JacobiP

Notations

Traditional name

Jacobi polynomial

Traditional notation

$P_n^{(a,b)}(z)$

Mathematica StandardForm notation

`JacobiP[n, a, b, z]`

Primary definition

05.06.02.0001.01

$$P_n^{(a,b)}(z) = \frac{1}{n!} \sum_{k=0}^{n} \frac{(-n)_k\,(a+b+n+1)_k\,(a+k+1)_{n-k}}{k!} \left(\frac{1-z}{2}\right)^k /;\ n \in \mathbb{N}$$

Specific values

Specialized values

For fixed *n, a, b*

05.06.03.0001.01

$$P_n^{(a,b)}(0) = \frac{2^{-n}\,\Gamma(a+n+1)}{\Gamma(a+1)\,n!}\ {}_2F_1(-b-n, -n; a+1; -1)$$

05.06.03.0002.01

$$P_n^{(a,b)}(1) = \frac{\Gamma(a+n+1)}{n!\,\Gamma(a+1)}$$

05.06.03.0003.01

$$P_n^{(a,b)}(-1) = \frac{\Gamma(-b)}{\Gamma(-b-n)\,n!}$$

05.06.03.0025.01

$$P_n^{(a,b)}(-1) = \frac{(-1)^n\,(b+1)_n}{n!}$$

For fixed *n, a, z*

05.06.03.0005.01

$$P_n^{(a,a)}(z) = \frac{(a+1)_n}{(2a+1)_n} C_n^{a+\frac{1}{2}}(z)$$

05.06.03.0006.01

$$P_n^{(a,-a)}(z) = \frac{\Gamma(a+n+1)}{\Gamma(n+1)} \frac{(1+z)^{a/2}}{(1-z)^{a/2}} P_n^{-a}(z)$$

05.06.03.0007.01

$$P_n^{(a,-a)}(z) = \frac{\Gamma(a+n+1)}{\Gamma(n+1)} \frac{(z+1)^{a/2}}{(z-1)^{a/2}} \mathbb{P}_n^{-a}(z)$$

05.06.03.0008.01

$$P_n^{(a,-n)}(z) = \frac{2^{-n}\,\Gamma(a+n+1)}{\Gamma(a+1)\,\Gamma(n+1)} (z+1)^n$$

05.06.03.0009.01

$$P_n^{\left(a,-\frac{1}{2}\right)}(z) = \frac{\Gamma\left(a+\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(a+n+\frac{1}{2}\right)} C_{2n}^{a+\frac{1}{2}}\left(\frac{\sqrt{z+1}}{\sqrt{2}}\right)$$

05.06.03.0010.01

$$P_n^{\left(a,\frac{1}{2}\right)}(z) = \sqrt{\frac{2}{\pi}} \frac{\Gamma\left(a+\frac{1}{2}\right)\Gamma\left(n+\frac{3}{2}\right)}{\sqrt{z+1}\,\Gamma\left(a+n+\frac{3}{2}\right)} C_{2n+1}^{a+\frac{1}{2}}\left(\frac{\sqrt{z+1}}{\sqrt{2}}\right)$$

For fixed *n*, *b*, *z*

05.06.03.0011.01

$$P_n^{(-b,b)}(z) = \frac{\Gamma(n-b+1)}{\Gamma(n+1)} \frac{(1-z)^{b/2}}{(1+z)^{b/2}} P_n^b(z)$$

05.06.03.0012.01

$$P_n^{(-b,b)}(z) = \frac{\Gamma(n-b+1)}{\Gamma(n+1)} \frac{(z-1)^{b/2}}{(z+1)^{b/2}} \mathbb{P}_n^b(z)$$

05.06.03.0013.01

$$P_n^{(-m-n,b)}(z) = \tilde{\infty} \;/;\; m \in \mathbb{N}^+$$

For fixed *a*, *b*, *z*

05.06.03.0014.01

$$P_0^{(a,b)}(z) = 1$$

05.06.03.0015.01

$$P_1^{(a,b)}(z) = \frac{1}{2}((a+b+2)\,z + a - b)$$

05.06.03.0016.01

$$P_2^{(a,b)}(z) = \frac{1}{8}\left((3+a+b)(4+a+b)\,z^2 + 2\left(3a+a^2-b(3+b)\right)z - 4 + a^2 - b + b^2 - a(1+2b)\right)$$

05.06.03.0017.01

$$P_3^{(a,b)}(z) == \frac{1}{48}\left((a+b+4)(a+b+5)(a+b+6)z^3 + 3(a-b)(a+b+4)(a+b+5)z^2 + 3(a+b+4)\left(a^2-(2b+1)a+b^2-b-6\right)z + (a-b)\left(-16+a^2+(-3+b)b-a(3+2b)\right)\right)$$

05.06.03.0018.01

$$P_4^{(a,b)}(z) == \frac{1}{384}\left((a+b+5)(a+b+6)(a+b+7)(a+b+8)z^4 + 4(a-b)(a+b+5)(a+b+6)(a+b+7)z^3 + 6(a+b+5)(a+b+6)\left(a^2-(2b+1)a+b^2-b-8\right)z^2 + \left(4(a+b+5)\left(a^3-3(b+1)a^2+\left(3b^2-22\right)a+b\left(-b^2+3b+22\right)\right)\right)z + 144+42b-6b^3-37b^2+a^4-2(2b+3)a^3+\left(6b^2+6b-37\right)a^2+2\left(-2b^3+3b^2+43b+21\right)a+b^4\right)$$

05.06.03.0019.01

$$P_5^{(a,b)}(z) == \frac{1}{3840}\left((a+b+6)(a+b+7)(a+b+8)(a+b+9)(a+b+10)z^5 + 5(a-b)(a+b+6)(a+b+7)(a+b+8)(a+b+9)z^4 + 10(a+b+6)(a+b+7)(a+b+8)\left(a^2-(2b+1)a+b^2-b-10\right)z^3 + 10(a+b+6)(a+b+7)\left(a^3-3(b+1)a^2+\left(3b^2-28\right)a+b\left(-b^2+3b+28\right)\right)z^2 + 5(a+b+6)\left(a^4-2(2b+3)a^3+\left(6b^2+6b-49\right)a^2+\left(-4b^3+6b^2+110b+54\right)a+b^4-6b^3-49b^2+54b+240\right)z + a^5-5(b+2)a^4+5\left(2b^2+4b-13\right)a^3-5\left(2b^3-51b-50\right)a^2 + \left(5b^4-20b^3-255b^2+1024\right)a-b\left(b^4-10b^3-65b^2+250b+1024\right)\right)$$

05.06.03.0020.01

$$P_6^{(a,b)}(z) == \frac{1}{46080}\left((a+b+7)(a+b+8)(a+b+9)(a+b+10)(a+b+11)(a+b+12)z^6 + 6(a-b)(a+b+7)(a+b+8)(a+b+9)(a+b+10)(a+b+11)z^5 + 15(a+b+7)(a+b+8)(a+b+9)(a+b+10)\left(a^2-(2b+1)a+b^2-b-12\right)z^4 + 20(a+b+7)(a+b+8)(a+b+9)\left(a^3-3(b+1)a^2+\left(3b^2-34\right)a+b\left(-b^2+3b+34\right)\right)z^3 + 15(a+b+7)(a+b+8)\left(a^4-2(2b+3)a^3+\left(6b^2+6b-61\right)a^2+\left(-4b^3+6b^2+134b+66\right)a+b^4-6b^3-61b^2+66b+360\right)z^2 + 6(a+b+7)\left(a^5-5(b+2)a^4+5\left(2b^2+4b-17\right)a^3-5\left(2b^3-63b-62\right)a^2 + \left(5b^4-20b^3-315b^2+1584\right)a-b\left(b^4-10b^3-85b^2+310b+1584\right)\right)z + 64(a+1)(a+2)(a+3)(a+4)(a+5)(a+6) - 192(a+2)(a+3)(a+4)(a+5)(a+6)(a+b+7) + 240(a+3)(a+4)(a+5)(a+6)(a+b+7)(a+b+8) - 160(a+4)(a+5)(a+6)(a+b+7)(a+b+8)(a+b+9) + 60(a+5)(a+6)(a+b+7)(a+b+8)(a+b+9)(a+b+10) - 12(a+6)(a+b+7)(a+b+8)(a+b+9)(a+b+10)(a+b+11) + (a+b+7)(a+b+8)(a+b+9)(a+b+10)(a+b+11)(a+b+12)\right)$$

05.06.03.0021.01

$$P_n^{(a,b)}(z) == \frac{1}{2^n}\sum_{k=0}^{n}\binom{a+n}{k}\binom{b+n}{n-k}(z+1)^k(z-1)^{n-k}$$

For fixed *n*, *z*

05.06.03.0022.01

$$P_n^{\left(\frac{1}{2},\frac{1}{2}\right)}(z) == \frac{1}{(n+1)!}\left(\frac{3}{2}\right)_n U_n(z)$$

05.06.03.0023.01

$$P_n^{\left(-\frac{1}{2},-\frac{1}{2}\right)}(z) = \frac{1}{n!}\left(\frac{1}{2}\right)_n T_n(z)$$

05.06.03.0024.01

$$P_n^{(0,0)}(z) = P_n(z)$$

Values at infinities

05.06.03.0026.01

$$P_n^{(a,b)}(\infty) = (a+b+n+1)_n \infty \,/;\, n > 0 \bigwedge a+b+2n \notin \mathbb{Z}$$

05.06.03.0027.01

$$P_n^{(a,b)}(-\infty) = (-1)^n (a+b+n+1)_n \infty \,/;\, n > 0 \bigwedge a+b+2n \notin \mathbb{Z}$$

General characteristics

Domain and analyticity

The function $P_n^{(a,b)}(z)$is defined over $\mathbb{N} \otimes \mathbb{C} \otimes \mathbb{C} \otimes \mathbb{C}$. For fixed n, a, b, the function $P_n^{(a,b)}(z)$ is a polynomial in z of degree n. For fixed n, a, z, the function $P_n^{(a,b)}(z)$ is a polynomial in b of degree n. For fixed n, b, z, the function $P_n^{(a,b)}(z)$ is a polynomial in a of degree n.

05.06.04.0001.01

$$(n * a * b * z) \longrightarrow P_n^{(a,b)}(z) :: (\mathbb{N} \otimes \mathbb{C} \otimes \mathbb{C} \otimes \mathbb{C}) \longrightarrow \mathbb{C}$$

Symmetries and periodicities

Parity

05.06.04.0002.01

$$P_n^{(a,b)}(-z) = (-1)^n P_n^{(b,a)}(z)$$

Mirror symmetry

05.06.04.0003.01

$$P_n^{(\bar{a},\bar{b})}(\bar{z}) = \overline{P_n^{(a,b)}(z)}$$

Periodicity

No periodicity

Poles and essential singularities

With respect to *z*

For fixed a, b the function $P_n^{(a,b)}(z)$ is polynomial and has pole of order n at $z = \tilde{\infty}$.

