+\frac{2K(m)(k-r)}{p}\,\middle|\,m\right)\operatorname{cn}\left(z+\frac{2K(m)(k-s)}{p}\,\middle|\,m\right)+\operatorname{cn}\left(z+\frac{2K(m)(k+r)}{p}\,\middle|\,m\right)\operatorname{cn}\left(z+\frac{2K(m)(k+s)}{p}\,\middle|\,m\right)\right) = -\frac{2}{m}\operatorname{ds}\left(\frac{2rK(m)}{p}\,\middle|\,m\right)\operatorname{ds}\left(\frac{2sK(m)}{p}\,\middle|\,m\right)\sum_{k=0}^{p-1}\operatorname{cn}\left(z+\frac{2K(m)k}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{2K(m)k}{p}\,\middle|\,m\right) /;\; p\in\mathbb{N}^+ \land r\in\mathbb{N}^+ \land r<p \land s\in\mathbb{N}^+ \land s<r$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.26.16.0091.01

$$\sum_{k=0}^{p-1} \operatorname{sn}\left(z+\frac{2K(m)k}{p} \middle| m\right) \operatorname{cn}\left(z+\frac{2K(m)k}{p} \middle| m\right) \left(\operatorname{sn}\left(z+\frac{2K(m)(k-r)}{p} \middle| m\right) \operatorname{sn}\left(z+\frac{2K(m)(k-s)}{p} \middle| m\right) + \operatorname{sn}\left(z+\frac{2K(m)(k+r)}{p} \middle| m\right) \operatorname{sn}\left(z+\frac{2K(m)(k+s)}{p} \middle| m\right)\right) = \frac{2}{m} \operatorname{ns}\left(\frac{2rK(m)}{p} \middle| m\right) \operatorname{ns}\left(\frac{2sK(m)}{p} \middle| m\right) \sum_{k=0}^{p-1} \operatorname{cn}\left(z+\frac{2K(m)k}{p} \middle| m\right) \operatorname{sn}\left(z+\frac{2K(m)k}{p} \middle| m\right) /; p \in \mathbb{N}^+ \wedge r \in \mathbb{N}^+ \wedge r < p \wedge s \in \mathbb{N}^+ \wedge s < r$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.26.16.0092.01

$$\sum_{k=0}^{p-1} \operatorname{dn}\left(z+\frac{2K(m)k}{p} \middle| m\right) \operatorname{cn}\left(z+\frac{2K(m)k}{p} \middle| m\right) \left(\operatorname{dn}\left(z+\frac{2K(m)(k-s)}{p} \middle| m\right) \operatorname{sn}\left(z+\frac{2K(m)(k-r)}{p} \middle| m\right) + \operatorname{dn}\left(z+\frac{2K(m)(k+s)}{p} \middle| m\right) \operatorname{sn}\left(z+\frac{2K(m)(k+r)}{p} \middle| m\right)\right) = -2 \operatorname{ns}\left(\frac{2rK(m)}{p} \middle| m\right) \operatorname{cs}\left(\frac{2sK(m)}{p} \middle| m\right) \sum_{k=0}^{p-1} \operatorname{cn}\left(z+\frac{2K(m)k}{p} \middle| m\right) \operatorname{sn}\left(z+\frac{2K(m)k}{p} \middle| m\right) /; p \in \mathbb{N}^+ \wedge r \in \mathbb{N}^+ \wedge r < p \wedge s \in \mathbb{N}^+ \wedge s < r$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.26.16.0093.01

$$\sum_{k=0}^{p-1} \operatorname{dn}\left(z+\frac{2kK(m)}{p} \middle| m\right)^2 \operatorname{dn}\left(z+\frac{2(k+r)K(m)}{p} \middle| m\right)^2 = \frac{p}{2K(m)} \left(\int_0^{2K(m)} \operatorname{dn}(t \mid m)^2 \operatorname{dn}\left(t+\frac{2rK(m)}{p} \middle| m\right)^2 dt + 4E(m) \operatorname{cs}\left(\frac{2rK(m)}{p} \middle| m\right)^2\right) - 2 \operatorname{cs}\left(\frac{2rK(m)}{p} \middle| m\right)^2 \sum_{k=0}^{p-1} \operatorname{dn}\left(z+\frac{2kK(m)}{p} \middle| m\right)^2 /; p \in \mathbb{N}^+ \wedge r \in \mathbb{N}^+ \wedge r < p-1$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.26.16.0094.01

$$\sum_{k=0}^{p-1}\operatorname{cn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right)\left(\operatorname{cn}\left(z+\frac{2\,(k-r)\,K(m)}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{2\,(k-r)\,K(m)}{p}\,\middle|\,m\right)+\operatorname{cn}\left(z+\frac{2\,(k+r)\,K(m)}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{2\,(k+r)\,K(m)}{p}\,\middle|\,m\right)\right)=\frac{4}{m^2}\operatorname{ns}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\operatorname{ds}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\sum_{k=0}^{p-1}\operatorname{dn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right)^2+\frac{p}{2\,K(m)}\left(\int_0^{2\,K(m)}\operatorname{cn}(t\,|\,m)\operatorname{sn}(t\,|\,m)\left(\operatorname{cn}\left(t+\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\operatorname{sn}\left(t+\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)+\operatorname{cn}\left(t-\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\operatorname{sn}\left(t-\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\right)dt-\frac{8}{m^2}\operatorname{ns}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\operatorname{ds}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)E(m)\right)\;/;\;p\in\mathbb{N}^+\land r\in\mathbb{N}^+\land r<p-1$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.26.16.0095.01

$$\sum_{k=0}^{p-1}\operatorname{cn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right)\left(\operatorname{cn}\left(z+\frac{2\,(k-r)\,K(m)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{2\,(k-r)\,K(m)}{p}\,\middle|\,m\right)+\operatorname{cn}\left(z+\frac{2\,(k+r)\,K(m)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{2\,(k+r)\,K(m)}{p}\,\middle|\,m\right)\right)=-\frac{4}{m}\operatorname{cs}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\operatorname{ds}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\sum_{k=0}^{p-1}\operatorname{dn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right)^2+\frac{p}{2\,K(m)}\left(\int_0^{2\,K(m)}\operatorname{cn}(t\,|\,m)\operatorname{dn}(t\,|\,m)\left(\operatorname{cn}\left(t+\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\operatorname{dn}\left(t+\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)+\operatorname{cn}\left(t-\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\operatorname{dn}\left(t-\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\right)dt+\frac{8}{m}E(m)\operatorname{cs}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\operatorname{ds}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\right)\;/;\;p\in\mathbb{N}^+\land r\in\mathbb{N}^+\land r<p-1$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.26.16.0096.01

$$\sum_{k=0}^{p-1}\operatorname{sn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right)\left(\operatorname{dn}\left(z+\frac{2\,(k-r)\,K(m)}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{2\,(k-r)\,K(m)}{p}\,\middle|\,m\right)+\operatorname{dn}\left(z+\frac{2\,(k+r)\,K(m)}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{2\,(k+r)\,K(m)}{p}\,\middle|\,m\right)\right)=\frac{4}{m}\operatorname{cs}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\operatorname{ns}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\sum_{k=0}^{p-1}\operatorname{dn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right)^2+\frac{p}{2\,K(m)}\left(\int_0^{2\,K(m)}\operatorname{sn}(t\,|\,m)\operatorname{dn}(t\,|\,m)\left(\operatorname{dn}\left(t+\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\operatorname{sn}\left(t+\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)+\operatorname{dn}\left(t-\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\operatorname{sn}\left(t-\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\right)dt-\frac{8}{m}\operatorname{cs}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\operatorname{ns}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)E(m)\right)\;/;\;p\in\mathbb{N}^+\land r\in\mathbb{N}^+\land r<p-1$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.26.16.0097.01

$$\sum_{k=0}^{p-1} \operatorname{dn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right)^3 \left(\operatorname{dn}\left(z+\frac{2\,(k-r)\,K(m)}{p}\,\middle|\,m\right)+\operatorname{dn}\left(z+\frac{2\,(k+r)\,K(m)}{p}\,\middle|\,m\right)\right) = 2\operatorname{ds}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\operatorname{ns}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\sum_{k=0}^{p-1}\operatorname{dn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right)^2 + \frac{p}{2\,K(m)}\left(\int_0^{2\,K(m)}\operatorname{dn}(t\mid m)^3\left(\operatorname{dn}\left(t+\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)+\operatorname{dn}\left(t-\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\right)dt - 4\operatorname{ns}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\operatorname{ds}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)E(m)\right) /;\ p\in\mathbb{N}^+ \land r\in\mathbb{N}^+ \land r<p-1$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.26.16.0098.01

$$\sum_{k=0}^{p-1} \operatorname{sn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right)^3 \left(\operatorname{sn}\left(z+\frac{2\,(k-r)\,K(m)}{p}\,\middle|\,m\right)+\operatorname{sn}\left(z+\frac{2\,(k+r)\,K(m)}{p}\,\middle|\,m\right)\right) = \frac{2}{m^2}\operatorname{cs}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\operatorname{ds}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\sum_{k=0}^{p-1}\operatorname{dn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right)^2 + \frac{p}{2\,K(m)}\left(\int_0^{2\,K(m)}\operatorname{sn}(t\mid m)^3\left(\operatorname{sn}\left(t+\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)+\operatorname{sn}\left(t-\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\right)dt - \frac{4}{m^2}\operatorname{cs}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\operatorname{ds}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)E(m)\right) /;\ p\in\mathbb{N}^+ \land r\in\mathbb{N}^+ \land r<p-1$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.26.16.0099.01

$$\sum_{k=0}^{p-1} \operatorname{cn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right)^3 \left(\operatorname{cn}\left(z+\frac{2\,(k-r)\,K(m)}{p}\,\middle|\,m\right)+\operatorname{cn}\left(z+\frac{2\,(k+r)\,K(m)}{p}\,\middle|\,m\right)\right) = \frac{p}{2\,K(m)}\left(\int_0^{2\,K(m)}\operatorname{cn}(t\mid m)^3\left(\operatorname{cn}\left(t+\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)+\operatorname{cn}\left(t-\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\right)dt - \frac{4}{m^2}\operatorname{ns}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\operatorname{cs}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)E(m)\right) + \frac{2}{m^2}\operatorname{cs}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\operatorname{ns}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\sum_{k=0}^{p-1}\operatorname{dn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right)^2 /;\ p\in\mathbb{N}^+ \land r\in\mathbb{N}^+ \land r<p-1$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.26.16.0100.01

$$\sum_{k=0}^{p-1}\operatorname{cn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right)\left(\operatorname{sn}\left(z+\frac{4\,(k-r)\,K(m)}{p}\,\middle|\,m\right)^2+\operatorname{sn}\left(z+\frac{4\,(k+r)\,K(m)}{p}\,\middle|\,m\right)^2\right)=\frac{2}{m}\left(\operatorname{ns}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right)^2+\operatorname{cs}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right)\operatorname{ds}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right)\right)\sum_{k=0}^{p-1}\operatorname{cn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right)\,/;\,p\in\mathbb{N}^+\land r\in\mathbb{N}^+\land r<p-1$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.26.16.0101.01

$$\sum_{k=0}^{p-1}\operatorname{sn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right)\left(\operatorname{cn}\left(z+\frac{4\,(k-r)\,K(m)}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{4\,(k-r)\,K(m)}{p}\,\middle|\,m\right)+\operatorname{cn}\left(z+\frac{4\,(k+r)\,K(m)}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{4\,(k+r)\,K(m)}{p}\,\middle|\,m\right)\right)=\frac{2}{m}\operatorname{ns}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right)\left(\operatorname{cs}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right)+\operatorname{ds}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right)\right)\sum_{k=0}^{p-1}\operatorname{cn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right)\,/;\,p\in\mathbb{N}^+\land r\in\mathbb{N}^+\land r<p-1$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.26.16.0102.01

$$\sum_{k=0}^{p-1}\operatorname{cn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right)\left(\operatorname{sn}\left(z+\frac{4\,(k-r)\,K(m)}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{4\,(k-s)\,K(m)}{p}\,\middle|\,m\right)+\operatorname{sn}\left(z+\frac{4\,(k+r)\,K(m)}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{4\,(k+s)\,K(m)}{p}\,\middle|\,m\right)\right)=\frac{2}{m}\operatorname{ns}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right)\operatorname{ns}\left(\frac{4\,s\,K(m)}{p}\,\middle|\,m\right)\sum_{k=0}^{p-1}\operatorname{cn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right)\,/;\,p\in\mathbb{N}^+\land r\in\mathbb{N}^+\land r<p\land s\in\mathbb{N}^+\land s<r$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.26.16.0103.01

$$\sum_{k=0}^{p-1} \operatorname{cn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right) \operatorname{dn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right) \left(\operatorname{cn}\left(z+\frac{4\,(k-r)\,K(m)}{p}\,\middle|\,m\right) \operatorname{cn}\left(z+\frac{4\,(k-s)\,K(m)}{p}\,\middle|\,m\right)+\operatorname{cn}\left(z+\frac{4\,(k+r)\,K(m)}{p}\,\middle|\,m\right) \operatorname{cn}\left(z+\frac{4\,(k+s)\,K(m)}{p}\,\middle|\,m\right)\right) = -\frac{2}{m} \operatorname{ds}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right) \operatorname{ds}\left(\frac{4\,s\,K(m)}{p}\,\middle|\,m\right) \sum_{k=0}^{p-1} \operatorname{cn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right) \operatorname{dn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right) /;$$

$$p \in \mathbb{N}^+ \land r \in \mathbb{N}^+ \land r < p \land s \in \mathbb{N}^+ \land s < r$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.26.16.0104.01

$$\sum_{k=0}^{p-1} \operatorname{cn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right) \operatorname{dn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right) \left(\operatorname{dn}\left(z+\frac{4\,(k-r)\,K(m)}{p}\,\middle|\,m\right) \operatorname{dn}\left(z+\frac{4\,(k-s)\,K(m)}{p}\,\middle|\,m\right)+\operatorname{dn}\left(z+\frac{4\,(k+r)\,K(m)}{p}\,\middle|\,m\right) \operatorname{dn}\left(z+\frac{4\,(k+s)\,K(m)}{p}\,\middle|\,m\right)\right) = -2 \operatorname{cs}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right) \operatorname{cs}\left(\frac{4\,s\,K(m)}{p}\,\middle|\,m\right) \sum_{k=0}^{p-1} \operatorname{cn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right) \operatorname{dn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right) /;$$

$$p \in \mathbb{N}^+ \land r \in \mathbb{N}^+ \land r < p \land s \in \mathbb{N}^+ \land s < r$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.26.16.0105.01

$$\sum_{k=0}^{p-1} \operatorname{cn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right) \operatorname{sn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right) \left(\operatorname{dn}\left(z+\frac{4\,(k-r)\,K(m)}{p}\,\middle|\,m\right) \operatorname{sn}\left(z+\frac{4\,(k-s)\,K(m)}{p}\,\middle|\,m\right)+\operatorname{dn}\left(z+\frac{4\,(k+r)\,K(m)}{p}\,\middle|\,m\right) \operatorname{sn}\left(z+\frac{4\,(k+s)\,K(m)}{p}\,\middle|\,m\right)\right) = \frac{2}{m} \operatorname{cs}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right) \operatorname{ns}\left(\frac{4\,s\,K(m)}{p}\,\middle|\,m\right) \sum_{k=0}^{p-1} \operatorname{cn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right) \operatorname{dn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right) /;$$

$$p \in \mathbb{N}^+ \land r \in \mathbb{N}^+ \land r < p \land s \in \mathbb{N}^+ \land s < r$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.26.16.0106.01

$$\sum_{k=0}^{p-1}\operatorname{cn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right)^2 \left(\operatorname{dn}\left(z+\frac{4\,(k-r)\,K(m)}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{4\,(k-r)\,K(m)}{p}\,\middle|\,m\right)+\operatorname{dn}\left(z+\frac{4\,(k+r)\,K(m)}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{4\,(k+r)\,K(m)}{p}\,\middle|\,m\right)\right) = -\frac{2}{m}\left(\operatorname{ds}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right)^2+\operatorname{cs}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right)\operatorname{ns}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right)\right)\sum_{k=0}^{p-1}\operatorname{dn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right) /;\ p\in\mathbb{N}^+\land r\in\mathbb{N}^+\land r<p$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.26.16.0107.01

$$\sum_{k=0}^{p-1}\operatorname{cn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right) \left(\operatorname{cn}\left(z+\frac{4\,(k-r)\,K(m)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{4\,(k-r)\,K(m)}{p}\,\middle|\,m\right)+\operatorname{cn}\left(z+\frac{4\,(k+r)\,K(m)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{4\,(k+r)\,K(m)}{p}\,\middle|\,m\right)\right) = -\frac{2}{m}\operatorname{ds}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right)\left(\operatorname{cs}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right)+\operatorname{ns}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right)\right)\sum_{k=0}^{p-1}\operatorname{dn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right) /;\ p\in\mathbb{N}^+\land r\in\mathbb{N}^+\land r<p$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.26.16.0108.01

