JacobiSN

Notations

Traditional name

Jacobi elliptic function sn

Traditional notation

$\operatorname{sn}(z \mid m)$

Mathematica StandardForm notation

`JacobiSN[z, m]`

Primary definition

09.36.02.0001.01

$$\operatorname{sn}(z \mid m) = \sin(\operatorname{am}(z \mid m))$$

Specific values

Specialized values

For fixed *z*

Case *m* = 0

09.36.03.0001.01

$$\operatorname{sn}(z \mid 0) = \sin(z)$$

09.36.03.0002.01

$$\operatorname{sn}\left(z + \frac{\pi}{2} \,\middle|\, 0\right) = \cos(z)$$

09.36.03.0026.01

$$\operatorname{sn}\left(z + \frac{\pi k}{2} \,\middle|\, 0\right) = \sin\left(z + \frac{\pi k}{2}\right) /; k \in \mathbb{Z}$$

Case *m* = 1

09.36.03.0003.01

$$\operatorname{sn}(z \mid 1) = \tanh(z)$$

09.36.03.0004.01

$$\operatorname{sn}\left(z+\frac{\pi i}{2} \,\middle|\, 1\right) = \coth(z)$$

09.36.03.0027.01

$$\operatorname{sn}\left(z+\frac{i \pi k}{2} \,\middle|\, 1\right) = \tanh\left(z+\frac{i \pi k}{2}\right) /; k \in \mathbb{Z}$$

For fixed *m*

Values at quarter-period points in the fundamental period parallelogram

09.36.03.0005.01

$$\operatorname{sn}(0 \mid m) = 0$$

09.36.03.0006.01

$$\operatorname{sn}(K(m) \mid m) = 1$$

09.36.03.0007.01

$$\operatorname{sn}(2 K(m) \mid m) = 0$$

09.36.03.0008.01

$$\operatorname{sn}(3 K(m) \mid m) = -1$$

09.36.03.0009.01

$$\operatorname{sn}(4 K(m) \mid m) = 0$$

09.36.03.0010.01

$$\operatorname{sn}(i K(1-m) \mid m) = \tilde{\infty}$$

09.36.03.0011.01

$$\operatorname{sn}(2 i K(1-m) \mid m) = 0$$

09.36.03.0012.01

$$\operatorname{sn}(3 i K(1-m) \mid m) = \tilde{\infty}$$

09.36.03.0013.01

$$\operatorname{sn}(4 i K(1-m) \mid m) = 0$$

09.36.03.0014.01

$$\operatorname{sn}(K(m) + i K(1-m) \mid m) = \frac{1}{\sqrt{m}}$$

09.36.03.0015.01

$$\operatorname{sn}(2 K(m) + i K(1-m) \mid m) = \tilde{\infty}$$

09.36.03.0016.01

$$\operatorname{sn}(3 K(m) + i K(1-m) \mid m) = -\frac{1}{\sqrt{m}}$$

09.36.03.0017.01

$$\operatorname{sn}(4 K(m) + i K(1-m) \mid m) = \tilde{\infty}$$

09.36.03.0018.01

$$\operatorname{sn}(2 r K(m) + (2 s+1) i K(1-m) \mid m) = \tilde{\infty} /; \{r, s\} \in \mathbb{Z}$$

09.36.03.0019.01

$$\operatorname{sn}(K(m) + 2 i K(1-m) \mid m) = 1$$

09.36.03.0020.01

$$\operatorname{sn}(2K(m) + 2 i K(1-m) \mid m) == 0$$

09.36.03.0021.01

$$\operatorname{sn}(3K(m) + 2 i K(1-m) \mid m) == -1$$

09.36.03.0022.01

$$\operatorname{sn}(4K(m) + 2 i K(1-m) \mid m) == 0$$

Values at half-quarter-period points

09.36.03.0023.01

$$\operatorname{sn}\left(\frac{K(m)}{2} \middle| m\right) == \frac{1}{\sqrt{1+\sqrt{1-m}}}$$

09.36.03.0024.01

$$\operatorname{sn}\left(\frac{i K(1-m)}{2} \middle| m\right) == \frac{i}{\sqrt[4]{m}}$$

09.36.03.0025.01

$$\operatorname{sn}\left(\frac{K(m)}{2} + \frac{i K(1-m)}{2} \middle| m\right) == \frac{1}{\sqrt{2}\sqrt[4]{m}}\left(\sqrt{1+\sqrt{m}} + i\sqrt{1-\sqrt{m}}\right)$$

General characteristics

Domain and analyticity

$\operatorname{sn}(z \mid m)$ is a meromorphic function of z which is defined over $\mathbb{C}^2$.

09.36.04.0001.01

$$(z * m) \longrightarrow \operatorname{sn}(z \mid m) :: (\mathbb{C} \otimes \mathbb{C}) \longrightarrow \mathbb{C}$$

Symmetries and periodicities

Parity

$\operatorname{sn}(z \mid m)$ is an odd function with respect to z.

09.36.04.0002.01

$$\operatorname{sn}(-z \mid m) == -\operatorname{sn}(z \mid m)$$

Mirror symmetry

09.36.04.0003.01

$$\operatorname{sn}(\bar{z} \mid \bar{m}) == \overline{\operatorname{sn}(z \mid m)}$$

Periodicity

$\operatorname{sn}(z \mid m)$ is a doubly periodic function with respect to z with periods $2 i K(1-m)$ and $4K(m)$.

09.36.04.0004.01

$$\operatorname{sn}(z + 2K(m) \mid m) == -\operatorname{sn}(z \mid m)$$

09.36.04.0005.01

$\mathrm{sn}(z + 4K(m) \mid m) = \mathrm{sn}(z \mid m)$

09.36.04.0006.01

$\mathrm{sn}(z + 2iK(1-m) \mid m) = \mathrm{sn}(z \mid m)$

09.36.04.0007.01

$\mathrm{sn}(z + 2iK(1-m) + 2K(m) \mid m) = -\mathrm{sn}(z \mid m)$

09.36.04.0008.01

$\mathrm{sn}(z + 2isK(1-m) + 2rK(m) \mid m) = (-1)^r \mathrm{sn}(z \mid m) \;/;\; \{r, s\} \in \mathbb{Z}$

Poles and essential singularities

With respect to *z*

For fixed m, the function sn$(z \mid m)$ has an infinite set of singular points:

a) $z = 2rK(m) + (2s+1)iK(1-m)$, $\{r, s\} \in \mathbb{Z}$, are the simple poles with residues $\frac{(-1)^r}{\sqrt{m}}$;

b) $z = \tilde{\infty}$ is an essential singular point.

09.36.04.0009.01

$\mathcal{Sing}_z(\mathrm{sn}(z \mid m)) = \{\{\{(2s+1)iK(1-m) + 2rK(m), 1\} \;/;\; \{r, s\} \in \mathbb{Z}\}, \{\tilde{\infty}, \infty\}\}$

09.36.04.0010.01

$$\mathrm{res}_z(\mathrm{sn}(z \mid m))((2s+1)iK(1-m) + 2rK(m)) = \frac{(-1)^r}{\sqrt{m}} \;/;\; \{r, s\} \in \mathbb{Z}$$

Branch points

With respect to *m*

For fixed z, the function sn$(z \mid m)$ is a meromorphic function in m that has no branch points.

09.36.04.0013.01

$\mathcal{BP}_m(\mathrm{sn}(z \mid m)) = \{\}$

P. Walker

With respect to *z*

For fixed m, the function sn$(z \mid m)$ does not have branch points.

09.36.04.0011.01

$\mathcal{BP}_z(\mathrm{sn}(z \mid m)) = \{\}$

Branch cuts

With respect to *m*

For fixed z, the function sn$(z \mid m)$ is a meromorphic function in m that has no branch cuts.

09.36.04.0014.01

$\mathcal{BC}_m(\mathrm{sn}(z \mid m)) = \{\}$

P. Walker

With respect to z

For fixed m, the function sn$(z \mid m)$ does not have branch cuts.

09.36.04.0012.01

$$\mathcal{BC}_z(\mathrm{sn}(z \mid m)) = \{\}$$

Series representations

Generalized power series

Expansions at $z = 0$

09.36.06.0006.01

$$\mathrm{sn}(z \mid m) \propto z - \frac{1+m}{6} z^3 + \frac{(1 + 14\, m + m^2)\, z^5}{120} + \ldots \;/;\; (z \to 0)$$

09.36.06.0001.02

$$\mathrm{sn}(z \mid m) \propto z - \frac{1+m}{6} z^3 + \frac{(1 + 14\, m + m^2)\, z^5}{120} - \frac{(1 + 135\, m + 135\, m^2 + m^3)\, z^7}{5040} + \frac{(1 + 1228\, m + 5478\, m^2 + 1228\, m^3 + m^4)\, z^9}{362\,880} + O(z^{11})$$

09.36.06.0007.01

$$\mathrm{sn}(z \mid m) = \sum_{k=0}^{\infty} \frac{(-1)^k\, \mathrm{sn}_k(m)\, z^{2k+1}}{(2k+1)!} \;/;\; \mathrm{sn}_0(m) = 1 \bigwedge \mathrm{sn}_n(m) = \sum_{j=0}^{n} \sum_{k=0}^{n} \binom{2n}{2j} \mathrm{cn}_j(m)\, \mathrm{dn}_k(m)\, \delta_{j+k-n} \bigwedge \mathrm{cn}_0(m) = 1 \bigwedge$$
$$\mathrm{cn}_n(m) = \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} \binom{2n-1}{2j+1} \mathrm{sn}_j(m)\, \mathrm{dn}_k(m)\, \delta_{j+k-n+1} \bigwedge \mathrm{dn}_0(m) = 1 \bigwedge \mathrm{dn}_n(m) = m \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} \binom{2n-1}{2j+1} \mathrm{sn}_j(m)\, \mathrm{cn}_k(m)\, \delta_{j+k-n+1}$$

09.36.06.0008.01

$$\mathrm{sn}(z \mid m) \propto z\, (1 + O(z^2))$$

Expansions at $z = 2\, r\, K(m) + (2\, s + 1)\, i\, K(1 - m)$

09.36.06.0009.01

$$\mathrm{sn}(z \mid m) \propto \frac{(-1)^r}{\sqrt{m}} \left(\frac{1}{z - z_0} + \frac{1}{6} (m+1)(z - z_0) + \frac{1}{360} (7\, m^2 - 22\, m + 7)(z - z_0)^3 + \ldots \right) /;$$
$$(z \to z_0) \bigwedge z_0 = 2\, r\, K(m) + (2\, s + 1)\, i\, K(1 - m) \bigwedge r \in \mathbb{Z} \bigwedge s \in \mathbb{Z}$$

09.36.06.0010.01

$$\mathrm{sn}(z \mid m) = \frac{(-1)^r}{\sqrt{m}} \sum_{k=0}^{\infty} (k+1) \sum_{r=0}^{k} \frac{(-1)^r}{r+1} \binom{k}{r} p_{r,k}\, (z - z_0)^{2k-1} \;/;$$
$$z_0 = 2\, r\, K(m) + (2\, s + 1)\, i\, K(1 - m) \bigwedge r \in \mathbb{Z} \bigwedge s \in \mathbb{Z} \bigwedge p_{j,0} = 1 \bigwedge p_{j,k} = \frac{1}{k} \sum_{i=1}^{k} \frac{(j\, i + i - k)\, (-1)^i\, \mathrm{sn}_i(m)\, p_{j,k-i}}{(2\, i + 1)!} \bigwedge$$
$$k \in \mathbb{N}^+ \bigwedge \mathrm{sn}_0(m) = 1 \bigwedge \mathrm{sn}_n(m) = \sum_{j=0}^{n} \sum_{k=0}^{n} \binom{2n}{2j} \mathrm{cn}_j(m)\, \mathrm{dn}_k(m)\, \delta_{j+k-n} \bigwedge \mathrm{cn}_0(m) = 1 \bigwedge$$
$$\mathrm{cn}_n(m) = \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} \binom{2n-1}{2j+1} \mathrm{sn}_j(m)\, \mathrm{dn}_k(m)\, \delta_{j+k-n+1} \bigwedge \mathrm{dn}_0(m) = 1 \bigwedge \mathrm{dn}_n(m) = m \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} \binom{2n-1}{2j+1} \mathrm{sn}_j(m)\, \mathrm{cn}_k(m)\, \delta_{j+k-n+1}$$

09.36.06.0011.01

$$\operatorname{sn}(z \mid m) \propto \frac{(-1)^r}{\sqrt{m}\,(z-z_0)}\left(1+O\left((z-z_0)^2\right)\right) /;\ z_0 = 2\,r\,K(m) + (2\,s+1)\,i\,K(1-m) \bigwedge r \in \mathbb{Z} \bigwedge s \in \mathbb{Z}$$

Expansions at $m == 0$

09.36.06.0012.01

$$\operatorname{sn}(z \mid m) \propto \sin(z) + \frac{1}{8}\cos(z)\,(\sin(2\,z) - 2\,z)\,m + \frac{1}{256}\left(-24\,z\cos(z) - 12\,z\cos(3\,z) + 2\left(-4\,z^2 + 9\cos(2\,z) + \cos(4\,z) + 8\right)\sin(z)\right)m^2 + \ldots /;\ (m \to 0)$$

09.36.06.0013.01

$$\begin{aligned}
\operatorname{sn}(z \mid m) \propto\ & \sin(z) + \frac{1}{8}\cos(z)\,(\sin(2\,z) - 2\,z)\,m + \\
& \frac{1}{256}\left(-24\,z\cos(z) - 12\,z\cos(3\,z) + 2\left(-4\,z^2 + 9\cos(2\,z) + \cos(4\,z) + 8\right)\sin(z)\right)m^2 + \frac{1}{12\,288} \\
& \left(32\,z\left(z^2 - 21\right)\cos(z) - 468\,z\cos(3\,z) - 60\,z\cos(5\,z) - 3\left(88\,z^2 - 67\right)\sin(z) - 6\left(36\,z^2 - 41\right)\sin(3\,z) + 48\sin(5\,z) + 3\sin(7\,z)\right) \\
& m^3 + \frac{1}{196\,608}\left(4\,z\left(128\,z^2 - 1845\right)\cos(z) + 72\,z\left(12\,z^2 - 83\right)\cos(3\,z) - 1260\,z\cos(5\,z) - 84\,z\cos(7\,z) + \right. \\
& \left.\left(32\,z^4 - 3120\,z^2 + 2214\right)\sin(z) - 81\left(48\,z^2 - 35\right)\sin(3\,z) + \left(690 - 600\,z^2\right)\sin(5\,z) + 72\sin(7\,z) + 3\sin(9\,z)\right)m^4 + \\
& \frac{1}{15\,728\,640}\left(-8\,z\left(16\,z^4 - 4580\,z^2 + 55\,245\right)\cos(z) + 1440\,z\left(69\,z^2 - 272\right)\cos(3\,z) + 200\,z\left(100\,z^2 - 531\right)\cos(5\,z) - \right. \\
& 12\,180\,z\cos(7\,z) - 540\,z\cos(9\,z) + 30\left(544\,z^4 - 392\cos(6\,z)\,z^2 - 19\,208\,z^2 + 8\left(108\,z^4 - 3174\,z^2 + 1925\right)\cos(2\,z) + \right. \\
& \left.\left.\left(3739 - 5592\,z^2\right)\cos(4\,z) + 475\cos(6\,z) + 33\cos(8\,z) + \cos(10\,z) + 12\,104\right)\sin(z)\right)m^5 + \\
& \frac{1}{754\,974\,720}\left(1\,860\,000\cos(5\,z)\,z^3 - 12\left(832\,z^4 - 129\,240\,z^2 + 1\,397\,055\right)\cos(z)\,z - \right. \\
& 648\left(144\,z^4 - 8180\,z^2 + 24\,385\right)\cos(3\,z)\,z - 5\,028\,300\cos(5\,z)\,z + 840\left(196\,z^2 - 909\right)\cos(7\,z)\,z - \\
& 59\,940\cos(9\,z)\,z - 1980\cos(11\,z)\,z - 8\left(32\,z^6 - 21\,180\,z^4 + 955\,575\,z^2 - 622\,845\right)\sin(z) + \\
& 270\left(4032\,z^4 - 48\,564\,z^2 + 25\,081\right)\sin(3\,z) + 15\left(20\,000\,z^4 - 301\,800\,z^2 + 139\,557\right)\sin(5\,z) - \\
& \left.720\left(833\,z^2 - 465\right)\sin(7\,z) - 90\left(324\,z^2 - 359\right)\sin(9\,z) + 1800\sin(11\,z) + 45\sin(13\,z)\right)m^6 + \frac{1}{84\,557\,168\,640} \\
& \left(4\,z\left(256\,z^6 - 337\,344\,z^4 + 38\,446\,800\,z^2 - 384\,987\,645\right)\cos(z) - 756\,z\left(28\,512\,z^4 - 801\,960\,z^2 + 2\,009\,245\right)\cos(3\,z) - \right. \\
& 2100\,z\left(4000\,z^4 - 135\,400\,z^2 + 257\,937\right)\cos(5\,z) + 8820\,z\left(5096\,z^2 - 11\,175\right)\cos(7\,z) + 22\,680\,z\left(108\,z^2 - 461\right)\cos(9\,z) - \\
& 623\,700\,z\cos(11\,z) - 16\,380\,z\cos(13\,z) - 14\left(3968\,z^6 - 1\,331\,760\,z^4 + 51\,350\,220\,z^2 - 32\,480\,415\right)\sin(z) - \\
& 63\left(20\,736\,z^6 - 2\,453\,760\,z^4 + 21\,261\,240\,z^2 - 10\,022\,255\right)\sin(3\,z) + \\
& 315\left(240\,000\,z^4 - 1\,750\,000\,z^2 + 668\,241\right)\sin(5\,z) + 735\left(10\,976\,z^4 - 133\,560\,z^2 + 51\,573\right)\sin(7\,z) - \\
& \left.5040\left(1701\,z^2 - 874\right)\sin(9\,z) - 630\left(484\,z^2 - 529\right)\sin(11\,z) + 15\,120\sin(13\,z) + 315\sin(15\,z)\right)m^7 + \\
& \frac{1}{1\,352\,914\,698\,240}\left(12\,z\left(3072\,z^6 - 1\,950\,368\,z^4 + 183\,014\,720\,z^2 - 1\,730\,146\,845\right)\cos(z) + \right. \\
& 648\,z\left(3456\,z^6 - 758\,016\,z^4 + 14\,749\,280\,z^2 - 32\,740\,365\right)\cos(3\,z) - \\
& 21\,000\,z\left(16\,400\,z^4 - 257\,560\,z^2 + 392\,841\right)\cos(5\,z) - 588\,z\left(76\,832\,z^4 - 1\,977\,640\,z^2 + 2\,906\,355\right)\cos(7\,z) + \\
& 11\,340\,z\left(10\,152\,z^2 - 19\,451\right)\cos(9\,z) + 9240\,z\left(484\,z^2 - 1953\right)\cos(11\,z) - 868\,140\,z\cos(13\,z) - \\
& 18\,900\,z\cos(15\,z) + \left(512\,z^8 - 1\,211\,392\,z^6 + 285\,670\,560\,z^4 - 9\,875\,050\,920\,z^2 + 6\,089\,965\,245\right)\sin(z) - \\
& 63\left(787\,968\,z^6 - 44\,102\,880\,z^4 + 310\,448\,520\,z^2 - 136\,673\,455\right)\sin(3\,z) - \\
& 70\left(400\,000\,z^6 - 26\,280\,000\,z^4 + 130\,369\,500\,z^2 - 43\,550\,109\right)\sin(5\,z) +
\end{aligned}$$

$$5880\left(60368 z^4-334950 z^2+102195\right)\sin(7 z)+315\left(69984 z^4-745848 z^2+253307\right)\sin(9 z)-$$
$$55440\left(275 z^2-133\right)\sin(11 z)-630\left(676 z^2-731\right)\sin(13 z)+17640\sin(15 z)+315\sin(17 z)\Big)\, m^8+$$
$$\frac{1}{194819716546560}\Big(-8 z\left(256 z^8-1008000 z^6+434436912 z^4-35584337880 z^2+321447804615\right)\cos(z)+$$
$$3888 z\left(222912 z^6-22302000 z^4+342903750 z^2-695415875\right)\cos(3 z)+$$
$$900 z\left(800000 z^6-92400000 z^4+959477400 z^2-1251408501\right)\cos(5 z)-$$
$$5292 z\left(3764768 z^4-42904400 z^2+48678885\right)\cos(7 z)-20412 z\left(69984 z^4-1517400 z^2+1876715\right)\cos(9 z)+$$
$$41580 z\left(53240 z^2-92781\right)\cos(11 z)+98280 z\left(676 z^2-2619\right)\cos(13 z)-10376100 z\cos(15 z)-$$
$$192780 z\cos(17 z)+18\left(10496 z^8-11495680 z^6+2165834160 z^4-69005718180 z^2+41633079075\right)\sin(z)+$$
$$162\left(186624 z^8-69745536 z^6+2616757920 z^4-15922895940 z^2+6643081375\right)\sin(3 z)-$$
$$630\left(18400000 z^6-542880000 z^4+2102352300 z^2-635484717\right)\sin(5 z)-$$
$$126\left(15059072 z^6-719147520 z^4+2605542660 z^2-673078005\right)\sin(7 z)+$$
$$45360\left(227448 z^4-1059399 z^2+274924\right)\sin(9 z)+945\left(468512 z^4-4562184 z^2+1409247\right)\sin(11 z)-$$
$$45360\left(4901 z^2-2264\right)\sin(13 z)-28350\left(180 z^2-193\right)\sin(15 z)+181440\sin(17 z)+2835\sin(19 z)\Big)\, m^9+$$
$$\frac{1}{15585577323724800}\Big(-160 z\left(2944 z^8-5196384 z^6+1745402148 z^4-129145913190 z^2+1124016300855\right)\cos(z)-$$
$$131220 z\left(1536 z^8-917760 z^6+59976224 z^4-780845800 z^2+1475445055\right)\cos(3 z)+$$
$$540000 z\left(340000 z^6-17170300 z^4+136270295 z^2-158031489\right)\cos(5 z)+$$
$$17640 z\left(2151296 z^6-174254976 z^4+1268918700 z^2-1197550395\right)\cos(7 z)-$$
$$918540 z\left(443232 z^4-4107600 z^2+3812645\right)\cos(9 z)-$$
$$41580 z\left(468512 z^4-9026600 z^2+9864255\right)\cos(11 z)+737100 z\left(28392 z^2-46115\right)\cos(13 z)+$$
$$1701000 z\left(300 z^2-1127\right)\cos(15 z)-66509100 z\cos(17 z)-1077300 z\cos(19 z)+$$
$$\left(-4096 z^{10}+25320960 z^8-18304957440 z^6+2944262714400 z^4-88003507762800 z^2+52079055504525\right)$$
$$\sin(z)+1215\left(5971968 z^8-988533504 z^6+28639981440 z^4-156265593960 z^2+62399823955\right)\sin(3 z)+$$
$$225\left(40000000 z^8-7554400000 z^6+143925600000 z^4-467583076800 z^2+130698250767\right)\sin(5 z)-$$
$$945\left(542126592 z^6-11186739200 z^4+30686626320 z^2-6994157055\right)\sin(7 z)-$$
$$17010\left(2519424 z^6-98210880 z^4+289458900 z^2-61983655\right)\sin(9 z)+$$
$$113400\left(1171280 z^4-4826206 z^2+1109661\right)\sin(11 z)+$$
$$4725\left(913952 z^4-8335080 z^2+2394237\right)\sin(13 z)-680400\left(2475 z^2-1103\right)\sin(15 z)-$$
$$28350\left(1156 z^2-1231\right)\sin(17 z)+1020600\sin(19 z)+14175\sin(21 z)\Big)\, m^{10}+O\left(m^{11}\right)$$