05.06.04.0004.01

$$\mathcal{Sing}_z\left(P_n^{(a,b)}(z)\right) = \{\{\tilde{\infty}, n\}\}$$

With respect to *b*

For fixed n, a, z, the function $P_n^{(a,b)}(z)$ has only one singular point at $b = \tilde{\infty}$. It is an essential singular point.

05.06.04.0005.01

$$\mathcal{S}ing_b\left(P_n^{(a,b)}(z)\right) = \{\{\tilde{\infty}, \infty\}\}$$

With respect to *a*

For fixed n, b, z, the function $P_n^{(a,b)}(z)$ has an infinite set of singular points:

a) $a = -n - k$ /; $k \in \mathbb{N}^+$, are the simple poles with residues $\frac{(-1)^{k-1}\,\Gamma(k-b)}{(k-1)!\,\Gamma(-b-n)\,\Gamma(n+1)}\,P_{-b+k-1}^{(-k-n,b)}(z)$;

b) $a = \tilde{\infty}$ is the point of convergence of poles, which is an essential singular point.

05.06.04.0006.01

$$\mathcal{S}ing_a\left(P_n^{(a,b)}(z)\right) = \left\{\left\{\{-n-k, 1\} /; k \in \mathbb{N}^+\right\}, \{\tilde{\infty}, \infty\}\right\}$$

05.06.04.0007.01

$$\text{res}_a\left(P_n^{(a,b)}(z)\right)(-n-k) = \frac{(-1)^{k-1}\,\Gamma(k-b)}{(k-1)!\,\Gamma(-b-n)\,\Gamma(n+1)}\,P_{-b+k-1}^{(-k-n,b)}(z) /; k \in \mathbb{N}^+$$

Branch points

With respect to *z*

For fixed n, a, b, the function $P_n^{(a,b)}(z)$ does not have branch points.

05.06.04.0008.01

$$\mathcal{BP}_z\left(P_n^{(a,b)}(z)\right) = \{\}$$

With respect to *b*

For fixed n, a, z, the function $P_n^{(a,b)}(z)$ does not have branch points.

05.06.04.0009.01

$$\mathcal{BP}_b\left(P_n^{(a,b)}(z)\right) = \{\}$$

With respect to *a*

For fixed n, b, z, the function $P_n^{(a,b)}(z)$ does not have branch points.

05.06.04.0010.01

$$\mathcal{BP}_a\left(P_n^{(a,b)}(z)\right) = \{\}$$

Branch cuts

With respect to *z*

For fixed n, a, b the function $P_n^{(a,b)}(z)$ does not have branch cuts.

05.06.04.0011.01

$$\mathcal{BC}_z\left(P_n^{(a,b)}(z)\right) = \{\}$$

With respect to *b*

For fixed n, a, z, the function $P_n^{(a,b)}(z)$ does not have branch cuts.

05.06.04.0012.01

$$\mathcal{BC}_b\left(P_n^{(a,b)}(z)\right) = \{\}$$

With respect to *a*

For fixed n, b, z, the function $P_n^{(a,b)}(z)$ does not have branch cuts.

05.06.04.0013.01

$$\mathcal{BC}_a\left(P_n^{(a,b)}(z)\right) = \{\}$$

Series representations

Generalized power series

Expansions at generic point $a == a_0$

For the function itself

05.06.06.0021.01

$$P_n^{(a,b)}(z) \propto P_n^{(a_0,b)}(z) + \sum_{k=0}^{n-1} \frac{1}{b+k+n+a_0+1} \left(P_n^{(a_0,b)}(z) + \frac{(b+2k+a_0+1)\,(b+k+1)_{n-k}}{(n-k)\,(b+k+a_0+1)_{n-k}} P_k^{(a_0,b)}(z) \right) (a-a_0) + \ldots /; (a \to a_0)$$

05.06.06.0022.01

$$P_n^{(a,b)}(z) \propto P_n^{(a_0,b)}(z) + \sum_{k=0}^{n-1} \frac{1}{b+k+n+a_0+1} \left(P_n^{(a_0,b)}(z) + \frac{(b+2k+a_0+1)\,(b+k+1)_{n-k}}{(n-k)\,(b+k+a_0+1)_{n-k}} P_k^{(a_0,b)}(z) \right) (a-a_0) + O\left((a-a_0)^2\right)$$

05.06.06.0023.01

$$P_n^{(a,b)}(z) = \frac{1}{n!} \sum_{k=0}^{\infty} \sum_{h=0}^{n} \frac{(-n)_h}{k!\,h!} \left(\frac{1-z}{2}\right)^h$$
$$\sum_{i=0}^{h} \sum_{j=0}^{n-h} (-1)^{i+j+n}\, j!\,(h+a_0+1)^{j-k}\,(b+n+a_0+1)^i\, S_h^{(i)}\, S_{n-h}^{(j)}\, {}_2\tilde{F}_1\left(-i, -k;\, j-k+1;\, \frac{h+a_0+1}{b+n+a_0+1}\right) (a-a_0)^k$$

05.06.06.0024.01

$$P_n^{(a,b)}(z) = \frac{1}{n!} \sum_{k=0}^{\infty} \sum_{h=0}^{n} \frac{(-n)_h}{k!\,h!} \sum_{s=0}^{k} \binom{k}{s}$$
$$\sum_{i=0}^{h} (-1)^{h+i}\, S_h^{(i)}\,(i-s+1)_s\,(b+n+a_0+1)^{i-s} \sum_{j=0}^{n-h} (-1)^{-h+j+n}\, S_{n-h}^{(j)}\,(j-k+s+1)_{k-s}\,(h+a_0+1)^{j-k+s} \left(\frac{1-z}{2}\right)^h (a-a_0)^k$$

05.06.06.0025.01

$$P_n^{(a,b)}(z) \propto P_n^{(a_0,b)}(z)\,(1+O(a-a_0))$$

Expansions at generic point $b == b_0$

For the function itself

05.06.06.0026.01

$$P_n^{(a,b)}(z) \propto P_n^{(a,b_0)}(z) + \sum_{k=0}^{n-1} \frac{1}{a+k+n+b_0+1} \left(P_n^{(a,b_0)}(z) + \frac{(-1)^{n-k} (a+2k+b_0+1)(a+k+1)_{n-k}}{(n-k)(a+k+b_0+1)_{n-k}} P_k^{(a,b_0)}(z) \right) (b-b_0) + \ldots /; (b \to b_0)$$

05.06.06.0027.01

$$P_n^{(a,b)}(z) \propto P_n^{(a,b_0)}(z) + \sum_{k=0}^{n-1} \frac{1}{a+k+n+b_0+1} \left(P_n^{(a,b_0)}(z) + \frac{(-1)^{n-k} (a+2k+b_0+1)(a+k+1)_{n-k}}{(n-k)(a+k+b_0+1)_{n-k}} P_k^{(a,b_0)}(z) \right) (b-b_0) + O\left((b-b_0)^2\right)$$

05.06.06.0028.01

$$P_n^{(a,b)}(z) = \frac{1}{n!} \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{h=0}^{n} \frac{(-n)_h}{h!} \sum_{j=0}^{h} (-1)^{h+j} S_h^{(j)} (j-k+1)_k (a+n+b_0+1)^{j-k} (a+h+1)_{n-h} \left(\frac{1-z}{2}\right)^h (b-b_0)^k$$

05.06.06.0029.01

$$P_n^{(a,b)}(z) \propto P_n^{(a,b_0)}(z)\,(1+O(b-b_0))$$

Expansions at generic point $z == z_0$

For the function itself

05.06.06.0030.01

$$P_n^{(a,b)}(z) \propto P_n^{(a,b)}(z_0) + \frac{a+b+n+1}{2} P_{n-1}^{(a+1,b+1)}(z_0)(z-z_0) + \frac{(a+b+n+1)(a+b+n+2)}{8} P_{n-2}^{(a+2,b+2)}(z_0)(z-z_0)^2 + \ldots /; (z \to z_0)$$

05.06.06.0031.01

$$P_n^{(a,b)}(z) \propto P_n^{(a,b)}(z_0) + \frac{a+b+n+1}{2} P_{n-1}^{(a+1,b+1)}(z_0)(z-z_0) + \frac{(a+b+n+1)(a+b+n+2)}{8} P_{n-2}^{(a+2,b+2)}(z_0)(z-z_0)^2 + O\left((z-z_0)^3\right)$$

05.06.06.0032.01

$$P_n^{(a,b)}(z) = \sum_{k=0}^{\infty} \frac{2^{-k}(a+b+n+1)_k}{k!} P_{n-k}^{(a+k,b+k)}(z_0)(z-z_0)^k$$

05.06.06.0033.01

$$P_n^{(a,b)}(z) = \frac{\Gamma(a+n+1)}{n!} \sum_{k=0}^{\infty} \frac{(z_0-1)^{-k}}{k!} {}_3\tilde{F}_2\left(1, -n, a+b+n+1; a+1, 1-k; \frac{1-z_0}{2}\right)(z-z_0)^k$$

05.06.06.0034.01

$$P_n^{(a,b)}(z) \propto P_n^{(a,b)}(z_0)\,(1+O(z-z_0))$$

Expansions at $z == 0$

For the function itself

05.06.06.0001.01

$$P_n^{(a,b)}(z) \propto 2^{-n} \Bigg({}_2F_1(-n, -b-n; a+1; -1) + \frac{n(a+b+n+1)}{a+1} {}_2F_1(1-n, -b-n; a+2; -1)\, z + \frac{(n-1)\,n\,(a+b+n+1)(a+b+n+2)}{2(a+1)(a+2)} {}_2F_1(2-n, -b-n; a+3; -1)\, z^2 + \ldots \Bigg) /; (z \to 0)$$

05.06.06.0002.01

$$P_n^{(a,b)}(z) == \frac{(a+1)_n}{n!} \tilde{F}_{1\times0\times0}^{2\times0\times0}\left(\begin{matrix} -n,\ a+b+n+1;;; \\ a+1;;; \end{matrix} -\frac{z}{2}, \frac{1}{2}\right)$$

05.06.06.0003.01

$$P_n^{(a,b)}(z) == \frac{2^{-n}(a+1)_n}{n!} \sum_{j=0}^{n} \frac{(-n)_j (a+b+n+1)_j}{(a+1)_j\, j!}\, {}_2F_1(-b-n,\ j-n;\ a+j+1;\ -1)\,(-z)^j$$

05.06.06.0004.01

$$P_n^{(a,b)}(z) == \frac{1}{2^n} \sum_{k=0}^{n} \binom{a+n}{k} \binom{b+n}{n-k} (z+1)^k (z-1)^{n-k}$$