$$\sum_{k=0}^{p-1}\operatorname{cn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right)^2\operatorname{dn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right)\left(\operatorname{sn}\left(z+\frac{4\,(k-r)\,K(m)}{p}\,\middle|\,m\right)+\operatorname{sn}\left(z+\frac{4\,(k+r)\,K(m)}{p}\,\middle|\,m\right)\right) = \frac{2}{m}\operatorname{cs}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right)\operatorname{ds}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right)\sum_{k=0}^{p-1}\operatorname{dn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right) /;\ p\in\mathbb{N}^+\land r\in\mathbb{N}^+\land r<p$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.26.16.0109.01

$$\sum_{k=0}^{p-1}\operatorname{sn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right) \left(\operatorname{cn}\left(z+\frac{4\,(k-r)\,K(m)}{p}\,\middle|\,m\right)\operatorname{cn}\left(z+\frac{4\,(k+s)\,K(m)}{p}\,\middle|\,m\right)+\operatorname{cn}\left(z+\frac{4\,(k+r)\,K(m)}{p}\,\middle|\,m\right)\operatorname{cn}\left(z+\frac{4\,(k+s)\,K(m)}{p}\,\middle|\,m\right)\right) = -\frac{2}{m}\operatorname{ds}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right)\operatorname{ds}\left(\frac{4\,s\,K(m)}{p}\,\middle|\,m\right)\sum_{k=0}^{p-1}\operatorname{dn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right) /;\ p\in\mathbb{N}^+\land r\in\mathbb{N}^+\land r<p\land s\in\mathbb{N}^+\land s<r$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.26.16.0110.01

$$\sum_{k=0}^{p-1} \operatorname{sn}\left(z+\frac{4kK(m)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{4kK(m)}{p}\,\middle|\,m\right)\left(\operatorname{dn}\left(z+\frac{4(k-r)K(m)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{4(k-s)K(m)}{p}\,\middle|\,m\right)+\operatorname{dn}\left(z+\frac{4(k+r)K(m)}{p}\right)\operatorname{dn}\left(z+\frac{4(k+s)K(m)}{p}\,\middle|\,m\right)\right) = -2\operatorname{cs}\left(\frac{4rK(m)}{p}\,\middle|\,m\right)\operatorname{cs}\left(\frac{4sK(m)}{p}\,\middle|\,m\right)\sum_{k=0}^{p-1}\operatorname{dn}\left(z+\frac{4kK(m)}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{4kK(m)}{p}\,\middle|\,m\right)\;/;\; p\in\mathbb{N}^+\land r\in\mathbb{N}^+\land r<p\land s\in\mathbb{N}^+\land s<r$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.26.16.0111.01

$$\sum_{k=0}^{p-1} \operatorname{sn}\left(z+\frac{4kK(m)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{4kK(m)}{p}\,\middle|\,m\right)\left(\operatorname{sn}\left(z+\frac{4(k-r)K(m)}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{4(k-s)K(m)}{p}\,\middle|\,m\right)+\operatorname{sn}\left(z+\frac{4(k+r)K(m)}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{4(k+s)K(m)}{p}\,\middle|\,m\right)\right) = \frac{2}{m}\operatorname{ns}\left(\frac{2rK(m)}{p}\,\middle|\,m\right)\operatorname{ns}\left(\frac{2sK(m)}{p}\,\middle|\,m\right)\sum_{k=0}^{p-1}\operatorname{dn}\left(z+\frac{4kK(m)}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{4kK(m)}{p}\,\middle|\,m\right)\;/;\; p\in\mathbb{N}^+\land r\in\mathbb{N}^+\land r<p\land s\in\mathbb{N}^+\land s<r$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.26.16.0112.01

$$\sum_{k=0}^{p-1} \operatorname{cn}\left(z+\frac{4kK(m)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{4kK(m)}{p}\,\middle|\,m\right)\left(\operatorname{cn}\left(z+\frac{4(k-s)K(m)}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{4(k-r)K(m)}{p}\,\middle|\,m\right)+\operatorname{cn}\left(z+\frac{4(k+s)K(m)}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{4(k+r)K(m)}{p}\,\middle|\,m\right)\right) = -\frac{2}{m}\operatorname{ns}\left(\frac{4rK(m)}{p}\,\middle|\,m\right)\operatorname{ds}\left(\frac{4sK(m)}{p}\,\middle|\,m\right)\sum_{k=0}^{p-1}\operatorname{dn}\left(z+\frac{4kK(m)}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{4kK(m)}{p}\,\middle|\,m\right)\;/;\; p\in\mathbb{N}^+\land r\in\mathbb{N}^+\land r<p\land s\in\mathbb{N}^+\land s<r$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.26.16.0113.01

$$\sum_{k=0}^{p-1}(-1)^k \operatorname{dn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right)^2 \left(\operatorname{cn}\left(z+\frac{2\,(k-r)\,K(m)}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{2\,(k-r)\,K(m)}{p}\,\middle|\,m\right)+\operatorname{cn}\left(z+\frac{2\,(k+r)\,K(m)}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{2\,(k+r)\,K(m)}{p}\,\middle|\,m\right)\right) = 2\left(\operatorname{cs}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)^2-\operatorname{ds}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\operatorname{ns}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\right)\sum_{k=0}^{p-1}(-1)^k \operatorname{cn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right) /;\ \frac{p}{2}\in\mathbb{N}^+\bigwedge r\in\mathbb{N}^+\bigwedge \gcd(p,r)=1$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.26.16.0114.01

$$\sum_{k=0}^{p-1}(-1)^k \operatorname{dn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{2\,(k+r)\,K(m)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{2\,(k+2\,r)\,K(m)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{2\,(k+3\,r)\,K(m)}{p}\,\middle|\,m\right) = 2\left(\operatorname{cs}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right)\operatorname{cs}\left(\frac{6\,r\,K(m)}{p}\,\middle|\,m\right)\operatorname{cs}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)+\operatorname{cs}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)^2\operatorname{cs}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right)\right)\sum_{k=0}^{p-1}(-1)^k \operatorname{Z}\left(\operatorname{am}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right)\middle|\,m\right) /;\ \frac{p}{2}\in\mathbb{N}^+\bigwedge r\in\mathbb{N}^+\bigwedge r<p\bigwedge \gcd(p,r)=1\bigwedge 1-m>0$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.26.16.0115.01

$$\sum_{k=0}^{p-1}(-1)^k \operatorname{dn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right)^3\left(\operatorname{dn}\left(z+\frac{2\,(k-r)\,K(m)}{p}\,\middle|\,m\right)+\operatorname{dn}\left(z+\frac{2\,(k+r)\,K(m)}{p}\,\middle|\,m\right)\right) = 2\operatorname{ns}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\operatorname{ds}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\sum_{k=0}^{p-1}(-1)^k \operatorname{dn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right)^2 /;\ \frac{p}{2}\in\mathbb{N}^+\bigwedge r\in\mathbb{N}^+\bigwedge \gcd(p,r)=1$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.26.16.0116.01

$$\sum_{k=0}^{p-1}(-1)^k \operatorname{cn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right)^3\left(\operatorname{cn}\left(z+\frac{2\,(k-r)\,K(m)}{p}\,\middle|\,m\right)+\operatorname{cn}\left(z+\frac{2\,(k+r)\,K(m)}{p}\,\middle|\,m\right)\right) = \frac{2}{m^2}\operatorname{cs}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\operatorname{ns}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\sum_{k=0}^{p-1}(-1)^k \operatorname{dn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right)^2 /;\ \frac{p}{2}\in\mathbb{N}^+\bigwedge r\in\mathbb{N}^+\bigwedge \gcd(p,r)=1$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.26.16.0117.01

$$\sum_{k=0}^{p-1} (-1)^k \operatorname{sn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right)^3 \left(\operatorname{sn}\left(z+\frac{2\,(k-r)\,K(m)}{p}\,\middle|\,m\right)+\operatorname{sn}\left(z+\frac{2\,(k+r)\,K(m)}{p}\,\middle|\,m\right)\right) = \frac{2}{m^2}\operatorname{cs}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\operatorname{ds}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\sum_{k=0}^{p-1}(-1)^k \operatorname{dn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right)^2 \;/;\; \frac{p}{2}\in\mathbb{N}^+ \bigwedge r\in\mathbb{N}^+ \bigwedge \gcd(p,\,r)=1$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

Sums over products of five Jacobi functions

09.26.16.0118.01

$$\sum_{k=0}^{p-1} \operatorname{dn}\left(z+\frac{2\,K(m)\,k}{p}\,\middle|\,m\right)^4 \left(\operatorname{dn}\left(z+\frac{2\,K(m)\,(k-r)}{p}\,\middle|\,m\right)+\operatorname{dn}\left(z+\frac{2\,K(m)\,(k+r)}{p}\,\middle|\,m\right)\right) = 2\operatorname{ds}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\operatorname{ns}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\sum_{k=0}^{p-1}\operatorname{dn}\left(z+\frac{2\,K(m)\,k}{p}\,\middle|\,m\right)^3 + 2\operatorname{cs}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)^2 \left(\operatorname{cs}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)^2 - \operatorname{ns}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\operatorname{ds}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\right)\sum_{k=0}^{p-1}\operatorname{dn}\left(z+\frac{2\,K(m)\,k}{p}\,\middle|\,m\right) \;/;\; p\in\mathbb{N}^+ \land r\in\mathbb{N}^+ \land r<p$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.26.16.0119.01

$$\sum_{k=0}^{p-1} \operatorname{dn}\left(z+\frac{2\,K(m)\,k}{p}\,\middle|\,m\right)^3 \left(\operatorname{dn}\left(z+\frac{2\,K(m)\,(k-r)}{p}\,\middle|\,m\right)^2+\operatorname{dn}\left(z+\frac{2\,K(m)\,(k+r)}{p}\,\middle|\,m\right)^2\right) = -2\operatorname{cs}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)^2 \sum_{k=0}^{p-1}\operatorname{dn}\left(z+\frac{2\,K(m)\,k}{p}\,\middle|\,m\right)^3 + 2\left(\operatorname{ds}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)^2\operatorname{cs}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)^2 + \operatorname{ds}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)^2\operatorname{ns}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)^2 + \operatorname{ns}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)^2\operatorname{cs}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)^2 - 3\operatorname{ns}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\operatorname{ds}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\operatorname{cs}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)^2\right)\sum_{k=0}^{p-1}\operatorname{dn}\left(z+\frac{2\,K(m)\,k}{p}\,\middle|\,m\right) \;/;\; p\in\mathbb{N}^+ \land r\in\mathbb{N}^+ \land r<p$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.26.16.0120.01

$$\sum_{k=0}^{p-1} \operatorname{cn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right)\left(\operatorname{dn}\left(z+\frac{2\,(k-r)\,K(m)}{p}\,\middle|\,m\right)^2+\operatorname{dn}\left(z+\frac{2\,(k+r)\,K(m)}{p}\,\middle|\,m\right)^2\right) = -2\operatorname{cs}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)^2 \sum_{k=0}^{p-1}\operatorname{cn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right) \;/;\; p\in\mathbb{N}^+ \land r\in\mathbb{N}^+ \land r<p-1$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.26.16.0121.01

$$\sum_{k=0}^{p-1} \operatorname{cn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right) \operatorname{sn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right)^2 \operatorname{dn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right)\left(\operatorname{sn}\left(z+\frac{2\,(k-r)\,K(m)}{p}\,\middle|\,m\right)+\operatorname{sn}\left(z+\frac{2\,(k+r)\,K(m)}{p}\,\middle|\,m\right)\right) = -\frac{2}{m} \operatorname{cs}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right) \operatorname{ds}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right) \sum_{k=0}^{p-1} \operatorname{cn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right) \operatorname{sn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right) \operatorname{dn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right) /;\; p\in\mathbb{N}^+ \land r\in\mathbb{N}^+ \land r<p-1$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.26.16.0122.01

$$\sum_{k=0}^{p-1} \operatorname{sn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right) \operatorname{cn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right)^2 \operatorname{dn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right)\left(\operatorname{cn}\left(z+\frac{2\,(k-r)\,K(m)}{p}\,\middle|\,m\right)+\operatorname{cn}\left(z+\frac{2\,(k+r)\,K(m)}{p}\,\middle|\,m\right)\right) = \frac{2}{m} \operatorname{cs}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right) \operatorname{ns}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right) \sum_{k=0}^{p-1} \operatorname{cn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right) \operatorname{sn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right) \operatorname{dn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right) /;\; p\in\mathbb{N}^+ \land r\in\mathbb{N}^+ \land r<p-1$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.26.16.0123.01

$$\sum_{k=0}^{p-1} \operatorname{sn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right)^4 \left(\operatorname{sn}\left(z+\frac{4\,(k-r)\,K(m)}{p}\,\middle|\,m\right)+\operatorname{sn}\left(z+\frac{4\,(k+r)\,K(m)}{p}\,\middle|\,m\right)\right) = -\frac{2}{m} \operatorname{cs}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right) \operatorname{ds}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right) \sum_{k=0}^{p-1} \operatorname{sn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right)^3 + \frac{2}{m^2} \operatorname{ns}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right)^2 \left(\operatorname{ns}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right)^2 - \operatorname{cs}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right) \operatorname{ds}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right)\right) \sum_{k=0}^{p-1} \operatorname{sn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right) /;\; p\in\mathbb{N}^+ \land r\in\mathbb{N}^+ \land r<p$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.26.16.0124.01

$$\sum_{k=0}^{p-1}\operatorname{sn}\left(z+\frac{4kK(m)}{p}\,\middle|\,m\right)^3\left(\operatorname{sn}\left(z+\frac{4(k-r)K(m)}{p}\,\middle|\,m\right)^2+\operatorname{sn}\left(z+\frac{4(k+r)K(m)}{p}\,\middle|\,m\right)^2\right)=\frac{2}{m}\operatorname{ns}\left(\frac{4rK(m)}{p}\,\middle|\,m\right)^2\sum_{k=0}^{p-1}\operatorname{sn}\left(z+\frac{4kK(m)}{p}\,\middle|\,m\right)^3+\frac{2}{m^2}\left(\operatorname{cs}\left(\frac{4rK(m)}{p}\,\middle|\,m\right)^2\operatorname{ns}\left(\frac{4rK(m)}{p}\,\middle|\,m\right)^2+\operatorname{ds}\left(\frac{4rK(m)}{p}\,\middle|\,m\right)^2\operatorname{ns}\left(\frac{4rK(m)}{p}\,\middle|\,m\right)^2+\operatorname{cs}\left(\frac{4rK(m)}{p}\,\middle|\,m\right)^2\operatorname{ds}\left(\frac{4rK(m)}{p}\,\middle|\,m\right)^2-3\operatorname{cs}\left(\frac{4rK(m)}{p}\,\middle|\,m\right)\operatorname{ds}\left(\frac{4rK(m)}{p}\,\middle|\,m\right)\operatorname{ns}\left(\frac{4rK(m)}{p}\,\middle|\,m\right)^2\right)\sum_{k=0}^{p-1}\operatorname{sn}\left(z+\frac{4kK(m)}{p}\,\middle|\,m\right)\;/;\;p\in\mathbb{N}^+\land r\in\mathbb{N}^+\land r<p$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.26.16.0125.01

$$\sum_{k=0}^{p-1}\operatorname{dn}\left(z+\frac{4kK(m)}{p}\,\middle|\,m\right)^2\operatorname{sn}\left(z+\frac{4kK(m)}{p}\,\middle|\,m\right)^2\left(\operatorname{cn}\left(z+\frac{4(k-r)K(m)}{p}\,\middle|\,m\right)+\operatorname{cn}\left(z+\frac{4(k+r)K(m)}{p}\,\middle|\,m\right)\right)=-2\operatorname{ns}\left(\frac{4rK(m)}{p}\,\middle|\,m\right)\operatorname{cs}\left(\frac{4rK(m)}{p}\,\middle|\,m\right)\sum_{k=0}^{p-1}\operatorname{cn}\left(z+\frac{4kK(m)}{p}\,\middle|\,m\right)^3+\frac{2}{m}\operatorname{cs}\left(\frac{4rK(m)}{p}\,\middle|\,m\right)\operatorname{ns}\left(\frac{4rK(m)}{p}\,\middle|\,m\right)^3\left(m\operatorname{sn}\left(\frac{4rK(m)}{p}\,\middle|\,m\right)^2+\operatorname{cn}\left(\frac{4rK(m)}{p}\,\middle|\,m\right)^2-\operatorname{cn}\left(\frac{4rK(m)}{p}\,\middle|\,m\right)\right)\sum_{k=0}^{p-1}\operatorname{cn}\left(z+\frac{4kK(m)}{p}\,\middle|\,m\right)\;/;\;p\in\mathbb{N}^+\land r\in\mathbb{N}^+\land r<p$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.26.16.0126.01