09.36.06.0014.01

$$\operatorname{sn}(z\mid m)\propto\sin(z)\,(1+O(m))$$

Expansions at $m == 1$

09.36.06.0015.01

$$\operatorname{sn}(z\mid m)\propto\tanh(z)+\frac{1}{4}\left(z\operatorname{sech}^2(z)-\tanh(z)\right)(m-1)-$$
$$\frac{1}{512}\left(72 z\cosh(z)+4\left(8 z^2-5\right)\sinh(z)-19\sinh(3 z)+\sinh(5 z)\right)\operatorname{sech}^3(z)\,(m-1)^2+\ldots\;/;\;(m\to 1)$$

09.36.06.0016.01

$$\operatorname{sn}(z\mid m)\propto\tanh(z)+\frac{1}{4}\left(z\operatorname{sech}^2(z)-\tanh(z)\right)(m-1)-$$
$$\frac{1}{512}\left(72 z\cosh(z)+4\left(8 z^2-5\right)\sinh(z)-19\sinh(3 z)+\sinh(5 z)\right)\operatorname{sech}^3(z)\,(m-1)^2+$$

$$\frac{1}{12\,288}\Big(-16\,z\,(8\,z^2-39)+2\,z\,(32\,z^2+303)\cosh(2\,z)-24\,z\cosh(4\,z)-$$
$$6\,z\cosh(6\,z)+3\,(112\,z^2-115)\sinh(2\,z)-150\sinh(4\,z)+15\sinh(6\,z)\Big)\operatorname{sech}^4(z)\,(m-1)^3\;+$$
$$\frac{1}{1\,572\,864}\Big(96\,z\,(152\,z^2-975)\cosh(z)-16\,z\,(304\,z^2+1725)\cosh(3\,z)+3000\,z\cosh(5\,z)+600\,z\cosh(7\,z)+$$
$$(5632\,z^4-19\,392\,z^2+17\,715)\sinh(z)+(-512\,z^4-19\,776\,z^2+25\,059)\sinh(3\,z)-$$
$$3\,(160\,z^2-2113)\sinh(5\,z)-6\,(16\,z^2+167)\sinh(7\,z)+3\sinh(9\,z)\Big)\operatorname{sech}^5(z)\;\;(m-1)^4+$$
$$\frac{1}{62\,914\,560}\Big(6\,z\,(5632\,z^4-50\,560\,z^2+259\,695)-4\,z\,(6656\,z^4+51\,840\,z^2-497\,775)\cosh(2\,z)+$$
$$8\,z\,(128\,z^4+11\,800\,z^2+45\,135)\cosh(4\,z)-120\,z\,(16\,z^2+717)\cosh(6\,z)-10\,z\,(32\,z^2+1413)\cosh(8\,z)+$$
$$60\,z\cosh(10\,z)-30\,(5120\,z^4-24\,320\,z^2+23\,841)\sinh(2\,z)+30\,(512\,z^4+13\,000\,z^2-17\,193)\sinh(4\,z)+$$
$$45\,(480\,z^2-1753)\sinh(6\,z)+45\,(80\,z^2+439)\sinh(8\,z)-135\sinh(10\,z)\Big)\operatorname{sech}^6(z)\,(m-1)^5-$$
$$\frac{1}{24\,159\,191\,040}\Big(15\,360\,z\,(928\,z^4-10\,290\,z^2+54\,723)\cosh(z)-480\,z\,(18\,560\,z^4+50\,960\,z^2-787\,377)\cosh(3\,z)+$$
$$48\,z\,(7424\,z^4+330\,400\,z^2+721\,995)\cosh(5\,z)-1680\,z\,(560\,z^2+12\,423)\cosh(7\,z)-240\,z\,(560\,z^2+11\,931)\cosh(9\,z)+$$
$$29\,520\,z\cosh(11\,z)+4\,(1\,236\,992\,z^6-8\,613\,120\,z^4+33\,029\,280\,z^2-29\,634\,165)\sinh(z)-$$
$$3\,(311\,296\,z^6+10\,296\,320\,z^4-68\,224\,320\,z^2+67\,579\,665)\sinh(3\,z)+$$
$$(16\,384\,z^6+3\,609\,600\,z^4+78\,102\,720\,z^2-95\,850\,225)\sinh(5\,z)+30\,(1792\,z^4+215\,040\,z^2-265\,557)\sinh(7\,z)+$$
$$30\,(256\,z^4+29\,952\,z^2+121\,377)\sinh(9\,z)-45\,(128\,z^2+885)\sinh(11\,z)+45\sinh(13\,z)\Big)\;\operatorname{sech}^7(z)\,(m-1)^6+$$
$$\frac{1}{1\,352\,914\,698\,240}\Big(-32\,z\,(1\,236\,992\,z^6-16\,466\,688\,z^4+137\,981\,760\,z^2-645\,664\,635)+$$
$$6\,z\,(6\,504\,448\,z^6+32\,657\,408\,z^4-860\,762\,560\,z^2+4\,958\,239\,335)\cosh(2\,z)-$$
$$3840\,z\,(1024\,z^6+82\,880\,z^4+93\,758\,z^2-2\,528\,085)\cosh(4\,z)+$$
$$2\,z\,(16\,384\,z^6+6\,289\,920\,z^4+175\,230\,720\,z^2+6\,228\,495)\cosh(6\,z)-672\,z\,(256\,z^4+65\,440\,z^2+1\,024\,365)\cosh(8\,z)-$$
$$42\,z\,(512\,z^4+125\,120\,z^2+1\,899\,015)\cosh(10\,z)+6720\,z\,(8\,z^2+207)\cosh(12\,z)-$$
$$1890\,z\cosh(14\,z)+35\,(6\,823\,936\,z^6-55\,971\,840\,z^4+257\,347\,584\,z^2-226\,081\,791)\sinh(2\,z)-$$
$$28\,(1\,949\,696\,z^6+28\,876\,800\,z^4-273\,056\,400\,z^2+260\,531\,505)\sinh(4\,z)+$$
$$7\,(139\,264\,z^6+16\,972\,800\,z^4+338\,054\,400\,z^2-352\,864\,395)\sinh(6\,z)+840\,(5120\,z^4+284\,832\,z^2-72\,189)\sinh(8\,z)+$$
$$105\,(5120\,z^4+264\,960\,z^2+917\,943)\sinh(10\,z)-1260\,(368\,z^2+1157)\sinh(12\,z)+4095\sinh(14\,z)\Big)\operatorname{sech}^8(z)\,(m-1)^7\;+$$
$$\frac{1}{173\,173\,081\,374\,720}\Big(16\,800\,z\,(372\,736\,z^6-5\,562\,240\,z^4+53\,071\,720\,z^2-241\,268\,799)\cosh(z)-$$
$$6048\,z\,(905\,216\,z^6-1\,580\,160\,z^4-59\,648\,680\,z^2+367\,609\,745)\cosh(3\,z)+$$
$$168\,z\,(3\,620\,864\,z^6+137\,978\,880\,z^4+64\,018\,880\,z^2-3\,076\,030\,125)\cosh(5\,z)-$$
$$24\,z\,(212\,992\,z^6+35\,374\,080\,z^4+633\,644\,480\,z^2-2\,522\,909\,025)\cosh(7\,z)+$$
$$30\,240\,z\,(1152\,z^4+127\,432\,z^2+1\,640\,373)\cosh(9\,z)+1680\,z\,(2304\,z^4+222\,224\,z^2+2\,905\,971)\cosh(11\,z)-$$
$$2520\,z\,(4352\,z^2+49\,343)\cosh(13\,z)+430\,920\,z\cosh(15\,z)+$$
$$2\,(1\,023\,606\,784\,z^8-11\,283\,980\,288\,z^6+64\,624\,062\,720\,z^4-276\,612\,940\,800\,z^2+226\,518\,668\,775)\sinh(z)-$$
$$9\,(62\,521\,344\,z^8+1\,935\,130\,624\,z^6-20\,219\,404\,800\,z^4+114\,095\,358\,720\,z^2-96\,412\,306\,235)\sinh(3\,z)+$$
$$2\,(16\,187\,392\,z^8+2\,528\,296\,960\,z^6+22\,150\,947\,840\,z^4-312\,004\,672\,560\,z^2+275\,867\,986\,905)\sinh(5\,z)+$$
$$(-131\,072\,z^8-96\,108\,544\,z^6-8\,858\,572\,800\,z^4-167\,025\,852\,000\,z^2+138\,260\,046\,645)\sinh(7\,z)-$$
$$63\,(16\,384\,z^6+7\,687\,680\,z^4+293\,400\,000\,z^2+74\,361\,395)\sinh(9\,z)-$$
$$224\,(512\,z^6+221\,040\,z^4+7\,951\,050\,z^2+25\,759\,215)\sinh(11\,z)+105\,(8192\,z^4+516\,384\,z^2+1\,063\,449)\sinh(13\,z)-$$

$$315\left(288 z^2+1691\right) \sinh(15 z)+315 \sinh(17 z)\big) \operatorname{sech}^9(z)\,(m-1)^8+\frac{1}{12\,468\,461\,858\,979\,840}$$

$$\Big(20 z\left(1\,023\,606\,784 z^8-18\,373\,533\,696 z^6+185\,116\,651\,776 z^4-1\,591\,235\,301\,600 z^2+6\,590\,667\,330\,135\right)-$$
$$4 z\left(5\,782\,503\,424 z^8+11\,574\,558\,720 z^6-826\,602\,803\,712 z^4+11\,241\,612\,285\,120 z^2-50\,828\,437\,643\,445\right) \cosh(2 z)+$$
$$2 z\left(1\,914\,699\,776 z^8+142\,726\,938\,624 z^6-663\,875\,218\,944 z^4-6\,913\,553\,623\,200 z^2+43\,597\,401\,570\,765\right) \cosh(4 z)-$$
$$8 z\left(16\,449\,536 z^8+4\,428\,693\,504 z^6+112\,644\,725\,760 z^4+33\,210\,581\,040 z^2-1\,635\,148\,968\,705\right) \cosh(6 z)+$$
$$4 z\left(65\,536 z^8+71\,589\,888 z^6+7\,153\,429\,248 z^4+66\,253\,556\,880 z^2-1\,097\,919\,700\,605\right) \cosh(8 z)-$$
$$2880 z\left(1024 z^6+782\,880 z^4+58\,308\,600 z^2+676\,097\,415\right) \cosh(10 z)-$$
$$18 z\left(16\,384 z^6+10\,633\,728 z^4+693\,621\,600 z^2+9\,057\,217\,365\right) \cosh(12 z)+$$
$$756 z\left(8192 z^4+1\,027\,360 z^2+7\,440\,975\right) \cosh(14 z)-22\,680 z\left(72 z^2+1493\right) \cosh(16 z)+22\,680 z \cosh(18 z)-$$
$$63\left(2\,332\,819\,456 z^8-29\,091\,840\,000 z^6+179\,510\,599\,680 z^4-871\,387\,205\,760 z^2+682\,578\,963\,495\right) \sinh(2 z)+$$
$$63\left(833\,355\,776 z^8+9\,007\,718\,400 z^6-129\,806\,760\,960 z^4+914\,279\,256\,720 z^2-730\,307\,555\,235\right) \sinh(4 z)-$$
$$9\left(354\,680\,832 z^8+25\,156\,812\,800 z^6+136\,480\,081\,920 z^4-3\,095\,763\,602\,400 z^2+2\,456\,267\,583\,735\right) \sinh(6 z)+$$
$$72\left(180\,224 z^8+66\,483\,200 z^6+5\,102\,603\,520 z^4+92\,641\,079\,790 z^2-56\,820\,978\,585\right) \sinh(8 z)+$$
$$315\left(409\,600 z^6+75\,909\,120 z^4+2\,395\,379\,520 z^2+1\,350\,388\,863\right) \sinh(10 z)+$$
$$315\left(40\,960 z^6+5\,698\,560 z^4+188\,126\,064 z^2+611\,766\,981\right) \sinh(12 z)-7560\left(14\,336 z^4+382\,788 z^2+602\,157\right)$$
$$\sinh(14 z)+5670\left(2232 z^2+5525\right) \sinh(16 z)-48\,195 \sinh(18 z)\Big) \operatorname{sech}^{10}(z)\,(m-1)^9-\frac{1}{7\,979\,815\,589\,747\,097\,600}$$

$$\Big(240 z\left(72\,741\,552\,128 z^8-1\,430\,099\,066\,880 z^6+15\,175\,126\,987\,776 z^4-143\,545\,963\,216\,320 z^2+567\,588\,084\,861\,735\right)$$
$$\cosh(z)-480 z\big(39\,405\,420\,544 z^8-218\,540\,298\,240 z^6-$$
$$1\,371\,786\,038\,784 z^4+39\,228\,923\,137\,320 z^2-175\,326\,548\,057\,325\big) \cosh(3 z)+$$
$$26\,880 z\left(134\,815\,744 z^8+4\,081\,820\,160 z^6-28\,787\,057\,856 z^4-175\,581\,265\,545 z^2+1\,053\,553\,433\,850\right) \cosh(5 z)-$$
$$480 z\left(268\,140\,544 z^8+32\,222\,699\,520 z^6+631\,247\,395\,968 z^4+298\,989\,472\,320 z^2-3\,869\,987\,716\,665\right) \cosh(7 z)+$$
$$1120 z\left(229\,376 z^8+101\,376\,000 z^6+7\,125\,757\,056 z^4-5\,253\,474\,240 z^2-1\,884\,308\,786\,235\right) \cosh(9 z)-$$
$$31\,680 z\left(112\,640 z^6+29\,022\,336 z^4+1\,873\,361\,700 z^2+20\,927\,561\,445\right) \cosh(11 z)-$$
$$5760 z\left(56\,320 z^6+6\,640\,704 z^4+522\,347\,490 z^2+8\,177\,342\,355\right) \cosh(13 z)+$$
$$7560 z\left(999\,424 z^4+51\,359\,520 z^2+279\,537\,015\right) \cosh(15 z)-113\,400 z\left(19\,296 z^2+165\,199\right) \cosh(17 z)+$$
$$33\,112\,800 z \cosh(19 z)+2\big(2\,748\,011\,511\,808 z^{10}-41\,952\,187\,514\,880 z^8+345\,480\,654\,274\,560 z^6-$$
$$1\,807\,745\,271\,897\,600 z^4+8\,497\,718\,945\,704\,800 z^2-6\,288\,200\,451\,412\,425\big) \sinh(z)-$$
$$2\big(954\,606\,813\,184 z^{10}+26\,964\,933\,672\,960 z^8-452\,624\,165\,683\,200 z^6+3\,108\,972\,461\,011\,200 z^4-$$
$$17\,443\,458\,335\,167\,200 z^2+12\,988\,772\,629\,429\,275\big) \sinh(3 z)+10\big(20\,065\,550\,336 z^{10}+2\,815\,274\,188\,800 z^8+$$
$$12\,926\,866\,046\,976 z^6-297\,957\,576\,349\,440 z^4+2\,660\,842\,809\,545\,760 z^2-1\,977\,158\,789\,043\,165\big) \sinh(5 z)-$$
$$4\big(1\,062\,207\,488 z^{10}+453\,524\,520\,960 z^8+20\,777\,484\,349\,440 z^6+63\,238\,275\,072\,000 z^4-$$
$$2\,708\,553\,310\,722\,000 z^2+1\,874\,819\,891\,427\,525\big) \sinh(7 z)+4\big(1\,048\,576 z^{10}+1\,931\,673\,600 z^8+$$
$$493\,750\,333\,440 z^6+33\,127\,034\,572\,800 z^4+590\,265\,992\,458\,800 z^2-245\,614\,921\,985\,475\big) \sinh(9 z)+$$
$$45\left(1\,441\,792 z^8+1\,587\,052\,544 z^6+194\,838\,336\,000 z^4+5\,805\,285\,837\,120 z^2+4\,550\,752\,841\,985\right) \sinh(11 z)+$$
$$45\left(131\,072 z^8+100\,237\,312 z^6+7\,884\,817\,920 z^4+360\,300\,104\,640 z^2+1\,262\,320\,276\,455\right) \sinh(13 z)-$$
$$315\left(1\,048\,576 z^6+231\,298\,560 z^4+3\,856\,707\,360 z^2+5\,043\,389\,535\right) \sinh(15 z)+$$
$$14\,175\left(13\,824 z^4+666\,272 z^2+1\,016\,955\right) \sinh(17 z)-$$
$$14\,175\left(512 z^2+2753\right) \sinh(19 z)+14\,175 \sinh(21 z)\Big) \operatorname{sech}^{11}(z)\,(m-1)^{10}+O\left((m-1)^{11}\right)$$

09.36.06.0017.01

$$\operatorname{sn}(z \mid m) \propto \tanh(z)+O(m-1)$$

q-series

09.36.06.0002.01

$$\operatorname{sn}(z \mid m) = \frac{2\pi}{\sqrt{m}\,K(m)} \sum_{n=0}^{\infty} \frac{q(m)^{n+\frac{1}{2}}}{1-q(m)^{2n+1}} \sin\left((2n+1)\frac{\pi z}{2K(m)}\right)$$

09.36.06.0003.01

$$\log(\operatorname{sn}(z \mid m)) = 2\log(\vartheta_3(0, q(m))) + \log\left(\sin\left(\frac{\pi z}{2K(m)}\right)\right) - 4\sum_{r=1}^{\infty} \frac{q(m)^r}{r(1+q(m)^r)} \sin^2\left(\frac{r\pi z}{2K(m)}\right)$$

Other series representations

09.36.06.0004.01

$$\operatorname{sn}(z \mid m) = \frac{\pi}{2\sqrt{m}\,K(1-m)} \sum_{k=-\infty}^{\infty} (-1)^k \tanh\left(\pi \frac{K(m)}{K(1-m)}\left(k + \frac{z}{2K(m)}\right)\right)$$

09.36.06.0005.01

$$\operatorname{sn}(z \mid m) \propto \frac{(-1)^r}{\sqrt{m}\,(z - i(2s+1)K(1-m) - 2rK(m))} + O(1) \;/;\; (z \to (2s+1)\,i\,K(1-m) + 2rK(m)) \bigwedge \{r, s\} \in \mathbb{Z}$$

Product representations

09.36.08.0001.01

$$\operatorname{sn}(z \mid m) = 2\frac{\sqrt[4]{q(m)}}{\sqrt[4]{m}} \sin\left(\frac{\pi z}{2K(m)}\right) \prod_{k=1}^{\infty} \frac{1 - 2q(m)^{2k}\cos\left(\frac{\pi z}{K(m)}\right) + q(m)^{4k}}{1 - 2q(m)^{2k-1}\cos\left(\frac{\pi z}{K(m)}\right) + q(m)^{4k-2}}$$