05.06.06.0005.01

$$P_n^{(a,b)}(z) \propto \frac{2^{-n}\,\Gamma(a+n+1)}{\Gamma(a+1)\, n!}\, {}_2F_1(-n,\ -b-n;\ a+1;\ -1)\,(1+O(z))$$

Expansions at $z == 1$

For the function itself

05.06.06.0006.02

$$P_n^{(a,b)}(z) \propto \frac{(a+1)_n}{n!}\left(1 + \frac{n(a+b+n+1)}{2(a+1)}(z-1) - \frac{(1-n)\,n\,(a+b+n+1)(a+b+n+2)}{8(a+1)(a+2)}(z-1)^2 + \ldots\right) /;\ (z \to 1)$$

05.06.06.0007.01

$$P_n^{(a,b)}(z) == \frac{\Gamma(a+n+1)}{n!} \sum_{k=0}^{n} \frac{(-n)_k (a+b+n+1)_k}{\Gamma(a+k+1)\, k!} \left(\frac{1-z}{2}\right)^k$$

05.06.06.0008.01

$$P_n^{(a,b)}(z) == \frac{\Gamma(a+n+1)}{n!}\, {}_2\tilde{F}_1\left(-n,\ a+b+n+1;\ a+1;\ \frac{1-z}{2}\right)$$

05.06.06.0009.01

$$P_n^{(a,b)}(z) == \frac{1}{n!} \sum_{k=0}^{n} \frac{(-n)_k (a+b+n+1)_k (a+k+1)_{n-k}}{k!} \left(\frac{1-z}{2}\right)^k$$

05.06.06.0010.02

$$P_n^{(a,b)}(z) \propto \frac{(a+1)_n}{n!}(1+O(z-1)) /;\ (z \to 1) \bigwedge a \notin \mathbb{N}^+$$

Expansions at $z == -1$

For the function itself

05.06.06.0011.02

$$P_n^{(a,b)}(z) \propto \frac{(-1)^n (b+1)_n}{n!}\left(1 - \frac{n(a+b+n+1)}{2(b+1)}(z+1) - \frac{(1-n)\,n\,(a+b+n+1)(a+b+n+2)}{8(b+1)(b+2)}(z+1)^2 - \ldots\right) /;$$
$$(z \to -1) \bigwedge b \notin \mathbb{Z}$$

05.06.06.0012.02

$$P_n^{(a,b)}(z) \propto \frac{(-1)^n (b+1)_n}{n!} \sum_{k=0}^{n} \frac{(-n)_k (a+b+n+1)_k}{(b+1)_k\, k!} \left(\frac{z+1}{2}\right)^k /; b \notin \mathbb{Z}$$

05.06.06.0013.02

$$P_n^{(a,b)}(z) \propto \frac{(-1)^n (b+1)_n}{n!} \, {}_2F_1\left(-n, a+b+n+1; b+1; \frac{z+1}{2}\right) /; b \notin \mathbb{Z}$$

05.06.06.0014.02

$$P_n^{(a,b)}(z) \propto \frac{(-1)^n (b+1)_n}{n!} (1 + O(z+1)) /; (z \to -1) \bigwedge b \notin \mathbb{Z}$$

Expansions at $z == \infty$

For the function itself

Expansions in $1/z$

05.06.06.0035.01

$$P_n^{(a,b)}(z) \propto \frac{2^{-n} (a+b+n+1)_n}{n!} z^n \left(1 - \frac{n(b-a)}{(a+b+2n)z} + \frac{n(n-1)\left(a^2 - (2b+1)a + (b-1)b - 2n\right)}{2(a+b+2n-1)(a+b+2n)z^2} + \ldots\right) /; (|z| \to \infty)$$

05.06.06.0036.01

$$P_n^{(a,b)}(z) \propto \frac{2^{-n} (a+b+n+1)_n}{n!} z^n \left(1 - \frac{n(b-a)}{(a+b+2n)z} + \frac{n(n-1)\left(a^2 - (2b+1)a + (b-1)b - 2n\right)}{2(a+b+2n-1)(a+b+2n)z^2} + O\left(\frac{1}{z^3}\right)\right)$$

05.06.06.0037.01

$$P_n^{(a,b)}(z) = \frac{\Gamma(a+n+1)(a+b+n+1)_n}{2^n} z^n \sum_{j=0}^{n} \frac{(-1)^j}{(n-j)!\, j!\, (-a-b-2n)_j} \, {}_2\tilde{F}_1(-j, -b-n; a-j+n+1; -1)\, z^{-j}$$

05.06.06.0038.01

$$P_n^{(a,b)}(z) \propto \frac{2^{-n} (a+b+n+1)_n}{n!} z^n \left(1 + O\left(\frac{1}{z}\right)\right) /; (|z| \to \infty)$$

Expansions in $1/(1-z)$

05.06.06.0015.01

$$P_n^{(a,b)}(z) \propto \frac{2^{-n} (a+b+n+1)_n (z-1)^n}{n!} \left(1 - \frac{2(-a-n)n}{(-a-b-2n)(1-z)} - \frac{2(1-n)(-a-n)(-a-n+1)n}{(-a-b-2n)(-a-b-2n+1)(1-z)^2} - \ldots\right) /;$$
$$(|z| \to \infty) \bigwedge \neg (a+b+2n \in \mathbb{Z} \bigwedge a+b+2n \leq 0)$$

05.06.06.0016.01

$$P_n^{(a,b)}(z) \propto \frac{2^{-n} (a+b+n+1)_n (z-1)^n}{n!} \sum_{k=0}^{n} \frac{(-n)_k (-a-n)_k}{(-a-b-2n)_k\, k!} \left(\frac{2}{1-z}\right)^k /; \neg (a+b+2n \in \mathbb{Z} \bigwedge a+b+2n \leq 0)$$

05.06.06.0017.01

$$P_n^{(a,b)}(z) \propto \frac{2^{-n} (a+b+n+1)_n}{n!} (z-1)^n \, {}_2F_1\left(-n, -a-n; -a-b-2n; \frac{2}{1-z}\right) /; \neg (a+b+2n \in \mathbb{Z} \bigwedge a+b+2n \leq 0)$$

05.06.06.0018.01

$$P_n^{(a,b)}(z) \propto \frac{2^{-n} (a+b+n+1)_n z^n}{n!} \left(1 + O\left(\frac{1}{z}\right)\right)$$

Expansions at $a == 0$

05.06.06.0039.01

$$P_n^{(a,b)}(z) \propto P_n^{(0,b)}(z) + \sum_{k=0}^{n-1} \frac{1}{b+k+n+1} \left(P_n^{(0,b)}(z) + \frac{(b+2k+1)(b+k+1)_{n-k}}{(n-k)(b+k+1)_{n-k}} P_k^{(0,b)}(z)\right) a + \frac{1}{2\,n!} \sum_{k=0}^{n-2} \frac{(-n)_{k+2}}{(k+2)!} \sum_{j=0}^{k} (-1)^{j+k+n} (j+1)(j+2)(b+n+1)^j (b+k+3)_{n-k-2} S_{k+2}^{(j+2)} \left(\frac{z+1}{2}\right)^{k+2} a^2 + \ldots /; (a \to 0)$$

05.06.06.0040.01

$$P_n^{(a,b)}(z) = \frac{1}{n!} \sum_{k=0}^{n} \frac{1}{k!} \sum_{m=0}^{n} \frac{(-n)_m}{m!} \sum_{j=0}^{m} (-1)^{j+m+n} (b+n+1)^{j-k} \left(\frac{z+1}{2}\right)^m (j-k+1)_k (b+m+1)_{n-m} S_m^{(j)} a^k$$

05.06.06.0041.01

$$P_n^{(a,b)}(z) \propto P_n^{(0,b)}(z) (1 + O(a))$$

Expansions at $a == \infty$

05.06.06.0019.01

$$P_n^{(a,b)}(z) \propto \frac{a^n}{n!} \left(\frac{z+1}{2}\right)^n \left(1 + \frac{n(2b(z-1)+z+n(3z-1)+1)}{2(z+1)a} + \frac{n!}{48(z+1)^2 (n-2)!\,a^2} \left(24 b^2 (z-1)^2 + 24 b (z+n(3z-1)+3)(z-1) + 4(z+1)^2 + 6(n-3nz)^2 + n(34z(z+2)-62)\right) + \ldots\right) /; (|a| \to \infty)$$

05.06.06.0042.01

$$P_n^{(a,b)}(z) = \frac{a^n}{n!} \sum_{k=0}^{n} \frac{1}{(n-k)!} \sum_{m=0}^{n} \frac{(-n)_m}{m!} \sum_{j=0}^{m} (-1)^{j+m+n} (b+n+1)^{j+k-n} \left(\frac{z+1}{2}\right)^m (j+k-n+1)_{n-k} (b+m+1)_{n-m} S_m^{(j)} a^{-k}$$

05.06.06.0043.01

$$P_n^{(a,b)}(z) \propto \frac{a^n}{n!} \left(\frac{z+1}{2}\right)^n \left(1 + O\left(\frac{1}{a}\right)\right) /; (|a| \to \infty)$$

Expansions at $b == 0$

05.06.06.0044.01

$$P_n^{(a,b)}(z) \propto P_n^{(a,0)}(z) + \sum_{k=0}^{n-1} \frac{1}{a+k+n+1} \left(P_n^{(a,0)}(z) + \frac{(-1)^{n-k} (a+2k+1)(a+k+1)_{n-k}}{(n-k)(a+k+1)_{n-k}} P_k^{(a,0)}(z)\right) b + \frac{1}{2\,n!} \sum_{k=0}^{n-2} \frac{(-n)_{k+2}}{(k+2)!} \sum_{j=0}^{k} (-1)^{j+k} S_{k+2}^{(j+2)} (j+1)(j+2)(a+b+n+1)^j (a+k+3)_{n-k-2} \left(\frac{1-z}{2}\right)^{k+2} b^2 + \ldots /; (b \to 0)$$

05.06.06.0045.01

$$P_n^{(a,b)}(z) = \frac{1}{n!} \sum_{k=0}^{n} \frac{1}{k!} \sum_{m=0}^{n} \frac{(-n)_m}{m!} \sum_{j=0}^{m} (-1)^j (a+n+1)^{j-k} \left(\frac{z-1}{2}\right)^m (j-k+1)_k (a+m+1)_{n-m} S_m^{(j)} b^k$$

05.06.06.0046.01

$$P_n^{(a,b)}(z) \propto P_n^{(a,0)}(z) (1 + O(b))$$

Expansions at $b == \infty$

05.06.06.0020.01

$$P_n^{(a,b)}(z) \propto \frac{b^n}{n!}\left(\frac{z-1}{2}\right)^n\left(1+\frac{n\,(3\,z\,n+n+z+2\,a\,(z+1)-1)}{2\,b\,(z-1)}+\frac{(n-1)\,n}{24\,b^2\,(z-1)^2}\right.$$
$$\left.\left(2\,(z-1)^2+12\,a^2\,(z+1)^2+3\,(3\,z\,n+n)^2+12\,a\,(z+1)\,(3\,z\,n+n+z-3)+n\,(17\,(z-2)\,z-31)\right)+\ldots\right)\,/;\,(|b|\to\infty)$$