$$\sum_{k=0}^{p-1}\operatorname{cn}\left(z+\frac{4kK(m)}{p}\,\middle|\,m\right)^3\left(\operatorname{cn}\left(z+\frac{4(k-r)K(m)}{p}\,\middle|\,m\right)^2+\operatorname{cn}\left(z+\frac{4(k+r)K(m)}{p}\,\middle|\,m\right)^2\right)=-\frac{2}{m}\operatorname{ds}\left(\frac{4rK(m)}{p}\,\middle|\,m\right)^2\sum_{k=0}^{p-1}\operatorname{cn}\left(z+\frac{4kK(m)}{p}\,\middle|\,m\right)^3+\frac{2}{m^2}\left(\operatorname{cs}\left(\frac{4rK(m)}{p}\,\middle|\,m\right)^2\operatorname{ds}\left(\frac{4rK(m)}{p}\,\middle|\,m\right)^2+\operatorname{ns}\left(\frac{4rK(m)}{p}\,\middle|\,m\right)^2\operatorname{ds}\left(\frac{4rK(m)}{p}\,\middle|\,m\right)^2+\operatorname{cs}\left(\frac{4rK(m)}{p}\,\middle|\,m\right)^2\operatorname{ns}\left(\frac{4rK(m)}{p}\,\middle|\,m\right)^2-3\operatorname{cs}\left(\frac{4rK(m)}{p}\,\middle|\,m\right)\operatorname{ns}\left(\frac{4rK(m)}{p}\,\middle|\,m\right)\operatorname{ds}\left(\frac{4rK(m)}{p}\,\middle|\,m\right)^2\right)\sum_{k=0}^{p-1}\operatorname{cn}\left(z+\frac{4kK(m)}{p}\,\middle|\,m\right)\;/;\;p\in\mathbb{N}^+\land r\in\mathbb{N}^+\land r<p$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.26.16.0127.01

$$\sum_{k=0}^{p-1}(-1)^k \operatorname{cn}\left(z+\frac{2kK(m)}{p}\,\middle|\,m\right)^2 \operatorname{dn}\left(z+\frac{2kK(m)}{p}\,\middle|\,m\right) \operatorname{sn}\left(z+\frac{2kK(m)}{p}\,\middle|\,m\right)\left(\operatorname{cn}\left(z+\frac{2(k-r)K(m)}{p}\,\middle|\,m\right)+\operatorname{cn}\left(z+\frac{2(k+r)K(m)}{p}\,\middle|\,m\right)\right) = -\frac{4}{m^2}\operatorname{ds}\left(\frac{2rK(m)}{p}\,\middle|\,m\right)^2 \operatorname{cs}\left(\frac{2rK(m)}{p}\,\middle|\,m\right)\operatorname{ns}\left(\frac{2rK(m)}{p}\,\middle|\,m\right)\sum_{k=0}^{p-1}(-1)^k Z\left(\operatorname{am}\left(z+\frac{2kK(m)}{p}\,\middle|\,m\right)\,\middle|\,m\right) + \frac{2}{m}\operatorname{cs}\left(\frac{2rK(m)}{p}\,\middle|\,m\right)\operatorname{ns}\left(\frac{2rK(m)}{p}\,\middle|\,m\right)\sum_{k=0}^{p-1}(-1)^k \operatorname{cn}\left(z+\frac{2kK(m)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{2kK(m)}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{2kK(m)}{p}\,\middle|\,m\right) /;\ \frac{p}{2}\in\mathbb{N}^+ \bigwedge r\in\mathbb{N}^+ \bigwedge r<p \bigwedge \gcd(p,r)=1 \bigwedge 1-m>0$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

Sums over products of six Jacobi functions

09.26.16.0128.01

$$\sum_{k=0}^{p-1}\operatorname{dn}\left(z+\frac{2K(m)k}{p}\,\middle|\,m\right)^4 \left(\operatorname{cn}\left(z+\frac{2K(m)(k-r)}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{2K(m)(k-r)}{p}\,\middle|\,m\right)+\operatorname{cn}\left(z+\frac{2K(m)(k+r)}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{2K(m)(k+r)}{p}\,\middle|\,m\right)\right) = -2\operatorname{ns}\left(\frac{2rK(m)}{p}\,\middle|\,m\right)\operatorname{ds}\left(\frac{2rK(m)}{p}\,\middle|\,m\right)\sum_{k=0}^{p-1}\operatorname{sn}\left(z+\frac{2K(m)k}{p}\,\middle|\,m\right)\operatorname{cn}\left(z+\frac{2K(m)k}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{2K(m)k}{p}\,\middle|\,m\right)^2 + 2\operatorname{cs}\left(\frac{2rK(m)}{p}\,\middle|\,m\right)^2\left(\operatorname{cs}\left(\frac{2rK(m)}{p}\,\middle|\,m\right)^2+3\operatorname{ds}\left(\frac{2rK(m)}{p}\,\middle|\,m\right)\operatorname{ns}\left(\frac{2rK(m)}{p}\,\middle|\,m\right)\right)\sum_{k=0}^{p-1}\operatorname{sn}\left(z+\frac{2K(m)k}{p}\,\middle|\,m\right)\operatorname{cn}\left(z+\frac{2K(m)k}{p}\,\middle|\,m\right) /;\ p\in\mathbb{N}^+ \land r\in\mathbb{N}^+ \land r<p$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.26.16.0129.01

$$\sum_{k=0}^{p-1} \mathrm{dn}\left(z+\frac{2K(m)k}{p} \middle| m\right)^4 \left(\mathrm{cn}\left(z+\frac{2K(m)(k-s)}{p} \middle| m\right)\mathrm{sn}\left(z+\frac{2K(m)(k-r)}{p} \middle| m\right)+\mathrm{cn}\left(z+\frac{2K(m)(k+s)}{p} \middle| m\right)\mathrm{sn}\left(z+\frac{2K(m)(k+r)}{p} \middle| m\right)\right) = -2\,\mathrm{ns}\left(\frac{2rK(m)}{p} \middle| m\right)\mathrm{ds}\left(\frac{2sK(m)}{p} \middle| m\right)\sum_{k=0}^{p-1}\mathrm{sn}\left(z+\frac{2K(m)k}{p} \middle| m\right)\mathrm{cn}\left(z+\frac{2K(m)k}{p} \middle| m\right)\mathrm{dn}\left(z+\frac{2K(m)k}{p} \middle| m\right)^2 + 2\left(\mathrm{cs}\left(\frac{2rK(m)}{p} \middle| m\right)\mathrm{ds}\left(\frac{2rK(m)}{p} \middle| m\right)\mathrm{cs}\left(\frac{2sK(m)}{p} \middle| m\right)\mathrm{ns}\left(\frac{2sK(m)}{p} \middle| m\right)+\mathrm{ns}\left(\frac{2rK(m)}{p} \middle| m\right)\mathrm{ds}\left(\frac{2sK(m)}{p} \middle| m\right)\left(\mathrm{cs}\left(\frac{2rK(m)}{p} \middle| m\right)^2+\mathrm{cs}\left(\frac{2sK(m)}{p} \middle| m\right)^2\right)\right) \sum_{k=0}^{p-1}\mathrm{sn}\left(z+\frac{2K(m)k}{p} \middle| m\right)\mathrm{cn}\left(z+\frac{2K(m)k}{p} \middle| m\right) /;\ p\in\mathbb{N}^+ \land r\in\mathbb{N}^+ \land r<p \land s\in\mathbb{N}^+ \land s<r$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.26.16.0130.01

$$\sum_{k=0}^{p-1} \mathrm{dn}\left(z+\frac{2kK(m)}{p} \middle| m\right)^3 \left(\mathrm{dn}\left(z+\frac{2(k-r)K(m)}{p} \middle| m\right)^3+\mathrm{dn}\left(z+\frac{2(k+r)K(m)}{p} \middle| m\right)^3\right) = \frac{p}{2K(m)}\left(24E(m)\,\mathrm{ds}\left(\frac{2rK(m)}{p} \middle| m\right)\mathrm{ns}\left(\frac{2rK(m)}{p} \middle| m\right)\mathrm{cs}\left(\frac{2rK(m)}{p} \middle| m\right)^2 + \int_0^{2K(m)}\mathrm{dn}(t\mid m)^3\left(\mathrm{dn}\left(t+\frac{2rK(m)}{p} \middle| m\right)^3+\mathrm{dn}\left(t-\frac{2rK(m)}{p} \middle| m\right)^3\right)dt\right) - 12\,\mathrm{cs}\left(\frac{2rK(m)}{p} \middle| m\right)^2\mathrm{ns}\left(\frac{2rK(m)}{p} \middle| m\right)\mathrm{ds}\left(\frac{2rK(m)}{p} \middle| m\right)\sum_{k=0}^{p-1}\mathrm{dn}\left(z+\frac{2kK(m)}{p} \middle| m\right)^2 /;\ p\in\mathbb{N}^+ \land r\in\mathbb{N}^+ \land r<p-1$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.26.16.0131.01

$$\sum_{k=0}^{p-1} \mathrm{sn}\left(z+\frac{2kK(m)}{p} \middle| m\right)^3 \left(\mathrm{sn}\left(z+\frac{2(k-r)K(m)}{p} \middle| m\right)^3+\mathrm{sn}\left(z+\frac{2(k+r)K(m)}{p} \middle| m\right)^3\right) = \frac{12}{m^3}\mathrm{ns}\left(\frac{2rK(m)}{p} \middle| m\right)^2\mathrm{cs}\left(\frac{2rK(m)}{p} \middle| m\right)\mathrm{ds}\left(\frac{2rK(m)}{p} \middle| m\right)\sum_{k=0}^{p-1}\mathrm{dn}\left(z+\frac{2kK(m)}{p} \middle| m\right)^2 + \frac{p}{2K(m)}\left(\int_0^{2K(m)}\mathrm{sn}(t\mid m)^3\left(\mathrm{sn}\left(t+\frac{2rK(m)}{p} \middle| m\right)^3+\mathrm{sn}\left(t-\frac{2rK(m)}{p} \middle| m\right)^3\right)dt - \frac{24}{m^3}\mathrm{ns}\left(\frac{2rK(m)}{p} \middle| m\right)^2\mathrm{ds}\left(\frac{2rK(m)}{p} \middle| m\right)\mathrm{cs}\left(\frac{2rK(m)}{p} \middle| m\right)E(m)\right) /;\ p\in\mathbb{N}^+ \land r\in\mathbb{N}^+ \land r<p-1$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.26.16.0132.01

$$\sum_{k=0}^{p-1} \operatorname{cn}\left(z+\frac{2\,k\,K(m)}{p} \,\middle|\, m\right)^3 \left(\operatorname{cn}\left(z+\frac{2\,(k-r)\,K(m)}{p} \,\middle|\, m\right)^3 + \operatorname{cn}\left(z+\frac{2\,(k+r)\,K(m)}{p} \,\middle|\, m\right)^3\right) = -\frac{12}{m^3} \operatorname{ds}\left(\frac{2\,r\,K(m)}{p} \,\middle|\, m\right)^2 \operatorname{cs}\left(\frac{2\,r\,K(m)}{p} \,\middle|\, m\right) \operatorname{ns}\left(\frac{2\,r\,K(m)}{p} \,\middle|\, m\right) \sum_{k=0}^{p-1} \operatorname{dn}\left(z+\frac{2\,k\,K(m)}{p} \,\middle|\, m\right)^2 + \frac{p}{2\,K(m)} \left(\int_0^{2\,K(m)} \operatorname{cn}(t \mid m)^3 \left(\operatorname{cn}\left(t+\frac{2\,r\,K(m)}{p} \,\middle|\, m\right)^3 + \operatorname{cn}\left(t-\frac{2\,r\,K(m)}{p} \,\middle|\, m\right)^3\right) dt + \frac{24}{m^3} E(m) \operatorname{cs}\left(\frac{2\,r\,K(m)}{p} \,\middle|\, m\right) \operatorname{ns}\left(\frac{2\,r\,K(m)}{p} \,\middle|\, m\right) \operatorname{ds}\left(\frac{2\,r\,K(m)}{p} \,\middle|\, m\right)^2\right) /;\, p \in \mathbb{N}^+ \land r \in \mathbb{N}^+ \land r < p-1$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.26.16.0133.01

$$\sum_{k=0}^{p-1} \operatorname{cn}\left(z+\frac{4\,k\,K(m)}{p} \,\middle|\, m\right) \operatorname{dn}\left(z+\frac{4\,k\,K(m)}{p} \,\middle|\, m\right) \operatorname{sn}\left(z+\frac{4\,k\,K(m)}{p} \,\middle|\, m\right) \left(\operatorname{sn}\left(z+\frac{4\,(k-r)\,K(m)}{p} \,\middle|\, m\right)^3 + \operatorname{sn}\left(z+\frac{4\,(k+r)\,K(m)}{p} \,\middle|\, m\right)^3\right) = -\frac{2}{m^2} \left(\operatorname{cs}\left(\frac{4\,r\,K(m)}{p} \,\middle|\, m\right)^2 \operatorname{ds}\left(\frac{4\,r\,K(m)}{p} \,\middle|\, m\right)^2 + \operatorname{ns}\left(\frac{4\,r\,K(m)}{p} \,\middle|\, m\right)^2 \operatorname{ds}\left(\frac{4\,r\,K(m)}{p} \,\middle|\, m\right)^2 + \operatorname{cs}\left(\frac{4\,r\,K(m)}{p} \,\middle|\, m\right)^2 \operatorname{ns}\left(\frac{4\,r\,K(m)}{p} \,\middle|\, m\right)^2 + 3 \operatorname{cs}\left(\frac{4\,r\,K(m)}{p} \,\middle|\, m\right) \operatorname{ns}\left(\frac{4\,r\,K(m)}{p} \,\middle|\, m\right)^2 \operatorname{ds}\left(\frac{4\,r\,K(m)}{p} \,\middle|\, m\right)\right) \sum_{k=0}^{p-1} \operatorname{cn}\left(z+\frac{4\,k\,K(m)}{p} \,\middle|\, m\right) \operatorname{dn}\left(z+\frac{4\,k\,K(m)}{p} \,\middle|\, m\right) /;\, p \in \mathbb{N}^+ \land r \in \mathbb{N}^+ \land r < p$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.26.16.0134.01

$$\sum_{k=0}^{p-1} \operatorname{sn}\left(z+\frac{4\,k\,K(m)}{p} \,\middle|\, m\right)^4 \left(\operatorname{cn}\left(z+\frac{4\,(k-r)\,K(m)}{p} \,\middle|\, m\right) \operatorname{dn}\left(z+\frac{4\,(k-r)\,K(m)}{p} \,\middle|\, m\right) + \operatorname{cn}\left(z+\frac{4\,(k+r)\,K(m)}{p} \,\middle|\, m\right) \operatorname{dn}\left(z+\frac{4\,(k+r)\,K(m)}{p} \,\middle|\, m\right)\right) = \frac{2}{m} \operatorname{cs}\left(\frac{4\,r\,K(m)}{p} \,\middle|\, m\right) \operatorname{ds}\left(\frac{4\,r\,K(m)}{p} \,\middle|\, m\right) \sum_{k=0}^{p-1} \operatorname{cn}\left(z+\frac{4\,k\,K(m)}{p} \,\middle|\, m\right) \operatorname{dn}\left(z+\frac{4\,k\,K(m)}{p} \,\middle|\, m\right) \operatorname{sn}\left(z+\frac{4\,k\,K(m)}{p} \,\middle|\, m\right)^2 + \frac{2}{m^2} \operatorname{ns}\left(\frac{4\,r\,K(m)}{p} \,\middle|\, m\right)^2 \left(\operatorname{ns}\left(\frac{4\,r\,K(m)}{p} \,\middle|\, m\right)^2 + 3 \operatorname{cs}\left(\frac{4\,r\,K(m)}{p} \,\middle|\, m\right) \operatorname{ds}\left(\frac{4\,r\,K(m)}{p} \,\middle|\, m\right)\right) \sum_{k=0}^{p-1} \operatorname{cn}\left(z+\frac{4\,k\,K(m)}{p} \,\middle|\, m\right) \operatorname{dn}\left(z+\frac{4\,k\,K(m)}{p} \,\middle|\, m\right) /;\, p \in \mathbb{N}^+ \land r \in \mathbb{N}^+ \land r < p$$