Differential equations

Ordinary nonlinear differential equations

With respect to *m*

09.36.13.0003.01

$$m^3 z^2 w(m)^{12} + \left(-3 z^2 m^3 - 3 z^2 m^2 - 1\right) w(m)^{10} + \left(3 z^2 m^3 + 9 z^2 m^2 + 3 z^2 m + 4\right) w(m)^8 + \left(-z^2 m^3 - 9 z^2 m^2 - 9 z^2 m - z^2 - 6\right) w(m)^6 + \left(3 m^2 z^2 + 9 m z^2 + 3 z^2 + 4\right) w(m)^4 + \left(-3 m z^2 - 3 z^2 - 1\right) w(m)^2 + \left(-64 (m-1)^2 m^4 w(m)^6 + 64 (m-1)^2 m^3 (m+1) w(m)^4 - 16 (m-1)^2 m^2 (m+1)^2 w(m)^2\right) w'(m)^4 + \left(64 (m-1) m^4 w(m)^7 - 32 (m-1) m^2 (m+2)(3m-1) w(m)^5 + 32 (m-1) m \left(m^3 + 7 m^2 - m - 1\right) w(m)^3 - 32 (m-1) m (m+1)(2m-1) w(m)\right) w'(m)^3 + z^2 + \left(-16 m^2 \left(m^2 - m + 1\right) w(m)^8 + 8 m (m+1)\left(4 m^2 - m + 1\right) w(m)^6 - 16 (m+1)\left(m^3 + 5 m^2 - 4 m + 1\right) w(m)^4 + 8\left(7 m^3 + 12 m^2 - 15 m + 4\right) w(m)^2 - 16 (2m-1)^2\right) w'(m)^2 + \left(-16 (m-1)^2 m^4 w(m)^8 + 32 (m-1)^2 m^3 (m+1) w(m)^6 - 16 (m-1)^2 m^2 \left(m^2 + 4 m + 1\right) w(m)^4 + 32 (m-1)^2 m^2 (m+1) w(m)^2 - 16 (m-1)^2 m^2\right) w''(m)^2 + \left(-8 m^2 w(m)^9 + 8 (m+1)(3m-1) w(m)^7 - 24\left(m^2 + 2m - 1\right) w(m)^5 + 8\left(m^2 + 6m - 3\right) w(m)^3 - 8 (2m-1) w(m)\right) w'(m) + \left(-8 (m-1) m^2 w(m)^9 + 8 (m-1) m (3m+1) w(m)^7 - 24 (m-1) m (m+1) w(m)^5 + 8 (m-1) m (m+3) w(m)^3 - 8 (m-1) m\, w(m) + \left(64 (m-1)^2 m^4 w(m)^7 - 96 (m-1)^2 m^3 (m+1) w(m)^5 + 32 (m-1)^2 m^2 \left(m^2 + 4m + 1\right) w(m)^3 - 32 (m-1)^2 m^2 (m+1) w(m)\right) w'(m)^2 + \left(-32 (m-1) m^4 w(m)^8 + 32 (m-1) m^2 \left(2 m^2 + 3m - 1\right) w(m)^6 - 32 (m-1) m \left(m^3 + 6 m^2 - 1\right) w(m)^4 + 32 (m-1) m \left(3 m^2 + 3m - 2\right) w(m)^2 - 32 (m-1) m (2m-1)\right) w'(m)\right) w''(m) == 0 \;/;\; w(m) == \operatorname{sn}(z \mid m)$$

With respect to z

09.36.13.0001.01

$$w'(z)^2 == \left(1 - w(z)^2\right)\left(1 - m\, w(z)^2\right) \;/;\; w(z) == \operatorname{sn}(z \mid m)$$

09.36.13.0002.01

$$w''(z) - w(z)\left(2 m\, w(z)^2 - m - 1\right) == 0 \;/;\; w(z) == \operatorname{sn}(z \mid m)$$

Partial differential equations

09.36.13.0004.01

$$4 m\, w(z, m)^4 w^{(1,0)}(z, m)^4 - w(z, m)^2 \left(8 (m-1) m\, w^{(1,0)}(z, m)\, w^{(1,1)}(z, m) + w^{(2,0)}(z, m)\left(w^{(2,0)}(z, m) - 8 (m-1) m\, w^{(0,1)}(z, m)\right)\right) w^{(1,0)}(z, m)^2 + 2 (m-1)\left(w^{(1,0)}(z, m)\, w^{(1,1)}(z, m) - w^{(0,1)}(z, m)\, w^{(2,0)}(z, m)\right) \left(2 (m-1) m\, w^{(1,0)}(z, m)\, w^{(1,1)}(z, m) + w^{(2,0)}(z, m)\left(w^{(2,0)}(z, m) - 2 (m-1) m\, w^{(0,1)}(z, m)\right)\right) == 0 \;/;\; w(z, m) == \operatorname{sn}(z \mid m)$$

Transformations

Transformations and argument simplifications

Argument involving basic arithmetic operations

09.36.16.0001.01

$$\operatorname{sn}(i z \mid m) == i \frac{\operatorname{sn}(z \mid 1-m)}{\operatorname{cn}(z \mid 1-m)}$$

09.36.16.0002.01

$$\operatorname{sn}(i z \mid m) == i \operatorname{sc}(z \mid 1-m)$$

09.36.16.0003.01

$$\operatorname{sn}(z \mid 1-m) == -i \frac{\operatorname{sn}(i z \mid m)}{\operatorname{cn}(i z \mid m)}$$

09.36.16.0004.01

$$\operatorname{sn}(z \mid 1-m) = -i \operatorname{sc}(i z \mid m)$$

09.36.16.0005.01

$$\operatorname{sn}(i z \mid 1-m) = i \frac{\operatorname{sn}(z \mid m)}{\operatorname{cn}(z \mid m)}$$

09.36.16.0006.01

$$\operatorname{sn}(i z \mid 1-m) = i \operatorname{sc}(z \mid m)$$

09.36.16.0021.01

$$\operatorname{sn}(x + i y \mid m) = (\operatorname{sn}(x \mid m) \operatorname{dn}(y \mid 1-m) + i \operatorname{cn}(x \mid m) \operatorname{dn}(x \mid m) \operatorname{cn}(y \mid 1-m) \operatorname{sn}(y \mid 1-m)) / \left(\operatorname{cn}(y \mid 1-m)^2 + m \operatorname{sn}(x \mid m)^2 \operatorname{sn}(y \mid 1-m)^2\right) /; \{x, y\} \in \mathbb{R}$$

09.36.16.0022.01

$$\operatorname{sn}\left(\sqrt{1-m}\, z \,\middle|\, \frac{m}{m-1}\right) = \sqrt{1-m}\, \frac{\operatorname{sn}(z \mid m)}{\operatorname{dn}(z \mid m)}$$

09.36.16.0023.01

$$\operatorname{sn}\left(\sqrt{1-m}\, z \,\middle|\, \frac{m}{m-1}\right) = \sqrt{1-m}\, \operatorname{sd}(z \mid m)$$

09.36.16.0024.01

$$\operatorname{sn}\left(\sqrt{m}\, z \,\middle|\, \frac{1}{m}\right) = \sqrt{m}\, \operatorname{sn}(z \mid m)$$

09.36.16.0025.01

$$\operatorname{sn}\left(i \sqrt{1-m}\, z \,\middle|\, \frac{1}{1-m}\right) = i \sqrt{1-m}\, \frac{\operatorname{sn}(z \mid m)}{\operatorname{cn}(z \mid m)}$$

09.36.16.0026.01

$$\operatorname{sn}\left(i \sqrt{1-m}\, z \,\middle|\, \frac{1}{1-m}\right) = i \sqrt{1-m}\, \operatorname{sc}(z \mid m)$$

09.36.16.0027.01

$$\operatorname{sn}\left(i \sqrt{m}\, z \,\middle|\, \frac{m-1}{m}\right) = i \sqrt{m}\, \frac{\operatorname{sn}(z \mid m)}{\operatorname{dn}(z \mid m)}$$

09.36.16.0028.01

$$\operatorname{sn}\left(i \sqrt{m}\, z \,\middle|\, \frac{m-1}{m}\right) = i \sqrt{m}\, \operatorname{sd}(z \mid m)$$

Landen's transformation:

09.36.16.0029.01

$$\operatorname{sn}\left(\left(1+\sqrt{1-m}\right) z \,\middle|\, \left(\frac{1-\sqrt{1-m}}{1+\sqrt{1-m}}\right)^2\right) = \left(1+\sqrt{1-m}\right) \frac{\operatorname{sn}(z \mid m) \operatorname{cn}(z \mid m)}{\operatorname{dn}(z \mid m)}$$

Gauss' transformation:

09.36.16.0030.01

$$\operatorname{sn}\left(\left(1+\sqrt{m}\right) z \,\middle|\, \frac{4\sqrt{m}}{\left(1+\sqrt{m}\right)^2}\right) = \left(1+\sqrt{m}\right) \frac{\operatorname{sn}(z \mid m)}{1+\sqrt{m}\, \operatorname{sn}(z \mid m)^2}$$

n th degree transformations:

09.36.16.0031.01

$$\operatorname{sn}\left(\frac{z}{M} \,\middle|\, l\right) = \frac{1}{M} \operatorname{sn}(z \mid m) \prod_{r=1}^{\frac{n-1}{2}} \left(1 - \frac{\operatorname{sn}(z \mid m)^2}{\operatorname{sn}\left(\frac{2 r K(m)}{n} \,\middle|\, m\right)^2}\right) \left(1 - m \operatorname{sn}\left(\frac{2 r K(m)}{n} \,\middle|\, m\right)^2 \operatorname{sn}(z \mid m)^2\right)^{-1} /;$$

$$\frac{n+1}{2} \in \mathbb{Z}^+ \bigwedge l = m^n \prod_{r=1}^{\frac{n-1}{2}} \operatorname{sn}\left(\frac{(2 r - 1) K(m)}{n} \,\middle|\, m\right)^8 \bigwedge M = \prod_{r=1}^{\frac{n-1}{2}} \frac{\operatorname{sn}\left(\frac{(2 r-1) K(m)}{n} \,\middle|\, m\right)^2}{\operatorname{sn}\left(\frac{2 r K(m)}{n} \,\middle|\, m\right)^2}$$

09.36.16.0032.01

$$\operatorname{sn}\left(\frac{z}{M} + \frac{K(m)}{n M} \,\middle|\, l\right) = \prod_{r=1}^{\frac{n}{2}} \left(1 - \frac{\operatorname{sn}(z \mid m)^2}{\operatorname{sn}\left(\frac{(2 r-1) K(m)}{n} \,\middle|\, m\right)^2}\right) \left(1 - m \operatorname{sn}\left(\frac{(2 r - 1) K(m)}{n} \,\middle|\, m\right)^2 \operatorname{sn}(z \mid m)^2\right)^{-1} /;$$

$$\frac{n}{2} \in \mathbb{Z}^+ \bigwedge l = m^n \prod_{r=1}^{\frac{n}{2}} \operatorname{sn}\left(\frac{(2 r - 1) K(m)}{n} \,\middle|\, m\right)^8 \bigwedge M = \prod_{r=1}^{\frac{n}{2}} \frac{\operatorname{sn}\left(\frac{(2 r-1) K(m)}{n} \,\middle|\, m\right)^2}{\operatorname{sn}\left(\frac{2 r K(m)}{n} \,\middle|\, m\right)^2}$$

Argument involving half-periods

09.36.16.0008.01

$$\operatorname{sn}(z + K(m) \mid m) = \operatorname{cd}(z \mid m)$$

09.36.16.0165.01

$$\operatorname{sn}(z - K(m) \mid m) = -\operatorname{cd}(z \mid m)$$

09.36.16.0020.01

$$\operatorname{sn}(z + 3 K(m) \mid m) = -\operatorname{cd}(z \mid m)$$

09.36.16.0166.01

$$\operatorname{sn}(z + (2 r + 1) K(m) \mid m) = (-1)^r \operatorname{cd}(z \mid m) /; r \in \mathbb{Z}$$

09.36.16.0010.01

$$\operatorname{sn}(z + i K(1 - m) \mid m) = \frac{1}{\sqrt{m}} \operatorname{ns}(z \mid m)$$

09.36.16.0167.01

$$\operatorname{sn}(z - i K(1 - m) \mid m) = \frac{\operatorname{ns}(z \mid m)}{\sqrt{m}}$$

09.36.16.0168.01

$$\operatorname{sn}(z + 3 i K(1 - m) \mid m) = \frac{1}{\sqrt{m}} \operatorname{ns}(z \mid m) /; s \in \mathbb{Z}$$

09.36.16.0169.01

$$\operatorname{sn}(z + (2 s + 1) i K(1 - m) \mid m) = \frac{1}{\sqrt{m}} \operatorname{ns}(z \mid m) /; s \in \mathbb{Z}$$

09.36.16.0012.01

$$\operatorname{sn}(z + K(m) + i K(1 - m) \mid m) = \frac{1}{\sqrt{m}} \operatorname{dc}(z \mid m)$$

09.36.16.0170.01

$$\operatorname{sn}(z - i K(1-m) + K(m) \mid m) = \frac{\operatorname{dc}(z \mid m)}{\sqrt{m}}$$

09.36.16.0171.01

$$\operatorname{sn}(z + i K(1-m) - K(m) \mid m) = -\frac{\operatorname{dc}(z \mid m)}{\sqrt{m}}$$

09.36.16.0172.01

$$\operatorname{sn}(z - i K(1-m) - K(m) \mid m) = -\frac{\operatorname{dc}(z \mid m)}{\sqrt{m}}$$

09.36.16.0014.01

$$\operatorname{sn}(z + 3 K(m) + i K(1-m) \mid m) == -\frac{1}{\sqrt{m}} \operatorname{dc}(z \mid m)$$

09.36.16.0016.01

$$\operatorname{sn}(z + (4 r + 1) K(m) + i (2 s + 1) K(1-m) \mid m) == \frac{1}{\sqrt{m}} \operatorname{dc}(z \mid m) \;/;\; \{r, s\} \in \mathbb{Z}$$

09.36.16.0018.01

$$\operatorname{sn}(z + (4 r - 1) K(m) + i (2 s + 1) K(1-m) \mid m) == -\frac{1}{\sqrt{m}} \operatorname{dc}(z \mid m) \;/;\; \{r, s\} \in \mathbb{Z}$$

09.36.16.0173.01

$$\operatorname{sn}(z + (2 s + 1) i K(1-m) + (2 r + 1) K(m) \mid m) = \frac{(-1)^r \operatorname{dc}(z \mid m)}{\sqrt{m}} \;/;\; \{r, s\} \in \mathbb{Z}$$

Argument involving inverse Jacobi functions

09.36.16.0174.01

$$\operatorname{sn}\left(\operatorname{cd}^{-1}(z \mid m) \mid m\right)^2 = \frac{z^2 - 1}{m z^2 - 1}$$

09.36.16.0175.01

$$\operatorname{sn}\left(\operatorname{cn}^{-1}(z \mid m) \mid m\right)^2 = 1 - z^2$$

09.36.16.0176.01

$$\operatorname{sn}\left(\operatorname{cs}^{-1}(z \mid m) \mid m\right)^2 = \frac{1}{z^2 + 1}$$

09.36.16.0177.01

$$\operatorname{sn}\left(\operatorname{dc}^{-1}(z \mid m) \mid m\right)^2 = \frac{1 - z^2}{m - z^2}$$

09.36.16.0178.01

$$\operatorname{sn}\left(\operatorname{dn}^{-1}(z \mid m) \mid m\right)^2 = \frac{1 - z^2}{m}$$

09.36.16.0179.01

$$\operatorname{sn}\left(\operatorname{ds}^{-1}(z \mid m) \mid m\right)^2 = \frac{1}{z^2 + m}$$

09.36.16.0180.01

$$\operatorname{sn}\left(\operatorname{nc}^{-1}(z \mid m) \mid m\right)^2 = 1 - \frac{1}{z^2}$$

09.36.16.0181.01

$$\operatorname{sn}\left(\operatorname{nd}^{-1}(z \mid m) \mid m\right)^2 = \frac{z^2 - 1}{m\, z^2}$$

09.36.16.0182.01

$$\operatorname{sn}\left(\operatorname{ns}^{-1}(z \mid m) \mid m\right) = \frac{1}{z}$$

09.36.16.0183.01

$$\operatorname{sn}\left(\operatorname{sc}^{-1}(z \mid m) \mid m\right)^2 = \frac{z^2}{z^2 + 1}$$

09.36.16.0184.01

$$\operatorname{sn}\left(\operatorname{sd}^{-1}(z \mid m) \mid m\right)^2 = \frac{z^2}{m\, z^2 + 1}$$

Addition formulas

09.36.16.0033.01

$$\operatorname{sn}(u + v \mid m) == \frac{\operatorname{cn}(u \mid m) \operatorname{dn}(u \mid m) \operatorname{sn}(v \mid m) + \operatorname{cn}(v \mid m) \operatorname{dn}(v \mid m) \operatorname{sn}(u \mid m)}{1 - m \operatorname{sn}(u \mid m)^2 \operatorname{sn}(v \mid m)^2}$$

09.36.16.0034.01

$$\operatorname{sn}(u + v \mid m) + \operatorname{sn}(u - v \mid m) == \frac{2 \operatorname{sn}(u \mid m) \operatorname{cn}(v \mid m) \operatorname{dn}(v \mid m)}{1 - m \operatorname{sn}(u \mid m)^2 \operatorname{sn}(v \mid m)^2}$$

09.36.16.0035.01

$$\operatorname{sn}(u + v \mid m) - \operatorname{sn}(u - v \mid m) == \frac{2 \operatorname{sn}(v \mid m) \operatorname{cn}(u \mid m) \operatorname{dn}(u \mid m)}{1 - m \operatorname{sn}(u \mid m)^2 \operatorname{sn}(v \mid m)^2}$$

09.36.16.0036.01

$$\operatorname{sn}(u + v \mid m) \operatorname{sn}(u - v \mid m) == \frac{\operatorname{sn}(u \mid m)^2 - \operatorname{sn}(v \mid m)^2}{1 - m \operatorname{sn}(u \mid m)^2 \operatorname{sn}(v \mid m)^2}$$

09.36.16.0037.01

$$\operatorname{sn}(u - v \mid m) \operatorname{sn}(u + v \mid m) == \frac{1}{m} \left(\frac{\operatorname{dn}(v \mid m)^2 + m \operatorname{cn}(v \mid m)^2 \operatorname{sn}(u \mid m)^2}{1 - m \operatorname{sn}(u \mid m)^2 \operatorname{sn}(v \mid m)^2} - 1 \right)$$

09.36.16.0038.01

$$\operatorname{sn}(u + v \mid m) \operatorname{sn}(u - v \mid m) == \frac{1}{m} \left(1 - \frac{\operatorname{dn}(u \mid m)^2 + m \operatorname{cn}(u \mid m)^2 \operatorname{sn}(v \mid m)^2}{1 - m \operatorname{sn}(u \mid m)^2 \operatorname{sn}(v \mid m)^2} \right)$$

09.36.16.0039.01

$$\operatorname{sn}(u - v \mid m) \operatorname{sn}(u + v \mid m) == \frac{\operatorname{cn}(v \mid m)^2 + \operatorname{dn}(v \mid m)^2 \operatorname{sn}(u \mid m)^2}{1 - m \operatorname{sn}(u \mid m)^2 \operatorname{sn}(v \mid m)^2} - 1$$

09.36.16.0040.01

$$\operatorname{sn}(u+v\mid m)\operatorname{sn}(u-v\mid m)=1-\frac{\operatorname{cn}(u\mid m)^2+\operatorname{dn}(u\mid m)^2\operatorname{sn}(v\mid m)^2}{1-m\operatorname{sn}(u\mid m)^2\operatorname{sn}(v\mid m)^2}$$

09.36.16.0041.01

$$(1+\operatorname{sn}(u+v\mid m))(1+\operatorname{sn}(u-v\mid m))=\frac{(\operatorname{cn}(v\mid m)+\operatorname{dn}(v\mid m)\operatorname{sn}(u\mid m))^2}{1-m\operatorname{sn}(u\mid m)^2\operatorname{sn}(v\mid m)^2}$$

09.36.16.0042.01

$$(1+\operatorname{sn}(u+v\mid m))(1-\operatorname{sn}(u-v\mid m))=\frac{(\operatorname{cn}(u\mid m)+\operatorname{dn}(u\mid m)\operatorname{sn}(v\mid m))^2}{1-m\operatorname{sn}(u\mid m)^2\operatorname{sn}(v\mid m)^2}$$

09.36.16.0043.01

$$\left(1+\sqrt{m}\operatorname{sn}(u+v\mid m)\right)\left(1+\sqrt{m}\operatorname{sn}(u-v\mid m)\right)=\frac{\left(\operatorname{dn}(v\mid m)+\sqrt{m}\operatorname{cn}(v\mid m)\operatorname{sn}(u\mid m)\right)^2}{1-m\operatorname{sn}(u\mid m)^2\operatorname{sn}(v\mid m)^2}$$

09.36.16.0044.01

$$\left(1+\sqrt{m}\operatorname{sn}(u+v\mid m)\right)\left(1-\sqrt{m}\operatorname{sn}(u-v\mid m)\right)=\frac{\left(\operatorname{dn}(u\mid m)+\sqrt{m}\operatorname{cn}(u\mid m)\operatorname{sn}(v\mid m)\right)^2}{1-m\operatorname{sn}(u\mid m)^2\operatorname{sn}(v\mid m)^2}$$

09.36.16.0045.01

$$\operatorname{sn}(v\mid m)\operatorname{cn}(u\mid m)\operatorname{dn}(u+v\mid m)=\operatorname{dn}(v\mid m)\operatorname{sn}(u+v\mid m)-\operatorname{cn}(v\mid m)\operatorname{sn}(u\mid m)$$

09.36.16.0046.01

$$\operatorname{dn}(v\mid m)\operatorname{cn}(u\mid m)\operatorname{sn}(u+v\mid m)=\operatorname{dn}(u+v\mid m)\operatorname{sn}(v\mid m)+\operatorname{cn}(u+v\mid m)\operatorname{sn}(u\mid m)$$

09.36.16.0047.01

$$\operatorname{sn}(v\mid m)\operatorname{sn}(u\mid m)\operatorname{sn}(u+v\mid m)=-\frac{\operatorname{sn}(u+v\mid m)}{\operatorname{dn}(u+v\mid m)}(\operatorname{cn}(u+v\mid m)-\operatorname{cn}(v\mid m)\operatorname{cn}(u\mid m))$$

09.36.16.0048.01

$$\operatorname{sn}(v\mid m)\operatorname{sn}(u\mid m)\operatorname{sn}(u+v\mid m)=-\frac{\operatorname{sn}(u+v\mid m)}{m\operatorname{cn}(u+v\mid m)}(\operatorname{dn}(u+v\mid m)-\operatorname{dn}(v\mid m)\operatorname{dn}(u\mid m))$$

09.36.16.0049.01

$$\operatorname{sn}(u+v\mid m)\operatorname{cn}(v\mid m)\operatorname{dn}(u\mid m)=\operatorname{cn}(u+v\mid m)\operatorname{sn}(v\mid m)+\operatorname{dn}(u+v\mid m)\operatorname{sn}(u\mid m)$$

09.36.16.0050.01

$$\operatorname{cn}(u+v\mid m)\operatorname{cn}(v\mid m)\operatorname{dn}(u\mid m)=\operatorname{dn}(u+v\mid m)\operatorname{cn}(u\mid m)\operatorname{dn}(v\mid m)-(1-m)\operatorname{sn}(u+v\mid m)\operatorname{sn}(v\mid m)$$