05.06.06.0047.01

$$P_n^{(a,b)}(z) = \frac{b^n}{n!}\sum_{k=0}^{n}\frac{1}{(n-k)!}\sum_{m=0}^{n}\frac{(-n)_m}{m!}\sum_{j=0}^{m}(-1)^j\,(a+n+1)^{j+k-n}\left(\frac{z-1}{2}\right)^m\,(j+k-n+1)_{n-k}\,(a+m+1)_{n-m}\,S_m^{(j)}\,b^{-k}$$

05.06.06.0048.01

$$P_n^{(a,b)}(z) = (a+1)_n\,b^n\sum_{k=0}^{n}\sum_{s=0}^{k}\sum_{j=0}^{s}\frac{2^{k-n-s}\,(a+n+1)^j\,(z-1)^{n+s-k}\,B_{s-j}^{(n+s-k+1)}(n+s-k)}{j!\,(s-j)!\,(n-k)!\,(k-s)!\,(a+1)_{n-k+s}}\,b^{-k}$$

05.06.06.0049.01

$$P_n^{(a,b)}(z) \propto \frac{b^n}{n!}\left(\frac{z-1}{2}\right)^n\left(1+O\left(\frac{1}{b}\right)\right)\,/;\,(|b|\to\infty)$$

Expansions at $n == \infty$

05.06.06.0050.01

$$P_n^{(a,b)}(z) \propto$$
$$\frac{1}{n!}\left(\left(\left(\frac{1}{2}\right)_n\,2^{a+b}\right)\left(e^{i\,n\cos^{-1}(z)}\left(1+e^{-i\cos^{-1}(z)}\right)^{-b-\frac{1}{2}}\left(1-e^{-i\cos^{-1}(z)}\right)^{-a-\frac{1}{2}}+e^{-i\,n\cos^{-1}(z)}\left(1-e^{i\cos^{-1}(z)}\right)^{-a-\frac{1}{2}}\left(1+e^{i\cos^{-1}(z)}\right)^{-b-\frac{1}{2}}\right)\right)$$
$$(1+\ldots)\,/;\,(n\to\infty)$$

05.06.06.0051.01

$$P_n^{(a,b)}(z) \propto \frac{2^{a+b}}{n!}\left(\frac{1}{2}\right)_n\left(\left(1+z-i\sqrt{1-z^2}\right)^{-b-\frac{1}{2}}\left(1-z+i\sqrt{1-z^2}\right)^{-a-\frac{1}{2}}e^{i\,n\cos^{-1}(z)}+\right.$$
$$\left.\left(1+z+i\sqrt{1-z^2}\right)^{-b-\frac{1}{2}}\left(1-z-i\sqrt{1-z^2}\right)^{-a-\frac{1}{2}}e^{-i\,n\cos^{-1}(z)}\right)(1+\ldots)\,/;\,(n\to\infty)$$

Integral representations

On the real axis

Of the direct function

05.06.07.0001.01

$$P_n^{(a,b)}(z) == \frac{\Gamma(a+n+1)}{2^n\,\Gamma(n+1)\,\Gamma(a+b+n+1)\,\Gamma(-b-n)}\int_0^1 t^{a+b+n}\,(1-t)^{-b-n-1}\,(z\,t-t+2)^n\,dt\,/;$$
$$\mathrm{Re}(b+n)<0\bigwedge\mathrm{Re}(a+b+n+1)>0\bigwedge|\arg(z+1)|<\pi$$

Integral representations of negative integer order

Rodrigues-type formula.

05.06.07.0002.01

$$P_n^{(a,b)}(z) = \frac{(-1)^n}{n!\, 2^n (1-z)^a (z+1)^b} \frac{\partial^n \left((1-z)^{a+n} (z+1)^{b+n}\right)}{\partial z^n}$$

Generating functions

05.06.11.0001.01

$$P_n^{(a,b)}(z) = \left([t^n] \frac{\left(\sqrt{t^2 - 2\,t\,z + 1} + 1 - t\right)^{-a} \left(\sqrt{t^2 - 2\,t\,z + 1} + 1 + t\right)^{-b}}{2^{-a-b} \sqrt{t^2 - 2\,t\,z + 1}} \right) /; -1 < z < 1$$

05.06.11.0002.01

$$P_n^{(\alpha,\beta)}(z) = [t^n]\left(\frac{(\alpha+1)_n (\beta+1)_n}{(a)_n (-a+\alpha+\beta+1)_n} {}_2F_1\left(a, -a+\alpha+\beta+1; \alpha+1; \frac{1}{2}\left(-t - \sqrt{t^2 - 2\,z\,t + 1} + 1\right)\right) \right.$$
$$\left. {}_2F_1\left(a, -a+\alpha+\beta+1; \beta+1; \frac{1}{2}\left(t - \sqrt{t^2 - 2\,z\,t + 1} + 1\right)\right)\right) /; -1 < z < 1$$

Differential equations

Ordinary linear differential equations and wronskians

For the direct function itself

05.06.13.0003.01

$$\left(1 - z^2\right) w''(z) + (b - a - (a + b + 2)\, z)\, w'(z) + n\,(n + 2\,\lambda)\, w(z) = 0 /; w(z) = c_1 P_n^{(a,b)}(z) + c_2\, G_{2,2}^{2,2}\left(\frac{1-z}{2} \,\middle|\, \begin{matrix} n+1, -a-b-n \\ 0, -a \end{matrix} \right)$$

05.06.13.0004.01

$$W_z\left(P_n^{(a,b)}(z), G_{2,2}^{2,2}\left(\frac{1-z}{2} \,\middle|\, \begin{matrix} n+1, -a-b-n \\ 0, -a \end{matrix} \right)\right) = \frac{(-1)^{n-1} 2^{a+b+1} \pi \csc(\pi\, a)\, \Gamma(b+n+1)}{n!} (1-z)^{-a-1} (z+1)^{-b-1}$$

05.06.13.0001.01

$$\left(1 - z^2\right) w''(z) + (b - a - (a + b + 2)\, z)\, w'(z) + n\,(n + 2\,\lambda)\, w(z) = 0 /; w(z) = c_1 P_n^{(a,b)}(z) + c_2 (1-z)^{-a} P_{a+n}^{(-a,b)}(z) \bigwedge a \notin \mathbb{Z}$$

05.06.13.0002.02

$$W_z\left(P_n^{(a,b)}(z), (1-z)^{-a} P_{a+n}^{(-a,b)}(z)\right) = \frac{2^{b+1} \sin(a\,\pi)}{\pi} (1-z)^{-a-1} (z+1)^{-b-1}$$

05.06.13.0005.01

$$w''(z) + \left(\frac{(a - b + (a + b + 2)\, g(z))\, g'(z)}{g(z)^2 - 1} - \frac{g''(z)}{g'(z)} \right) w'(z) - \frac{n\,(a + b + n + 1)\, g'(z)^2}{g(z)^2 - 1} w(z) = 0 /;$$
$$w(z) = c_1 P_n^{(a,b)}(g(z)) + c_2\, G_{2,2}^{2,2}\left(\frac{1-g(z)}{2} \,\middle|\, \begin{matrix} n+1, -a-b-n \\ 0, -a \end{matrix} \right)$$

05.06.13.0006.01

$$W_z\left(P_n^{(a,b)}(g(z)), G_{2,2}^{2,2}\left(\frac{1-g(z)}{2} \,\middle|\, \begin{matrix} n+1, -a-b-n \\ 0, -a \end{matrix} \right)\right) = -\frac{2^{a+b+1} \pi \csc(\pi\,(a+n))\, \Gamma(b+n+1)}{\Gamma(n+1)} (1-g(z))^{-a-1} (g(z)+1)^{-b-1} g'(z)$$

05.06.13.0007.01

$$w''(z)\,h(z)^2+\left(h(z)\left(\frac{(a-b+(a+b+2)\,g(z))\,h(z)\,g'(z)}{g(z)^2-1}-2\,h'(z)\right)-\frac{h(z)^2\,g''(z)}{g'(z)}\right)w'(z)-\left(-2\,h'(z)^2-\frac{h(z)\,g''(z)\,h'(z)}{g'(z)}+\frac{1}{g(z)^2-1}\,h(z)\,g'(z)\,(n\,(a+b+n+1)\,h(z)\,g'(z)+(a-b+(a+b+2)\,g(z))\,h'(z))+h(z)\,h''(z)\right)$$

$$w(z)=0\,/;\,w(z)=c_1\,h(z)\,P_n^{(a,b)}(g(z))\,+c_2\,h(z)\,G_{2,2}^{2,2}\!\left(\frac{1-g(z)}{2}\,\middle|\,\begin{matrix}n+1,\,-a-b-n\\0,\,-a\end{matrix}\right)$$

05.06.13.0008.01

$$W_z\!\left(h(z)\,P_n^{(a,b)}(g(z)),\,h(z)\,G_{2,2}^{2,2}\!\left(\frac{1-g(z)}{2}\,\middle|\,\begin{matrix}n+1,\,-a-b-n\\0,\,-a\end{matrix}\right)\right)=-\frac{2^{a+b+1}\,\pi\csc(\pi\,(a+n))\,\Gamma(b+n+1)}{\Gamma(n+1)}\,(1-g(z))^{-a-1}\,(g(z)+1)^{-b-1}\,h(z)^2\,g'(z)$$

05.06.13.0009.01

$$z^2\,w''(z)+\left(\frac{d\,r\,((a+b+2)\,d\,z^r+a-b)\,z^r}{d^2\,z^{2r}-1}-r-2\,s+1\right)z\,w'(z)+\frac{-(a-b)\,d\,r\,s\,z^r-d^2\,(s+r\,n)\,(r\,(a+b+n+1)-s)\,z^{2r}-s\,(r+s)}{d^2\,z^{2r}-1}\,w(z)=0\,/;$$

$$w(z)=c_1\,z^s\,P_n^{(a,b)}(d\,z^r)\,+c_2\,z^s\,G_{2,2}^{2,2}\!\left(\frac{1-d\,z^r}{2}\,\middle|\,\begin{matrix}n+1,\,-a-b-n\\0,\,-a\end{matrix}\right)$$

05.06.13.0010.01

$$W_z\!\left(z^s\,P_n^{(a,b)}(d\,z^r),\,z^s\,G_{2,2}^{2,2}\!\left(\frac{1-d\,z^r}{2}\,\middle|\,\begin{matrix}n+1,\,-a-b-n\\0,\,-a\end{matrix}\right)\right)=-\frac{2^{a+b+1}\,d\,\pi\,r\,z^{r+2s-1}}{\Gamma(n+1)}\,(1-d\,z^r)^{-a-1}\,(d\,z^r+1)^{-b-1}\csc(\pi\,(a+n))\,\Gamma(b+n+1)$$