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09.26.16.0135.01

$$\sum_{k=0}^{p-1} \operatorname{sn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right)^4 \left(\operatorname{cn}\left(z+\frac{4\,(k-s)\,K(m)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{4\,(k-r)\,K(m)}{p}\,\middle|\,m\right)+\operatorname{cn}\left(z+\frac{4\,(k+s)\,K(m)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{4\,(k+r)\,K(m)}{p}\,\middle|\,m\,\middle|\,m\right)\right) = \frac{2}{m}\operatorname{cs}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right)\operatorname{ds}\left(\frac{4\,s\,K(m)}{p}\,\middle|\,m\right)\sum_{k=0}^{p-1}\operatorname{cn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right)^2 + \frac{2}{m^2}\left(\operatorname{ds}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right)\operatorname{ns}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right)\operatorname{cs}\left(\frac{4\,s\,K(m)}{p}\,\middle|\,m\right)\operatorname{ns}\left(\frac{4\,s\,K(m)}{p}\,\middle|\,m\right)+\operatorname{cs}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right)\operatorname{ds}\left(\frac{4\,s\,K(m)}{p}\,\middle|\,m\right)\left(\operatorname{ns}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right)^2+\operatorname{ns}\left(\frac{4\,s\,K(m)}{p}\,\middle|\,m\right)^2\right)\right)\sum_{k=0}^{p-1}\operatorname{cn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right) /;\ p\in\mathbb{N}^+\land r\in\mathbb{N}^+\land r<p$$

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09.26.16.0136.01

$$\sum_{k=0}^{p-1}\operatorname{cn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right)\left(\operatorname{cn}\left(z+\frac{4\,(k-r)\,K(m)}{p}\,\middle|\,m\right)^3+\operatorname{cn}\left(z+\frac{4\,(k+r)\,K(m)}{p}\,\middle|\,m\right)^3\right) = -\frac{2}{m}\left(\operatorname{cs}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right)^2\operatorname{ds}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right)^2+\operatorname{ns}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right)^2\operatorname{ds}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right)^2+\operatorname{cs}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right)^2\operatorname{ns}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right)^2+3\operatorname{cs}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right)\operatorname{ns}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right)\operatorname{ds}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right)^2\right)\sum_{k=0}^{p-1}\operatorname{dn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right) /;\ p\in\mathbb{N}^+\land r\in\mathbb{N}^+\land r<p$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

Sums over products of seven Jacobi functions

09.26.16.0137.01

$$\sum_{k=0}^{p-1}\operatorname{sn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right)\operatorname{cn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right)^2\left(\operatorname{dn}\left(z+\frac{2\,(k-r)\,K(m)}{p}\,\middle|\,m\right)^3+\operatorname{dn}\left(z+\frac{2\,(k+r)\,K(m)}{p}\,\middle|\,m\right)^3\right) = -4\operatorname{cs}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)^2\operatorname{ns}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\operatorname{ds}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\sum_{k=0}^{p-1}\operatorname{cn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right) /;\ p\in\mathbb{N}^+\land r\in\mathbb{N}^+\land r<p-1$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.26.16.0138.01

$$\sum_{k=0}^{p-1} \operatorname{cn}\left(z+\frac{2 k K(m)}{p} \,\middle|\, m\right) \operatorname{dn}\left(z+\frac{2 k K(m)}{p} \,\middle|\, m\right) \operatorname{sn}\left(z+\frac{2 k K(m)}{p} \,\middle|\, m\right)^2 \left(\operatorname{sn}\left(z+\frac{2 (k-r) K(m)}{p} \,\middle|\, m\right)^3+\operatorname{sn}\left(z+\frac{2 (k+r) K(m)}{p} \,\middle|\, m\right)^3\right) = -\frac{4}{m^2} \operatorname{ns}\left(\frac{2 r K(m)}{p} \,\middle|\, m\right)^2 \operatorname{ds}\left(\frac{2 r K(m)}{p} \,\middle|\, m\right) \operatorname{cs}\left(\frac{2 r K(m)}{p} \,\middle|\, m\right) \sum_{k=0}^{p-1} \operatorname{cn}\left(z+\frac{2 k K(m)}{p} \,\middle|\, m\right) \operatorname{sn}\left(z+\frac{2 k K(m)}{p} \,\middle|\, m\right) \operatorname{dn}\left(z+\frac{2 k K(m)}{p} \,\middle|\, m\right) /;\ p \in \mathbb{N}^+ \land r \in \mathbb{N}^+ \land r < p-1$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.26.16.0139.01

$$\sum_{k=0}^{p-1} \operatorname{cn}\left(z+\frac{2 k K(m)}{p} \,\middle|\, m\right)^2 \operatorname{dn}\left(z+\frac{2 k K(m)}{p} \,\middle|\, m\right) \operatorname{sn}\left(z+\frac{2 k K(m)}{p} \,\middle|\, m\right) \left(\operatorname{cn}\left(z+\frac{2 (k-r) K(m)}{p} \,\middle|\, m\right)^3+\operatorname{cn}\left(z+\frac{2 (k+r) K(m)}{p} \,\middle|\, m\right)^3\right) = -\frac{4}{m^2} \operatorname{ds}\left(\frac{2 r K(m)}{p} \,\middle|\, m\right)^2 \operatorname{cs}\left(\frac{2 r K(m)}{p} \,\middle|\, m\right) \operatorname{ns}\left(\frac{2 r K(m)}{p} \,\middle|\, m\right) \sum_{k=0}^{p-1} \operatorname{cn}\left(z+\frac{2 k K(m)}{p} \,\middle|\, m\right) \operatorname{sn}\left(z+\frac{2 k K(m)}{p} \,\middle|\, m\right) \operatorname{dn}\left(z+\frac{2 k K(m)}{p} \,\middle|\, m\right) /;\ p \in \mathbb{N}^+ \land r \in \mathbb{N}^+ \land r < p-1$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.26.16.0140.01

$$\sum_{k=0}^{p-1} \operatorname{cn}\left(z+\frac{2 k K(m)}{p} \,\middle|\, m\right) \operatorname{dn}\left(z+\frac{2 k K(m)}{p} \,\middle|\, m\right) \operatorname{sn}\left(z+\frac{2 k K(m)}{p} \,\middle|\, m\right) \left(\operatorname{dn}\left(z+\frac{2 (k-r) K(m)}{p} \,\middle|\, m\right)^4+\operatorname{dn}\left(z+\frac{2 (k+r) K(m)}{p} \,\middle|\, m\right)^4\right) = 2 \left(\operatorname{cs}\left(\frac{2 r K(m)}{p} \,\middle|\, m\right)^4-\operatorname{ds}\left(\frac{2 r K(m)}{p} \,\middle|\, m\right)^2 \operatorname{cs}\left(\frac{2 r K(m)}{p} \,\middle|\, m\right)^2-\operatorname{ns}\left(\frac{2 r K(m)}{p} \,\middle|\, m\right)^2 \operatorname{cs}\left(\frac{2 r K(m)}{p} \,\middle|\, m\right)^2-\operatorname{ns}\left(\frac{2 r K(m)}{p} \,\middle|\, m\right)^2 \operatorname{ds}\left(\frac{2 r K(m)}{p} \,\middle|\, m\right)^2\right) \sum_{k=0}^{p-1} \operatorname{cn}\left(z+\frac{2 k K(m)}{p} \,\middle|\, m\right) \operatorname{sn}\left(z+\frac{2 k K(m)}{p} \,\middle|\, m\right) \operatorname{dn}\left(z+\frac{2 k K(m)}{p} \,\middle|\, m\right) /;\ p \in \mathbb{N}^+ \land r \in \mathbb{N}^+ \land r < p-1$$

Khare/Lakshminarayan/Sukhatme_2002

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Sums over products of arbitrarily many Jacobi functions

09.26.16.0141.01

$$\sum_{j=0}^{p-1}\prod_{k=0}^{l-1}\operatorname{dn}\left(z+\frac{2K(m)(j+kr)}{p}\,\middle|\,m\right)=\left(\left(\prod_{k=1}^{\frac{l-1}{2}}\operatorname{cs}\left(\frac{2krK(m)}{p}\,\middle|\,m\right)\right)^2+2(-1)^{\frac{l-1}{2}}\sum_{k=1}^{\frac{l-1}{2}}\prod_{n=1}^{l}\operatorname{If}\left[n==k,1,\operatorname{cs}\left(\frac{2(n-k)rK(m)}{p}\,\middle|\,m\right)\right]\right)\sum_{k=0}^{p-1}\operatorname{dn}\left(z+\frac{2K(m)k}{p}\,\middle|\,m\right)/;$$
$$p\in\mathbb{N}^+\bigwedge r\in\mathbb{N}^+\bigwedge r<p-1\bigwedge\frac{l-1}{2}\in\mathbb{N}\bigwedge l\leq p$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.26.16.0142.01

$$\sum_{j=0}^{p-1}\prod_{k=0}^{r-1}\operatorname{dn}\left(z+\frac{2K(m)(j+k)}{p}\,\middle|\,m\right)=\frac{p}{2K(m)}\int_0^{2K(m)}\prod_{k=0}^{r-1}\operatorname{dn}\left(t+\frac{2kK(m)}{p}\,\middle|\,m\right)dt\,/;\,p-3\in\mathbb{N}\bigwedge\frac{r}{2}\in\mathbb{N}^+\bigwedge r<p-1$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.26.16.0143.01

$$\sum_{j=0}^{p-1}\prod_{k=0}^{r-1}\operatorname{sn}\left(z+\frac{2K(m)(j+k)}{p}\,\middle|\,m\right)=\frac{p}{2K(m)}\int_0^{2K(m)}\prod_{k=0}^{r-1}\operatorname{sn}\left(t+\frac{2kK(m)}{p}\,\middle|\,m\right)dt\,/;\,p-3\in\mathbb{N}\bigwedge\frac{r}{2}\in\mathbb{N}^+\bigwedge r<p-1$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.26.16.0144.01

$$\sum_{j=0}^{p-1}\prod_{k=0}^{r-1}\operatorname{cn}\left(z+\frac{2K(m)(j+k)}{p}\,\middle|\,m\right)=\frac{p}{2K(m)}\int_0^{2K(m)}\prod_{k=0}^{r-1}\operatorname{cn}\left(t+\frac{2kK(m)}{p}\,\middle|\,m\right)dt\,/;\,p-3\in\mathbb{N}\bigwedge\frac{r}{2}\in\mathbb{N}^+\bigwedge r<p-1$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.26.16.0047.01

$$\sum_{k=0}^{p-1}\prod_{l=0}^{r-1}\operatorname{cn}\left(z+\frac{4(k+n_l)K(m)}{p}\,\middle|\,m\right)=\sum_{k=0}^{p-1}\prod_{l=0}^{r-1}\operatorname{cn}\left(\frac{4(k+n_l)K(m)}{p}\,\middle|\,m\right)/;$$
$$\frac{p-1}{2}\in\mathbb{N}^+\bigwedge\frac{r}{2}\in\mathbb{N}^+\bigwedge n_0==0\bigwedge n_l\in\mathbb{Z}\bigwedge 1\leq n_l<p\bigwedge n_l<n_{l+1}$$

Khare/Sukhatme_2002

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Identities

Functional identities

Univariate functional identities

09.26.17.0001.01

$$m\, w(z)^4 - 2\,(m-1)\, w(z)^2 + m + \left(m\, w(z)^4 - 2\, m\, w(z)^2 + m - 1\right) w(2\, z) - 1 == 0 \,/;\, w(z) == \mathrm{cn}(z \mid m)$$

Bivariate functional identities

09.26.17.0002.01

$$(g(x-y) + g(x+y))\left(g(x)^2 + g(y)^2\right) == 2\, g(x)\, g(y)\, (g(x-y)\, g(x+y) + 1) \,/;\, g(z) == \mathrm{cn}(c\, z \mid m)$$

Complex characteristics

Real part

09.26.19.0001.01

$$\mathrm{Re}(\mathrm{cn}(x + i\, y \mid m)) == \frac{\mathrm{cn}(x \mid m)\, \mathrm{cn}(y \mid 1-m)}{\mathrm{cn}(y \mid 1-m)^2 + m\, \mathrm{sn}(x \mid m)^2\, \mathrm{sn}(y \mid 1-m)^2} \,/;\, \{x, y, m\} \in \mathbb{R}$$

Imaginary part

09.26.19.0002.01

$$\mathrm{Im}(\mathrm{cn}(x + i\, y \mid m)) == -\frac{\mathrm{dn}(x \mid m)\, \mathrm{dn}(y \mid 1-m)\, \mathrm{sn}(x \mid m)\, \mathrm{sn}(y \mid 1-m)}{\mathrm{cn}(y \mid 1-m)^2 + m\, \mathrm{sn}(x \mid m)^2\, \mathrm{sn}(y \mid 1-m)^2} \,/;\, \{x, y, m\} \in \mathbb{R}$$

Absolute value

09.26.19.0003.01

$$|\mathrm{cn}(x + i\, y \mid m)| == \frac{\sqrt{(\mathrm{cn}(x \mid m)\, \mathrm{cn}(y \mid 1-m))^2 + (\mathrm{dn}(x \mid m)\, \mathrm{dn}(y \mid 1-m)\, \mathrm{sn}(x \mid m)\, \mathrm{sn}(y \mid 1-m))^2}}{\mathrm{cn}(y \mid 1-m)^2 + m\, \mathrm{sn}(x \mid m)^2\, \mathrm{sn}(y \mid 1-m)^2} \,/;\, \{x, y, m\} \in \mathbb{R}$$

Argument

09.26.19.0004.01

$$\arg(\mathrm{cn}(x + i\, y \mid m)) == \tan^{-1}(\mathrm{cn}(x \mid m)\, \mathrm{cn}(y \mid 1-m),\, -\mathrm{dn}(x \mid m)\, \mathrm{dn}(y \mid 1-m)\, \mathrm{sn}(x \mid m)\, \mathrm{sn}(y \mid 1-m)) \,/;\, \{x, y, m\} \in \mathbb{R}$$

Conjugate value

09.26.19.0005.01

$$\overline{\mathrm{cn}(x + i\, y \mid m)} == \frac{\mathrm{cn}(x \mid m)\, \mathrm{cn}(y \mid 1-m) + i\, \mathrm{dn}(x \mid m)\, \mathrm{dn}(y \mid 1-m)\, \mathrm{sn}(x \mid m)\, \mathrm{sn}(y \mid 1-m)}{\mathrm{cn}(y \mid 1-m)^2 + m\, \mathrm{sn}(x \mid m)^2\, \mathrm{sn}(y \mid 1-m)^2} \,/;\, \{x, y, m\} \in \mathbb{R}$$

Differentiation

Low-order differentiation

With respect to *z*

09.26.20.0001.01

$$\frac{\partial\, \mathrm{cn}(z \mid m)}{\partial z} == -\mathrm{sn}(z \mid m)\, \mathrm{dn}(z \mid m)$$

09.26.20.0002.01

$$\frac{\partial^2 \operatorname{cn}(z \mid m)}{\partial z^2} = \operatorname{cn}(z \mid m)\left(m \operatorname{sn}(z \mid m)^2 - \operatorname{dn}(z \mid m)^2\right)$$

09.26.20.0003.01

$$\frac{\partial^2 \operatorname{cn}(z \mid m)}{\partial z^2} = -2\, m \operatorname{cn}(z \mid m)^3 + (2\, m - 1) \operatorname{cn}(z \mid m)$$

With respect to *m*

09.26.20.0004.01

$$\frac{\partial \operatorname{cn}(z \mid m)}{\partial m} = \frac{1}{2\, m\, (1-m)} (\operatorname{sn}(z \mid m) \operatorname{dn}(z \mid m)\, ((m-1)\, z + E(\operatorname{am}(z \mid m) \mid m) - m \operatorname{sn}(z \mid m) \operatorname{cd}(z \mid m)))$$