09.36.16.0051.01

$$\operatorname{dn}(u+v\mid m)\operatorname{cn}(v\mid m)\operatorname{sn}(u\mid m)=\operatorname{sn}(u+v\mid m)\operatorname{dn}(u\mid m)-\operatorname{sn}(v\mid m)\operatorname{cn}(u\mid m)$$

09.36.16.0052.01

$$Z(\operatorname{am}(v\mid m)\mid m)-Z(\operatorname{am}(u+v\mid m)\mid m)+Z(\operatorname{am}(u\mid m)\mid m)=m\operatorname{sn}(v\mid m)\operatorname{sn}(u\mid m)\operatorname{sn}(u+v\mid m)$$

09.36.16.0053.01

$$Z(\operatorname{am}(v\mid m)\mid m)-Z(\operatorname{am}(u+v\mid m)\mid m)+Z(\operatorname{am}(u\mid m)\mid m)=\frac{m\operatorname{sn}(v\mid m)}{\operatorname{dn}(v\mid m)}(\operatorname{cn}(v\mid m)-\operatorname{cn}(u\mid m)\operatorname{cn}(u+v\mid m))$$

09.36.16.0054.01

$$Z(\operatorname{am}(v\mid m)\mid m)-Z(\operatorname{am}(u+v\mid m)\mid m)+Z(\operatorname{am}(u\mid m)\mid m)=\frac{\operatorname{sn}(v\mid m)}{\operatorname{cn}(v\mid m)}(\operatorname{dn}(v\mid m)-\operatorname{dn}(u\mid m)\operatorname{dn}(u+v\mid m))$$

Half-angle formulas

09.36.16.0055.01

$$\operatorname{sn}\left(\frac{z}{2} \middle| m\right)^2 = \frac{1-\operatorname{cn}(z \mid m)}{1+\operatorname{dn}(z \mid m)}$$

Multiple arguments

Double angle formulas

09.36.16.0056.01

$$\operatorname{sn}(2z \mid m) = \frac{2\operatorname{sn}(z \mid m)\operatorname{cn}(z \mid m)\operatorname{dn}(z \mid m)}{1-m\operatorname{sn}(z \mid m)^4}$$

09.36.16.0057.01

$$\operatorname{sn}(2z \mid m) = \frac{2\operatorname{sn}(z \mid m)\operatorname{cn}(z \mid m)\operatorname{dn}(z \mid m)}{\operatorname{cn}(z \mid m)^2+\operatorname{dn}(z \mid m)^2\operatorname{sn}(z \mid m)^2}$$

Multiple angle formulas

09.36.16.0058.01

$$\operatorname{sn}(n z, m) = (-1)^{\frac{n-1}{2}} m^{\frac{n^2-1}{4}} \prod_{\mu=-\frac{n-1}{2}}^{\frac{n-1}{2}} \prod_{\nu=-\frac{n-1}{2}}^{\frac{n-1}{2}} \operatorname{sn}\left(z+2\,\frac{\mu K(m)+i\,\nu K(1-m)}{n} \middle| m\right) /;\ \frac{n+1}{2} \in \mathbb{Z}^+$$

09.36.16.0059.01

$$\operatorname{sn}(n z \mid m) = \frac{1}{n}\sum_{r,s=0}^{n-1} \operatorname{sn}\left(z+\frac{4 r K(m)+2 i s K(1-m)}{n} \middle| m\right) /;\ \frac{n+1}{2} \in \mathbb{Z}^+$$

09.36.16.0060.01

$$\operatorname{sn}\left(\frac{2n}{\pi} K\left(\lambda\left(\frac{n}{\pi i}\log(q(m))\right)\right) z \middle| \lambda\left(\frac{n}{\pi i}\log(q(m))\right)\right) = \frac{\sqrt[4]{q(m)^n}}{q(m)^{n/4}} \frac{\left(\sqrt[4]{m}\right)^n}{\sqrt[4]{\lambda\left(\frac{n}{\pi i}\log(q(m))\right)}} \prod_{r=0}^{n-1} \operatorname{sn}\left(\frac{2K(m)}{\pi}\left(z+\frac{r\pi}{n}\right) \middle| m\right) /;\ n \in \mathbb{Z}^+$$

Products of a single Jacobi function

09.36.16.0064.01

$$\operatorname{cn}(z \mid m)\operatorname{cn}\left(z+\frac{4K(m)}{3} \middle| m\right)\operatorname{cn}\left(z+\frac{8K(m)}{3} \middle| m\right) = \frac{\operatorname{dn}\left(\frac{2K(m)}{3} \middle| m\right)^2}{1-\operatorname{dn}\left(\frac{2K(m)}{3} \middle| m\right)^2}\left(\operatorname{cn}(z \mid m)+\operatorname{cn}\left(z+\frac{4K(m)}{3} \middle| m\right)+\operatorname{cn}\left(z+\frac{8K(m)}{3} \middle| m\right)\right)$$

Khare/Sukhatme_2002

Khare/Sukhatme_JMP_2002

09.36.16.0068.01

$$m^{\frac{p-1}{2}} \prod_{k=0}^{p-1} \operatorname{sn}\left(z+\frac{4kK(m)}{p} \middle| m\right) = (-1)^{\frac{p-1}{2}} \left(\prod_{k=1}^{\frac{p-1}{2}} \operatorname{ns}\left(\frac{4kK(m)}{p} \middle| m\right)^2\right) \sum_{k=0}^{p-1} \operatorname{sn}\left(z+\frac{4kK(m)}{p} \middle| m\right) /;\ \frac{p-1}{2} \in \mathbb{N}$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.36.16.0069.01

$$m^{\frac{p-1}{2}} \prod_{k=0}^{p-1} \operatorname{cn}\left(z + \frac{4\,k\,K(m)}{p} \,\middle|\, m\right) = \left(\prod_{k=1}^{\frac{p-1}{2}} \operatorname{ds}\left(\frac{4\,k\,K(m)}{p} \,\middle|\, m\right)^2\right) \sum_{k=0}^{p-1} \operatorname{cn}\left(z + \frac{4\,k\,K(m)}{p} \,\middle|\, m\right) /;\ \frac{p-1}{2} \in \mathbb{N} \bigwedge r \in \mathbb{N}^+ \bigwedge r < p$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.36.16.0070.01

$$m^{p/2} \prod_{k=0}^{p-1} \operatorname{sn}\left(z + \frac{2\,k\,K(m)}{p} \,\middle|\, m\right) = \left(\prod_{k=1}^{p/2-1} \operatorname{ns}\left(\frac{2\,k\,K(m)}{p} \,\middle|\, m\right)^2\right) \sum_{k=0}^{p-1} (-1)^k \operatorname{Z}\left(\operatorname{am}\left(z + \frac{2\,k\,K(m)}{p} \,\middle|\, m\right) \,\middle|\, m\right) /;$$

$$\frac{p}{2} \in \mathbb{N}^+ \bigwedge r \in \mathbb{N}^+ \bigwedge r < p \bigwedge \gcd(p, r) = 1 \bigwedge 1 - m > 0$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.36.16.0071.01

$$m^{p/2} \prod_{k=0}^{p-1} \operatorname{cn}\left(z + \frac{2\,k\,K(m)}{p} \,\middle|\, m\right) = \sqrt{1-m}\ (-1)^{p/2} \left(\prod_{k=1}^{p/2-1} \operatorname{ds}\left(\frac{2\,k\,K(m)}{p} \,\middle|\, m\right)^2\right) \sum_{k=0}^{p-1} (-1)^k \operatorname{Z}\left(\operatorname{am}\left(z + \frac{2\,k\,K(m)}{p} \,\middle|\, m\right) \,\middle|\, m\right) /;$$

$$\frac{p}{2} \in \mathbb{N}^+ \bigwedge r \in \mathbb{N}^+ \bigwedge r < p \bigwedge \gcd(p, r) = 1 \bigwedge 1 - m > 0$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

Sums over products of two Jacobi functions

09.36.16.0061.01

$$\operatorname{cn}(z \mid m) \operatorname{cn}\left(z + \frac{4\,K(m)}{3} \,\middle|\, m\right) + \operatorname{cn}\left(z + \frac{4\,K(m)}{3} \,\middle|\, m\right) \operatorname{cn}\left(z + \frac{8\,K(m)}{3} \,\middle|\, m\right) + \operatorname{cn}\left(z + \frac{8\,K(m)}{3} \,\middle|\, m\right) \operatorname{cn}(z \mid m) =$$

$$-\operatorname{dn}\left(\frac{2\,K(m)}{3} \,\middle|\, m\right) \left(\operatorname{dn}\left(\frac{2\,K(m)}{3} \,\middle|\, m\right) + 2\right) \Big/ \left(\operatorname{dn}\left(\frac{2\,K(m)}{3} \,\middle|\, m\right) + 1\right)^2$$

Khare/Sukhatme_2002

Khare/Sukhatme_JMP_2002

09.36.16.0062.01

$$\sum_{k=0}^{p-1} \operatorname{cn}\left(z + \frac{4\,k\,K(m)}{p} \,\middle|\, m\right) \operatorname{cn}\left(z + \frac{4\,(k+1)\,K(m)}{p} \,\middle|\, m\right) = \sum_{k=0}^{p-1} \operatorname{cn}\left(\frac{4\,k\,K(m)}{p} \,\middle|\, m\right) \operatorname{cn}\left(\frac{4\,(k+1)\,K(m)}{p} \,\middle|\, m\right) /;\ \frac{p-1}{2} \in \mathbb{N}^+$$

Khare/Sukhatme_2002

Khare/Sukhatme_JMP_2002

09.36.16.0063.01

$$\sum_{k=0}^{p-1} \mathrm{cn}\left(z + \frac{4\,k\,K(m)}{p} \,\middle|\, m\right) \mathrm{cn}\left(z + \frac{4\,(k+n)\,K(m)}{p} \,\middle|\, m\right) = \sum_{k=0}^{p-1} \mathrm{cn}\left(\frac{4\,k\,K(m)}{p} \,\middle|\, m\right) \mathrm{cn}\left(\frac{4\,(k+n)\,K(m)}{p} \,\middle|\, m\right) /; \frac{p-1}{2} \in \mathbb{N}^+ \bigwedge n \in \mathbb{Z} \bigwedge 1 \leq n \leq \frac{p+1}{2}$$

Khare/Sukhatme_2002

Khare/Sukhatme_JMP_2002

09.36.16.0072.01

$$\sum_{k=0}^{p-1} \mathrm{sn}\left(z + \frac{2\,k\,K(m)}{p} \,\middle|\, m\right) \left(\mathrm{cn}\left(z + \frac{2\,K(m)\,(k-r)}{p} \,\middle|\, m\right) + \mathrm{cn}\left(z + \frac{2\,K(m)\,(k+r)}{p} \,\middle|\, m\right)\right) = 0 /; p \in \mathbb{N}^+ \land r \in \mathbb{N}^+ \land r < p-1$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.36.16.0073.01

$$\sum_{k=0}^{p-1} \mathrm{dn}\left(z + \frac{2\,K(m)\,k}{p} \,\middle|\, m\right) \mathrm{dn}\left(z + \frac{2\,K(m)\,(k+r)}{p} \,\middle|\, m\right) = p\,\mathrm{dn}\left(\frac{2\,r\,K(m)}{p} \,\middle|\, m\right) \left(1 - \frac{\left|\mathrm{cn}\left(\frac{2\,r\,K(m)}{p} \,\middle|\, m\right)\right| Z\left(\sin^{-1}\left(\mathrm{sn}\left(\frac{2\,K(m)}{p} \,\middle|\, m\right)\right) \,\middle|\, m\right)}{\left|\mathrm{sn}\left(\frac{2\,r\,K(m)}{p} \,\middle|\, m\right)\right| \mathrm{dn}\left(\frac{2\,r\,K(m)}{p} \,\middle|\, m\right)}\right) /; p \in \mathbb{N}^+ \land r \in \mathbb{N}^+ \land r < p-1 \land m \in \mathbb{R} \land m < 1$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.36.16.0074.01

$$\sum_{k=0}^{p-1} \mathrm{sn}\left(z + \frac{2\,K(m)\,k}{p} \,\middle|\, m\right) \mathrm{sn}\left(z + \frac{2\,K(m)\,(k+r)}{p} \,\middle|\, m\right) = \frac{p\,\mathrm{cn}\left(\frac{2\,r\,K(m)}{p} \,\middle|\, m\right) Z\left(\sin^{-1}\left(\mathrm{sn}\left(\frac{2\,r\,K(m)}{p} \,\middle|\, m\right)\right) \,\middle|\, m\right)}{m \left|\mathrm{sn}\left(\frac{2\,r\,K(m)}{p} \,\middle|\, m\right)\right| \left|\mathrm{cn}\left(\frac{2\,r\,K(m)}{p} \,\middle|\, m\right)\right|} /; p-2 \in \mathbb{N} \land r \in \mathbb{N}^+ \land r < p-1 \land m \in \mathbb{R} \land m < 1$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.36.16.0075.01

$$\sum_{k=0}^{p-1} \mathrm{cn}\left(z + \frac{2\,K(m)\,k}{p} \,\middle|\, m\right) \mathrm{cn}\left(z + \frac{2\,K(m)\,(k+r)}{p} \,\middle|\, m\right) = p\,\mathrm{cn}\left(\frac{2\,r\,K(m)}{p} \,\middle|\, m\right) \left(1 - \frac{\mathrm{dn}\left(\frac{2\,r\,K(m)}{p} \,\middle|\, m\right) Z\left(\sin^{-1}\left(\mathrm{sn}\left(\frac{2\,r\,K(m)}{p} \,\middle|\, m\right)\right) \,\middle|\, m\right)}{m \left|\mathrm{sn}\left(\frac{2\,r\,K(m)}{p} \,\middle|\, m\right)\right| \left|\mathrm{cn}\left(\frac{2\,r\,K(m)}{p} \,\middle|\, m\right)\right|}\right) /; p-2 \in \mathbb{N} \land r \in \mathbb{N}^+ \land r < p-1 \land m \in \mathbb{R} \land m < 1$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.36.16.0076.01

$$\sum_{k=0}^{p-1} \mathrm{cn}\left(z + \frac{4\,k\,K(m)}{p} \,\middle|\, m\right) \left(\mathrm{dn}\left(z + \frac{4\,(k-r)\,K(m)}{p} \,\middle|\, m\right) + \mathrm{dn}\left(z + \frac{4\,(k+r)\,K(m)}{p} \,\middle|\, m\right)\right) = 0 /; p \in \mathbb{N}^+ \land r \in \mathbb{N}^+ \land r < p-1$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.36.16.0077.01

$$\sum_{k=0}^{p-1}\operatorname{dn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right)\left(\operatorname{sn}\left(z+\frac{4\,(k-r)\,K(m)}{p}\,\middle|\,m\right)+\operatorname{sn}\left(z+\frac{4\,(k+r)\,K(m)}{p}\,\middle|\,m\right)\right)=0\;/;\;p\in\mathbb{N}^+\land r\in\mathbb{N}^+\land r<p$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.36.16.0078.01

$$\sum_{k=0}^{p-1}(-1)^k\operatorname{sn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right)\left(\operatorname{cn}\left(z+\frac{2\,(k-r)\,K(m)}{p}\,\middle|\,m\right)+\operatorname{cn}\left(z+\frac{2\,(k+r)\,K(m)}{p}\,\middle|\,m\right)\right)=0\;/;$$
$$\frac{p}{2}\in\mathbb{N}^+\bigwedge r\in\mathbb{N}^+\bigwedge\gcd(p,r)=1$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.36.16.0079.01

$$\sum_{k=0}^{p-1}(-1)^k\operatorname{dn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{2\,(k+r)\,K(m)}{p}\,\middle|\,m\right)=-2\operatorname{cs}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\sum_{k=0}^{p-1}(-1)^k\operatorname{Z}\left(\operatorname{am}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right)\middle|\,m\right)\;/;$$
$$\frac{p}{2}\in\mathbb{N}^+\bigwedge r\in\mathbb{N}^+\bigwedge r<p\bigwedge\gcd(p,r)=1\bigwedge 1-m>0$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.36.16.0080.01

$$\sum_{k=0}^{p-1}(-1)^k\operatorname{sn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{2\,(k+r)\,K(m)}{p}\,\middle|\,m\right)=\frac{2}{m}\operatorname{ns}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\sum_{k=0}^{p-1}(-1)^k\operatorname{Z}\left(\operatorname{am}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right)\middle|\,m\right)\;/;$$
$$\frac{p}{2}\in\mathbb{N}^+\bigwedge r\in\mathbb{N}^+\bigwedge r<p\bigwedge\gcd(p,r)=1\bigwedge 1-m>0$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.36.16.0081.01

$$\sum_{k=0}^{p-1}(-1)^k\operatorname{cn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right)\operatorname{cn}\left(z+\frac{2\,(k+r)\,K(m)}{p}\,\middle|\,m\right)=-\frac{2}{m}\operatorname{ds}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\sum_{k=0}^{p-1}(-1)^k\operatorname{Z}\left(\operatorname{am}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right)\middle|\,m\right)\;/;$$
$$\frac{p}{2}\in\mathbb{N}^+\bigwedge r\in\mathbb{N}^+\bigwedge r<p\bigwedge\gcd(p,r)=1\bigwedge 1-m>0$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

Sums over products of three Jacobi functions

09.36.16.0066.01

$$\frac{\sum_{k=0}^{p-1} \operatorname{cn}\left(z + \frac{4 k K(m)}{p} \middle| m\right) \operatorname{cn}\left(z + \frac{4 K(m) (k+n_1)}{p} \middle| m\right) \operatorname{cn}\left(z + \frac{4 K(m) (k+n_2)}{p} \middle| m\right)}{\sum_{k=0}^{p-1} \operatorname{cn}\left(z + \frac{4 k K(m)}{p} \middle| m\right)} = \frac{\sum_{k=0}^{p-1} \operatorname{cn}\left(\frac{4 k K(m)}{p} \middle| m\right) \operatorname{cn}\left(\frac{4 (k+n_1) K(m)}{p} \middle| m\right) \operatorname{cn}\left(\frac{4 (k+n_2) K(m)}{p} \middle| m\right)}{\sum_{k=0}^{p-1} \operatorname{cn}\left(\frac{4 k K(m)}{p} \middle| m\right)} /; \frac{p-1}{2} \in \mathbb{N}^+ \bigwedge n_1 \in \mathbb{Z} \bigwedge n_2 \in \mathbb{Z} \bigwedge 1 \leq n_1 < n_2 < p$$

Khare/Sukhatme_2002

Khare/Sukhatme_JMP_2002

09.36.16.0065.01

$$\operatorname{cn}(z \mid m)^2 \left(\operatorname{cn}\left(z + \frac{4 K(m)}{3} \middle| m\right) + \operatorname{cn}\left(z + \frac{8 K(m)}{3} \middle| m\right)\right) + \operatorname{cn}\left(z + \frac{4 K(m)}{3} \middle| m\right)^2 \left(\operatorname{cn}\left(z + \frac{8 K(m)}{3} \middle| m\right) + \operatorname{cn}(z \mid m)\right) + \operatorname{cn}\left(z + \frac{8 K(m)}{3} \middle| m\right)^2 \left(\operatorname{cn}(z \mid m) + \operatorname{cn}\left(z + \frac{4 K(m)}{3} \middle| m\right)\right) = -2 \operatorname{dn}\left(\frac{2 K(m)}{3} \middle| m\right) \left(\operatorname{dn}\left(\frac{2 K(m)}{3} \middle| m\right)^2 + \operatorname{dn}\left(\frac{2 K(m)}{3} \middle| m\right) + 1\right) \Big/ \left(\left(1 + \operatorname{dn}\left(\frac{2 K(m)}{3} \middle| m\right)\right) \left(1 - \operatorname{dn}\left(\frac{2 K(m)}{3} \middle| m\right)^2\right)\right) \left(\operatorname{cn}(z \mid m) + \operatorname{cn}\left(z + \frac{4 K(m)}{3} \middle| m\right) + \operatorname{cn}\left(z + \frac{8 K(m)}{3} \middle| m\right)\right)$$