05.06.13.0011.01

$$w''(z)+\left(\frac{d\,((a+b+1)\,d\,r^z+a-b)\,r^z+1}{d^2\,r^{2z}-1}\log(r)-2\log(s)\right)w'(z)+\left(\log^2(s)+\log(r)\log(s)-\frac{d\,r^z\log(r)}{d^2\,r^{2z}-1}\,(d\,n\,(a+b+n+1)\log(r)\,r^z+((a+b+2)\,d\,r^z+a-b)\log(s))\right)w(z)=0\,/;$$

$$w(z)=c_1\,s^z\,P_n^{(a,b)}(d\,r^z)+c_2\,s^z\,G_{2,2}^{2,2}\!\left(\frac{1-d\,r^z}{2}\,\middle|\,\begin{matrix}n+1,\,-a-b-n\\0,\,-a\end{matrix}\right)$$

05.06.13.0012.01

$$W_z\!\left(s^z\,P_n^{(a,b)}(d\,r^z),\,s^z\,G_{2,2}^{2,2}\!\left(\frac{1-d\,r^z}{2}\,\middle|\,\begin{matrix}n+1,\,-a-b-n\\0,\,-a\end{matrix}\right)\right)=-\frac{1}{\Gamma(n+1)}\,2^{a+b+1}\,d\,\pi\,r^z\,(1-d\,r^z)^{-a-1}\,(d\,r^z+1)^{-b-1}\,s^{2z}\csc(\pi\,(a+n))\,\Gamma(b+n+1)\log(r)$$

Transformations

Transformations and argument simplifications

Argument involving basic arithmetic operations

05.06.16.0001.01

$$P_n^{(a,b)}(-z) = (-1)^n P_n^{(b,a)}(z)$$

Identities

Recurrence identities

Consecutive neighbors

With respect to *n*

05.06.17.0001.01

$$P_n^{(a,b)}(z) = \frac{(a+b+2n+3)\left(a^2-b^2+z(a+b+2n+2)(a+b+2n+4)\right)}{2(a+n+1)(b+n+1)(a+b+2n+4)} P_{n+1}^{(a,b)}(z) - \frac{(n+2)(a+b+n+2)(a+b+2n+2)}{(a+n+1)(b+n+1)(a+b+2n+4)} P_{n+2}^{(a,b)}(z)$$

05.06.17.0002.01

$$P_n^{(a,b)}(z) = \frac{(a+b+2n-1)\left(a^2-b^2+z(a+b+2n-2)(a+b+2n)\right)}{2n(a+b+n)(a+b+2n-2)} P_{n-1}^{(a,b)}(z) - \frac{(a+n-1)(b+n-1)(a+b+2n)}{n(a+b+n)(a+b+2n-2)} P_{n-2}^{(a,b)}(z)$$

With respect to *a*

05.06.17.0016.01

$$P_n^{(a,b)}(z) = \frac{-(z-1)b+a(3-z)-2(z+(z-1)n-2)}{2(a+n+1)} P_n^{(a+1,b)}(z) + \frac{(a+b+n+2)(z-1)}{2(a+n+1)} P_n^{(a+2,b)}(z)$$

05.06.17.0017.01

$$P_n^{(a,b)}(z) = \frac{a(z-3)+b(z-1)+2zn-2n+2}{(a+b+n)(z-1)} P_n^{(a-1,b)}(z) + \frac{2(a+n-1)}{(a+b+n)(z-1)} P_n^{(a-2,b)}(z)$$

With respect to *b*

05.06.17.0018.01

$$P_n^{(a,b)}(z) = \left((a(z+1)+b(z+3)+2(nz+z+n+2)) P_n^{(a,b+1)}(z) - (z+1)(a+b+n+2) P_n^{(a,b+2)}(z)\right) \big/ (2(b+n+1))$$

05.06.17.0019.01

$$P_n^{(a,b)}(z) = \left((a(z+1)+b(z+3)+2(zn+n-1)) P_n^{(a,b-1)}(z) - 2(b+n-1) P_n^{(a,b-2)}(z)\right) \big/ ((z+1)(a+b+n))$$

Distant neighbors

With respect to *n*

05.06.17.0020.01

$$P_n^{(a,b)}(z) == C_m(n, a, b, z)\, P_{n+m}^{(a,b)}(z) - \frac{(m+n+1)\,(a+b+m+n+1)\,(a+b+2\,m+2\,n)}{(a+m+n)\,(b+m+n)\,(a+b+2\,m+2\,n+2)}\, C_{m-1}(n, a, b, z)\, P_{n+m+1}^{(a,b)}(z)\, /;$$

$$C_0(n, a, b, z) == 1 \bigwedge C_1(n, a, b, z) == \frac{(a+b+2\,n+3)\left(a^2-b^2+z\,(a+b+2\,n+2)\,(a+b+2\,n+4)\right)}{2\,(a+n+1)\,(b+n+1)\,(a+b+2\,n+4)} \bigwedge$$

$$C_m(n, a, b, z) == \frac{(a+b+2\,m+2\,n+1)\left(a^2-b^2+(a+b+2\,m+2\,n+2)\,z\,(a+b+2\,m+2\,n)\right)}{2\,(a+m+n)\,(b+m+n)\,(a+b+2\,m+2\,n+2)}\, C_{m-1}(n, a, b, z) - \frac{(m+n)\,(a+b+m+n)\,(a+b+2\,m+2\,n-2)}{(a+m+n-1)\,(b+m+n-1)\,(a+b+2\,m+2\,n)}\, C_{m-2}(n, a, b, z) \bigwedge m \in \mathbb{N}^+$$

05.06.17.0021.01

$$P_n^{(a,b)}(z) == \frac{(a-m+n)\,(b-m+n)\,(a+b-2\,m+2\,n+2)}{(m-n-1)\,(a+b-m+n+1)\,(a+b-2\,m+2\,n)}\, C_{m-1}(n, a, b, z)\, P_{n-m-1}^{(a,b)}(z) + C_m(n, a, b, z)\, P_{n-m}^{(a,b)}(z)\, /;$$

$$C_0(n, a, b, z) == 1 \bigwedge C_1(n, a, b, z) == \frac{(a+b+2\,n-1)\left(a^2-b^2+z\,(a+b+2\,n-2)\,(a+b+2\,n)\right)}{2\,n\,(a+b+n)\,(a+b+2\,n-2)} \bigwedge$$

$$C_m(n, a, b, z) == \frac{(a+b-2\,m+2\,n+1)\left(a^2-b^2+z\,(a+b-2\,m+2\,n)\,(a+b-2\,m+2\,n+2)\right)}{2\,(-m+n+1)\,(a+b-m+n+1)\,(a+b-2\,m+2\,n)}\, C_{m-1}(n, a, b, z) + \frac{(a-m+n+1)\,(b-m+n+1)\,(a+b-2\,m+2\,n+4)}{(m-n-2)\,(a+b-m+n+2)\,(a+b-2\,m+2\,n+2)}\, C_{m-2}(n, a, b, z) \bigwedge m \in \mathbb{N}^+$$

Functional identities

Relations between contiguous functions

Recurrence relations

05.06.17.0003.01

$$2\,(a+b+2\,n+2)\,(a+n)\,(b+n)\, P_{n-1}^{(a,b)}(z) + 2\,(a+b+n+1)\,(n+1)\,(a+b+2\,n)\, P_{n+1}^{(a,b)}(z) == \left((a+b+2\,n+1)\left(a^2-b^2\right)+z\,(a+b+2\,n)_3\right) P_n^{(a,b)}(z)$$

05.06.17.0004.01

$$P_n^{(a,b)}(z) == \left(2\left((a+n)\,(b+n)\,(a+b+2\,n+2)\, P_{n-1}^{(a,b)}(z) + (n+1)\,(a+b+n+1)\,(a+b+2\,n)\, P_{n+1}^{(a,b)}(z)\right)\right) \Big/ \left((a+b+2\,n+1)\left(a^2-b^2+z\,(a+b+2\,n)\,(a+b+2\,n+2)\right)\right)$$

05.06.17.0005.01

$$P_n^{(a,b-1)}(z) - P_n^{(a-1,b)}(z) == P_{n-1}^{(a,b)}(z)$$

05.06.17.0006.01

$$z\, P_n^{(a,b)}(z) == \frac{2\,(a+n)\,(b+n)}{(a+b+2\,n)\,(a+b+2\,n+1)}\, P_{n-1}^{(a,b)}(z) + \frac{2\,(n+1)\,(a+b+n+1)}{(a+b+2\,n+1)\,(a+b+2\,n+2)}\, P_{n+1}^{(a,b)}(z) + \frac{b^2-a^2}{(a+b+2\,n)\,(a+b+2\,n+2)}\, P_n^{(a,b)}(z)$$

Normalized recurrence relation

05.06.17.0007.01

$$z\,p(n,z) == p(n+1,z) + \frac{\left(b^2-a^2\right) p(n,z)}{(a+b+2n)(a+b+2n+2)} + \frac{4n(a+n)(b+n)(a+b+n)}{(a+b+2n-1)(a+b+2n)^2(a+b+2n+1)}\,p(n-1,z)\;/;$$

$$p(n,z) == \frac{2^n\,\Gamma(n+1)}{(a+b+n+1)_n}\,P_n^{(a,b)}(z)$$

Additional relations between contiguous functions

05.06.17.0008.01

$$P_n^{(a,b)}(z) - P_n^{(a-1,b)}(z) == \frac{z+1}{2}\,P_{n-1}^{(a,b+1)}(z)$$

05.06.17.0022.01

$$P_n^{(a,b)}(z) == \frac{(a+b+(a-b+2)z)\,P_n^{(a+1,b-1)}(z) + (z-1)(b+n-1)\,P_n^{(a+2,b-2)}(z)}{(z+1)(a+n+1)}$$

05.06.17.0023.01

$$P_n^{(a,b)}(z) == \frac{(z+1)(a+n-1)\,P_n^{(a-2,b+2)}(z) - (z\,a+a+b-b\,z-2z)\,P_n^{(a-1,b+1)}(z)}{(z-1)(b+n+1)}$$

05.06.17.0024.01

$$P_n^{(a,b)}(z) = \frac{a+b+n+1}{b+n+1}\,P_{n+1}^{(a,b)}(z) - \frac{a+b+2n+2}{b+n+1}\,P_{n+1}^{(a-1,b)}(z)$$

05.06.17.0025.01

$$P_n^{(a,b)}(z) = \frac{z+1}{2}\,P_n^{(a,b+1)}(z) - \frac{z-1}{2}\,P_n^{(a+1,b)}(z)$$

05.06.17.0026.01

$$P_n^{(a,b)}(z) = \frac{a+b+n+1}{a+n+1}\,P_n^{(a+1,b)}(z) - \frac{b+n}{a+n+1}\,P_n^{(a+1,b-1)}(z)$$