09.26.20.0005.01

$$\frac{\partial^2 \operatorname{cn}(z \mid m)}{\partial m^2} = \frac{1}{4\, (m-1)^2\, m^2} \Bigg(\operatorname{cn}(z \mid m) \Big(-\left(((m-1)\, z + E(\operatorname{am}(z \mid m) \mid m))^2 - 3\, m \operatorname{cd}(z \mid m) \operatorname{sn}(z \mid m)\, ((m-1)\, z + E(\operatorname{am}(z \mid m) \mid m)) + 2\, m^2 \operatorname{cd}(z \mid m)^2 \operatorname{sn}(z \mid m)^2\right) \operatorname{dn}(z \mid m)^2 + m \operatorname{sn}(z \mid m)^2 \left(\operatorname{sc}(z \mid m)\, (-m\, z + z - E(\operatorname{am}(z \mid m) \mid m) + m \operatorname{cd}(z \mid m) \operatorname{sn}(z \mid m)) + \sqrt{1 - m \operatorname{sn}(z \mid m)^2}\right) \operatorname{dn}(z \mid m) + m\, ((m-1)\, z + E(\operatorname{am}(z \mid m) \mid m)) \operatorname{sn}(z \mid m)^2\, ((m-1)\, z + E(\operatorname{am}(z \mid m) \mid m) - m \operatorname{cd}(z \mid m) \operatorname{sn}(z \mid m))\Big) - \operatorname{dn}(z \mid m) \operatorname{sn}(z \mid m) \Big(-2\, z\, m^2 + 2 \operatorname{cd}(z \mid m) \operatorname{sn}(z \mid m)\, m^2 + 4\, z\, m - 3\, E(\operatorname{am}(z \mid m) \mid m)\, m - F(\operatorname{am}(z \mid m) \mid m)\, m - (m-1)\, ((m-1)\, z + E(\operatorname{am}(z \mid m) \mid m)) \operatorname{nd}(z \mid m) \operatorname{sd}(z \mid m) \operatorname{sn}(z \mid m)\, m + z \operatorname{dn}(z \mid m) \sqrt{1 - m \operatorname{sn}(z \mid m)^2}\; m - 2\, z + E(\operatorname{am}(z \mid m) \mid m) + F(\operatorname{am}(z \mid m) \mid m) - z \operatorname{dn}(z \mid m) \sqrt{1 - m \operatorname{sn}(z \mid m)^2} + E(\operatorname{am}(z \mid m) \mid m) \operatorname{dn}(z \mid m) \sqrt{1 - m \operatorname{sn}(z \mid m)^2}\Big)\Bigg)$$

Symbolic differentiation

With respect to *z*

09.26.20.0008.01

$$\frac{\partial^n \operatorname{cn}(z \mid m)}{\partial z^n} = \operatorname{cn}(z \mid m)\, \delta_n - \sum_{j=0}^{n-1} \binom{n-1}{j} \frac{\partial^j \operatorname{sn}(z \mid m)}{\partial z^j} \frac{\partial^{-j+n-1} \operatorname{dn}(z \mid m)}{\partial z^{-j+n-1}} \;/;\; n \in \mathbb{N}$$

09.26.20.0006.02

$$\frac{\partial^n \operatorname{cn}(z \mid m)}{\partial z^n} = \frac{2^{1-n}\, \pi^{n+1}}{\sqrt{m}\; K(m)^{n+1}} \sum_{k=0}^{\infty} \frac{(2\, k+1)^n\, q(m)^{k+\frac{1}{2}}}{q(m)^{2\, k+1} + 1} \cos\left(\frac{\pi\, n}{2} + \frac{(2\, k+1)\, \pi\, z}{2\, K(m)}\right) /;\; n \in \mathbb{N}$$

Fractional integro-differentiation

With respect to *z*

09.26.20.0007.01

$$\frac{\partial^\alpha \operatorname{cn}(z \mid m)}{\partial z^\alpha} = \frac{2^{\alpha+1}\, \pi^{3/2}\, z^{-\alpha}}{\sqrt{m}\; K(m)} \sum_{k=0}^{\infty} \frac{q(m)^{k+\frac{1}{2}}}{1 + q(m)^{2\, k+1}}\, {}_1\tilde{F}_2\left(1; \frac{1-\alpha}{2}, 1 - \frac{\alpha}{2}; -\frac{(2\, k+1)^2\, \pi^2\, z^2}{16\, K(m)^2}\right)$$

Integration

Indefinite integration

Involving only one direct function

09.26.21.0001.01

$$\int \operatorname{cn}(z \mid m)\, dz = \frac{\cos^{-1}(\operatorname{dn}(z \mid m)) \operatorname{sn}(z \mid m)}{\sqrt{1 - \operatorname{dn}(z \mid m)^2}}$$

Involving functions of the direct function

Involving elementary functions of the direct function

Involving powers of the direct function

09.26.21.0002.01

$$\int \operatorname{cn}(z \mid m)^2\, dz = z - \frac{z}{m} + \frac{E(\operatorname{am}(z \mid m) \mid m) \left(\operatorname{cn}(z \mid m)^2 + \frac{1}{m} - 1\right)}{\operatorname{dn}(z \mid m) \sqrt{1 - m \operatorname{sn}(z \mid m)^2}}$$

09.26.21.0003.01

$$\int \operatorname{cn}(z \mid m)^3\, dz = \frac{\operatorname{sn}(z \mid m)}{2\, m} \left(\frac{(2\, m - 1) \cos^{-1}(\operatorname{dn}(z \mid m))}{\sqrt{1 - \operatorname{dn}(z \mid m)^2}} + \operatorname{dn}(z \mid m)\right)$$

09.26.21.0004.01

$$\int \frac{dz}{\operatorname{cn}(z \mid m)} = \frac{1}{\sqrt{1 - m}} \log\left(\frac{\operatorname{dn}(z \mid m) + \sqrt{1 - m}\, \operatorname{sn}(z \mid m)}{\operatorname{cn}(z \mid m)}\right)$$

09.26.21.0005.01

$$\int \frac{dz}{\operatorname{cn}(z \mid m)^2} = z + \frac{\operatorname{dn}(z \mid m) \operatorname{sn}(z \mid m)}{(1 - m) \operatorname{cn}(z \mid m)} - \frac{E(\operatorname{am}(z \mid m) \mid m) \left(m \operatorname{cn}(z \mid m)^2 - m + 1\right)}{(1 - m) \left(\operatorname{dn}(z \mid m) \sqrt{1 - m \operatorname{sn}(z \mid m)^2}\right)}$$

09.26.21.0006.01

$$\int \frac{1}{\operatorname{cn}(z \mid m)^3}\, dz = \frac{1}{2\,(1 - m)} \left(\frac{\operatorname{dn}(z \mid m) \operatorname{sn}(z \mid m)}{\operatorname{cn}(z \mid m)^2} - \frac{2\, m - 1}{\sqrt{1 - m}} \log\left(\frac{\operatorname{dn}(z \mid m) + \sqrt{1 - m}\, \operatorname{sn}(z \mid m)}{\operatorname{cn}(z \mid m)}\right)\right)$$

Definite integration

Involving functions of the direct function

Involving elementary functions of the direct function

Involving products of the direct function

09.26.21.0007.01

$$\int_0^{2K(m)} m^2 \operatorname{cn}(t \mid m)^3 \operatorname{cn}(a+t \mid m) \, dt = \frac{2}{m^2} \left(E(m) \operatorname{cs}(a \mid m) \operatorname{ns}(a \mid m) + K(m) \left(Z(\operatorname{am}(a \mid m) \mid m) \operatorname{ds}(a \mid m)^3 + m^2 \operatorname{cn}(a \mid m) - \operatorname{cs}(a \mid m) \operatorname{ns}(a \mid m)\right)\right)$$

Khare/Lakshminarayan/Sukhatme_2003

09.26.21.0008.01

$$\int_0^{2K(m)} \operatorname{cn}(t \mid m) \operatorname{cn}(a+t \mid m) \operatorname{cn}(b+t \mid m) \operatorname{cn}(c+t \mid m) \, dt = \frac{1}{m^2} \left((2K(m)) \left(\operatorname{cn}(a \mid m) \operatorname{cn}(b \mid m) \operatorname{cn}(c \mid m) m^2 + \operatorname{ds}(b-a \mid m) \operatorname{ds}(c-a \mid m) \operatorname{ds}(a \mid m) Z(\operatorname{am}(a \mid m) \mid m) - \operatorname{ds}(b \mid m) \operatorname{ds}(b-a \mid m) \operatorname{ds}(c-b \mid m) Z(\operatorname{am}(b \mid m) \mid m) + \operatorname{ds}(c-a \mid m) \operatorname{ds}(c-b \mid m) \operatorname{ds}(c \mid m) Z(\operatorname{am}(c \mid m) \mid m)\right)\right)$$

Khare/Lakshminarayan/Sukhatme_2003

Involving direct function and elliptic functions

Involving Jacobi functions

Involving dn

09.26.21.0009.01

$$\int_0^{2K(m)} \operatorname{dn}(t \mid m) \operatorname{cn}(t \mid m) \operatorname{dn}(a+t \mid m) \operatorname{cn}(a+t \mid m) \, dt = \frac{1}{m} \left(2K(m) \left(2 \operatorname{cs}(a \mid m) \operatorname{ds}(a \mid m) - \left(\operatorname{cs}(a \mid m)^2 + \operatorname{ds}(a \mid m)^2\right) \operatorname{ns}(a \mid m) Z(\operatorname{am}(a \mid m) \mid m)\right) - 4 \operatorname{cs}(a \mid m) \operatorname{ds}(a \mid m) E(m)\right)$$

Khare/Lakshminarayan/Sukhatme_2003

Involving sn

09.26.21.0010.01

$$\int_0^{2K(m)} \operatorname{sn}(t \mid m) \operatorname{cn}(t \mid m) \operatorname{sn}(a+t \mid m) \operatorname{cn}(a+t \mid m) \, dt = \frac{1}{m^2} \left(2 \left(K(m) \left(\operatorname{dn}(a \mid m)^2 + 1\right) (\operatorname{cs}(a \mid m) \operatorname{ns}(a \mid m) Z(\operatorname{am}(a \mid m) \mid m) - \operatorname{ds}(a \mid m)) \operatorname{ns}(a \mid m) + 2 E(m) \operatorname{ds}(a \mid m) \operatorname{ns}(a \mid m)\right)\right)$$

Khare/Lakshminarayan/Sukhatme_2003

09.26.21.0011.01

$$\int_0^{2K(m)} \operatorname{cn}(t \mid m) \operatorname{sn}(a+t \mid m) \operatorname{cn}(b+t \mid m) \operatorname{sn}(c+t \mid m) \, dt = \frac{2K(m)}{m^2} \left(\operatorname{cn}(b \mid m) \operatorname{sn}(a \mid m) \operatorname{sn}(c \mid m) m^2 - \operatorname{ds}(a \mid m) \operatorname{ds}(b-a \mid m) \operatorname{ns}(c-a \mid m) Z(\operatorname{am}(a \mid m) \mid m) + \operatorname{ds}(b \mid m) \operatorname{ns}(b-a \mid m) \operatorname{ns}(c-b \mid m) Z(\operatorname{am}(b \mid m) \mid m) - \operatorname{ds}(c \mid m) \operatorname{ns}(c-a \mid m) \operatorname{ds}(c-b \mid m) Z(\operatorname{am}(c \mid m) \mid m)\right)$$

Khare/Lakshminarayan/Sukhatme_2003

Representations through equivalent functions

With inverse function

09.26.27.0001.01

$$\operatorname{cn}\left(\operatorname{cn}^{-1}(z \mid m) \,\middle|\, m\right) = z$$

With related functions

Involving am

09.26.27.0002.01

$$\operatorname{cn}(z \mid m) = \cos(\operatorname{am}(z \mid m))$$

Involving one other Jacobi elliptic function

Involving cd

09.26.27.0005.01

$$\operatorname{cn}(z \mid m)^2 = \frac{(m-1)\operatorname{cd}(z \mid m)^2}{m \operatorname{cd}(z \mid m)^2 - 1}$$

Involving cs

09.26.27.0008.01

$$\operatorname{cn}(z \mid m)^2 = \frac{\operatorname{cs}(z \mid m)^2}{\operatorname{cs}(z \mid m)^2 + 1}$$

Involving dc

09.26.27.0011.01

$$\operatorname{cn}(z \mid m)^2 = \frac{1-m}{\operatorname{dc}(z \mid m)^2 - m}$$

Involving dn

09.26.27.0012.01

$$\operatorname{cn}(z \mid m)^2 = \frac{\operatorname{dn}(z \mid m)^2 - 1}{m} + 1$$

Involving ds

09.26.27.0013.01

$$\operatorname{cn}(z \mid m)^2 = \frac{\operatorname{ds}(z \mid m)^2 + m - 1}{\operatorname{ds}(z \mid m)^2 + m}$$

Involving nc

09.26.27.0014.01

$$\operatorname{cn}(z \mid m) == \frac{1}{\operatorname{nc}(z \mid m)}$$

09.26.27.0015.01

$$\operatorname{cn}(z \mid m) == \operatorname{nc}(i\, z \mid 1 - m)$$

Involving nd

09.26.27.0016.01

$$\operatorname{cn}(z \mid m)^2 == \frac{(m-1)\operatorname{nd}(z \mid m)^2 + 1}{m \operatorname{nd}(z \mid m)^2}$$

Involving ns

09.26.27.0018.01

$$\operatorname{cn}(z \mid m)^2 == 1 - \frac{1}{\operatorname{ns}(z \mid m)^2}$$

Involving sc

09.26.27.0020.01

$$\operatorname{cn}(z \mid m)^2 == \frac{1}{\operatorname{sc}(z \mid m)^2 + 1}$$

Involving sd

09.26.27.0021.01

$$\operatorname{cn}(z \mid m)^2 == \frac{(m-1)\operatorname{sd}(z \mid m)^2 + 1}{m \operatorname{sd}(z \mid m)^2 + 1}$$

Involving sn

09.26.27.0022.01

$$\operatorname{cn}(z \mid m)^2 == 1 - \operatorname{sn}(z \mid m)^2$$

Involving two other Jacobi elliptic functions

Involving cd and dn

09.26.27.0003.01

$$\operatorname{cn}(z \mid m) == \operatorname{cd}(z \mid m) \operatorname{dn}(z \mid m)$$

Involving cd and nc

09.26.27.0032.01

$$\operatorname{cn}(z \mid m) = \frac{(m-1)\operatorname{cd}(z \mid m)^2 \operatorname{nc}(z \mid m)}{m \operatorname{cd}(z \mid m)^2 - 1}$$

Involving cd and nd

09.26.27.0004.01

$$\operatorname{cn}(z \mid m) = \frac{\operatorname{cd}(z \mid m)}{\operatorname{nd}(z \mid m)}$$

Involving cs and nc

09.26.27.0033.01

$$\operatorname{cn}(z \mid m) = \frac{\operatorname{cs}(z \mid m)^2 \operatorname{nc}(z \mid m)}{\operatorname{cs}(z \mid m)^2 + 1}$$

Involving cs and ns

09.26.27.0006.01

$$\operatorname{cn}(z \mid m) = \frac{\operatorname{cs}(z \mid m)}{\operatorname{ns}(z \mid m)}$$

09.26.27.0034.01

$$\operatorname{cn}(z \mid m) = \frac{\operatorname{cs}(z \mid m) \operatorname{ns}(z \mid m)}{\operatorname{cs}(z \mid m)^2 + 1}$$

Involving cs and sn

09.26.27.0007.01

$$\operatorname{cn}(z \mid m) = \operatorname{cs}(z \mid m) \operatorname{sn}(z \mid m)$$

Involving dc and dn

09.26.27.0009.01

$$\operatorname{cn}(z \mid m) = \frac{\operatorname{dn}(z \mid m)}{\operatorname{dc}(z \mid m)}$$

09.26.27.0035.01

$$\operatorname{cn}(z \mid m) = \frac{\operatorname{dc}(z \mid m) \left(\operatorname{dn}(z \mid m)^2 + m - 1\right)}{m \operatorname{dn}(z \mid m)}$$

Involving dc and nc

09.26.27.0036.01

$$\operatorname{cn}(z \mid m) = \frac{(m-1) \operatorname{nc}(z \mid m)}{m - \operatorname{dc}(z \mid m)^2}$$

Involving dc and nd

09.26.27.0010.01

$$\operatorname{cn}(z \mid m) = \frac{1}{\operatorname{dc}(z \mid m) \operatorname{nd}(z \mid m)}$$

09.26.27.0037.01

$$\operatorname{cn}(z \mid m) = \frac{(m-1)\operatorname{dc}(z \mid m)\operatorname{nd}(z \mid m)}{m - \operatorname{dc}(z \mid m)^2}$$

Involving dn and nc

09.26.27.0038.01

$$\operatorname{cn}(z \mid m) = \frac{\left(\operatorname{dn}(z \mid m)^2 + m - 1\right)\operatorname{nc}(z \mid m)}{m}$$

Involving ds and nc

09.26.27.0039.01

$$\operatorname{cn}(z \mid m) = \frac{\left(\operatorname{ds}(z \mid m)^2 + m - 1\right)\operatorname{nc}(z \mid m)}{\operatorname{ds}(z \mid m)^2 + m}$$

Involving nc and nd

09.26.27.0040.01

$$\operatorname{cn}(z \mid m) = \frac{\operatorname{nc}(z \mid m)\left(m\operatorname{nd}(z \mid m)^2 - \operatorname{nd}(z \mid m)^2 + 1\right)}{m\operatorname{nd}(z \mid m)^2}$$

Involving nc and ns

09.26.27.0041.01

$$\operatorname{cn}(z \mid m) = \frac{\operatorname{nc}(z \mid m)(\operatorname{ns}(z \mid m) - 1)(\operatorname{ns}(z \mid m) + 1)}{\operatorname{ns}(z \mid m)^2}$$