Khare/Sukhatme_2002

Khare/Sukhatme_JMP_2002

09.36.16.0082.01

$$\sum_{k=0}^{p-1} \operatorname{dn}\left(z + \frac{2 K(m) k}{p} \middle| m\right)^2 \left(\operatorname{dn}\left(z + \frac{2 K(m) (k-r)}{p} \middle| m\right) + \operatorname{dn}\left(z + \frac{2 K(m) (k+r)}{p} \middle| m\right)\right) = 2 \left(\operatorname{ds}\left(\frac{2 r K(m)}{p} \middle| m\right) \operatorname{ns}\left(\frac{2 r K(m)}{p} \middle| m\right) - \operatorname{cs}\left(\frac{2 r K(m)}{p} \middle| m\right)^2\right) \sum_{k=0}^{p-1} \operatorname{dn}\left(z + \frac{2 K(m) k}{p} \middle| m\right) /; p \in \mathbb{N}^+ \bigwedge r \in \mathbb{N}^+ \bigwedge r < p$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.36.16.0083.01

$$\sum_{k=0}^{p-1} \operatorname{cn}\left(z + \frac{2 K(m) k}{p} \middle| m\right) \left(\operatorname{cn}\left(z + \frac{2 K(m) (k-r)}{p} \middle| m\right) \operatorname{dn}\left(z + \frac{2 K(m) (k-r)}{p} \middle| m\right) + \operatorname{cn}\left(z + \frac{2 K(m) (k+r)}{p} \middle| m\right) \operatorname{dn}\left(z + \frac{2 K(m) (k+r)}{p} \middle| m\right)\right) = -\frac{2}{m} \operatorname{cs}\left(\frac{2 r K(m)}{p} \middle| m\right) \left(\operatorname{ds}\left(\frac{2 r K(m)}{p} \middle| m\right) - \operatorname{ns}\left(\frac{2 r K(m)}{p} \middle| m\right)\right) \sum_{k=0}^{p-1} \operatorname{dn}\left(z + \frac{2 K(m) k}{p} \middle| m\right) /; p \in \mathbb{N}^+ \bigwedge r \in \mathbb{N}^+ \bigwedge r < p$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.36.16.0084.01

$$\sum_{k=0}^{p-1}\operatorname{sn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right)\left(\operatorname{dn}\left(z+\frac{2\,K(m)\,(k-r)}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{2\,K(m)\,(k-r)}{p}\,\middle|\,m\right)+\operatorname{dn}\left(z+\frac{2\,K(m)\,(k+r)}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{2\,K(m)\,(k+r)}{p}\,\middle|\,m\right)\right)= -\frac{2}{m}\operatorname{cs}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\left(\operatorname{ds}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)-\operatorname{ns}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\right)\sum_{k=0}^{p-1}\operatorname{dn}\left(z+\frac{2\,K(m)\,k}{p}\,\middle|\,m\right)\;/;\;p\in\mathbb{N}^+\land r\in\mathbb{N}^+\land r<p$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.36.16.0085.01

$$\sum_{k=0}^{p-1}\operatorname{dn}\left(z+\frac{2\,K(m)\,k}{p}\,\middle|\,m\right)\left(\operatorname{dn}\left(z+\frac{2\,K(m)\,(k-r)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{2\,K(m)\,(k-s)}{p}\,\middle|\,m\right)+\operatorname{dn}\left(z+\frac{2\,K(m)\,(k+r)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{2\,K(m)\,(k+s)}{p}\,\middle|\,m\right)\right)= -2\left(\operatorname{cs}\left(\frac{2\,(r-s)\,K(m)}{p}\,\middle|\,m\right)\left(\operatorname{cs}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)-\operatorname{cs}\left(\frac{2\,s\,K(m)}{p}\,\middle|\,m\right)\right)+\operatorname{cs}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\operatorname{cs}\left(\frac{2\,s\,K(m)}{p}\,\middle|\,m\right)\right) \sum_{k=0}^{p-1}\operatorname{dn}\left(z+\frac{2\,K(m)\,k}{p}\,\middle|\,m\right)\;/;\;p\in\mathbb{N}^+\land r\in\mathbb{N}^+\land r<p\land s\in\mathbb{N}^+\land s<r$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.36.16.0086.01

$$\sum_{k=0}^{p-1}\operatorname{dn}\left(z+\frac{2\,K(m)\,k}{p}\,\middle|\,m\right)\left(\operatorname{cn}\left(z+\frac{2\,K(m)\,(k-r)}{p}\,\middle|\,m\right)\operatorname{cn}\left(z+\frac{2\,K(m)\,(k-s)}{p}\,\middle|\,m\right)+\operatorname{cn}\left(z+\frac{2\,K(m)\,(k+r)}{p}\,\middle|\,m\right)\operatorname{cn}\left(z+\frac{2\,K(m)\,(k+s)}{p}\,\middle|\,m\right)\right)= -\frac{2}{m}\left(\operatorname{ds}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\operatorname{ds}\left(\frac{2\,s\,K(m)}{p}\,\middle|\,m\right)+\operatorname{ds}\left(\frac{2\,(r-s)\,K(m)}{p}\,\middle|\,m\right)\left(\operatorname{cs}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)-\operatorname{cs}\left(\frac{2\,s\,K(m)}{p}\,\middle|\,m\right)\right)\right) \sum_{k=0}^{p-1}\operatorname{dn}\left(z+\frac{2\,K(m)\,k}{p}\,\middle|\,m\right)\;/;\;p\in\mathbb{N}^+\land r\in\mathbb{N}^+\land r<p\land s\in\mathbb{N}^+\land s<r$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.36.16.0087.01

$$\sum_{k=0}^{p-1} \operatorname{dn}\left(z+\frac{2K(m)k}{p} \middle| m\right) \left(\operatorname{sn}\left(z+\frac{2K(m)(k-r)}{p} \middle| m\right) \operatorname{sn}\left(z+\frac{2K(m)(k-s)}{p} \middle| m\right) + \operatorname{sn}\left(z+\frac{2K(m)(k+r)}{p} \middle| m\right) \operatorname{sn}\left(z+\frac{2K(m)(k+s)}{p} \middle| m\right)\right) = \frac{2}{m}\left(\operatorname{ns}\left(\frac{2rK(m)}{p} \middle| m\right) \operatorname{ns}\left(\frac{2sK(m)}{p} \middle| m\right) + \operatorname{ns}\left(\frac{2(r-s)K(m)}{p} \middle| m\right)\left(\operatorname{cs}\left(\frac{2rK(m)}{p} \middle| m\right) - \operatorname{cs}\left(\frac{2sK(m)}{p} \middle| m\right)\right)\right) \sum_{k=0}^{p-1} \operatorname{dn}\left(z+\frac{2K(m)k}{p} \middle| m\right) /;\ p \in \mathbb{N}^+ \bigwedge r \in \mathbb{N}^+ \bigwedge r < p \bigwedge s \in \mathbb{N}^+ \bigwedge s < r$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.36.16.0088.01

$$\sum_{k=0}^{p-1} \operatorname{cn}\left(z+\frac{2K(m)k}{p} \middle| m\right) \left(\operatorname{cn}\left(z+\frac{2K(m)(k-r)}{p} \middle| m\right) \operatorname{dn}\left(z+\frac{2K(m)(k-s)}{p} \middle| m\right) + \operatorname{cn}\left(z+\frac{2K(m)(k+r)}{p} \middle| m\right) \operatorname{dn}\left(z+\frac{2K(m)(k+s)}{p} \middle| m\right)\right) = -\frac{2}{m}\left(\left(\operatorname{cs}\left(\frac{2sK(m)}{p} \middle| m\right) + \operatorname{cs}\left(\frac{2(r-s)K(m)}{p} \middle| m\right)\right) \operatorname{ds}\left(\frac{2rK(m)}{p} \middle| m\right) - \operatorname{ds}\left(\frac{2(r-s)K(m)}{p} \middle| m\right) \operatorname{ds}\left(\frac{2sK(m)}{p} \middle| m\right)\right) \sum_{k=0}^{p-1} \operatorname{dn}\left(z+\frac{2K(m)k}{p} \middle| m\right) /;\ p \in \mathbb{N}^+ \bigwedge r \in \mathbb{N}^+ \bigwedge r < p \bigwedge s \in \mathbb{N}^+ \bigwedge s < r$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.36.16.0089.01

$$\sum_{k=0}^{p-1} \operatorname{sn}\left(z+\frac{2K(m)k}{p} \middle| m\right) \left(\operatorname{dn}\left(z+\frac{2K(m)(k-s)}{p} \middle| m\right) \operatorname{sn}\left(z+\frac{2K(m)(k-r)}{p} \middle| m\right) + \operatorname{dn}\left(z+\frac{2K(m)(k+s)}{p} \middle| m\right) \operatorname{sn}\left(z+\frac{2K(m)(k+r)}{p} \middle| m\right)\right) = \frac{2}{m}\left(\left(\operatorname{cs}\left(\frac{2sK(m)}{p} \middle| m\right) + \operatorname{cs}\left(\frac{2(r-s)K(m)}{p} \middle| m\right)\right) \operatorname{ns}\left(\frac{2rK(m)}{p} \middle| m\right) - \operatorname{ns}\left(\frac{2(r-s)K(m)}{p} \middle| m\right) \operatorname{ns}\left(\frac{2sK(m)}{p} \middle| m\right)\right) \sum_{k=0}^{p-1} \operatorname{dn}\left(z+\frac{2K(m)k}{p} \middle| m\right) /;\ p \in \mathbb{N}^+ \bigwedge r \in \mathbb{N}^+ \bigwedge r < p \bigwedge s \in \mathbb{N}^+ \bigwedge s < r$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.36.16.0090.01

$$\sum_{k=0}^{p-1} \operatorname{cn}\left(z+\frac{2K(m)k}{p} \middle| m\right) \operatorname{dn}\left(z+\frac{2K(m)k}{p} \middle| m\right) \left(\operatorname{sn}\left(z+\frac{2K(m)(k-r)}{p} \middle| m\right) + \operatorname{sn}\left(z+\frac{2K(m)(k+r)}{p} \middle| m\right)\right) = 0 /;\ p \in \mathbb{N}^+ \bigwedge r \in \mathbb{N}^+ \bigwedge r < p-1$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.36.16.0091.01

$$\sum_{k=0}^{p-1} \operatorname{dn}\left(z+\frac{2\,K(m)\,k}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{2\,K(m)\,k}{p}\,\middle|\,m\right)\left(\operatorname{cn}\left(z+\frac{2\,K(m)\,(k-r)}{p}\,\middle|\,m\right)+\operatorname{cn}\left(z+\frac{2\,K(m)\,(k+r)}{p}\,\middle|\,m\right)\right)=0\;/;\; p\in\mathbb{N}^+\land r\in\mathbb{N}^+\land r<p-1$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.36.16.0092.01

$$\sum_{k=0}^{p-1} \operatorname{cn}\left(z+\frac{2\,K(m)\,k}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{2\,K(m)\,k}{p}\,\middle|\,m\right)\left(\operatorname{dn}\left(z+\frac{2\,K(m)\,(k-r)}{p}\,\middle|\,m\right)+\operatorname{dn}\left(z+\frac{2\,K(m)\,(k+r)}{p}\,\middle|\,m\right)\right)=0\;/;\; p\in\mathbb{N}^+\land r\in\mathbb{N}^+\land r<p-1$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.36.16.0093.01

$$\sum_{k=0}^{p-1} \operatorname{cn}\left(z+\frac{2\,K(m)\,k}{p}\,\middle|\,m\right)\left(\operatorname{dn}\left(z+\frac{2\,K(m)\,(k-s)}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{2\,K(m)\,(k-r)}{p}\,\middle|\,m\right)+\operatorname{dn}\left(z+\frac{2\,K(m)\,(k+s)}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{2\,K(m)\,(k+r)}{p}\,\middle|\,m\right)\right)=0\;/;\; p\in\mathbb{N}^+\land r\in\mathbb{N}^+\land r<p-1\land s\in\mathbb{N}^+\land s<p-1$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.36.16.0094.01

$$\sum_{k=0}^{p-1} \operatorname{sn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right)^2\left(\operatorname{sn}\left(z+\frac{4\,(k-r)\,K(m)}{p}\,\middle|\,m\right)+\operatorname{sn}\left(z+\frac{4\,(k+r)\,K(m)}{p}\,\middle|\,m\right)\right)=\frac{2}{m}\left(\operatorname{ns}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right)^2-\operatorname{ds}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right)\operatorname{cs}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right)\right)\sum_{k=0}^{p-1}\operatorname{sn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right)\;/;\; p\in\mathbb{N}^+\land r\in\mathbb{N}^+\land r<p-1$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.36.16.0095.01

$$\sum_{k=0}^{p-1} \operatorname{cn}\left(z + \frac{4\,k\,K(m)}{p} \,\middle|\, m\right) \left(\operatorname{cn}\left(z + \frac{4\,(k-r)\,K(m)}{p} \,\middle|\, m\right) \operatorname{sn}\left(z + \frac{4\,(k-r)\,K(m)}{p} \,\middle|\, m\right) + \operatorname{cn}\left(z + \frac{4\,(k+r)\,K(m)}{p} \,\middle|\, m\right) \operatorname{sn}\left(z + \frac{4\,(k+r)\,K(m)}{p} \,\middle|\, m\right)\right) = \frac{2}{m} \operatorname{ns}\left(\frac{4\,r\,K(m)}{p} \,\middle|\, m\right) \left(\operatorname{cs}\left(\frac{4\,r\,K(m)}{p} \,\middle|\, m\right) - \operatorname{ds}\left(\frac{4\,r\,K(m)}{p} \,\middle|\, m\right)\right) \sum_{k=0}^{p-1} \operatorname{sn}\left(z + \frac{4\,k\,K(m)}{p} \,\middle|\, m\right) /;\, p \in \mathbb{N}^+ \land r \in \mathbb{N}^+ \land r < p - 1$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.36.16.0096.01

$$\sum_{k=0}^{p-1} \operatorname{dn}\left(z + \frac{4\,k\,K(m)}{p} \,\middle|\, m\right) \left(\operatorname{dn}\left(z + \frac{4\,(k-r)\,K(m)}{p} \,\middle|\, m\right) \operatorname{sn}\left(z + \frac{4\,(k-r)\,K(m)}{p} \,\middle|\, m\right) + \operatorname{dn}\left(z + \frac{4\,(k+r)\,K(m)}{p} \,\middle|\, m\right) \operatorname{sn}\left(z + \frac{4\,(k+r)\,K(m)}{p} \,\middle|\, m\right)\right) = 2 \operatorname{ns}\left(\frac{8\,r\,K(m)}{p} \,\middle|\, m\right) \left(\operatorname{ds}\left(\frac{8\,r\,K(m)}{p} \,\middle|\, m\right) - \operatorname{cs}\left(\frac{8\,r\,K(m)}{p} \,\middle|\, m\right)\right) \sum_{k=0}^{p-1} \operatorname{sn}\left(z + \frac{4\,k\,K(m)}{p} \,\middle|\, m\right) /;\, p \in \mathbb{N}^+ \land r \in \mathbb{N}^+ \land r < p - 1$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.36.16.0097.01

$$\sum_{k=0}^{p-1} \operatorname{sn}\left(z + \frac{4\,k\,K(m)}{p} \,\middle|\, m\right) \left(\operatorname{cn}\left(z + \frac{4\,(k-r)\,K(m)}{p} \,\middle|\, m\right) \operatorname{cn}\left(z + \frac{4\,(k-s)\,K(m)}{p} \,\middle|\, m\right) + \operatorname{cn}\left(z + \frac{4\,(k+r)\,K(m)}{p} \,\middle|\, m\right) \operatorname{cn}\left(z + \frac{4\,(k+s)\,K(m)}{p} \,\middle|\, m\right)\right) = -\frac{2}{m} \left(\operatorname{ds}\left(\frac{4\,r\,K(m)}{p} \,\middle|\, m\right) \operatorname{ds}\left(\frac{4\,s\,K(m)}{p} \,\middle|\, m\right) + \operatorname{ds}\left(\frac{4\,(r-s)\,K(m)}{p} \,\middle|\, m\right) \left(\operatorname{ns}\left(\frac{4\,r\,K(m)}{p} \,\middle|\, m\right) - \operatorname{ns}\left(\frac{4\,s\,K(m)}{p} \,\middle|\, m\right)\right)\right) \sum_{k=0}^{p-1} \operatorname{sn}\left(z + \frac{4\,k\,K(m)}{p} \,\middle|\, m\right) /;\, p \in \mathbb{N}^+ \land r \in \mathbb{N}^+ \land r < p \land s \in \mathbb{N}^+ \land s < r$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.36.16.0098.01

$$\sum_{k=0}^{p-1} \operatorname{sn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right) \left(\operatorname{dn}\left(z+\frac{4\,(k-r)\,K(m)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{4\,(k-s)\,K(m)}{p}\,\middle|\,m\right)+\operatorname{dn}\left(z+\frac{4\,(k+r)\,K(m)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{4\,(k+s)\,K(m)}{p}\,\middle|\,m\right)\right) = -2\left(\operatorname{cs}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right)\operatorname{cs}\left(\frac{4\,s\,K(m)}{p}\,\middle|\,m\right)+\operatorname{cs}\left(\frac{4\,(r-s)\,K(m)}{p}\,\middle|\,m\right)\left(\operatorname{ns}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right)-\operatorname{ns}\left(\frac{4\,s\,K(m)}{p}\,\middle|\,m\right)\right)\right) \sum_{k=0}^{p-1} \operatorname{sn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right) /;\; p\in\mathbb{N}^+ \land r\in\mathbb{N}^+ \land r<p \land s\in\mathbb{N}^+ \land s<r$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.36.16.0099.01

$$\sum_{k=0}^{p-1} \operatorname{dn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right) \left(\operatorname{cn}\left(z+\frac{4\,(k-r)\,K(m)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{4\,(k-r)\,K(m)}{p}\,\middle|\,m\right)+\operatorname{cn}\left(z+\frac{4\,(k+r)\,K(m)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{4\,(k+r)\,K(m)}{p}\,\middle|\,m\right)\right) = -2\operatorname{ds}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right)\left(\operatorname{cs}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right)-\operatorname{ns}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right)\right)\sum_{k=0}^{p-1} \operatorname{cn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right) /;\; p\in\mathbb{N}^+ \land r\in\mathbb{N}^+ \land r<p$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.36.16.0100.01

$$\sum_{k=0}^{p-1} \operatorname{sn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right) \left(\operatorname{cn}\left(z+\frac{4\,(k-r)\,K(m)}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{4\,(k-r)\,K(m)}{p}\,\middle|\,m\right)+\operatorname{cn}\left(z+\frac{4\,(k+r)\,K(m)}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{4\,(k+r)\,K(m)}{p}\,\middle|\,m\right)\right) = -\frac{2}{m}\operatorname{ds}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right)\left(\operatorname{cs}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right)-\operatorname{ns}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right)\right)\sum_{k=0}^{p-1} \operatorname{cn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right) /;\; p\in\mathbb{N}^+ \land r\in\mathbb{N}^+ \land r<p$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.36.16.0101.01

$$\sum_{k=0}^{p-1} \operatorname{cn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right) \left(\operatorname{sn}\left(z+\frac{4\,(k-r)\,K(m)}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{4\,(k-s)\,K(m)}{p}\,\middle|\,m\right)+\operatorname{sn}\left(z+\frac{4\,(k+r)\,K(m)}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{4\,(k+s)\,K(m)}{p}\,\middle|\,m\right)\right) = \frac{2}{m}\left(\left(\operatorname{ds}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right)-\operatorname{ds}\left(\frac{4\,s\,K(m)}{p}\,\middle|\,m\right)\right)\operatorname{ns}\left(\frac{4\,(r-s)\,K(m)}{p}\,\middle|\,m\right)+\operatorname{ns}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right)\operatorname{ns}\left(\frac{4\,s\,K(m)}{p}\,\middle|\,m\right)\right) \sum_{k=0}^{p-1}\operatorname{cn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right) /;\; p\in\mathbb{N}^+\land r\in\mathbb{N}^+\land r<p\land s\in\mathbb{N}^+\land s<r$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.36.16.0102.01

$$\sum_{k=0}^{p-1} \operatorname{cn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right) \left(\operatorname{dn}\left(z+\frac{4\,(k-r)\,K(m)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{4\,(k-s)\,K(m)}{p}\,\middle|\,m\right)+\operatorname{dn}\left(z+\frac{4\,(k+r)\,K(m)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{4\,(k+s)\,K(m)}{p}\,\middle|\,m\right)\right) = -2\left(\operatorname{cs}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right)\operatorname{cs}\left(\frac{4\,s\,K(m)}{p}\,\middle|\,m\right)+\operatorname{cs}\left(\frac{4\,(r-s)\,K(m)}{p}\,\middle|\,m\right)\left(\operatorname{ds}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right)-\operatorname{ds}\left(\frac{4\,s\,K(m)}{p}\,\middle|\,m\right)\right)\right) \sum_{k=0}^{p-1}\operatorname{cn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right) /;\; p\in\mathbb{N}^+\land r\in\mathbb{N}^+\land r<p\land s\in\mathbb{N}^+\land s<r$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.36.16.0103.01

$$\sum_{k=0}^{p-1} (-1)^k\operatorname{dn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{2\,(k+r)\,K(m)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{2\,(k+2\,r)\,K(m)}{p}\,\middle|\,m\right) = -\left(\operatorname{cs}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)^2+2\operatorname{cs}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\operatorname{cs}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right)\right)\sum_{k=0}^{p-1}(-1)^k\operatorname{dn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right) /;\; \frac{p}{2}\in\mathbb{N}^+\bigwedge r\in\mathbb{N}^+\bigwedge\gcd(p,r)=1$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.36.16.0104.01

$$\sum_{k=0}^{p-1}(-1)^k \operatorname{dn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right)^2 \left(\operatorname{dn}\left(z+\frac{2\,(k-r)\,K(m)}{p}\,\middle|\,m\right)+\operatorname{dn}\left(z+\frac{2\,(k+r)\,K(m)}{p}\,\middle|\,m\right)\right) = 2\left(\operatorname{cs}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)^2+\operatorname{ds}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\operatorname{ns}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\right)\sum_{k=0}^{p-1}(-1)^k \operatorname{dn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right) /;$$

$$\frac{p}{2}\in\mathbb{N}^+ \bigwedge r\in\mathbb{N}^+ \bigwedge \gcd(p,\,r)=1$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.36.16.0105.01

$$\sum_{k=0}^{p-1}(-1)^k \operatorname{cn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right) \left(\operatorname{cn}\left(z+\frac{2\,(k-r)\,K(m)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{2\,(k-r)\,K(m)}{p}\,\middle|\,m\right)+\operatorname{cn}\left(z+\frac{2\,(k+r)\,K(m)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{2\,(k+r)\,K(m)}{p}\,\middle|\,m\right)\right) = -\frac{2}{m}\operatorname{cs}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\left(\operatorname{ds}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)+\operatorname{ns}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\right)\sum_{k=0}^{p-1}(-1)^k \operatorname{dn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right) /;$$

$$\frac{p}{2}\in\mathbb{N}^+ \bigwedge r\in\mathbb{N}^+ \bigwedge \gcd(p,\,r)=1$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.36.16.0106.01

$$\sum_{k=0}^{p-1}(-1)^k \operatorname{sn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right) \left(\operatorname{dn}\left(z+\frac{2\,(k-r)\,K(m)}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{2\,(k-r)\,K(m)}{p}\,\middle|\,m\right)+\operatorname{dn}\left(z+\frac{2\,(k+r)\,K(m)}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{2\,(k+r)\,K(m)}{p}\,\middle|\,m\right)\right) = \frac{2}{m}\operatorname{cs}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\left(\operatorname{ds}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)+\operatorname{ns}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\right)\sum_{k=0}^{p-1}(-1)^k \operatorname{dn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right) /;$$

$$\frac{p}{2}\in\mathbb{N}^+ \bigwedge r\in\mathbb{N}^+ \bigwedge \gcd(p,\,r)=1$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