Expansion with respect to parameters

05.06.17.0013.01

$$P_n^{(a,b)}(z) == \sum_{k=0}^{n} \frac{(a+b+n+1)_k\,(a+k+1)_{n-k}\,\Gamma(k+\alpha+\beta+1)}{\Gamma(n-k+1)\,\Gamma(2k+\alpha+\beta+1)}\,{}_3F_2(k-n,\,a+b+k+n+1,\,k+\alpha+1;\,a+k+1,\,2k+\alpha+\beta+2;\,1)\,P_k^{(\alpha,\beta)}(z)$$

05.06.17.0014.01

$$P_n^{(a,b)}(z) == \sum_{k=0}^{n} \frac{(b+2k+\alpha+1)\,\Gamma(b+n+1)\,\Gamma(a+b+k+n+1)\,\Gamma(b+k+\alpha+1)\,(a-\alpha)_{n-k}}{\Gamma(b+k+1)\,\Gamma(a+b+n+1)\,\Gamma(b+k+n+\alpha+2)\,(n-k)!}\,P_k^{(\alpha,b)}(z)\;/;\;n\in\mathbb{N}$$

05.06.17.0015.01

$$P_n^{(a,b)}(z) == \sum_{k=0}^{n} (-1)^{n-k}\,(a+2k+\delta+1)\,\Gamma(a+n+1)\,\Gamma(a+b+k+n+1)\,\Gamma(a+k+\delta+1)$$

$$(b-\delta)_{n-k} \Big/ \left(\Gamma(a+k+1)\,\Gamma(a+b+n+1)\,\Gamma(a+k+n+\delta+2)\,(n-k)!\right) P_k^{(a,\delta)}(z)\;/;\;n\in\mathbb{N}$$

Relations of special kind

05.06.17.0009.01

$$P_n^{(a,b)}(z) = \frac{\Gamma(-a-b-n)\,\Gamma(a+n+1)}{\Gamma(-b-n)\,\Gamma(n+1)}\,P_{-a-b-n-1}^{(a,b)}(z)$$

05.06.17.0010.01

$$P_n^{(a,b)}(z) = \frac{\Gamma(a+n+1)\,\Gamma(b+n+1)}{\Gamma(n+1)\,\Gamma(a+b+n+1)}\left(\frac{z+1}{2}\right)^{-b} P_{b+n}^{(a,-b)}(z)$$

05.06.17.0011.01

$$P_n^{(a,b)}(z) = \frac{\Gamma(a+n+1)\,\Gamma(b+n+1)}{\Gamma(n+1)\,\Gamma(a+b+n+1)}\left(\frac{z-1}{2}\right)^{-a} P_{a+n}^{(-a,b)}(z)\;/;\; a\in\mathbb{Z}$$

05.06.17.0012.01

$$P_n^{(a,b)}(z) = \left(\frac{z-1}{2}\right)^{-a}\left(\frac{z+1}{2}\right)^{-b} P_{a+b+n}^{(-a,-b)}(z)\;/;\; a\in\mathbb{Z}$$

Complex characteristics

Real part

05.06.19.0001.01

$$\mathrm{Re}\left(P_n^{(a,b)}(x+i\,y)\right) = \sum_{j=0}^{\left\lfloor \frac{n}{2}\right\rfloor} \frac{(-1)^j\,(a+b+n+1)_{2j}\,y^{2j}}{2^{2j}\,(2j)!}\,P_{n-2j}^{(a+2j,b+2j)}(x)\;/;\; x\in\mathbb{R}\land y\in\mathbb{R}\land a\in\mathbb{R}\land b\in\mathbb{R}$$

Imaginary part

05.06.19.0002.01

$$\mathrm{Im}\left(P_n^{(a,b)}(x+i\,y)\right) = \sum_{j=0}^{\left\lfloor \frac{n-1}{2}\right\rfloor} \frac{(-1)^j\,(a+b+n+1)_{2j+1}\,y^{2j+1}}{2^{2j+1}\,(2j+1)!}\,P_{-2j+n-1}^{(a+2j+1,b+2j+1)}(x)\;/;\; x\in\mathbb{R}\land y\in\mathbb{R}\land a\in\mathbb{R}\land b\in\mathbb{R}$$

Argument

05.06.19.0003.01

$$\arg\left(P_n^{(a,b)}(x+i\,y)\right) = \tan^{-1}\left(\sum_{j=0}^{\left\lfloor \frac{n}{2}\right\rfloor} \frac{(-1)^j\,(a+b+n+1)_{2j}}{2^{2j}\,(2j)!}\,P_{n-2j}^{(a+2j,b+2j)}(x)\,y^{2j},\right.$$
$$\left.\sum_{j=0}^{\left\lfloor \frac{n-1}{2}\right\rfloor} \frac{(-1)^j\,(a+b+n+1)_{2j+1}}{2^{2j+1}\,(2j+1)!}\,P_{-2j+n-1}^{(a+2j+1,b+2j+1)}(x)\,y^{2j+1}\right)\;/;\; x\in\mathbb{R}\land y\in\mathbb{R}\land a\in\mathbb{R}\land b\in\mathbb{R}$$

Conjugate value

05.06.19.0004.01

$$\overline{P_n^{(a,b)}(x+i\,y)} = \sum_{j=0}^{\left\lfloor \frac{n}{2}\right\rfloor} \frac{(-1)^j\,(a+b+n+1)_{2j}}{2^{2j}\,(2j)!}\,P_{n-2j}^{(a+2j,b+2j)}(x)\,y^{2j} - i\sum_{j=0}^{\left\lfloor \frac{n-1}{2}\right\rfloor} \frac{(-1)^j\,(a+b+n+1)_{2j+1}}{2^{2j+1}\,(2j+1)!}\,P_{-2j+n-1}^{(a+2j+1,b+2j+1)}(x)\,y^{2j+1}\;/;$$
$$x\in\mathbb{R}\land y\in\mathbb{R}\land a\in\mathbb{R}\land b\in\mathbb{R}$$

Differentiation

Low-order differentiation

With respect to *a*

05.06.20.0001.01

$$\frac{\partial P_n^{(a,b)}(z)}{\partial a} = \frac{\Gamma(a+n+1)\,(\psi(a+n+1)-\psi(a+1))}{\Gamma(n+1)}\,{}_2\tilde{F}_1\left(-n,\, a+b+n+1;\, a+1;\, \frac{1-z}{2}\right) - \frac{(1-z)\,\Gamma(a+n+1)}{2\,(a+1)\,\Gamma(n)\,\Gamma(a+2)}\left((a+1)\,F_{2\times0\times1}^{2\times1\times2}\left(\begin{matrix}1-n,\, a+b+n+2;\, 1;\, 1,\, a+b+n+1;\\ 2,\, a+2;;\, a+b+n+2;\end{matrix}\; \frac{1-z}{2},\, \frac{1-z}{2}\right) - (a+b+n+1)\,F_{2\times0\times1}^{2\times1\times2}\left(\begin{matrix}1-n,\, a+b+n+2;\, 1;\, 1,\, a+1;\\ 2,\, a+2;;\, a+2;\end{matrix}\; \frac{1-z}{2},\, \frac{1-z}{2}\right)\right)$$

05.06.20.0010.01

$$\frac{\partial P_n^{(a,b)}(z)}{\partial a} = \sum_{k=0}^{n-1} \frac{1}{a+b+k+n+1}\left(P_n^{(a,b)}(z) + \frac{(a+b+2k+1)\;(b+k+1)_{n-k}}{(n-k)\,(a+b+k+1)_{n-k}}\,P_k^{(a,b)}(z)\right)$$

05.06.20.0013.01

$$\frac{\partial^2 P_n^{(a,b)}(z)}{\partial a^2} = \frac{1}{n!}\sum_{k=0}^{n-2}\frac{(-n)_{k+2}}{(k+2)!}\sum_{j=0}^{k}(-1)^{j+k+n}\,(j+1)\,(j+2)\,(a+b+n+1)^j\,(b+k+3)_{n-k-2}\,S_{k+2}^{(j+2)}\left(\frac{z+1}{2}\right)^{k+2}$$

With respect to *b*

05.06.20.0002.01

$$\frac{\partial P_n^{(a,b)}(z)}{\partial b} = -\frac{\Gamma(a+n+1)\,(1-z)}{2\,\Gamma(n)\,\Gamma(a+2)}\,F_{2\times0\times1}^{2\times1\times2}\left(\begin{matrix}a+b+n+2,\, 1-n;\, 1;\, 1,\, a+b+n+1;\\ 2,\, a+2;;\, a+b+n+2;\end{matrix}\; \frac{1-z}{2},\, \frac{1-z}{2}\right)$$

05.06.20.0011.01

$$\frac{\partial P_n^{(a,b)}(z)}{\partial b} - \sum_{k=0}^{n-1}\frac{1}{a+b+k+n+1}\left(P_n^{(a,b)}(z) + \frac{\left((-1)^{n-k}\,(a+b+2k+1)\,(a+k+1)_{n-k}\right)}{(n-k)\,(a+b+k+1)_{n-k}}\,P_k^{(a,b)}(z)\right)$$

05.06.20.0014.01

$$\frac{\partial^2 P_n^{(a,b)}(z)}{\partial b^2} = \frac{1}{n!}\sum_{k=0}^{n-2}\frac{(-n)_{k+2}}{(k+2)!}\sum_{j=0}^{k}(-1)^{j+k}\,S_{k+2}^{(j+2)}\,(j+1)\,(j+2)\,(a+b+n+1)^j\,(a+k+3)_{n-k-2}\left(\frac{1-z}{2}\right)^{k+2}$$

With respect to *z*

Forward shift operator:

05.06.20.0003.01

$$\frac{\partial P_n^{(a,b)}(z)}{\partial z} = \frac{a+b+n+1}{2}\,P_{n-1}^{(a+1,b+1)}(z)$$

05.06.20.0004.01

$$\frac{\partial^2 P_n^{(a,b)}(z)}{\partial z^2} = \frac{1}{4}\,(a+b+n+1)\,(a+b+n+2)\,P_{n-2}^{(a+2,b+2)}(z)$$

Backward shift operator:

05.06.20.0005.01

$$\left(1-z^2\right)\frac{\partial P_n^{(a,b)}(z)}{\partial z}+(b-a-(a+b)\,z)\,P_n^{(a,b)}(z)=-2\,(n+1)\,P_{n+1}^{(a-1,b-1)}(z)$$

05.06.20.0006.01

$$\frac{\partial\left((1-z)^a\,(z+1)^b\,P_n^{(a,b)}(z)\right)}{\partial z}=-2\,(n+1)\,(1-z)^{a-1}\,(z+1)^{b-1}\,P_{n+1}^{(a-1,b-1)}(z)$$