Involving nc and sc

09.26.27.0042.01

$$\operatorname{cn}(z \mid m) = \frac{\operatorname{nc}(z \mid m)}{\operatorname{sc}(z \mid m)^2 + 1}$$

Involving nc and sd

09.26.27.0043.01

$$\operatorname{cn}(z \mid m) = \frac{\operatorname{nc}(z \mid m)\left(m\operatorname{sd}(z \mid m)^2 - \operatorname{sd}(z \mid m)^2 + 1\right)}{m\operatorname{sd}(z \mid m)^2 + 1}$$

Involving nc and sn

09.26.27.0044.01

$$\operatorname{cn}(z \mid m) = -\operatorname{nc}(z \mid m)(\operatorname{sn}(z \mid m) - 1)(\operatorname{sn}(z \mid m) + 1)$$

Involving ns and sc

09.26.27.0017.01

$$\operatorname{cn}(z \mid m) = \frac{1}{\operatorname{sc}(z \mid m) \operatorname{ns}(z \mid m)}$$

09.26.27.0045.01

$$\operatorname{cn}(z \mid m) = \frac{(\operatorname{ns}(z \mid m) - 1)(\operatorname{ns}(z \mid m) + 1) \operatorname{sc}(z \mid m)}{\operatorname{ns}(z \mid m)}$$

09.26.27.0046.01

$$\operatorname{cn}(z \mid m) = \frac{\operatorname{ns}(z \mid m) \operatorname{sc}(z \mid m)}{\operatorname{sc}(z \mid m)^2 + 1}$$

Involving sc and sn

09.26.27.0019.01

$$\operatorname{cn}(z \mid m) = \frac{\operatorname{sn}(z \mid m)}{\operatorname{sc}(z \mid m)}$$

09.26.27.0047.01

$$\operatorname{cn}(z \mid m) = -\frac{\operatorname{sc}(z \mid m)(\operatorname{sn}(z \mid m) - 1)(\operatorname{sn}(z \mid m) + 1)}{\operatorname{sn}(z \mid m)}$$

Involving three other Jacobi elliptic functions

09.26.27.0048.01

$$\operatorname{cn}(z \mid m) = \frac{\operatorname{cs}(z \mid m)^2 \operatorname{dc}(z \mid m)}{\left(\operatorname{cs}(z \mid m)^2 + 1\right) \operatorname{dn}(z \mid m)}$$

09.26.27.0049.01

$$\operatorname{cn}(z \mid m) = \frac{\operatorname{cs}(z \mid m) \operatorname{ds}(z \mid m)}{\left(\operatorname{cs}(z \mid m)^2 + 1\right) \operatorname{dn}(z \mid m)}$$

09.26.27.0050.01

$$\operatorname{cn}(z \mid m) = \frac{\operatorname{dc}(z \mid m) \operatorname{dn}(z \mid m) \left(\operatorname{ds}(z \mid m)^2 + m - 1\right)}{\operatorname{ds}(z \mid m)^2}$$

09.26.27.0051.01

$$\operatorname{cn}(z \mid m) = -\frac{(\operatorname{dn}(z \mid m) - \operatorname{ds}(z \mid m))(\operatorname{dn}(z \mid m) + \operatorname{ds}(z \mid m)) \operatorname{nc}(z \mid m)}{\operatorname{ds}(z \mid m)^2}$$

09.26.27.0052.01

$$\operatorname{cn}(z \mid m) = \frac{\operatorname{cd}(z \mid m)^2 \operatorname{ds}(z \mid m)^2 \operatorname{nc}(z \mid m)}{\operatorname{cd}(z \mid m)^2 \operatorname{ds}(z \mid m)^2 + 1}$$

09.26.27.0053.01

$$\operatorname{cn}(z \mid m) = \frac{\operatorname{dc}(z \mid m) \operatorname{dn}(z \mid m) + m \operatorname{nc}(z \mid m) - \operatorname{nc}(z \mid m)}{m}$$

09.26.27.0054.01

$$\operatorname{cn}(z \mid m) = -\frac{\operatorname{cd}(z \mid m) \left(-\operatorname{cs}(z \mid m)^2 + m - 1\right) \operatorname{nd}(z \mid m)}{\operatorname{cs}(z \mid m)^2 + 1}$$

09.26.27.0055.01

$$\operatorname{cn}(z \mid m) = \frac{\operatorname{cs}(z \mid m)^2 \operatorname{dc}(z \mid m) \operatorname{nd}(z \mid m)}{\operatorname{cs}(z \mid m)^2 + 1}$$

09.26.27.0056.01

$$\operatorname{cn}(z \mid m) = \frac{\operatorname{cs}(z \mid m) \operatorname{ds}(z \mid m) \operatorname{nd}(z \mid m)}{\operatorname{cs}(z \mid m)^2 + 1}$$

09.26.27.0057.01

$$\operatorname{cn}(z \mid m) = \frac{\operatorname{dc}(z \mid m) \operatorname{ds}(z \mid m)^2 \operatorname{nd}(z \mid m)}{\operatorname{dc}(z \mid m)^2 + \operatorname{ds}(z \mid m)^2}$$

09.26.27.0058.01

$$\operatorname{cn}(z \mid m) = \frac{\operatorname{dc}(z \mid m) (\operatorname{dn}(z \mid m) + m \operatorname{nd}(z \mid m) - \operatorname{nd}(z \mid m))}{m}$$

09.26.27.0059.01

$$\operatorname{cn}(z \mid m) = \frac{\operatorname{nc}(z \mid m) (\operatorname{ds}(z \mid m) \operatorname{nd}(z \mid m) - 1) (\operatorname{ds}(z \mid m) \operatorname{nd}(z \mid m) + 1)}{\operatorname{ds}(z \mid m)^2 \operatorname{nd}(z \mid m)^2}$$

09.26.27.0060.01

$$\operatorname{cn}(z \mid m) = \frac{\operatorname{dc}(z \mid m) (\operatorname{ds}(z \mid m) \operatorname{nd}(z \mid m) - 1) (\operatorname{ds}(z \mid m) \operatorname{nd}(z \mid m) + 1)}{\operatorname{ds}(z \mid m)^2 \operatorname{nd}(z \mid m)}$$

09.26.27.0061.01

$$\operatorname{cn}(z \mid m) = -\frac{\operatorname{cs}(z \mid m) (\operatorname{dn}(z \mid m) - 1) (\operatorname{dn}(z \mid m) + 1) \operatorname{ns}(z \mid m)}{m}$$

09.26.27.0062.01

$$\operatorname{cn}(z \mid m) = \frac{\operatorname{dc}(z \mid m) \operatorname{ds}(z \mid m) \operatorname{ns}(z \mid m)}{\operatorname{dc}(z \mid m)^2 + \operatorname{ds}(z \mid m)^2}$$

09.26.27.0063.01

$$\operatorname{cn}(z \mid m) = \frac{\operatorname{dc}(z \mid m) (\operatorname{ns}(z \mid m) - 1) (\operatorname{ns}(z \mid m) + 1)}{\operatorname{dn}(z \mid m) \operatorname{ns}(z \mid m)^2}$$

09.26.27.0064.01

$$\operatorname{cn}(z \mid m) = \frac{\operatorname{dc}(z \mid m) \operatorname{nd}(z \mid m) (\operatorname{ns}(z \mid m) - 1) (\operatorname{ns}(z \mid m) + 1)}{\operatorname{ns}(z \mid m)^2}$$

09.26.27.0065.01

$$\operatorname{cn}(z \mid m) = \frac{\operatorname{dc}(z \mid m) (\operatorname{ns}(z \mid m) - 1) (\operatorname{ns}(z \mid m) + 1)}{\operatorname{ds}(z \mid m) \operatorname{ns}(z \mid m)}$$

09.26.27.0066.01

$$\operatorname{cn}(z \mid m) = \frac{\operatorname{cd}(z \mid m) \operatorname{nd}(z \mid m) \left(\operatorname{ns}(z \mid m)^2 - m\right)}{\operatorname{ns}(z \mid m)^2}$$

09.26.27.0067.01

$$\operatorname{cn}(z \mid m) = \frac{\operatorname{nc}(z \mid m) \operatorname{ns}(z \mid m) - \operatorname{sc}(z \mid m)}{\operatorname{ns}(z \mid m)}$$

09.26.27.0068.01

$$\operatorname{cn}(z \mid m) = \frac{\left(\operatorname{dn}(z \mid m)^2 + m - 1\right) \operatorname{ds}(z \mid m) \operatorname{sc}(z \mid m)}{m \operatorname{dn}(z \mid m)}$$

09.26.27.0069.01

$$\operatorname{cn}(z \mid m) = \frac{\operatorname{dn}(z \mid m) \left(\operatorname{ds}(z \mid m)^2 + m - 1\right) \operatorname{sc}(z \mid m)}{\operatorname{ds}(z \mid m)}$$

09.26.27.0070.01

$$\operatorname{cn}(z \mid m) = \frac{(\operatorname{ds}(z \mid m) \operatorname{nd}(z \mid m) - 1) (\operatorname{ds}(z \mid m) \operatorname{nd}(z \mid m) + 1) \operatorname{sc}(z \mid m)}{\operatorname{ds}(z \mid m) \operatorname{nd}(z \mid m)}$$

09.26.27.0071.01

$$\operatorname{cn}(z \mid m) = \frac{\left(\operatorname{dn}(z \mid m)^2 + m - 1\right) \operatorname{ns}(z \mid m) \operatorname{sc}(z \mid m)}{m}$$

09.26.27.0072.01

$$\operatorname{cn}(z \mid m) = \frac{\operatorname{cs}(z \mid m) \operatorname{nc}(z \mid m)}{\operatorname{cs}(z \mid m) + \operatorname{sc}(z \mid m)}$$

09.26.27.0073.01

$$\operatorname{cn}(z \mid m) = \frac{\operatorname{ns}(z \mid m)}{\operatorname{cs}(z \mid m) + \operatorname{sc}(z \mid m)}$$

09.26.27.0074.01

$$\operatorname{cn}(z \mid m) = \frac{\operatorname{dc}(z \mid m)}{\operatorname{dn}(z \mid m) \left(\operatorname{sc}(z \mid m)^2 + 1\right)}$$

09.26.27.0075.01

$$\operatorname{cn}(z \mid m) = \frac{\operatorname{dc}(z \mid m) \operatorname{nd}(z \mid m)}{\operatorname{sc}(z \mid m)^2 + 1}$$

09.26.27.0076.01

$$\operatorname{cn}(z \mid m) = \frac{\operatorname{ds}(z \mid m) \operatorname{sc}(z \mid m)}{\operatorname{dn}(z \mid m) \left(\operatorname{sc}(z \mid m)^2 + 1\right)}$$

09.26.27.0077.01

$$\operatorname{cn}(z \mid m) = \frac{\operatorname{ds}(z \mid m) \operatorname{nd}(z \mid m) \operatorname{sc}(z \mid m)}{\operatorname{sc}(z \mid m)^2 + 1}$$

09.26.27.0078.01

$$\operatorname{cn}(z \mid m) = -\frac{\operatorname{cd}(z \mid m) \operatorname{nd}(z \mid m) \left(m \operatorname{sc}(z \mid m)^2 - \operatorname{sc}(z \mid m)^2 - 1\right)}{\operatorname{sc}(z \mid m)^2 + 1}$$

09.26.27.0079.01

$$\operatorname{cn}(z \mid m) = \frac{\operatorname{nd}(z \mid m) \operatorname{sc}(z \mid m)}{\left(\operatorname{sc}(z \mid m)^2 + 1\right) \operatorname{sd}(z \mid m)}$$

09.26.27.0080.01

$$\operatorname{cn}(z \mid m) = \frac{\operatorname{cs}(z \mid m) \left(\operatorname{cs}(z \mid m)^2 - m + 1\right) \operatorname{nd}(z \mid m) \operatorname{sd}(z \mid m)}{\operatorname{cs}(z \mid m)^2 + 1}$$

09.26.27.0081.01

$$\operatorname{cn}(z \mid m) = \frac{\operatorname{dc}(z \mid m)\,(\operatorname{ns}(z \mid m) - 1)\,(\operatorname{ns}(z \mid m) + 1)\,\operatorname{sd}(z \mid m)}{\operatorname{ns}(z \mid m)}$$

09.26.27.0082.01

$$\operatorname{cn}(z \mid m) = \frac{\operatorname{cd}(z \mid m)\left(\operatorname{ns}(z \mid m)^2 - m\right)\operatorname{sd}(z \mid m)}{\operatorname{ns}(z \mid m)}$$

09.26.27.0083.01

$$\operatorname{cn}(z \mid m) = -\frac{\operatorname{dn}(z \mid m)\left(\operatorname{dn}(z \mid m)^2 + m - 1\right)\operatorname{sc}(z \mid m)\,\operatorname{sd}(z \mid m)}{(\operatorname{dn}(z \mid m) - 1)\,(\operatorname{dn}(z \mid m) + 1)}$$

09.26.27.0084.01

$$\operatorname{cn}(z \mid m) = \frac{(m - 1)\operatorname{nd}(z \mid m)\,\operatorname{sc}(z \mid m)\,\operatorname{sd}(z \mid m)}{\operatorname{nd}(z \mid m)^2 - \operatorname{sc}(z \mid m)^2 - 1}$$

09.26.27.0085.01

$$\operatorname{cn}(z \mid m) = -\frac{\operatorname{nd}(z \mid m)\left(m \operatorname{sc}(z \mid m)^2 - \operatorname{sc}(z \mid m)^2 - 1\right)\operatorname{sd}(z \mid m)}{\operatorname{sc}(z \mid m)\left(\operatorname{sc}(z \mid m)^2 + 1\right)}$$

09.26.27.0086.01

$$\operatorname{cn}(z \mid m) = \frac{\operatorname{nc}(z \mid m)\,(\operatorname{nd}(z \mid m) - \operatorname{sd}(z \mid m))\,(\operatorname{nd}(z \mid m) + \operatorname{sd}(z \mid m))}{\operatorname{nd}(z \mid m)^2}$$

09.26.27.0087.01

$$\operatorname{cn}(z \mid m) = \frac{\operatorname{dc}(z \mid m)\,(\operatorname{nd}(z \mid m) - \operatorname{sd}(z \mid m))\,(\operatorname{nd}(z \mid m) + \operatorname{sd}(z \mid m))}{\operatorname{nd}(z \mid m)}$$

09.26.27.0088.01

$$\operatorname{cn}(z \mid m) = \frac{\operatorname{sc}(z \mid m)\,(\operatorname{nd}(z \mid m) - \operatorname{sd}(z \mid m))\,(\operatorname{nd}(z \mid m) + \operatorname{sd}(z \mid m))}{\operatorname{nd}(z \mid m)\,\operatorname{sd}(z \mid m)}$$

09.26.27.0089.01

$$\operatorname{cn}(z \mid m) = -\operatorname{nc}(z \mid m)\,(\operatorname{dn}(z \mid m)\,\operatorname{sd}(z \mid m) - 1)\,(\operatorname{dn}(z \mid m)\,\operatorname{sd}(z \mid m) + 1)$$

09.26.27.0090.01

$$\operatorname{cn}(z \mid m) = \frac{\operatorname{cd}(z \mid m)^2 \operatorname{nc}(z \mid m)}{\operatorname{cd}(z \mid m)^2 + \operatorname{sd}(z \mid m)^2}$$

09.26.27.0091.01

$$\operatorname{cn}(z \mid m) = \operatorname{dc}(z \mid m)\,\operatorname{dn}(z \mid m)\left(m \operatorname{sd}(z \mid m)^2 - \operatorname{sd}(z \mid m)^2 + 1\right)$$

09.26.27.0092.01

$$\operatorname{cn}(z \mid m) = \frac{\operatorname{dn}(z \mid m)\,\operatorname{sc}(z \mid m)\left(m \operatorname{sd}(z \mid m)^2 - \operatorname{sd}(z \mid m)^2 + 1\right)}{\operatorname{sd}(z \mid m)}$$

09.26.27.0093.01

$$\operatorname{cn}(z \mid m) = \frac{\operatorname{dc}(z \mid m)\,\operatorname{ns}(z \mid m)\,\operatorname{sd}(z \mid m)}{\operatorname{dc}(z \mid m)^2 \operatorname{sd}(z \mid m)^2 + 1}$$

09.26.27.0094.01

$$\operatorname{cn}(z \mid m) = \operatorname{sc}(z \mid m)\,(\operatorname{ns}(z \mid m) - \operatorname{sn}(z \mid m))$$

09.26.27.0095.01

$$\operatorname{cn}(z \mid m) = \frac{\operatorname{dc}(z \mid m) \left(\operatorname{ds}(z \mid m)^2 + m - 1\right) \operatorname{sn}(z \mid m)}{\operatorname{ds}(z \mid m)}$$