Sums over products of four Jacobi functions

09.36.16.0107.01

$$\sum_{k=0}^{p-1} \operatorname{dn}\left(z+\frac{2K(m)k}{p}\,\middle|\,m\right)^2 \left(\operatorname{cn}\left(z+\frac{2K(m)(k-r)}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{2K(m)(k-r)}{p}\,\middle|\,m\right)+\operatorname{cn}\left(z+\frac{2K(m)(k+r)}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{2K(m)(k+r)}{p}\,\middle|\,m\right)\right) = -2\left(\operatorname{cs}\left(\frac{2rK(m)}{p}\,\middle|\,m\right)^2+\operatorname{ds}\left(\frac{2rK(m)}{p}\,\middle|\,m\right)\operatorname{ns}\left(\frac{2rK(m)}{p}\,\middle|\,m\right)\right)\sum_{k=0}^{p-1}\operatorname{cn}\left(z+\frac{2K(m)k}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{2K(m)k}{p}\,\middle|\,m\right) /;\ p\in\mathbb{N}^+ \land r\in\mathbb{N}^+ \land r<p$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.36.16.0108.01

$$\sum_{k=0}^{p-1} \operatorname{sn}\left(z+\frac{2K(m)k}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{2K(m)k}{p}\,\middle|\,m\right) \left(\operatorname{cn}\left(z+\frac{2K(m)(k-r)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{2K(m)(k-r)}{p}\,\middle|\,m\right)+\operatorname{cn}\left(z+\frac{2K(m)(k+r)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{2K(m)(k+r)}{p}\,\middle|\,m\right)\right) = -2\operatorname{cs}\left(\frac{2rK(m)}{p}\,\middle|\,m\right)\left(\operatorname{ds}\left(\frac{2rK(m)}{p}\,\middle|\,m\right)+\operatorname{ns}\left(\frac{2rK(m)}{p}\,\middle|\,m\right)\right)\sum_{k=0}^{p-1}\operatorname{cn}\left(z+\frac{2K(m)k}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{2K(m)k}{p}\,\middle|\,m\right) /;\ p\in\mathbb{N}^+ \land r\in\mathbb{N}^+ \land r<p$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.36.16.0109.01

$$\sum_{k=0}^{p-1} \operatorname{sn}\left(z+\frac{2K(m)k}{p}\,\middle|\,m\right)\operatorname{cn}\left(z+\frac{2K(m)k}{p}\,\middle|\,m\right) \left(\operatorname{dn}\left(z+\frac{2K(m)(k-r)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{2K(m)(k-s)}{p}\,\middle|\,m\right)+\operatorname{dn}\left(z+\frac{2K(m)(k+r)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{2K(m)(k+s)}{p}\,\middle|\,m\right)\right) = -2\operatorname{cs}\left(\frac{2rK(m)}{p}\,\middle|\,m\right)\operatorname{cs}\left(\frac{2sK(m)}{p}\,\middle|\,m\right)\sum_{k=0}^{p-1}\operatorname{cn}\left(z+\frac{2K(m)k}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{2K(m)k}{p}\,\middle|\,m\right) /;\ p\in\mathbb{N}^+ \land r\in\mathbb{N}^+ \land r<p \land s\in\mathbb{N}^+ \land s<r$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.36.16.0110.01

$$\sum_{k=0}^{p-1} \operatorname{sn}\left(z+\frac{2K(m)k}{p}\,\middle|\,m\right)\operatorname{cn}\left(z+\frac{2K(m)k}{p}\,\middle|\,m\right)\left(\operatorname{cn}\left(z+\frac{2K(m)(k-r)}{p}\,\middle|\,m\right)\operatorname{cn}\left(z+\frac{2K(m)(k-s)}{p}\,\middle|\,m\right)+\operatorname{cn}\left(z+\frac{2K(m)(k+r)}{p}\,\middle|\,m\right)\operatorname{cn}\left(z+\frac{2K(m)(k+s)}{p}\,\middle|\,m\right)\right)= -\frac{2}{m}\operatorname{ds}\left(\frac{2rK(m)}{p}\,\middle|\,m\right)\operatorname{ds}\left(\frac{2sK(m)}{p}\,\middle|\,m\right)\sum_{k=0}^{p-1}\operatorname{cn}\left(z+\frac{2K(m)k}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{2K(m)k}{p}\,\middle|\,m\right)\;/;\; p\in\mathbb{N}^+\land r\in\mathbb{N}^+\land r<p\land s\in\mathbb{N}^+\land s<r$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.36.16.0111.01

$$\sum_{k=0}^{p-1} \operatorname{sn}\left(z+\frac{2K(m)k}{p}\,\middle|\,m\right)\operatorname{cn}\left(z+\frac{2K(m)k}{p}\,\middle|\,m\right)\left(\operatorname{sn}\left(z+\frac{2K(m)(k-r)}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{2K(m)(k-s)}{p}\,\middle|\,m\right)+\operatorname{sn}\left(z+\frac{2K(m)(k+r)}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{2K(m)(k+s)}{p}\,\middle|\,m\right)\right)= \frac{2}{m}\operatorname{ns}\left(\frac{2rK(m)}{p}\,\middle|\,m\right)\operatorname{ns}\left(\frac{2sK(m)}{p}\,\middle|\,m\right)\sum_{k=0}^{p-1}\operatorname{cn}\left(z+\frac{2K(m)k}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{2K(m)k}{p}\,\middle|\,m\right)\;/;\; p\in\mathbb{N}^+\land r\in\mathbb{N}^+\land r<p\land s\in\mathbb{N}^+\land s<r$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.36.16.0112.01

$$\sum_{k=0}^{p-1} \operatorname{dn}\left(z+\frac{2K(m)k}{p}\,\middle|\,m\right)\operatorname{cn}\left(z+\frac{2K(m)k}{p}\,\middle|\,m\right)\left(\operatorname{dn}\left(z+\frac{2K(m)(k-s)}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{2K(m)(k-r)}{p}\,\middle|\,m\right)+\operatorname{dn}\left(z+\frac{2K(m)(k+s)}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{2K(m)(k+r)}{p}\,\middle|\,m\right)\right)= -2\operatorname{ns}\left(\frac{2rK(m)}{p}\,\middle|\,m\right)\operatorname{cs}\left(\frac{2sK(m)}{p}\,\middle|\,m\right)\sum_{k=0}^{p-1}\operatorname{cn}\left(z+\frac{2K(m)k}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{2K(m)k}{p}\,\middle|\,m\right)\;/;\; p\in\mathbb{N}^+\land r\in\mathbb{N}^+\land r<p\land s\in\mathbb{N}^+\land s<r$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.36.16.0113.01

$$\sum_{k=0}^{p-1} \mathrm{dn}\left(z + \frac{2\,k\,K(m)}{p} \,\middle|\, m\right)^2 \mathrm{dn}\left(z + \frac{2\,(k+r)\,K(m)}{p} \,\middle|\, m\right)^2 = \frac{p}{2\,K(m)} \left(\int_0^{2\,K(m)} \mathrm{dn}(t \mid m)^2\, \mathrm{dn}\left(t + \frac{2\,r\,K(m)}{p} \,\middle|\, m\right)^2 dt + 4\,E(m)\, \mathrm{cs}\left(\frac{2\,r\,K(m)}{p} \,\middle|\, m\right)^2\right) - 2\,\mathrm{cs}\left(\frac{2\,r\,K(m)}{p} \,\middle|\, m\right)^2 \sum_{k=0}^{p-1} \mathrm{dn}\left(z + \frac{2\,k\,K(m)}{p} \,\middle|\, m\right)^2 \text{/; } p \in \mathbb{N}^+ \land r \in \mathbb{N}^+ \land r < p-1$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.36.16.0114.01

$$\sum_{k=0}^{p-1} \mathrm{cn}\left(z + \frac{2\,k\,K(m)}{p} \,\middle|\, m\right) \mathrm{sn}\left(z + \frac{2\,k\,K(m)}{p} \,\middle|\, m\right) \left(\mathrm{cn}\left(z + \frac{2\,(k-r)\,K(m)}{p} \,\middle|\, m\right) \mathrm{sn}\left(z + \frac{2\,(k-r)\,K(m)}{p} \,\middle|\, m\right) + \mathrm{cn}\left(z + \frac{2\,(k+r)\,K(m)}{p} \,\middle|\, m\right) \mathrm{sn}\left(z + \frac{2\,(k+r)\,K(m)}{p} \,\middle|\, m\right)\right) = \frac{4}{m^2}\, \mathrm{ns}\left(\frac{2\,r\,K(m)}{p} \,\middle|\, m\right) \mathrm{ds}\left(\frac{2\,r\,K(m)}{p} \,\middle|\, m\right) \sum_{k=0}^{p-1} \mathrm{dn}\left(z + \frac{2\,k\,K(m)}{p} \,\middle|\, m\right)^2 + \frac{p}{2\,K(m)} \left(\int_0^{2\,K(m)} \mathrm{cn}(t \mid m)\, \mathrm{sn}(t \mid m) \left(\mathrm{cn}\left(t + \frac{2\,r\,K(m)}{p} \,\middle|\, m\right) \mathrm{sn}\left(t + \frac{2\,r\,K(m)}{p} \,\middle|\, m\right) + \mathrm{cn}\left(t - \frac{2\,r\,K(m)}{p} \,\middle|\, m\right) \mathrm{sn}\left(t - \frac{2\,r\,K(m)}{p} \,\middle|\, m\right)\right) dt - \frac{8}{m^2}\, \mathrm{ns}\left(\frac{2\,r\,K(m)}{p} \,\middle|\, m\right) \mathrm{ds}\left(\frac{2\,r\,K(m)}{p} \,\middle|\, m\right) E(m)\right) \text{/; } p \in \mathbb{N}^+ \land r \in \mathbb{N}^+ \land r < p-1$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.36.16.0115.01

$$\sum_{k=0}^{p-1} \mathrm{cn}\left(z + \frac{2\,k\,K(m)}{p} \,\middle|\, m\right) \mathrm{dn}\left(z + \frac{2\,k\,K(m)}{p} \,\middle|\, m\right) \left(\mathrm{cn}\left(z + \frac{2\,(k-r)\,K(m)}{p} \,\middle|\, m\right) \mathrm{dn}\left(z + \frac{2\,(k-r)\,K(m)}{p} \,\middle|\, m\right) + \mathrm{cn}\left(z + \frac{2\,(k+r)\,K(m)}{p} \,\middle|\, m\right) \mathrm{dn}\left(z + \frac{2\,(k+r)\,K(m)}{p} \,\middle|\, m\right)\right) = -\frac{4}{m}\, \mathrm{cs}\left(\frac{2\,r\,K(m)}{p} \,\middle|\, m\right) \mathrm{ds}\left(\frac{2\,r\,K(m)}{p} \,\middle|\, m\right) \sum_{k=0}^{p-1} \mathrm{dn}\left(z + \frac{2\,k\,K(m)}{p} \,\middle|\, m\right)^2 + \frac{p}{2\,K(m)} \left(\int_0^{2\,K(m)} \mathrm{cn}(t \mid m)\, \mathrm{dn}(t \mid m) \left(\mathrm{cn}\left(t + \frac{2\,r\,K(m)}{p} \,\middle|\, m\right) \mathrm{dn}\left(t + \frac{2\,r\,K(m)}{p} \,\middle|\, m\right) + \mathrm{cn}\left(t - \frac{2\,r\,K(m)}{p} \,\middle|\, m\right) \mathrm{dn}\left(t - \frac{2\,r\,K(m)}{p} \,\middle|\, m\right)\right) dt + \frac{8}{m}\, E(m)\, \mathrm{cs}\left(\frac{2\,r\,K(m)}{p} \,\middle|\, m\right) \mathrm{ds}\left(\frac{2\,r\,K(m)}{p} \,\middle|\, m\right)\right) \text{/; } p \in \mathbb{N}^+ \land r \in \mathbb{N}^+ \land r < p-1$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.36.16.0116.01

$$\sum_{k=0}^{p-1}\operatorname{sn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right)\left(\operatorname{dn}\left(z+\frac{2\,(k-r)\,K(m)}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{2\,(k-r)\,K(m)}{p}\,\middle|\,m\right)+\operatorname{dn}\left(z+\frac{2\,(k+r)\,K(m)}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{2\,(k+r)\,K(m)}{p}\,\middle|\,m\right)\right)=\frac{4}{m}\operatorname{cs}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\operatorname{ns}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\sum_{k=0}^{p-1}\operatorname{dn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right)^2+\frac{p}{2\,K(m)}\left(\int_0^{2\,K(m)}\operatorname{sn}(t\mid m)\operatorname{dn}(t\mid m)\left(\operatorname{dn}\left(t+\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\operatorname{sn}\left(t+\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)+\operatorname{dn}\left(t-\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\operatorname{sn}\left(t-\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\right)dt-\frac{8}{m}\operatorname{cs}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\operatorname{ns}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)E(m)\right)\text{/; }p\in\mathbb{N}^+\land r\in\mathbb{N}^+\land r<p-1$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.36.16.0117.01

$$\sum_{k=0}^{p-1}\operatorname{dn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right)^3\left(\operatorname{dn}\left(z+\frac{2\,(k-r)\,K(m)}{p}\,\middle|\,m\right)+\operatorname{dn}\left(z+\frac{2\,(k+r)\,K(m)}{p}\,\middle|\,m\right)\right)=2\operatorname{ds}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\operatorname{ns}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\sum_{k=0}^{p-1}\operatorname{dn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right)^2+\frac{p}{2\,K(m)}\left(\int_0^{2\,K(m)}\operatorname{dn}(t\mid m)^3\left(\operatorname{dn}\left(t+\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)+\operatorname{dn}\left(t-\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\right)dt-4\operatorname{ns}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\operatorname{ds}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)E(m)\right)\text{/; }p\in\mathbb{N}^+\land r\in\mathbb{N}^+\land r<p-1$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.36.16.0118.01

$$\sum_{k=0}^{p-1}\operatorname{sn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right)^3\left(\operatorname{sn}\left(z+\frac{2\,(k-r)\,K(m)}{p}\,\middle|\,m\right)+\operatorname{sn}\left(z+\frac{2\,(k+r)\,K(m)}{p}\,\middle|\,m\right)\right)=\frac{2}{m^2}\operatorname{cs}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\operatorname{ds}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\sum_{k=0}^{p-1}\operatorname{dn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right)^2+\frac{p}{2\,K(m)}\left(\int_0^{2\,K(m)}\operatorname{sn}(t\mid m)^3\left(\operatorname{sn}\left(t+\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)+\operatorname{sn}\left(t-\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\right)dt-\frac{4}{m^2}\operatorname{cs}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\operatorname{ds}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)E(m)\right)\text{/; }p\in\mathbb{N}^+\land r\in\mathbb{N}^+\land r<p-1$$

Khare/Lakshminarayan/Sukhatme_2002

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09.36.16.0119.01

$$\sum_{k=0}^{p-1} \operatorname{cn}\left(z+\frac{2kK(m)}{p}\,\middle|\,m\right)^3 \left(\operatorname{cn}\left(z+\frac{2(k-r)K(m)}{p}\,\middle|\,m\right)+\operatorname{cn}\left(z+\frac{2(k+r)K(m)}{p}\,\middle|\,m\right)\right) = \frac{p}{2K(m)}\left(\int_0^{2K(m)} \operatorname{cn}(t\,|\,m)^3 \left(\operatorname{cn}\left(t+\frac{2rK(m)}{p}\,\middle|\,m\right)+\operatorname{cn}\left(t-\frac{2rK(m)}{p}\,\middle|\,m\right)\right)dt - \frac{4}{m^2}\operatorname{ns}\left(\frac{2rK(m)}{p}\,\middle|\,m\right)\operatorname{cs}\left(\frac{2rK(m)}{p}\,\middle|\,m\right)E(m)\right) + \frac{2}{m^2}\operatorname{cs}\left(\frac{2rK(m)}{p}\,\middle|\,m\right)\operatorname{ns}\left(\frac{2rK(m)}{p}\,\middle|\,m\right)\sum_{k=0}^{p-1}\operatorname{dn}\left(z+\frac{2kK(m)}{p}\,\middle|\,m\right)^2 \;/;\; p\in\mathbb{N}^+ \land r\in\mathbb{N}^+ \land r<p-1$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.36.16.0120.01

$$\sum_{k=0}^{p-1} \operatorname{cn}\left(z+\frac{4kK(m)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{4kK(m)}{p}\,\middle|\,m\right)\left(\operatorname{sn}\left(z+\frac{4(k-r)K(m)}{p}\,\middle|\,m\right)^2+\operatorname{sn}\left(z+\frac{4(k+r)K(m)}{p}\,\middle|\,m\right)^2\right) = \frac{2}{m}\left(\operatorname{ns}\left(\frac{4rK(m)}{p}\,\middle|\,m\right)^2+\operatorname{cs}\left(\frac{4rK(m)}{p}\,\middle|\,m\right)\operatorname{ds}\left(\frac{4rK(m)}{p}\,\middle|\,m\right)\right) \sum_{k=0}^{p-1}\operatorname{cn}\left(z+\frac{4kK(m)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{4kK(m)}{p}\,\middle|\,m\right) \;/;\; p\in\mathbb{N}^+ \land r\in\mathbb{N}^+ \land r<p-1$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.36.16.0121.01

$$\sum_{k=0}^{p-1} \operatorname{sn}\left(z+\frac{4kK(m)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{4kK(m)}{p}\,\middle|\,m\right)\left(\operatorname{cn}\left(z+\frac{4(k-r)K(m)}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{4(k-r)K(m)}{p}\,\middle|\,m\right)+\operatorname{cn}\left(z+\frac{4(k+r)K(m)}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{4(k+r)K(m)}{p}\,\middle|\,m\right)\right) = \frac{2}{m}\operatorname{ns}\left(\frac{4rK(m)}{p}\,\middle|\,m\right)\left(\operatorname{cs}\left(\frac{4rK(m)}{p}\,\middle|\,m\right)+\operatorname{ds}\left(\frac{4rK(m)}{p}\,\middle|\,m\right)\right)\sum_{k=0}^{p-1}\operatorname{cn}\left(z+\frac{4kK(m)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{4kK(m)}{p}\,\middle|\,m\right) \;/;\; p\in\mathbb{N}^+ \land r\in\mathbb{N}^+ \land r<p-1$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.36.16.0122.01

$$\sum_{k=0}^{p-1} \operatorname{cn}\left(z+\frac{4kK(m)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{4kK(m)}{p}\,\middle|\,m\right)\left(\operatorname{sn}\left(z+\frac{4(k-r)K(m)}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{4(k-s)K(m)}{p}\,\middle|\,m\right)+\operatorname{sn}\left(z+\frac{4(k+r)K(m)}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{4(k+s)K(m)}{p}\,\middle|\,m\right)\right) = \frac{2}{m}\operatorname{ns}\left(\frac{4rK(m)}{p}\,\middle|\,m\right)\operatorname{ns}\left(\frac{4sK(m)}{p}\,\middle|\,m\right)\sum_{k=0}^{p-1}\operatorname{cn}\left(z+\frac{4kK(m)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{4kK(m)}{p}\,\middle|\,m\right) \;/;\; p\in\mathbb{N}^+ \land r\in\mathbb{N}^+ \land r<p \land s\in\mathbb{N}^+ \land s<r$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.36.16.0123.01

$$\sum_{k=0}^{p-1} \operatorname{cn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right) \left(\operatorname{cn}\left(z+\frac{4\,(k-r)\,K(m)}{p}\,\middle|\,m\right)\operatorname{cn}\left(z+\frac{4\,(k-s)\,K(m)}{p}\,\middle|\,m\right)+\operatorname{cn}\left(z+\frac{4\,(k+r)\,K(m)}{p}\,\middle|\,m\right)\operatorname{cn}\left(z+\frac{4\,(k+s)\,K(m)}{p}\,\middle|\,m\right)\right) = -\frac{2}{m}\operatorname{ds}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right)\operatorname{ds}\left(\frac{4\,s\,K(m)}{p}\,\middle|\,m\right)\sum_{k=0}^{p-1}\operatorname{cn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right) /;$$

$p\in\mathbb{N}^+ \land r\in\mathbb{N}^+ \land r<p \land s\in\mathbb{N}^+ \land s<r$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.36.16.0124.01

$$\sum_{k=0}^{p-1} \operatorname{cn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right) \left(\operatorname{dn}\left(z+\frac{4\,(k-r)\,K(m)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{4\,(k-s)\,K(m)}{p}\,\middle|\,m\right)+\operatorname{dn}\left(z+\frac{4\,(k+r)\,K(m)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{4\,(k+s)\,K(m)}{p}\,\middle|\,m\right)\right) = -2\operatorname{cs}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right)\operatorname{cs}\left(\frac{4\,s\,K(m)}{p}\,\middle|\,m\right)\sum_{k=0}^{p-1}\operatorname{cn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right) /;$$

$p\in\mathbb{N}^+ \land r\in\mathbb{N}^+ \land r<p \land s\in\mathbb{N}^+ \land s<r$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.36.16.0125.01

$$\sum_{k=0}^{p-1} \operatorname{cn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right) \left(\operatorname{dn}\left(z+\frac{4\,(k-r)\,K(m)}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{4\,(k-s)\,K(m)}{p}\,\middle|\,m\right)+\operatorname{dn}\left(z+\frac{4\,(k+r)\,K(m)}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{4\,(k+s)\,K(m)}{p}\,\middle|\,m\right)\right) = \frac{2}{m}\operatorname{cs}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right)\operatorname{ns}\left(\frac{4\,s\,K(m)}{p}\,\middle|\,m\right)\sum_{k=0}^{p-1}\operatorname{cn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right) /;$$