Symbolic differentiation

With respect to *a*

05.06.20.0015.01

$$\frac{\partial^m P_n^{(a,b)}(z)}{\partial a^m}=\frac{1}{n!}\sum_{k=0}^{n}\frac{(-n)_k}{k!}\sum_{j=0}^{k}(-1)^{j+k+n}\,S_k^{(j)}\,(j-m+1)_m\,(a+b+n+1)^{j-m}\,(b+k+1)_{n-k}\left(\frac{z+1}{2}\right)^k\;/;\;m\in\mathbb{N}$$

With respect to *b*

05.06.20.0016.01

$$\frac{\partial^m P_n^{(a,b)}(z)}{\partial b^m}=\frac{1}{n!}\sum_{k=0}^{n}\frac{(-n)_k}{k!}\sum_{j=0}^{k}(-1)^{j+k}\,S_k^{(j)}\,(j-m+1)_m\,(a+b+n+1)^{j-m}\,(a+k+1)_{n-k}\left(\frac{1-z}{2}\right)^k\;/;\;m\in\mathbb{N}$$

With respect to *z*

05.06.20.0007.01

$$\frac{\partial^m P_n^{(a,b)}(z)}{\partial z^m}=2^{-m}\,(a+b+n+1)_m\,P_{n-m}^{(a+m,b+m)}(z)\;/;\;m\in\mathbb{N}^+$$

05.06.20.0008.01

$$\frac{\partial^m P_n^{(a,b)}(z)}{\partial z^m}=\frac{\Gamma(a+n+1)\,(z-1)^{-m}}{\Gamma(n+1)}\,{}_3\tilde{F}_2\!\left(1,\,-n,\,a+b+n+1;\,a+1,\,1-m;\,\frac{1-z}{2}\right)/;\;m\in\mathbb{N}^+$$

05.06.20.0012.01

$$\frac{\partial^m P_n^{(a,b)}(z)}{\partial z^m}=2^{-m}\,(n+a+b+1)_m\sum_{k=0}^{n-m}\frac{(n+m+a+b+1)_k\,(k+m+a+1)_{n-m-k}\,\Gamma(k+a+b+1)}{\Gamma(n-k-m+1)\,\Gamma(2\,k+a+b+1)}$$
$${}_3F_2(k-n+m,\,k+n+m+a+b+1,\,k+a+1;\,k+m+a+1,\,2\,k+a+b+2;\,1)\,P_k^{(a,b)}(z)\;/;\;m\in\mathbb{N}$$

Fractional integro-differentiation

With respect to *a*

05.06.20.0017.01

$$\frac{\partial^\alpha P_n^{(a,b)}(z)}{\partial a^\alpha}=\frac{1}{n!}\sum_{k=0}^{n}\frac{1}{\Gamma(k-\alpha+1)}\sum_{m=0}^{n}\frac{(-n)_m}{m!}\sum_{j=0}^{m}(-1)^{j+m+n}\,(b+n+1)^{j-k}\left(\frac{z+1}{2}\right)^m(j-k+1)_k\,(b+m+1)_{n-m}\,S_m^{(j)}\,a^{k-\alpha}$$

With respect to *b*

05.06.20.0018.01

$$\frac{\partial^\alpha P_n^{(a,b)}(z)}{\partial b^\alpha}=\frac{1}{n!}\sum_{k=0}^{n}\frac{1}{\Gamma(k-\alpha+1)}\sum_{m=0}^{n}\frac{(-n)_m}{m!}\sum_{j=0}^{m}(-1)^{j}\,(a+n+1)^{j-k}\left(\frac{z-1}{2}\right)^m(j-k+1)_k\,(a+m+1)_{n-m}\,S_m^{(j)}\,b^{k-\alpha}$$

With respect to z

05.06.20.0009.01

$$\frac{\partial^\alpha P_n^{(a,b)}(z)}{\partial z^\alpha} = \frac{\Gamma(a+n+1)}{\Gamma(n+1)} z^{-\alpha} \tilde{F}_{1\times1\times0}^{2\times1\times0}\left(\begin{matrix} -n, a+b+n+1; 1; ; \\ a+1; 1-\alpha; ; \end{matrix} -\frac{z}{2}, \frac{1}{2}\right)$$

Integration

Indefinite integration

Involving only one direct function

05.06.21.0001.01

$$\int P_n^{(a,b)}(z)\, dz = \frac{2}{a+b+n} P_{n+1}^{(a-1,b-1)}(z)$$

Involving one direct function and elementary functions

Involving power function

05.06.21.0002.01

$$\int z^{\alpha-1} P_n^{(a,b)}(z)\, dz = \frac{(a+1)_n z^\alpha}{\alpha\,\Gamma(n+1)} \tilde{F}_{1\times1\times0}^{2\times1\times0}\left(\begin{matrix} -n, a+b+n+1; \alpha; ; \\ a+1; \alpha+1; ; \end{matrix} -\frac{z}{2}, \frac{1}{2}\right)$$

Involving algebraic functions

05.06.21.0003.01

$$\int (z-1)^c P_n^{(a,b)}(z)\, dz = \frac{(z-1)^{c+1}\,\Gamma(c+1)\,\Gamma(a+n+1)}{\Gamma(n+1)} {}_3\tilde{F}_2\left(-n, a+b+n+1, c+1; a+1, c+2; \frac{1-z}{2}\right)$$

05.06.21.0004.01

$$\int (1-z)^a (z+1)^b P_n^{(a,b)}(z)\, dz = -\frac{(1-z)^{a+1} (z+1)^{b+1} P_{n-1}^{(a+1,b+1)}(z)}{2\,n}$$

Definite integration

Involving the direct function

Orthogonality:

05.06.21.0005.01

$$\int_{-1}^{1} (1-t)^a (t+1)^b P_m^{(a,b)}(t)\, P_n^{(a,b)}(t)\, dt = \frac{2^{a+b+1}\,\Gamma(a+n+1)\,\Gamma(b+n+1)}{n!\,(a+b+2\,n+1)\,\Gamma(a+b+n+1)} \delta_{m,n} \;/;\; \mathrm{Re}(a) > -1 \land \mathrm{Re}(b) > -1$$

05.06.21.0006.01

$$\int_{-1}^{1} (1-t)^a (t+1)^c P_n^{(a,b)}(t)\, dt = \frac{2^{a+c+1}\,\Gamma(c+1)\,\Gamma(a+n+1)\,\Gamma(-b+c+1)}{n!\,\Gamma(-b+c-n+1)\,\Gamma(a+c+n+2)} \;/;\; \mathrm{Re}(a) > -1 \land \mathrm{Re}(b) > -1 \land \mathrm{Re}(c) > -1$$

05.06.21.0007.01

$$\int_{-1}^{1}(1-t)^a\,(t+1)^b\,e^{\beta\,(t+1)}\,P_m^{(a,b)}(t)\,dt = \beta^m\,2^{a+b+m+1}\,\frac{\Gamma(a+1)\,\Gamma(b+m+1)}{\Gamma(a+b+2\,m+2)}\,P_m^{(a,b)}(1)\;{}_1F_1(b+m+1;\,a+b+2\,m+2;\,2\,\beta)\,/;\;\mathrm{Re}(a)>-1\bigwedge\mathrm{Re}(b)>-1$$

Summation

Infinite summation

05.06.23.0001.01

$$\sum_{n=0}^{\infty}P_n^{(a,b)}(z)\,w^n = \frac{\left(\sqrt{w^2-2\,z\,w+1}+1-w\right)^{-a}\left(\sqrt{w^2-2\,z\,w+1}+1+w\right)^{-b}}{2^{-a-b}\sqrt{w^2-2\,z\,w+1}}\,/;\;-1<z<1\land|w|<1$$

05.06.23.0003.01

$$\sum_{k=0}^{\infty}\frac{(a)_k\,(-a+\alpha+\beta+1)_k}{(\alpha+1)_k\,(\beta+1)_k}\,P_k^{(\alpha,\beta)}(z)\,t^k = {}_2F_1\!\left(a,\,-a+\alpha+\beta+1;\,\alpha+1;\,\frac{1}{2}\left(-t-\sqrt{t^2-2\,z\,t+1}+1\right)\right){}_2F_1\!\left(a,\,-a+\alpha+\beta+1;\,\beta+1;\,\frac{1}{2}\left(t-\sqrt{t^2-2\,z\,t+1}+1\right)\right)$$

05.06.23.0002.01

$$\sum_{n=0}^{\infty}\frac{n!\,(a+b+2\,n+1)\,\Gamma(a+b+n+1)}{\Gamma(a+n+1)\,\Gamma(b+n+1)}\,P_n^{(a,b)}(x)\,P_n^{(a,b)}(y) = 2^{a+b+1}\,(1-x)^{-\frac{a}{2}}\,(x+1)^{-\frac{b}{2}}\,(1-y)^{-\frac{a}{2}}\,(y+1)^{-\frac{b}{2}}\,\delta(x-y)\,/;\;-1<x<1\bigwedge-1<y<1\bigwedge\mathrm{Re}(a)>-1\bigwedge\mathrm{Re}(b)>-1$$

Operations

Limit operation

05.06.25.0001.01

$$\lim_{n\to\infty}\frac{1}{n^a}\left(\frac{2}{z}\right)^{-a}P_n^{(a,b)}\!\left(\cos\!\left(\frac{z}{n}\right)\right) = J_a(z)$$

05.06.25.0002.01

$$\lim_{a\to\infty}a^{-\frac{n}{2}}\,P_n^{(a,a)}\!\left(\frac{z}{\sqrt{a}}\right) = \frac{H_n(z)}{2^n\,\Gamma(n+1)}$$

05.06.25.0003.01

$$\lim_{b\to\infty}P_n^{(0,b)}\!\left(1-\frac{2\,z}{b}\right) = L_n(z)$$

05.06.25.0004.01

$$\lim_{b\to\infty}P_n^{(a,b)}\!\left(1-\frac{2\,z}{b}\right) = L_n^a(z)$$

Orthogonality, completeness, and Fourier expansions

The set of functions $P_n^{(a,b)}(x)$, $n = 0, 1, \ldots$, forms a complete, orthogonal (with weight $\frac{n!\,(a+b+2\,n+1)\,\Gamma(a+b+n+1)}{2^{a+b+1}\,\Gamma(a+n+1)\,\Gamma(b+n+1)}\,(1-x)^a\,(1+x)^b$) system on the interval $(-1, 1)$.