09.26.27.0096.01

$$\operatorname{cn}(z \mid m) = \frac{(m-1) \operatorname{cd}(z \mid m)^2 \operatorname{sc}(z \mid m) \operatorname{sn}(z \mid m)}{(\operatorname{cd}(z \mid m) - 1) (\operatorname{cd}(z \mid m) + 1)}$$

09.26.27.0097.01

$$\operatorname{cn}(z \mid m) = -\frac{(m-1) \operatorname{sc}(z \mid m) \operatorname{sn}(z \mid m)}{(\operatorname{dc}(z \mid m) - 1) (\operatorname{dc}(z \mid m) + 1)}$$

09.26.27.0098.01

$$\operatorname{cn}(z \mid m) = -\frac{\left(\operatorname{dn}(z \mid m)^2 + m - 1\right) \operatorname{sc}(z \mid m) \operatorname{sn}(z \mid m)}{(\operatorname{dn}(z \mid m) - 1) (\operatorname{dn}(z \mid m) + 1)}$$

09.26.27.0099.01

$$\operatorname{cn}(z \mid m) = \left(\operatorname{ds}(z \mid m)^2 + m - 1\right) \operatorname{sc}(z \mid m) \operatorname{sn}(z \mid m)$$

09.26.27.0100.01

$$\operatorname{cn}(z \mid m) = \frac{\left(m \operatorname{nd}(z \mid m)^2 - \operatorname{nd}(z \mid m)^2 + 1\right) \operatorname{sc}(z \mid m) \operatorname{sn}(z \mid m)}{(\operatorname{nd}(z \mid m) - 1) (\operatorname{nd}(z \mid m) + 1)}$$

09.26.27.0101.01

$$\operatorname{cn}(z \mid m) = \frac{(m-1) \operatorname{cd}(z \mid m) \operatorname{sd}(z \mid m) \operatorname{sn}(z \mid m)}{(\operatorname{cd}(z \mid m) - 1) (\operatorname{cd}(z \mid m) + 1)}$$

09.26.27.0102.01

$$\operatorname{cn}(z \mid m) = -\frac{(m-1) \operatorname{dc}(z \mid m) \operatorname{sd}(z \mid m) \operatorname{sn}(z \mid m)}{(\operatorname{dc}(z \mid m) - 1) (\operatorname{dc}(z \mid m) + 1)}$$

09.26.27.0103.01

$$\operatorname{cn}(z \mid m) = \frac{\operatorname{sc}(z \mid m) \left(m \operatorname{sd}(z \mid m)^2 - \operatorname{sd}(z \mid m)^2 + 1\right) \operatorname{sn}(z \mid m)}{\operatorname{sd}(z \mid m)^2}$$

09.26.27.0104.01

$$\operatorname{cn}(z \mid m) = \frac{\operatorname{dc}(z \mid m) \left(m \operatorname{sd}(z \mid m)^2 - \operatorname{sd}(z \mid m)^2 + 1\right) \operatorname{sn}(z \mid m)}{\operatorname{sd}(z \mid m)}$$

09.26.27.0105.01

$$\operatorname{cn}(z \mid m) = -\frac{\operatorname{dc}(z \mid m) (\operatorname{sn}(z \mid m) - 1) (\operatorname{sn}(z \mid m) + 1)}{\operatorname{dn}(z \mid m)}$$

09.26.27.0106.01

$$\operatorname{cn}(z \mid m) = -\operatorname{dc}(z \mid m) \operatorname{nd}(z \mid m) (\operatorname{sn}(z \mid m) - 1) (\operatorname{sn}(z \mid m) + 1)$$

09.26.27.0107.01

$$\operatorname{cn}(z \mid m) = -\frac{\operatorname{dc}(z \mid m) \operatorname{sd}(z \mid m) (\operatorname{sn}(z \mid m) - 1) (\operatorname{sn}(z \mid m) + 1)}{\operatorname{sn}(z \mid m)}$$

09.26.27.0108.01

$$\operatorname{cn}(z \mid m) = \operatorname{nc}(z \mid m) - \operatorname{sc}(z \mid m) \operatorname{sn}(z \mid m)$$

09.26.27.0109.01

$$\operatorname{cn}(z \mid m) = -\operatorname{cd}(z \mid m) \operatorname{nd}(z \mid m) \left(m \operatorname{sn}(z \mid m)^2 - 1\right)$$

09.26.27.0110.01

$$\operatorname{cn}(z \mid m) = -\frac{\operatorname{cd}(z \mid m) \operatorname{sd}(z \mid m) \left(m \operatorname{sn}(z \mid m)^2 - 1\right)}{\operatorname{sn}(z \mid m)}$$

Involving four other Jacobi elliptic functions

09.26.27.0111.01

$$\operatorname{cn}(z \mid m) = \frac{\operatorname{cd}(z \mid m) \operatorname{ds}(z \mid m)^2 \operatorname{nc}(z \mid m)}{\operatorname{cd}(z \mid m) \operatorname{ds}(z \mid m)^2 + \operatorname{dc}(z \mid m)}$$

09.26.27.0112.01

$$\operatorname{cn}(z \mid m) = \frac{\operatorname{ds}(z \mid m)^2 \operatorname{nc}(z \mid m) - \operatorname{dc}(z \mid m) \operatorname{dn}(z \mid m)}{\operatorname{ds}(z \mid m)^2}$$

09.26.27.0113.01

$$\operatorname{cn}(z \mid m) = \frac{\operatorname{ds}(z \mid m)^2 \operatorname{nd}(z \mid m)}{\operatorname{dc}(z \mid m) + \operatorname{cs}(z \mid m) \operatorname{ds}(z \mid m)}$$

09.26.27.0114.01

$$\operatorname{cn}(z \mid m) = \frac{\operatorname{dc}(z \mid m) \left(\operatorname{ds}(z \mid m)^2 \operatorname{nd}(z \mid m) - \operatorname{dn}(z \mid m)\right)}{\operatorname{ds}(z \mid m)^2}$$

09.26.27.0115.01

$$\operatorname{cn}(z \mid m) = \frac{\operatorname{ds}(z \mid m)^2 \operatorname{nc}(z \mid m) \operatorname{nd}(z \mid m) - \operatorname{dc}(z \mid m)}{\operatorname{ds}(z \mid m)^2 \operatorname{nd}(z \mid m)}$$

09.26.27.0116.01

$$\operatorname{cn}(z \mid m) = \frac{\operatorname{ds}(z \mid m) \operatorname{ns}(z \mid m)}{\operatorname{dc}(z \mid m) + \operatorname{cs}(z \mid m) \operatorname{ds}(z \mid m)}$$

09.26.27.0117.01

$$\operatorname{cn}(z \mid m) = \frac{\operatorname{dc}(z \mid m) (\operatorname{ds}(z \mid m) \operatorname{ns}(z \mid m) - \operatorname{dn}(z \mid m))}{\operatorname{ds}(z \mid m)^2}$$

09.26.27.0118.01

$$\operatorname{cn}(z \mid m) = \frac{\operatorname{nc}(z \mid m) (\operatorname{ds}(z \mid m) \operatorname{ns}(z \mid m) - \operatorname{dn}(z \mid m))}{\operatorname{ds}(z \mid m) \operatorname{ns}(z \mid m)}$$

09.26.27.0119.01

$$\operatorname{cn}(z \mid m) = \frac{\operatorname{ds}(z \mid m) \operatorname{nc}(z \mid m) \operatorname{ns}(z \mid m) - \operatorname{dc}(z \mid m)}{\operatorname{ds}(z \mid m) \operatorname{ns}(z \mid m)}$$

09.26.27.0120.01

$$\operatorname{cn}(z \mid m) = \frac{\left(\operatorname{ds}(z \mid m)^2 \operatorname{nd}(z \mid m) - \operatorname{dn}(z \mid m)\right) \operatorname{sc}(z \mid m)}{\operatorname{ds}(z \mid m)}$$

09.26.27.0121.01

$$\operatorname{cn}(z \mid m) = \frac{(\operatorname{ds}(z \mid m) \operatorname{ns}(z \mid m) - \operatorname{dn}(z \mid m)) \operatorname{sc}(z \mid m)}{\operatorname{ds}(z \mid m)}$$

09.26.27.0122.01

$$\operatorname{cn}(z \mid m) = \frac{\operatorname{cs}(z \mid m)\operatorname{dc}(z \mid m)}{\operatorname{dn}(z \mid m)(\operatorname{cs}(z \mid m) + \operatorname{sc}(z \mid m))}$$

09.26.27.0123.01

$$\operatorname{cn}(z \mid m) = \frac{\operatorname{ds}(z \mid m)}{\operatorname{dn}(z \mid m)(\operatorname{cs}(z \mid m) + \operatorname{sc}(z \mid m))}$$

09.26.27.0124.01

$$\operatorname{cn}(z \mid m) = \frac{\operatorname{cs}(z \mid m)\operatorname{dc}(z \mid m)\operatorname{nd}(z \mid m)}{\operatorname{cs}(z \mid m) + \operatorname{sc}(z \mid m)}$$

09.26.27.0125.01

$$\operatorname{cn}(z \mid m) = \frac{\operatorname{ds}(z \mid m)\operatorname{nd}(z \mid m)}{\operatorname{cs}(z \mid m) + \operatorname{sc}(z \mid m)}$$

09.26.27.0126.01

$$\operatorname{cn}(z \mid m) = \frac{\operatorname{cd}(z \mid m)\operatorname{ds}(z \mid m)\operatorname{nc}(z \mid m)}{\operatorname{cd}(z \mid m)\operatorname{ds}(z \mid m) + \operatorname{sc}(z \mid m)}$$

09.26.27.0127.01

$$\operatorname{cn}(z \mid m) = \frac{\operatorname{cd}(z \mid m)\operatorname{nd}(z \mid m)(\operatorname{cs}(z \mid m) - m\operatorname{sc}(z \mid m) + \operatorname{sc}(z \mid m))}{\operatorname{cs}(z \mid m) + \operatorname{sc}(z \mid m)}$$

09.26.27.0128.01

$$\operatorname{cn}(z \mid m) = \frac{\operatorname{ds}(z \mid m)\operatorname{nc}(z \mid m) - \operatorname{dn}(z \mid m)\operatorname{sc}(z \mid m)}{\operatorname{ds}(z \mid m)}$$

09.26.27.0129.01

$$\operatorname{cn}(z \mid m) = \frac{m\operatorname{nc}(z \mid m) - \operatorname{nc}(z \mid m) + \operatorname{dn}(z \mid m)\operatorname{ds}(z \mid m)\operatorname{sc}(z \mid m)}{m}$$

09.26.27.0130.01

$$\operatorname{cn}(z \mid m) = \frac{\operatorname{sc}(z \mid m)\left(\operatorname{ds}(z \mid m)\operatorname{nd}(z \mid m)^2 - \operatorname{sd}(z \mid m)\right)}{\operatorname{nd}(z \mid m)}$$

09.26.27.0131.01

$$\operatorname{cn}(z \mid m) = \frac{\operatorname{dc}(z \mid m)(\operatorname{nd}(z \mid m)\operatorname{ns}(z \mid m) - \operatorname{sd}(z \mid m))}{\operatorname{ns}(z \mid m)}$$

09.26.27.0132.01

$$\operatorname{cn}(z \mid m) = \frac{\operatorname{nc}(z \mid m)(\operatorname{nd}(z \mid m)\operatorname{ns}(z \mid m) - \operatorname{sd}(z \mid m))}{\operatorname{nd}(z \mid m)\operatorname{ns}(z \mid m)}$$

09.26.27.0133.01

$$\operatorname{cn}(z \mid m) = \frac{\operatorname{sc}(z \mid m)(\operatorname{nd}(z \mid m)\operatorname{ns}(z \mid m) - \operatorname{sd}(z \mid m))}{\operatorname{nd}(z \mid m)}$$

09.26.27.0134.01

$$\operatorname{cn}(z \mid m) = -\frac{\left(\operatorname{dn}(z \mid m)^2 + m - 1\right)\operatorname{sc}(z \mid m)\operatorname{sd}(z \mid m)}{\operatorname{dn}(z \mid m) - \operatorname{nd}(z \mid m)}$$

09.26.27.0135.01

$$\operatorname{cn}(z \mid m) = \frac{(\operatorname{dn}(z \mid m) + m\operatorname{nd}(z \mid m) - \operatorname{nd}(z \mid m))\operatorname{sc}(z \mid m)\operatorname{sd}(z \mid m)}{(\operatorname{nd}(z \mid m) - 1)(\operatorname{nd}(z \mid m) + 1)}$$

09.26.27.0136.01

$$\operatorname{cn}(z \mid m) = -\frac{\left(-\operatorname{cs}(z \mid m)^2 + m - 1\right) \operatorname{nd}(z \mid m) \operatorname{sd}(z \mid m)}{\operatorname{cs}(z \mid m) + \operatorname{sc}(z \mid m)}$$

09.26.27.0137.01

$$\operatorname{cn}(z \mid m) = \frac{\operatorname{cd}(z \mid m) (\operatorname{nd}(z \mid m) \operatorname{ns}(z \mid m) - m \operatorname{sd}(z \mid m))}{\operatorname{ns}(z \mid m)}$$

09.26.27.0138.01

$$\operatorname{cn}(z \mid m) = \operatorname{dn}(z \mid m) \operatorname{sc}(z \mid m) (\operatorname{ds}(z \mid m) + m \operatorname{sd}(z \mid m) - \operatorname{sd}(z \mid m))$$

09.26.27.0139.01

$$\operatorname{cn}(z \mid m) = \frac{\operatorname{nc}(z \mid m) \operatorname{ns}(z \mid m) - \operatorname{dc}(z \mid m) \operatorname{sd}(z \mid m)}{\operatorname{ns}(z \mid m)}$$

09.26.27.0140.01

$$\operatorname{cn}(z \mid m) = \frac{\operatorname{cs}(z \mid m) \operatorname{dc}(z \mid m) \operatorname{nd}(z \mid m)}{\operatorname{cs}(z \mid m) + \operatorname{dc}(z \mid m) \operatorname{sd}(z \mid m)}$$

09.26.27.0141.01

$$\operatorname{cn}(z \mid m) = \frac{\operatorname{ns}(z \mid m)}{\operatorname{cs}(z \mid m) + \operatorname{dc}(z \mid m) \operatorname{sd}(z \mid m)}$$

09.26.27.0142.01

$$\operatorname{cn}(z \mid m) = \operatorname{sc}(z \mid m) (\operatorname{ns}(z \mid m) - \operatorname{dn}(z \mid m) \operatorname{sd}(z \mid m))$$

09.26.27.0143.01

$$\operatorname{cn}(z \mid m) = -\operatorname{dc}(z \mid m) \operatorname{sd}(z \mid m) (\operatorname{dn}(z \mid m) \operatorname{sd}(z \mid m) - \operatorname{ns}(z \mid m))$$

09.26.27.0144.01

$$\operatorname{cn}(z \mid m) = \frac{1}{\operatorname{ns}(z \mid m)} \left(\operatorname{sd}(z \mid m) \left(\operatorname{nc}(z \mid m) \operatorname{sd}(z \mid m) \operatorname{ns}(z \mid m)^3 - \operatorname{nc}(z \mid m) \operatorname{sd}(z \mid m) \operatorname{ns}(z \mid m) - m \operatorname{cd}(z \mid m)\right)\right)$$

09.26.27.0145.01

$$\operatorname{cn}(z \mid m) = \frac{\operatorname{nc}(z \mid m) \operatorname{nd}(z \mid m) - \operatorname{sc}(z \mid m) \operatorname{sd}(z \mid m)}{\operatorname{nd}(z \mid m)}$$

09.26.27.0146.01

$$\operatorname{cn}(z \mid m) = \frac{\operatorname{dc}(z \mid m) \operatorname{nd}(z \mid m)^2 - \operatorname{sc}(z \mid m) \operatorname{sd}(z \mid m)}{\operatorname{nd}(z \mid m)}$$

09.26.27.0147.01

$$\operatorname{cn}(z \mid m) = \frac{\operatorname{cd}(z \mid m) \operatorname{nc}(z \mid m)}{\operatorname{cd}(z \mid m) + \operatorname{sc}(z \mid m) \operatorname{sd}(z \mid m)}$$

09.26.27.0148.01

$$\operatorname{cn}(z \mid m) = \frac{\operatorname{nd}(z \mid m)}{\operatorname{cd}(z \mid m) + \operatorname{sc}(z \mid m) \operatorname{sd}(z \mid m)}$$