$p\in\mathbb{N}^+ \land r\in\mathbb{N}^+ \land r<p \land s\in\mathbb{N}^+ \land s<r$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.36.16.0126.01

$$\sum_{k=0}^{p-1} \operatorname{cn}\left(z+\frac{4\,k\,K(m)}{p} \,\middle|\, m\right)^2 \left(\operatorname{dn}\left(z+\frac{4\,(k-r)\,K(m)}{p} \,\middle|\, m\right) \operatorname{sn}\left(z+\frac{4\,(k-r)\,K(m)}{p} \,\middle|\, m\right)+\operatorname{dn}\left(z+\frac{4\,(k+r)\,K(m)}{p} \,\middle|\, m\right) \operatorname{sn}\left(z+\frac{4\,(k+r)\,K(m)}{p} \,\middle|\, m\right)\right) = -\frac{2}{m}\left(\operatorname{ds}\left(\frac{4\,r\,K(m)}{p} \,\middle|\, m\right)^2+\operatorname{cs}\left(\frac{4\,r\,K(m)}{p} \,\middle|\, m\right) \operatorname{ns}\left(\frac{4\,r\,K(m)}{p} \,\middle|\, m\right)\right) \sum_{k=0}^{p-1} \operatorname{dn}\left(z+\frac{4\,k\,K(m)}{p} \,\middle|\, m\right) \operatorname{sn}\left(z+\frac{4\,k\,K(m)}{p} \,\middle|\, m\right) /;\ p \in \mathbb{N}^+ \land r \in \mathbb{N}^+ \land r < p$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.36.16.0127.01

$$\sum_{k=0}^{p-1} \operatorname{cn}\left(z+\frac{4\,k\,K(m)}{p} \,\middle|\, m\right) \operatorname{sn}\left(z+\frac{4\,k\,K(m)}{p} \,\middle|\, m\right) \left(\operatorname{cn}\left(z+\frac{4\,(k-r)\,K(m)}{p} \,\middle|\, m\right) \operatorname{dn}\left(z+\frac{4\,(k-r)\,K(m)}{p} \,\middle|\, m\right)+\operatorname{cn}\left(z+\frac{4\,(k+r)\,K(m)}{p} \,\middle|\, m\right) \operatorname{dn}\left(z+\frac{4\,(k+r)\,K(m)}{p} \,\middle|\, m\right)\right) = -\frac{2}{m} \operatorname{ds}\left(\frac{4\,r\,K(m)}{p} \,\middle|\, m\right)\left(\operatorname{cs}\left(\frac{4\,r\,K(m)}{p} \,\middle|\, m\right)+\operatorname{ns}\left(\frac{4\,r\,K(m)}{p} \,\middle|\, m\right)\right) \sum_{k=0}^{p-1} \operatorname{dn}\left(z+\frac{4\,k\,K(m)}{p} \,\middle|\, m\right) \operatorname{sn}\left(z+\frac{4\,k\,K(m)}{p} \,\middle|\, m\right) /;\ p \in \mathbb{N}^+ \land r \in \mathbb{N}^+ \land r < p$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.36.16.0128.01

$$\sum_{k=0}^{p-1} \operatorname{cn}\left(z+\frac{4\,k\,K(m)}{p} \,\middle|\, m\right)^2 \operatorname{dn}\left(z+\frac{4\,k\,K(m)}{p} \,\middle|\, m\right) \left(\operatorname{sn}\left(z+\frac{4\,(k-r)\,K(m)}{p} \,\middle|\, m\right)+\operatorname{sn}\left(z+\frac{4\,(k+r)\,K(m)}{p} \,\middle|\, m\right)\right) = \frac{2}{m} \operatorname{cs}\left(\frac{4\,r\,K(m)}{p} \,\middle|\, m\right) \operatorname{ds}\left(\frac{4\,r\,K(m)}{p} \,\middle|\, m\right) \sum_{k=0}^{p-1} \operatorname{dn}\left(z+\frac{4\,k\,K(m)}{p} \,\middle|\, m\right) \operatorname{sn}\left(z+\frac{4\,k\,K(m)}{p} \,\middle|\, m\right) /;\ p \in \mathbb{N}^+ \land r \in \mathbb{N}^+ \land r < p$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.36.16.0129.01

$$\sum_{k=0}^{p-1} \operatorname{sn}\left(z+\frac{4\,k\,K(m)}{p} \,\middle|\, m\right) \operatorname{dn}\left(z+\frac{4\,k\,K(m)}{p} \,\middle|\, m\right) \left(\operatorname{cn}\left(z+\frac{4\,(k-r)\,K(m)}{p} \,\middle|\, m\right) \operatorname{cn}\left(z+\frac{4\,(k+s)\,K(m)}{p} \,\middle|\, m\right)+\operatorname{cn}\left(z+\frac{4\,(k+r)\,K(m)}{p} \,\middle|\, m\right) \operatorname{cn}\left(z+\frac{4\,(k+s)\,K(m)}{p} \,\middle|\, m\right)\right) = -\frac{2}{m} \operatorname{ds}\left(\frac{4\,r\,K(m)}{p} \,\middle|\, m\right) \operatorname{ds}\left(\frac{4\,s\,K(m)}{p} \,\middle|\, m\right) \sum_{k=0}^{p-1} \operatorname{dn}\left(z+\frac{4\,k\,K(m)}{p} \,\middle|\, m\right) \operatorname{sn}\left(z+\frac{4\,k\,K(m)}{p} \,\middle|\, m\right) /;\ p \in \mathbb{N}^+ \land r \in \mathbb{N}^+ \land r < p \land s \in \mathbb{N}^+ \land s < r$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.36.16.0130.01

$$\sum_{k=0}^{p-1} \operatorname{sn}\left(z + \frac{4\,k\,K(m)}{p} \,\middle|\, m\right) \operatorname{dn}\left(z + \frac{4\,k\,K(m)}{p} \,\middle|\, m\right) \left(\operatorname{dn}\left(z + \frac{4\,(k-r)\,K(m)}{p} \,\middle|\, m\right) \operatorname{dn}\left(z + \frac{4\,(k-s)\,K(m)}{p} \,\middle|\, m\right) + \operatorname{dn}\left(z + \frac{4\,(k+r)\,K(m)}{p}\right) \operatorname{dn}\left(z + \frac{4\,(k+s)\,K(m)}{p} \,\middle|\, m\right)\right) = -2 \operatorname{cs}\left(\frac{4\,r\,K(m)}{p} \,\middle|\, m\right) \operatorname{cs}\left(\frac{4\,s\,K(m)}{p} \,\middle|\, m\right) \sum_{k=0}^{p-1} \operatorname{dn}\left(z + \frac{4\,k\,K(m)}{p} \,\middle|\, m\right) \operatorname{sn}\left(z + \frac{4\,k\,K(m)}{p} \,\middle|\, m\right) /;$$

$$p \in \mathbb{N}^+ \land r \in \mathbb{N}^+ \land r < p \land s \in \mathbb{N}^+ \land s < r$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.36.16.0131.01

$$\sum_{k=0}^{p-1} \operatorname{sn}\left(z + \frac{4\,k\,K(m)}{p} \,\middle|\, m\right) \operatorname{dn}\left(z + \frac{4\,k\,K(m)}{p} \,\middle|\, m\right) \left(\operatorname{sn}\left(z + \frac{4\,(k-r)\,K(m)}{p} \,\middle|\, m\right) \operatorname{sn}\left(z + \frac{4\,(k-s)\,K(m)}{p} \,\middle|\, m\right) + \operatorname{sn}\left(z + \frac{4\,(k+r)\,K(m)}{p} \,\middle|\, m\right) \operatorname{sn}\left(z + \frac{4\,(k+s)\,K(m)}{p} \,\middle|\, m\right)\right) = \frac{2}{m} \operatorname{ns}\left(\frac{2\,r\,K(m)}{p} \,\middle|\, m\right) \operatorname{ns}\left(\frac{2\,s\,K(m)}{p} \,\middle|\, m\right) \sum_{k=0}^{p-1} \operatorname{dn}\left(z + \frac{4\,k\,K(m)}{p} \,\middle|\, m\right) \operatorname{sn}\left(z + \frac{4\,k\,K(m)}{p} \,\middle|\, m\right) /;$$

$$p \in \mathbb{N}^+ \land r \in \mathbb{N}^+ \land r < p \land s \in \mathbb{N}^+ \land s < r$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.36.16.0132.01

$$\sum_{k=0}^{p-1} \operatorname{cn}\left(z + \frac{4\,k\,K(m)}{p} \,\middle|\, m\right) \operatorname{dn}\left(z + \frac{4\,k\,K(m)}{p} \,\middle|\, m\right) \left(\operatorname{cn}\left(z + \frac{4\,(k-s)\,K(m)}{p} \,\middle|\, m\right) \operatorname{sn}\left(z + \frac{4\,(k-r)\,K(m)}{p} \,\middle|\, m\right) + \operatorname{cn}\left(z + \frac{4\,(k+s)\,K(m)}{p} \,\middle|\, m\right) \operatorname{sn}\left(z + \frac{4\,(k+r)\,K(m)}{p} \,\middle|\, m\right)\right) = -\frac{2}{m} \operatorname{ns}\left(\frac{4\,r\,K(m)}{p} \,\middle|\, m\right) \operatorname{ds}\left(\frac{4\,s\,K(m)}{p} \,\middle|\, m\right) \sum_{k=0}^{p-1} \operatorname{dn}\left(z + \frac{4\,k\,K(m)}{p} \,\middle|\, m\right) \operatorname{sn}\left(z + \frac{4\,k\,K(m)}{p} \,\middle|\, m\right) /;$$

$$p \in \mathbb{N}^+ \land r \in \mathbb{N}^+ \land r < p \land s \in \mathbb{N}^+ \land s < r$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.36.16.0133.01

$$\sum_{k=0}^{p-1}(-1)^k \operatorname{dn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right)^2 \left(\operatorname{cn}\left(z+\frac{2\,(k-r)\,K(m)}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{2\,(k-r)\,K(m)}{p}\,\middle|\,m\right)+\operatorname{cn}\left(z+\frac{2\,(k+r)\,K(m)}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{2\,(k+r)\,K(m)}{p}\,\middle|\,m\right)\right) = 2\left(\operatorname{cs}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)^2-\operatorname{ds}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\operatorname{ns}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\right)\sum_{k=0}^{p-1}(-1)^k \operatorname{cn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right) /;\ \frac{p}{2}\in\mathbb{N}^+\bigwedge r\in\mathbb{N}^+\bigwedge \gcd(p,r)==1$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.36.16.0134.01

$$\sum_{k=0}^{p-1}(-1)^k \operatorname{dn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{2\,(k+r)\,K(m)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{2\,(k+2\,r)\,K(m)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{2\,(k+3\,r)\,K(m)}{p}\,\middle|\,m\right) = 2\left(\operatorname{cs}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right)\operatorname{cs}\left(\frac{6\,r\,K(m)}{p}\,\middle|\,m\right)\operatorname{cs}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)+\operatorname{cs}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)^2\operatorname{cs}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right)\right) \sum_{k=0}^{p-1}(-1)^k Z\left(\operatorname{am}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right)\,\middle|\,m\right) /;\ \frac{p}{2}\in\mathbb{N}^+\bigwedge r\in\mathbb{N}^+\bigwedge r<p\bigwedge \gcd(p,r)==1\bigwedge 1-m>0$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.36.16.0135.01

$$\sum_{k=0}^{p-1}(-1)^k \operatorname{dn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right)^3\left(\operatorname{dn}\left(z+\frac{2\,(k-r)\,K(m)}{p}\,\middle|\,m\right)+\operatorname{dn}\left(z+\frac{2\,(k+r)\,K(m)}{p}\,\middle|\,m\right)\right) = 2\operatorname{ns}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\operatorname{ds}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\sum_{k=0}^{p-1}(-1)^k \operatorname{dn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right)^2 /;\ \frac{p}{2}\in\mathbb{N}^+\bigwedge r\in\mathbb{N}^+\bigwedge \gcd(p,r)==1$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.36.16.0136.01

$$\sum_{k=0}^{p-1}(-1)^k \operatorname{cn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right)^3\left(\operatorname{cn}\left(z+\frac{2\,(k-r)\,K(m)}{p}\,\middle|\,m\right)+\operatorname{cn}\left(z+\frac{2\,(k+r)\,K(m)}{p}\,\middle|\,m\right)\right) = \frac{2}{m^2}\operatorname{cs}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\operatorname{ns}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\sum_{k=0}^{p-1}(-1)^k \operatorname{dn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right)^2 /;\ \frac{p}{2}\in\mathbb{N}^+\bigwedge r\in\mathbb{N}^+\bigwedge \gcd(p,r)==1$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.36.16.0137.01

$$\sum_{k=0}^{p-1}(-1)^k \operatorname{sn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right)^3\left(\operatorname{sn}\left(z+\frac{2\,(k-r)\,K(m)}{p}\,\middle|\,m\right)+\operatorname{sn}\left(z+\frac{2\,(k+r)\,K(m)}{p}\,\middle|\,m\right)\right)= \frac{2}{m^2}\operatorname{cs}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\operatorname{ds}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\sum_{k=0}^{p-1}(-1)^k \operatorname{dn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right)^2 \;/;\; \frac{p}{2}\in\mathbb{N}^+\bigwedge r\in\mathbb{N}^+\bigwedge \gcd(p,r)=1$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

Sums over products of five Jacobi functions

09.36.16.0138.01

$$\sum_{k=0}^{p-1}\operatorname{dn}\left(z+\frac{2\,K(m)\,k}{p}\,\middle|\,m\right)^4\left(\operatorname{dn}\left(z+\frac{2\,K(m)\,(k-r)}{p}\,\middle|\,m\right)+\operatorname{dn}\left(z+\frac{2\,K(m)\,(k+r)}{p}\,\middle|\,m\right)\right)= 2\operatorname{ds}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\operatorname{ns}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\sum_{k=0}^{p-1}\operatorname{dn}\left(z+\frac{2\,K(m)\,k}{p}\,\middle|\,m\right)^3+2\operatorname{cs}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)^2 \left(\operatorname{cs}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)^2-\operatorname{ns}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\operatorname{ds}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\right)\sum_{k=0}^{p-1}\operatorname{dn}\left(z+\frac{2\,K(m)\,k}{p}\,\middle|\,m\right) /;\; p\in\mathbb{N}^+\land r\in\mathbb{N}^+\land r<p$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.36.16.0139.01

$$\sum_{k=0}^{p-1}\operatorname{dn}\left(z+\frac{2\,K(m)\,k}{p}\,\middle|\,m\right)^3\left(\operatorname{dn}\left(z+\frac{2\,K(m)\,(k-r)}{p}\,\middle|\,m\right)^2+\operatorname{dn}\left(z+\frac{2\,K(m)\,(k+r)}{p}\,\middle|\,m\right)^2\right)= -2\operatorname{cs}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)^2\sum_{k=0}^{p-1}\operatorname{dn}\left(z+\frac{2\,K(m)\,k}{p}\,\middle|\,m\right)^3+ 2\left(\operatorname{ds}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)^2\operatorname{cs}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)^2+\operatorname{ds}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)^2\operatorname{ns}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)^2+\operatorname{ns}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)^2\operatorname{cs}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)^2- 3\operatorname{ns}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\operatorname{ds}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\operatorname{cs}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)^2\right)\sum_{k=0}^{p-1}\operatorname{dn}\left(z+\frac{2\,K(m)\,k}{p}\,\middle|\,m\right) /;\; p\in\mathbb{N}^+\land r\in\mathbb{N}^+\land r<p$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.36.16.0140.01

$$\sum_{k=0}^{p-1}\operatorname{cn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right)\left(\operatorname{dn}\left(z+\frac{2\,(k-r)\,K(m)}{p}\,\middle|\,m\right)^2+\operatorname{dn}\left(z+\frac{2\,(k+r)\,K(m)}{p}\,\middle|\,m\right)^2\right)= -2\operatorname{cs}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)^2\sum_{k=0}^{p-1}\operatorname{cn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right) /;\; p\in\mathbb{N}^+\land r\in\mathbb{N}^+\land r<p-1$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.36.16.0141.01

$$\sum_{k=0}^{p-1} \operatorname{cn}\left(z+\frac{2kK(m)}{p}\,\middle|\,m\right) \operatorname{sn}\left(z+\frac{2kK(m)}{p}\,\middle|\,m\right)^2 \operatorname{dn}\left(z+\frac{2kK(m)}{p}\,\middle|\,m\right)\left(\operatorname{sn}\left(z+\frac{2(k-r)K(m)}{p}\,\middle|\,m\right)+\operatorname{sn}\left(z+\frac{2(k+r)K(m)}{p}\,\middle|\,m\right)\right) = -\frac{2}{m}\operatorname{cs}\left(\frac{2rK(m)}{p}\,\middle|\,m\right)\operatorname{ds}\left(\frac{2rK(m)}{p}\,\middle|\,m\right)\sum_{k=0}^{p-1}\operatorname{cn}\left(z+\frac{2kK(m)}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{2kK(m)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{2kK(m)}{p}\,\middle|\,m\right) /; p\in\mathbb{N}^+ \land r\in\mathbb{N}^+ \land r<p-1$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.36.16.0142.01

$$\sum_{k=0}^{p-1} \operatorname{sn}\left(z+\frac{2kK(m)}{p}\,\middle|\,m\right) \operatorname{cn}\left(z+\frac{2kK(m)}{p}\,\middle|\,m\right)^2 \operatorname{dn}\left(z+\frac{2kK(m)}{p}\,\middle|\,m\right)\left(\operatorname{cn}\left(z+\frac{2(k-r)K(m)}{p}\,\middle|\,m\right)+\operatorname{cn}\left(z+\frac{2(k+r)K(m)}{p}\,\middle|\,m\right)\right) = \frac{2}{m}\operatorname{cs}\left(\frac{2rK(m)}{p}\,\middle|\,m\right)\operatorname{ns}\left(\frac{2rK(m)}{p}\,\middle|\,m\right)\sum_{k=0}^{p-1}\operatorname{cn}\left(z+\frac{2kK(m)}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{2kK(m)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{2kK(m)}{p}\,\middle|\,m\right) /; p\in\mathbb{N}^+ \land r\in\mathbb{N}^+ \land r<p-1$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.36.16.0143.01

$$\sum_{k=0}^{p-1} \operatorname{sn}\left(z+\frac{4kK(m)}{p}\,\middle|\,m\right)^4 \left(\operatorname{sn}\left(z+\frac{4(k-r)K(m)}{p}\,\middle|\,m\right)+\operatorname{sn}\left(z+\frac{4(k+r)K(m)}{p}\,\middle|\,m\right)\right) = -\frac{2}{m}\operatorname{cs}\left(\frac{4rK(m)}{p}\,\middle|\,m\right)\operatorname{ds}\left(\frac{4rK(m)}{p}\,\middle|\,m\right)\sum_{k=0}^{p-1}\operatorname{sn}\left(z+\frac{4kK(m)}{p}\,\middle|\,m\right)^3 + \frac{2}{m^2}\operatorname{ns}\left(\frac{4rK(m)}{p}\,\middle|\,m\right)^2 \left(\operatorname{ns}\left(\frac{4rK(m)}{p}\,\middle|\,m\right)^2 - \operatorname{cs}\left(\frac{4rK(m)}{p}\,\middle|\,m\right)\operatorname{ds}\left(\frac{4rK(m)}{p}\,\middle|\,m\right)\right)\sum_{k=0}^{p-1}\operatorname{sn}\left(z+\frac{4kK(m)}{p}\,\middle|\,m\right) /; p\in\mathbb{N}^+ \land r\in\mathbb{N}^+ \land r<p$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.36.16.0144.01

$$\sum_{k=0}^{p-1} \operatorname{sn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right)^3 \left(\operatorname{sn}\left(z+\frac{4\,(k-r)\,K(m)}{p}\,\middle|\,m\right)^2+\operatorname{sn}\left(z+\frac{4\,(k+r)\,K(m)}{p}\,\middle|\,m\right)^2\right) = \frac{2}{m}\operatorname{ns}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right)^2 \sum_{k=0}^{p-1} \operatorname{sn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right)^3 + \frac{2}{m^2}\left(\operatorname{cs}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right)^2 \operatorname{ns}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right)^2 + \operatorname{ds}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right)^2 \operatorname{ns}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right)^2 + \operatorname{cs}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right)^2 \operatorname{ds}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right)^2 - 3\operatorname{cs}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right) \operatorname{ds}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right) \operatorname{ns}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right)^2\right) \sum_{k=0}^{p-1} \operatorname{sn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right) /;\; p \in \mathbb{N}^+ \land r \in \mathbb{N}^+ \land r < p$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.36.16.0145.01

$$\sum_{k=0}^{p-1} \operatorname{dn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right)^2 \operatorname{sn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right)^2 \left(\operatorname{cn}\left(z+\frac{4\,(k-r)\,K(m)}{p}\,\middle|\,m\right)+\operatorname{cn}\left(z+\frac{4\,(k+r)\,K(m)}{p}\,\middle|\,m\right)\right) = -2\operatorname{ns}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right) \operatorname{cs}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right) \sum_{k=0}^{p-1} \operatorname{cn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right)^3 + \frac{2}{m}\operatorname{cs}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right) \operatorname{ns}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right)^3 \left(m\operatorname{sn}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right)^2 + \operatorname{cn}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right)^2 - \operatorname{cn}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right)\right) \sum_{k=0}^{p-1} \operatorname{cn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right) /;\; p \in \mathbb{N}^+ \land r \in \mathbb{N}^+ \land r < p$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.36.16.0146.01

$$\sum_{k=0}^{p-1} \operatorname{cn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right)^3 \left(\operatorname{cn}\left(z+\frac{4\,(k-r)\,K(m)}{p}\,\middle|\,m\right)^2+\operatorname{cn}\left(z+\frac{4\,(k+r)\,K(m)}{p}\,\middle|\,m\right)^2\right) = -\frac{2}{m}\operatorname{ds}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right)^2 \sum_{k=0}^{p-1} \operatorname{cn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right)^3 + \frac{2}{m^2}\left(\operatorname{cs}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right)^2 \operatorname{ds}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right)^2 + \operatorname{ns}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right)^2 \operatorname{ds}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right)^2 + \operatorname{cs}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right)^2 \operatorname{ns}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right)^2 - 3\operatorname{cs}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right) \operatorname{ns}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right) \operatorname{ds}\left(\frac{4\,r\,K(m)}{p}\,\middle|\,m\right)^2\right) \sum_{k=0}^{p-1} \operatorname{cn}\left(z+\frac{4\,k\,K(m)}{p}\,\middle|\,m\right) /;\; p \in \mathbb{N}^+ \land r \in \mathbb{N}^+ \land r < p$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.36.16.0147.01

$$\sum_{k=0}^{p-1}(-1)^k \operatorname{cn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right)^2 \operatorname{dn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right) \operatorname{sn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right)\left(\operatorname{cn}\left(z+\frac{2\,(k-r)\,K(m)}{p}\,\middle|\,m\right)+\operatorname{cn}\left(z+\frac{2\,(k+r)\,K(m)}{p}\,\middle|\,m\right)\right) = -\frac{4}{m^2}\operatorname{ds}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)^2 \operatorname{cs}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\operatorname{ns}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\sum_{k=0}^{p-1}(-1)^k Z\left(\operatorname{am}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right)\,\middle|\,m\right) + \frac{2}{m}\operatorname{cs}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\operatorname{ns}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\sum_{k=0}^{p-1}(-1)^k \operatorname{cn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right) /;\ \frac{p}{2}\in\mathbb{N}^+ \bigwedge r\in\mathbb{N}^+ \bigwedge r<p \bigwedge \gcd(p,\,r) == 1 \bigwedge 1-m>0$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