05.06.25.0005.01

$$\sum_{n=0}^{\infty}\left(\sqrt{\frac{n!\,(a+b+2\,n+1)\,\Gamma(a+b+n+1)}{2^{a+b+1}\,\Gamma(a+n+1)\,\Gamma(b+n+1)}}\,(1-x)^{a/2}\,(1+x)^{b/2}\,P_n^{(a,b)}(x)\right)\left(\sqrt{\frac{n!\,(a+b+2\,n+1)\,\Gamma(a+b+n+1)}{2^{a+b+1}\,\Gamma(a+n+1)\,\Gamma(b+n+1)}}\,(1-y)^{a/2}\,(1+y)^{b/2}\,P_n^{(a,b)}(y)\right) = \delta(x-y)\,/;\,-1<x<1\bigwedge -1<y<1\bigwedge \mathrm{Re}(a)>-1\bigwedge \mathrm{Re}(b)>-1$$

05.06.25.0006.01

$$\int_{-1}^{1}\left(\sqrt{\frac{m!\,(a+b+2\,m+1)\,\Gamma(a+b+m+1)}{2^{a+b+1}\,\Gamma(a+m+1)\,\Gamma(b+m+1)}}\,(1-t)^{a/2}\,(1+t)^{b/2}\,P_m^{(a,b)}(t)\right)\left(\sqrt{\frac{n!\,(a+b+2\,n+1)\,\Gamma(a+b+n+1)}{2^{a+b+1}\,\Gamma(a+n+1)\,\Gamma(b+n+1)}}\,(1-t)^{a/2}\,(1+t)^{b/2}\,P_n^{(a,b)}(t)\right)dt = \delta_{m,n}\,/;\,\mathrm{Re}(a)>-1\bigwedge \mathrm{Re}(b)>-1$$

Any sufficiently smooth function $f(x)$ can be expanded in the system $\left\{P_n^{(a,b)}(x)\right\}_{n=0,1,\ldots}$ as a generalized Fourier series, with its sum converging to $f(x)$ almost everywhere.

05.06.25.0007.01

$$f(x) = \sum_{n=0}^{\infty} c_n\,\psi_n(x)\,/;$$

$$c_n = \int_{-1}^{1}\psi_n(t)\,f(t)\,dt \bigwedge \psi_n(x) = \sqrt{\frac{n!\,(a+b+2\,n+1)\,\Gamma(a+b+n+1)}{2^{a+b+1}\,\Gamma(a+n+1)\,\Gamma(b+n+1)}}\,(1-x)^{a/2}\,(1+x)^{b/2}\,P_n^{(a,b)}(x)\bigwedge -1<x<1$$

Representations through more general functions

Through hypergeometric functions

Involving ${}_2\tilde{F}_1$

05.06.26.0001.01

$$P_n^{(a,b)}(z) = \frac{\Gamma(a+n+1)}{\Gamma(n+1)}\,{}_2\tilde{F}_1\!\left(-n,\,a+b+n+1;\,a+1;\,\frac{1-z}{2}\right)$$

Involving ${}_2F_1$

05.06.26.0002.01

$$P_n^{(a,b)}(z) = \frac{\Gamma(a+n+1)}{\Gamma(n+1)\,\Gamma(a+1)}\,{}_2F_1\!\left(-n,\,a+b+n+1;\,a+1;\,\frac{1-z}{2}\right)/;\,-a\notin\mathbb{N}^+$$

05.06.26.0003.01

$$P_n^{(a,b)}(z) = \frac{\Gamma(-b)}{\Gamma(n+1)\,\Gamma(-b-n)}\,{}_2F_1\!\left(-n,\,a+b+n+1;\,b+1;\,\frac{z+1}{2}\right)/;\,b \notin \mathbb{Z}$$

05.06.26.0004.01

$$P_n^{(a,b)}(z) = \frac{2^{-n}\,(a+b+n+1)_n}{\Gamma(n+1)}\,(z-1)^n\,{}_2F_1\!\left(-n,\,-a-n;\,-a-b-2n;\,\frac{2}{1-z}\right)/;\,a+b+2n \notin \mathbb{Z}$$

Through hypergeometric functions of two variables

05.06.26.0005.01

$$P_n^{(a,b)}(z) = \frac{(a+1)_n}{\Gamma(n+1)}\,\tilde{F}_{1\times 0\times 0}^{2\times 0\times 0}\!\left(\begin{matrix}-n,\,a+b+n+1;;; \\ a+1;;;\end{matrix}\;-\frac{z}{2},\,\frac{1}{2}\right)$$

Through Meijer *G*

Classical cases for the direct function itself

05.06.26.0006.01

$$P_n^{(a,b)}(z) = -\frac{1}{\pi}\lim_{m\to n}\frac{\sin(\pi m)\,\Gamma(a+m+1)}{\Gamma(a+b+m+1)}\,G_{2,2}^{1,2}\!\left(\frac{z-1}{2}\,\middle|\,\begin{matrix}m+1,\,-a-b-m \\ 0,\,-a\end{matrix}\right)/;\,n \in \mathbb{Z}$$

Classical cases involving algebraic functions

05.06.26.0007.01

$$(z+1)^b\,P_n^{(a,b)}(2z+1) = \frac{1}{\Gamma(n+1)\,\Gamma(-b-n)}\,G_{2,2}^{1,2}\!\left(z\,\middle|\,\begin{matrix}-a-n,\,b+n+1 \\ 0,\,-a\end{matrix}\right)$$

05.06.26.0008.01

$$(z+1)^b\,P_n^{(a,b)}\!\left(1+\frac{2}{z}\right) = \frac{1}{\Gamma(n+1)\,\Gamma(-b-n)}\,G_{2,2}^{2,1}\!\left(z\,\middle|\,\begin{matrix}b+1,\,a+b+1 \\ a+b+n+1,\,-n\end{matrix}\right)/;\,z \notin (-1,\,0)$$

05.06.26.0009.01

$$(z+1)^{-a-b-n-1}\,P_n^{(a,b)}\!\left(\frac{1-z}{1+z}\right) = \frac{1}{\Gamma(n+1)\,\Gamma(a+b+n+1)}\,G_{2,2}^{1,2}\!\left(z\,\middle|\,\begin{matrix}-a-b-n,\,-a-n \\ 0,\,-a\end{matrix}\right)/;\,z \notin (-\infty,\,-1)$$

05.06.26.0010.01

$$(z+1)^{-a-b-n-1}\,P_n^{(a,b)}\!\left(\frac{z-1}{z+1}\right) = \frac{1}{\Gamma(n+1)\,\Gamma(a+b+n+1)}\,G_{2,2}^{2,1}\!\left(z\,\middle|\,\begin{matrix}-a-b-n,\,-b-n \\ 0,\,-b\end{matrix}\right)/;\,z \notin (-1,\,0)$$

Classical cases involving unit step θ

05.06.26.0011.01

$$\theta(1-|z|)\,(1-z)^a\,P_n^{(a,b)}(2z-1) = \frac{\Gamma(a+n+1)}{\Gamma(n+1)}\,G_{2,2}^{2,0}\!\left(z\,\middle|\,\begin{matrix}a+n+1,\,-b-n \\ 0,\,-b\end{matrix}\right)/;\,z \notin (-1,\,0)$$

05.06.26.0012.01

$$\theta(|z|-1)\,(z-1)^a\,P_n^{(a,b)}(2z-1) = \frac{\Gamma(a+n+1)}{\Gamma(n+1)}\,G_{2,2}^{0,2}\!\left(z\,\middle|\,\begin{matrix}-b-n,\,a+n+1 \\ 0,\,-b\end{matrix}\right)$$

05.06.26.0013.01

$$\theta(1-|z|)\,(1-z)^a\,P_n^{(a,b)}\!\left(\frac{2}{z}-1\right) = \frac{\Gamma(a+n+1)}{\Gamma(n+1)}\,G_{2,2}^{2,0}\!\left(z\,\middle|\,\begin{matrix}a+1,\,a+b+1 \\ -n,\,a+b+n+1\end{matrix}\right)$$

05.06.26.0014.01

$$\theta(|z|-1)\,(z-1)^b\,P_n^{(b,b)}\left(\frac{2}{z}-1\right)=\frac{\Gamma(b+n+1)}{\Gamma(n+1)}\,G_{2,2}^{0,2}\left(z\,\middle|\,\begin{array}{c}b+1,\,2b+1\\-n,\,2b+n+1\end{array}\right)/;\,z\notin(-\infty,\,-1)$$

05.06.26.0015.01

$$\theta(1-|z|)\,(1-z)^{-a-b-n-1}\,P_n^{(a,b)}\left(\frac{1+z}{1-z}\right)=\frac{(-1)^n\,\Gamma(-a-b-n)}{n!}\,G_{2,2}^{2,0}\left(z\,\middle|\,\begin{array}{c}-a-n,\,-a-b-n\\0,\,-a\end{array}\right)$$

05.06.26.0016.01

$$\theta(|z|-1)\,(z-1)^{-a-b-n-1}\,P_n^{(a,b)}\left(\frac{z+1}{z-1}\right)=\frac{(-1)^n\,\Gamma(-a-b-n)}{n!}\,G_{2,2}^{0,2}\left(z\,\middle|\,\begin{array}{c}-b-n,\,-a-b-n\\0,\,-b\end{array}\right)$$

Theorems

Expansions in generalized Fourier series

$$f(x)=\sum_{k=0}^{\infty}c_k\,\psi_k(x)\,/;\quad c_k=\int_{-1}^{1}f(t)\,\psi_k(t)\,dt,$$

$$\psi_k(x)=\sqrt{\frac{n!\,(a+b+2n+1)\,\Gamma(a+b+n+1)}{2^{a+b+1}\,\Gamma(a+n+1)\,\Gamma(b+n+1)}}\,(1-x)^{a/2}\,(1+x)^{b/2}\,P_k^{(a,b)}(x),\;\;k\in\mathbb{N}.$$

The quantum mechanical representation matrices of angular momentum

The quantum mechanical representation matrices $D_{m\,m'}^{L}(\alpha,\,\beta,\,\gamma)$ of angular momentum L are given by

$$D_{m\,m'}^{L}(\alpha,\,\beta,\,\gamma)=e^{i\,(m\,\alpha+m'\,\gamma)}\sqrt{\frac{(L+m')!\,(L-m')!}{(L+m)!\,(L-m)!}}\left(\cos\left(\frac{\beta}{2}\right)\right)^{m+m'}\left(\sin\left(\frac{\beta}{2}\right)\right)^{m-m'}P_{L-m'}^{(m-m',m+m')}(\cos(\beta))$$

where α, β, γ are the Euler angles and $L,\, m,\, m'\in\mathbb{N},\, -L\le m,\, m'\le L$.

The expected value of the number of real eigenvalues of a one matrix

The expected value r_n of the number of real eigenvalues of a $n\times n$ matrix whose matrix elements are random variables with Gaussian distribution (mean = 0, variance = 1) is

$$r_n=\left(1-(-1)^n\right)/2+\sqrt{2}\,P_{n-2}^{(1-n,3/2)}(3).$$

The equilibrium positions of *n* unit charges

The equilibrium positions of n unit charges, a charge q at -1, and a charge p at +1 interacting with potential $-\log(x)$ are the zeros of $P_n^{(2p-1,2q-1)}(x)$.

History

– C. J. Jacobi (1859)
– P. L. Chebyshev (1870).

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