09.26.27.0149.01

$$\operatorname{cn}(z \mid m) = -\frac{\operatorname{nd}(z \mid m) (-\operatorname{cd}(z \mid m) + m \operatorname{sc}(z \mid m) \operatorname{sd}(z \mid m) - \operatorname{sc}(z \mid m) \operatorname{sd}(z \mid m))}{\operatorname{sc}(z \mid m)^2 + 1}$$

09.26.27.0150.01

$$\operatorname{cn}(z \mid m) = \operatorname{dn}(z \mid m) (\operatorname{dc}(z \mid m) + m \operatorname{sc}(z \mid m) \operatorname{sd}(z \mid m) - \operatorname{sc}(z \mid m) \operatorname{sd}(z \mid m))$$

09.26.27.0151.01

$$\operatorname{cn}(z \mid m) = \operatorname{nc}(z \mid m) - \operatorname{dn}(z \mid m) \operatorname{sc}(z \mid m) \operatorname{sd}(z \mid m)$$

09.26.27.0152.01

$$\operatorname{cn}(z \mid m) = \frac{\operatorname{nc}(z \mid m) \operatorname{nd}(z \mid m) - \operatorname{dc}(z \mid m) \operatorname{sd}(z \mid m)^2}{\operatorname{nd}(z \mid m)}$$

09.26.27.0153.01

$$\operatorname{cn}(z \mid m) = \frac{\operatorname{cd}(z \mid m) \operatorname{nc}(z \mid m)}{\operatorname{dc}(z \mid m) \operatorname{sd}(z \mid m)^2 + \operatorname{cd}(z \mid m)}$$

09.26.27.0154.01

$$\operatorname{cn}(z \mid m) = \operatorname{dc}(z \mid m) \left(\operatorname{nd}(z \mid m) - \operatorname{dn}(z \mid m) \operatorname{sd}(z \mid m)^2\right)$$

09.26.27.0155.01

$$\operatorname{cn}(z \mid m) = -\frac{\operatorname{sc}(z \mid m) \left(\operatorname{dn}(z \mid m) \operatorname{sd}(z \mid m)^2 - \operatorname{nd}(z \mid m)\right)}{\operatorname{sd}(z \mid m)}$$

09.26.27.0156.01

$$\operatorname{cn}(z \mid m) = \operatorname{nc}(z \mid m) - \operatorname{dc}(z \mid m) \operatorname{dn}(z \mid m) \operatorname{sd}(z \mid m)^2$$

09.26.27.0157.01

$$\operatorname{cn}(z \mid m) = \operatorname{sc}(z \mid m) (\operatorname{ds}(z \mid m) \operatorname{nd}(z \mid m) - \operatorname{sn}(z \mid m))$$

09.26.27.0158.01

$$\operatorname{cn}(z \mid m) = \frac{\operatorname{dc}(z \mid m) (\operatorname{ns}(z \mid m) - \operatorname{sn}(z \mid m))}{\operatorname{ds}(z \mid m)}$$

09.26.27.0159.01

$$\operatorname{cn}(z \mid m) = \operatorname{dc}(z \mid m) \operatorname{sd}(z \mid m) (\operatorname{ns}(z \mid m) - \operatorname{sn}(z \mid m))$$

09.26.27.0160.01

$$\operatorname{cn}(z \mid m) = \frac{(m - 1) \operatorname{cd}(z \mid m) \operatorname{sc}(z \mid m) \operatorname{sn}(z \mid m)}{\operatorname{cd}(z \mid m) - \operatorname{dc}(z \mid m)}$$

09.26.27.0161.01

$$\operatorname{cn}(z \mid m) = -\frac{(\operatorname{dn}(z \mid m) + m \operatorname{nd}(z \mid m) - \operatorname{nd}(z \mid m)) \operatorname{sc}(z \mid m) \operatorname{sn}(z \mid m)}{\operatorname{dn}(z \mid m) - \operatorname{nd}(z \mid m)}$$

09.26.27.0162.01

$$\operatorname{cn}(z \mid m) = \frac{(m - 1) \operatorname{sd}(z \mid m) \operatorname{sn}(z \mid m)}{\operatorname{cd}(z \mid m) - \operatorname{dc}(z \mid m)}$$

09.26.27.0163.01

$$\operatorname{cn}(z \mid m) = \operatorname{dc}(z \mid m) (\operatorname{ds}(z \mid m) + m \operatorname{sd}(z \mid m) - \operatorname{sd}(z \mid m)) \operatorname{sn}(z \mid m)$$

09.26.27.0164.01

$$\operatorname{cn}(z \mid m) = \frac{(\operatorname{dc}(z \mid m) + m \operatorname{sc}(z \mid m) \operatorname{sd}(z \mid m) - \operatorname{sc}(z \mid m) \operatorname{sd}(z \mid m)) \operatorname{sn}(z \mid m)}{\operatorname{sd}(z \mid m)}$$

09.26.27.0165.01

$$\operatorname{cn}(z \mid m) = \frac{\operatorname{sc}(z \mid m) (m \operatorname{nd}(z \mid m) \operatorname{sd}(z \mid m) - \operatorname{nd}(z \mid m) \operatorname{sd}(z \mid m) + \operatorname{sn}(z \mid m))}{(\operatorname{nd}(z \mid m) - 1) (\operatorname{nd}(z \mid m) + 1)}$$

09.26.27.0166.01

$$\operatorname{cn}(z \mid m) = \operatorname{cd}(z \mid m) \operatorname{sd}(z \mid m) (\operatorname{ns}(z \mid m) - m \operatorname{sn}(z \mid m))$$

09.26.27.0167.01

$$\operatorname{cn}(z \mid m) = \frac{\operatorname{dc}(z \mid m)\,(\operatorname{dn}(z \mid m)\operatorname{ds}(z \mid m) + m\operatorname{sn}(z \mid m) - \operatorname{sn}(z \mid m))}{\operatorname{ds}(z \mid m)}$$

09.26.27.0168.01

$$\operatorname{cn}(z \mid m) = \operatorname{sc}(z \mid m)\,(\operatorname{dn}(z \mid m)\operatorname{ds}(z \mid m) + m\operatorname{sn}(z \mid m) - \operatorname{sn}(z \mid m))$$

09.26.27.0169.01

$$\operatorname{cn}(z \mid m) = \operatorname{dc}(z \mid m)\operatorname{nd}(z \mid m) - \operatorname{sc}(z \mid m)\operatorname{sn}(z \mid m)$$

09.26.27.0170.01

$$\operatorname{cn}(z \mid m) = -m\operatorname{cd}(z \mid m)\operatorname{nd}(z \mid m)\operatorname{sc}(z \mid m)^2 + \operatorname{cd}(z \mid m)\operatorname{nd}(z \mid m)\operatorname{sc}(z \mid m)^2 - \operatorname{sn}(z \mid m)\operatorname{sc}(z \mid m) + \operatorname{cd}(z \mid m)\operatorname{nd}(z \mid m)$$

09.26.27.0171.01

$$\operatorname{cn}(z \mid m) = \operatorname{dc}(z \mid m)\operatorname{dn}(z \mid m) + m\operatorname{sc}(z \mid m)\operatorname{sn}(z \mid m) - \operatorname{sc}(z \mid m)\operatorname{sn}(z \mid m)$$

09.26.27.0172.01

$$\operatorname{cn}(z \mid m) = \operatorname{dc}(z \mid m)\,(\operatorname{nd}(z \mid m) - \operatorname{sd}(z \mid m)\operatorname{sn}(z \mid m))$$

09.26.27.0173.01

$$\operatorname{cn}(z \mid m) = -\frac{\operatorname{sc}(z \mid m)\,(\operatorname{sd}(z \mid m)\operatorname{sn}(z \mid m) - \operatorname{nd}(z \mid m))}{\operatorname{sd}(z \mid m)}$$

09.26.27.0174.01

$$\operatorname{cn}(z \mid m) = \operatorname{cd}(z \mid m)\,(\operatorname{cd}(z \mid m)\operatorname{nc}(z \mid m) - m\operatorname{sd}(z \mid m)\operatorname{sn}(z \mid m))$$

09.26.27.0175.01

$$\operatorname{cn}(z \mid m) = \operatorname{cd}(z \mid m)\,(\operatorname{nd}(z \mid m) - m\operatorname{sd}(z \mid m)\operatorname{sn}(z \mid m))$$

09.26.27.0176.01

$$\operatorname{cn}(z \mid m) = \operatorname{nc}(z \mid m) - \operatorname{dc}(z \mid m)\operatorname{sd}(z \mid m)\operatorname{sn}(z \mid m)$$

Involving five other Jacobi elliptic functions

09.26.27.0177.01

$$\operatorname{cn}(z \mid m) = \operatorname{sc}(z \mid m)\,(\operatorname{ds}(z \mid m)\operatorname{nd}(z \mid m) - \operatorname{dn}(z \mid m)\operatorname{sd}(z \mid m))$$

09.26.27.0178.01

$$\operatorname{cn}(z \mid m) = \frac{1}{\operatorname{ns}(z \mid m)}\left(\operatorname{nc}(z \mid m)\operatorname{ns}(z \mid m)\operatorname{nd}(z \mid m)^2 - \operatorname{nc}(z \mid m)\operatorname{sd}(z \mid m)\operatorname{nd}(z \mid m) - m\operatorname{cd}(z \mid m)\operatorname{sd}(z \mid m)\right)$$

09.26.27.0179.01

$$\operatorname{cn}(z \mid m) = \operatorname{dc}(z \mid m)\operatorname{nd}(z \mid m) - \operatorname{dn}(z \mid m)\operatorname{sc}(z \mid m)\operatorname{sd}(z \mid m)$$

09.26.27.0180.01

$$\operatorname{cn}(z \mid m) = \operatorname{cd}(z \mid m)\operatorname{nd}(z \mid m) - m\operatorname{sc}(z \mid m)\operatorname{sd}(z \mid m)\operatorname{nd}(z \mid m) + \operatorname{sc}(z \mid m)\operatorname{sd}(z \mid m)\operatorname{nd}(z \mid m) - \operatorname{dn}(z \mid m)\operatorname{sc}(z \mid m)\operatorname{sd}(z \mid m)$$

09.26.27.0181.01

$$\operatorname{cn}(z \mid m) = \operatorname{sd}(z \mid m)\left(\operatorname{nc}(z \mid m)\operatorname{sd}(z \mid m)\operatorname{ns}(z \mid m)^2 - \operatorname{nc}(z \mid m)\operatorname{sd}(z \mid m) - m\operatorname{cd}(z \mid m)\operatorname{sn}(z \mid m)\right)$$

09.26.27.0182.01

$$\operatorname{cn}(z \mid m) = \operatorname{cd}(z \mid m)\operatorname{nd}(z \mid m) - m\operatorname{sc}(z \mid m)\operatorname{sd}(z \mid m)\operatorname{nd}(z \mid m) + \operatorname{sc}(z \mid m)\operatorname{sd}(z \mid m)\operatorname{nd}(z \mid m) - \operatorname{sc}(z \mid m)\operatorname{sn}(z \mid m)$$

09.26.27.0183.01

$$\operatorname{cn}(z \mid m) = \operatorname{nc}(z \mid m)\operatorname{nd}(z \mid m)^2 - \operatorname{nc}(z \mid m)\operatorname{sd}(z \mid m)^2 - m\operatorname{cd}(z \mid m)\operatorname{sd}(z \mid m)\operatorname{sn}(z \mid m)$$

Involving Weierstrass functions

09.26.27.0023.01

$$\operatorname{cn}(z \mid m) = \frac{\sigma_1\left(\frac{z}{\sqrt{e_1-e_3}};\, g_2,\, g_3\right)}{\sigma_3\left(\frac{z}{\sqrt{e_1-e_3}};\, g_2,\, g_3\right)} \;/;$$

$$\{\omega_1, \omega_2, \omega_3\} = \{\omega_1(g_2, g_3), -\omega_1(g_2, g_3) - \omega_3(g_2, g_3), \omega_3(g_2, g_3)\} \bigwedge m = \lambda\left(\frac{\omega_3}{\omega_1}\right) \bigwedge e_n = \wp(\omega_n;\, g_2,\, g_3) \bigwedge n \in \{1, 2, 3\}$$

09.26.27.0024.01

$$\operatorname{cn}(z \mid m)^2 = \frac{\wp\left(\frac{z}{\sqrt{e_1-e_3}};\, g_2,\, g_3\right) - e_1}{\wp\left(\frac{z}{\sqrt{e_1-e_3}};\, g_2,\, g_3\right) - e_3} \;/;$$

$$\{\omega_1, \omega_2, \omega_3\} = \{\omega_1(g_2, g_3), -\omega_1(g_2, g_3) - \omega_3(g_2, g_3), \omega_3(g_2, g_3)\} \bigwedge m = \lambda\left(\frac{\omega_3}{\omega_1}\right) \bigwedge e_n = \wp(\omega_n;\, g_2,\, g_3) \bigwedge n \in \{1, 2, 3\}$$

09.26.27.0025.01

$$\operatorname{cn}\left(z \,\middle|\, \frac{e_2 - e_3}{e_1 - e_3}\right) = \sqrt{\frac{\wp\left(\frac{z}{\sqrt{e_1-e_3}};\, g_2,\, g_3\right) - e_1}{\wp\left(\frac{z}{\sqrt{e_1-e_3}};\, g_2,\, g_3\right) - e_3}} \;/;$$

$$\{\omega_1, \omega_2, \omega_3\} = \{\omega_1(g_2, g_3), -\omega_1(g_2, g_3) - \omega_3(g_2, g_3), \omega_3(g_2, g_3)\} \wedge e_n = \wp(\omega_n;\, g_2,\, g_3) \wedge n \in \{1, 2, 3\}$$

Involving theta functions

09.26.27.0026.02

$$\operatorname{cn}(z \mid m) = (1-m)^{1/4}\, m^{-1/4}\, \frac{\vartheta_2\left(\frac{\pi z}{2 K(m)},\, q(m)\right)}{\vartheta_4\left(\frac{\pi z}{2 K(m)},\, q(m)\right)}$$

09.26.27.0027.01

$$\operatorname{cn}(z \mid m) = \frac{\vartheta_4(0,\, q(m))}{\vartheta_2(0,\, q(m))}\, \frac{\vartheta_2\left(\frac{z}{\vartheta_3(0,q(m))^2},\, q(m)\right)}{\vartheta_4\left(\frac{z}{\vartheta_3(0,q(m))^2},\, q(m)\right)}$$

09.26.27.0028.01

$$\operatorname{cn}(z \mid m) = \frac{\vartheta_c(z \mid m)}{\vartheta_n(z \mid m)}$$

09.26.27.0029.01

$$Z(\operatorname{am}(a \mid m) \mid m) - Z(\operatorname{am}(a+u \mid m) \mid m) + Z(\operatorname{am}(u \mid m) \mid m) = m \operatorname{sn}(a \mid m) \operatorname{sn}(u \mid m) \operatorname{sn}(a+u \mid m)$$

09.26.27.0030.01

$$Z(\operatorname{am}(a \mid m) \mid m) - Z(\operatorname{am}(a+u \mid m) \mid m) + Z(\operatorname{am}(u \mid m) \mid m) = \frac{m \operatorname{sn}(a \mid m)}{\operatorname{dn}(a \mid m)} (\operatorname{cn}(a \mid m) - \operatorname{cn}(u \mid m) \operatorname{cn}(a+u \mid m))$$

09.26.27.0031.01

$$Z(\operatorname{am}(a \mid m) \mid m) - Z(\operatorname{am}(a+u \mid m) \mid m) + Z(\operatorname{am}(u \mid m) \mid m) = \frac{\operatorname{sn}(a \mid m)}{\operatorname{cn}(a \mid m)} (\operatorname{dn}(a \mid m) - \operatorname{dn}(u \mid m) \operatorname{dn}(a+u \mid m))$$

Zeros

09.26.30.0001.01

$\mathrm{cn}((2r+2s+1)K(m)+2is K(1-m)\mid m) == 0 \,/;\, \{r,s\}\in\mathbb{Z}$

Theorems

The cosine theorem of spherical geometry

The cosine theorem of spherical geometry, $\cos(c) == \cos(a)\cos(b)+\sin(a)\sin(b)\cos(\gamma)$ for a spherical triangle with sides a, b, and c and angles α, β, and γ, can be rewritten as $\mathrm{cn}(w\mid m) == \mathrm{cn}(u\mid m)\,\mathrm{cn}(v\mid m)+\mathrm{sn}(u\mid m)\,\mathrm{sn}(v\mid m)\,\mathrm{sn}(w\mid m) \,/;\, w == u-v$ by making the substitution $a == \mathrm{am}(u, m)$, $b == \mathrm{am}(v, m)$, $c == \mathrm{am}(w, m)$, and $m == (\sin(\alpha)/\sin(a))^2$.

History

– C. G .J. Jacobi (1827)
– N. H. Abel (1827)
– C. Gudermann (1838) introduced the notations cn

Global References

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