Sums over products of six Jacobi functions

09.36.16.0148.01

$$\sum_{k=0}^{p-1}\operatorname{dn}\left(z+\frac{2\,K(m)\,k}{p}\,\middle|\,m\right)^4 \left(\operatorname{cn}\left(z+\frac{2\,K(m)\,(k-r)}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{2\,K(m)\,(k-r)}{p}\,\middle|\,m\right)+\operatorname{cn}\left(z+\frac{2\,K(m)\,(k+r)}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{2\,K(m)\,(k+r)}{p}\,\middle|\,m\right)\right) = -2\operatorname{ns}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\operatorname{ds}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\sum_{k=0}^{p-1}\operatorname{sn}\left(z+\frac{2\,K(m)\,k}{p}\,\middle|\,m\right)\operatorname{cn}\left(z+\frac{2\,K(m)\,k}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{2\,K(m)\,k}{p}\,\middle|\,m\right)^2 + 2\operatorname{cs}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)^2\left(\operatorname{cs}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)^2+3\operatorname{ds}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\operatorname{ns}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)\right)\sum_{k=0}^{p-1}\operatorname{sn}\left(z+\frac{2\,K(m)\,k}{p}\,\middle|\,m\right)\operatorname{cn}\left(z+\frac{2\,K(m)\,k}{p}\,\middle|\,m\right) /;\ p\in\mathbb{N}^+ \wedge r\in\mathbb{N}^+ \wedge r<p$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.36.16.0149.01

$$\sum_{k=0}^{p-1} \mathrm{dn}\left(z+\frac{2K(m)k}{p} \,\middle|\, m\right)^4 \left(\mathrm{cn}\left(z+\frac{2K(m)(k-s)}{p} \,\middle|\, m\right) \mathrm{sn}\left(z+\frac{2K(m)(k-r)}{p} \,\middle|\, m\right) + \mathrm{cn}\left(z+\frac{2K(m)(k+s)}{p} \,\middle|\, m\right) \mathrm{sn}\left(z+\frac{2K(m)(k+r)}{p} \,\middle|\, m\right)\right) = -2\,\mathrm{ns}\left(\frac{2rK(m)}{p} \,\middle|\, m\right) \mathrm{ds}\left(\frac{2sK(m)}{p} \,\middle|\, m\right) \sum_{k=0}^{p-1} \mathrm{sn}\left(z+\frac{2K(m)k}{p} \,\middle|\, m\right) \mathrm{cn}\left(z+\frac{2K(m)k}{p} \,\middle|\, m\right) \mathrm{dn}\left(z+\frac{2K(m)k}{p} \,\middle|\, m\right)^2 + 2\left(\mathrm{cs}\left(\frac{2rK(m)}{p} \,\middle|\, m\right) \mathrm{ds}\left(\frac{2rK(m)}{p} \,\middle|\, m\right) \mathrm{cs}\left(\frac{2sK(m)}{p} \,\middle|\, m\right) \mathrm{ns}\left(\frac{2sK(m)}{p} \,\middle|\, m\right) + \mathrm{ns}\left(\frac{2rK(m)}{p} \,\middle|\, m\right) \mathrm{ds}\left(\frac{2sK(m)}{p} \,\middle|\, m\right) \left(\mathrm{cs}\left(\frac{2rK(m)}{p} \,\middle|\, m\right)^2 + \mathrm{cs}\left(\frac{2sK(m)}{p} \,\middle|\, m\right)^2\right)\right) \sum_{k=0}^{p-1} \mathrm{sn}\left(z+\frac{2K(m)k}{p} \,\middle|\, m\right) \mathrm{cn}\left(z+\frac{2K(m)k}{p} \,\middle|\, m\right) /;\ p \in \mathbb{N}^+ \land r \in \mathbb{N}^+ \land r < p \land s \in \mathbb{N}^+ \land s < r$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.36.16.0150.01

$$\sum_{k=0}^{p-1} \mathrm{dn}\left(z+\frac{2kK(m)}{p} \,\middle|\, m\right)^3 \left(\mathrm{dn}\left(z+\frac{2(k-r)K(m)}{p} \,\middle|\, m\right)^3 + \mathrm{dn}\left(z+\frac{2(k+r)K(m)}{p} \,\middle|\, m\right)^3\right) = \frac{p}{2K(m)} \left(24E(m)\,\mathrm{ds}\left(\frac{2rK(m)}{p} \,\middle|\, m\right) \mathrm{ns}\left(\frac{2rK(m)}{p} \,\middle|\, m\right) \mathrm{cs}\left(\frac{2rK(m)}{p} \,\middle|\, m\right)^2 + \int_0^{2K(m)} \mathrm{dn}(t \mid m)^3 \left(\mathrm{dn}\left(t+\frac{2rK(m)}{p} \,\middle|\, m\right)^3 + \mathrm{dn}\left(t-\frac{2rK(m)}{p} \,\middle|\, m\right)^3\right) dt\right) - 12\,\mathrm{cs}\left(\frac{2rK(m)}{p} \,\middle|\, m\right)^2 \mathrm{ns}\left(\frac{2rK(m)}{p} \,\middle|\, m\right) \mathrm{ds}\left(\frac{2rK(m)}{p} \,\middle|\, m\right) \sum_{k=0}^{p-1} \mathrm{dn}\left(z+\frac{2kK(m)}{p} \,\middle|\, m\right)^2 /;\ p \in \mathbb{N}^+ \land r \in \mathbb{N}^+ \land r < p-1$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.36.16.0151.01

$$\sum_{k=0}^{p-1} \mathrm{sn}\left(z+\frac{2kK(m)}{p} \,\middle|\, m\right)^3 \left(\mathrm{sn}\left(z+\frac{2(k-r)K(m)}{p} \,\middle|\, m\right)^3 + \mathrm{sn}\left(z+\frac{2(k+r)K(m)}{p} \,\middle|\, m\right)^3\right) = \frac{12}{m^3}\,\mathrm{ns}\left(\frac{2rK(m)}{p} \,\middle|\, m\right)^2 \mathrm{cs}\left(\frac{2rK(m)}{p} \,\middle|\, m\right) \mathrm{ds}\left(\frac{2rK(m)}{p} \,\middle|\, m\right) \sum_{k=0}^{p-1} \mathrm{dn}\left(z+\frac{2kK(m)}{p} \,\middle|\, m\right)^2 + \frac{p}{2K(m)} \left(\int_0^{2K(m)} \mathrm{sn}(t \mid m)^3 \left(\mathrm{sn}\left(t+\frac{2rK(m)}{p} \,\middle|\, m\right)^3 + \mathrm{sn}\left(t-\frac{2rK(m)}{p} \,\middle|\, m\right)^3\right) dt - \frac{24}{m^3}\,\mathrm{ns}\left(\frac{2rK(m)}{p} \,\middle|\, m\right)^2 \mathrm{ds}\left(\frac{2rK(m)}{p} \,\middle|\, m\right) \mathrm{cs}\left(\frac{2rK(m)}{p} \,\middle|\, m\right) E(m)\right) /;\ p \in \mathbb{N}^+ \land r \in \mathbb{N}^+ \land r < p-1$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.36.16.0152.01

$$\sum_{k=0}^{p-1} \operatorname{cn}\left(z+\frac{2kK(m)}{p}\,\middle|\,m\right)^3 \left(\operatorname{cn}\left(z+\frac{2(k-r)K(m)}{p}\,\middle|\,m\right)^3 + \operatorname{cn}\left(z+\frac{2(k+r)K(m)}{p}\,\middle|\,m\right)^3\right) = -\frac{12}{m^3}\operatorname{ds}\left(\frac{2rK(m)}{p}\,\middle|\,m\right)^2 \operatorname{cs}\left(\frac{2rK(m)}{p}\,\middle|\,m\right)\operatorname{ns}\left(\frac{2rK(m)}{p}\,\middle|\,m\right)\sum_{k=0}^{p-1}\operatorname{dn}\left(z+\frac{2kK(m)}{p}\,\middle|\,m\right)^2 + \frac{p}{2K(m)}\left(\int_0^{2K(m)}\operatorname{cn}(t\mid m)^3\left(\operatorname{cn}\left(t+\frac{2rK(m)}{p}\,\middle|\,m\right)^3+\operatorname{cn}\left(t-\frac{2rK(m)}{p}\,\middle|\,m\right)^3\right)dt + \frac{24}{m^3}E(m)\operatorname{cs}\left(\frac{2rK(m)}{p}\,\middle|\,m\right)\operatorname{ns}\left(\frac{2rK(m)}{p}\,\middle|\,m\right)\operatorname{ds}\left(\frac{2rK(m)}{p}\,\middle|\,m\right)^2\right) /;\ p\in\mathbb{N}^+\land r\in\mathbb{N}^+\land r<p-1$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.36.16.0153.01

$$\sum_{k=0}^{p-1} \operatorname{cn}\left(z+\frac{4kK(m)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{4kK(m)}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{4kK(m)}{p}\,\middle|\,m\right)\left(\operatorname{sn}\left(z+\frac{4(k-r)K(m)}{p}\,\middle|\,m\right)^3+\operatorname{sn}\left(z+\frac{4(k+r)K(m)}{p}\,\middle|\,m\right)^3\right) = -\frac{2}{m^2}\left(\operatorname{cs}\left(\frac{4rK(m)}{p}\,\middle|\,m\right)^2\operatorname{ds}\left(\frac{4rK(m)}{p}\,\middle|\,m\right)^2+\operatorname{ns}\left(\frac{4rK(m)}{p}\,\middle|\,m\right)^2\operatorname{ds}\left(\frac{4rK(m)}{p}\,\middle|\,m\right)^2 + \operatorname{cs}\left(\frac{4rK(m)}{p}\,\middle|\,m\right)^2\operatorname{ns}\left(\frac{4rK(m)}{p}\,\middle|\,m\right)^2 + 3\operatorname{cs}\left(\frac{4rK(m)}{p}\,\middle|\,m\right)\operatorname{ns}\left(\frac{4rK(m)}{p}\,\middle|\,m\right)^2\operatorname{ds}\left(\frac{4rK(m)}{p}\,\middle|\,m\right)\right) \sum_{k=0}^{p-1}\operatorname{cn}\left(z+\frac{4kK(m)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{4kK(m)}{p}\,\middle|\,m\right) /;\ p\in\mathbb{N}^+\land r\in\mathbb{N}^+\land r<p$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.36.16.0154.01

$$\sum_{k=0}^{p-1} \operatorname{sn}\left(z+\frac{4kK(m)}{p}\,\middle|\,m\right)^4 \left(\operatorname{cn}\left(z+\frac{4(k-r)K(m)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{4(k-r)K(m)}{p}\,\middle|\,m\right)+\operatorname{cn}\left(z+\frac{4(k+r)K(m)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{4(k+r)K(m)}{p}\,\middle|\,m\right)\right) = \frac{2}{m}\operatorname{cs}\left(\frac{4rK(m)}{p}\,\middle|\,m\right)\operatorname{ds}\left(\frac{4rK(m)}{p}\,\middle|\,m\right)\sum_{k=0}^{p-1}\operatorname{cn}\left(z+\frac{4kK(m)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{4kK(m)}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{4kK(m)}{p}\,\middle|\,m\right)^2 + \frac{2}{m^2}\operatorname{ns}\left(\frac{4rK(m)}{p}\,\middle|\,m\right)^2\left(\operatorname{ns}\left(\frac{4rK(m)}{p}\,\middle|\,m\right)^2+3\operatorname{cs}\left(\frac{4rK(m)}{p}\,\middle|\,m\right)\operatorname{ds}\left(\frac{4rK(m)}{p}\,\middle|\,m\right)\right) \sum_{k=0}^{p-1}\operatorname{cn}\left(z+\frac{4kK(m)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{4kK(m)}{p}\,\middle|\,m\right) /;\ p\in\mathbb{N}^+\land r\in\mathbb{N}^+\land r<p$$

Khare/Lakshminarayan/Sukhatme_2002

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09.36.16.0155.01

$$\sum_{k=0}^{p-1} \operatorname{sn}\left(z+\frac{4kK(m)}{p}\,\middle|\,m\right)^4 \left(\operatorname{cn}\left(z+\frac{4(k-s)K(m)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{4(k-r)K(m)}{p}\,\middle|\,m\right)+\operatorname{cn}\left(z+\frac{4(k+s)K(m)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{4(k+r)K(m)}{p}\,\middle|\,m\,\middle|\,m\right)\right)= \frac{2}{m}\operatorname{cs}\left(\frac{4rK(m)}{p}\,\middle|\,m\right)\operatorname{ds}\left(\frac{4sK(m)}{p}\,\middle|\,m\right)\sum_{k=0}^{p-1}\operatorname{cn}\left(z+\frac{4kK(m)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{4kK(m)}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{4kK(m)}{p}\,\middle|\,m\right)^2+ \frac{2}{m^2}\left(\operatorname{ds}\left(\frac{4rK(m)}{p}\,\middle|\,m\right)\operatorname{ns}\left(\frac{4rK(m)}{p}\,\middle|\,m\right)\operatorname{cs}\left(\frac{4sK(m)}{p}\,\middle|\,m\right)\operatorname{ns}\left(\frac{4sK(m)}{p}\,\middle|\,m\right)+\operatorname{cs}\left(\frac{4rK(m)}{p}\,\middle|\,m\right)\operatorname{ds}\left(\frac{4sK(m)}{p}\,\middle|\,m\right)\left(\operatorname{ns}\left(\frac{4rK(m)}{p}\,\middle|\,m\right)^2+\operatorname{ns}\left(\frac{4sK(m)}{p}\,\middle|\,m\right)^2\right)\right)\sum_{k=0}^{p-1}\operatorname{cn}\left(z+\frac{4kK(m)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{4kK(m)}{p}\,\middle|\,m\right)\text{ /; } p\in\mathbb{N}^+\land r\in\mathbb{N}^+\land r<p$$

Khare/Lakshminarayan/Sukhatme_2002

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09.36.16.0156.01

$$\sum_{k=0}^{p-1}\operatorname{cn}\left(z+\frac{4kK(m)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{4kK(m)}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{4kK(m)}{p}\,\middle|\,m\right)\left(\operatorname{cn}\left(z+\frac{4(k-r)K(m)}{p}\,\middle|\,m\right)^3+\operatorname{cn}\left(z+\frac{4(k+r)K(m)}{p}\,\middle|\,m\right)^3\right)= -\frac{2}{m}\left(\operatorname{cs}\left(\frac{4rK(m)}{p}\,\middle|\,m\right)^2\operatorname{ds}\left(\frac{4rK(m)}{p}\,\middle|\,m\right)^2+\operatorname{ns}\left(\frac{4rK(m)}{p}\,\middle|\,m\right)^2\operatorname{ds}\left(\frac{4rK(m)}{p}\,\middle|\,m\right)^2+ \operatorname{cs}\left(\frac{4rK(m)}{p}\,\middle|\,m\right)^2\operatorname{ns}\left(\frac{4rK(m)}{p}\,\middle|\,m\right)^2+3\operatorname{cs}\left(\frac{4rK(m)}{p}\,\middle|\,m\right)\operatorname{ns}\left(\frac{4rK(m)}{p}\,\middle|\,m\right)\operatorname{ds}\left(\frac{4rK(m)}{p}\,\middle|\,m\right)^2\right) \sum_{k=0}^{p-1}\operatorname{dn}\left(z+\frac{4kK(m)}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{4kK(m)}{p}\,\middle|\,m\right)\text{ /; } p\in\mathbb{N}^+\land r\in\mathbb{N}^+\land r<p$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

Sums over products of seven Jacobi functions

09.36.16.0157.01

$$\sum_{k=0}^{p-1}\operatorname{sn}\left(z+\frac{2kK(m)}{p}\,\middle|\,m\right)\operatorname{cn}\left(z+\frac{2kK(m)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{2kK(m)}{p}\,\middle|\,m\right)^2 \left(\operatorname{dn}\left(z+\frac{2(k-r)K(m)}{p}\,\middle|\,m\right)^3+\operatorname{dn}\left(z+\frac{2(k+r)K(m)}{p}\,\middle|\,m\right)^3\right)=-4\operatorname{cs}\left(\frac{2rK(m)}{p}\,\middle|\,m\right)^2\operatorname{ns}\left(\frac{2rK(m)}{p}\,\middle|\,m\right) \operatorname{ds}\left(\frac{2rK(m)}{p}\,\middle|\,m\right)\sum_{k=0}^{p-1}\operatorname{cn}\left(z+\frac{2kK(m)}{p}\,\middle|\,m\right)\operatorname{sn}\left(z+\frac{2kK(m)}{p}\,\middle|\,m\right)\operatorname{dn}\left(z+\frac{2kK(m)}{p}\,\middle|\,m\right)\text{ /; } p\in\mathbb{N}^+\land r\in\mathbb{N}^+\land r<p-1$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.36.16.0158.01

$$\sum_{k=0}^{p-1} \operatorname{cn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right) \operatorname{dn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right) \operatorname{sn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right)^2 \left(\operatorname{sn}\left(z+\frac{2\,(k-r)\,K(m)}{p}\,\middle|\,m\right)^3+\operatorname{sn}\left(z+\frac{2\,(k+r)\,K(m)}{p}\,\middle|\,m\right)^3\right) = -\frac{4}{m^2} \operatorname{ns}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)^2 \operatorname{ds}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right) \operatorname{cs}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right) \sum_{k=0}^{p-1} \operatorname{cn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right) \operatorname{sn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right) \operatorname{dn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right) /;\ p\in\mathbb{N}^+ \land r\in\mathbb{N}^+ \land r<p-1$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.36.16.0159.01

$$\sum_{k=0}^{p-1} \operatorname{cn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right)^2 \operatorname{dn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right) \operatorname{sn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right) \left(\operatorname{cn}\left(z+\frac{2\,(k-r)\,K(m)}{p}\,\middle|\,m\right)^3+\operatorname{cn}\left(z+\frac{2\,(k+r)\,K(m)}{p}\,\middle|\,m\right)^3\right) = -\frac{4}{m^2} \operatorname{ds}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)^2 \operatorname{cs}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right) \operatorname{ns}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right) \sum_{k=0}^{p-1} \operatorname{cn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right) \operatorname{sn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right) \operatorname{dn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right) /;\ p\in\mathbb{N}^+ \land r\in\mathbb{N}^+ \land r<p-1$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.36.16.0160.01

$$\sum_{k=0}^{p-1} \operatorname{cn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right) \operatorname{dn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right) \operatorname{sn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right) \left(\operatorname{dn}\left(z+\frac{2\,(k-r)\,K(m)}{p}\,\middle|\,m\right)^4+\operatorname{dn}\left(z+\frac{2\,(k+r)\,K(m)}{p}\,\middle|\,m\right)^4\right) = 2\left(\operatorname{cs}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)^4-\operatorname{ds}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)^2 \operatorname{cs}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)^2-\operatorname{ns}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)^2 \operatorname{cs}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)^2-\operatorname{ns}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)^2 \operatorname{ds}\left(\frac{2\,r\,K(m)}{p}\,\middle|\,m\right)^2\right) \sum_{k=0}^{p-1} \operatorname{cn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right) \operatorname{sn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right) \operatorname{dn}\left(z+\frac{2\,k\,K(m)}{p}\,\middle|\,m\right) /;\ p\in\mathbb{N}^+ \land r\in\mathbb{N}^+ \land r<p-1$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

Sums over products of arbitrarily many Jacobi functions

09.36.16.0161.01

$$\sum_{j=0}^{p-1}\prod_{k=0}^{l-1}\operatorname{dn}\left(z+\frac{2K(m)(j+kr)}{p}\,\middle|\,m\right)=\left(\left(\prod_{k=1}^{\frac{l-1}{2}}\operatorname{cs}\left(\frac{2krK(m)}{p}\,\middle|\,m\right)\right)^2+2(-1)^{\frac{l-1}{2}}\sum_{k=1}^{\frac{l-1}{2}}\prod_{n=1}^{l}\operatorname{If}\left[n==k,\,1,\,\operatorname{cs}\left(\frac{2(n-k)rK(m)}{p}\,\middle|\,m\right)\right]\right)\sum_{k=0}^{p-1}\operatorname{dn}\left(z+\frac{2K(m)k}{p}\,\middle|\,m\right)\,/;\,p\in\mathbb{N}^+\bigwedge r\in\mathbb{N}^+\bigwedge r<p-1\bigwedge\frac{l-1}{2}\in\mathbb{N}\bigwedge l\leq p$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.36.16.0162.01

$$\sum_{j=0}^{p-1}\prod_{k=0}^{r-1}\operatorname{dn}\left(z+\frac{2K(m)(j+k)}{p}\,\middle|\,m\right)=\frac{p}{2K(m)}\int_0^{2K(m)}\prod_{k=0}^{r-1}\operatorname{dn}\left(t+\frac{2kK(m)}{p}\,\middle|\,m\right)dt\,/;\,p-3\in\mathbb{N}\bigwedge\frac{r}{2}\in\mathbb{N}^+\bigwedge r<p-1$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.36.16.0163.01

$$\sum_{j=0}^{p-1}\prod_{k=0}^{r-1}\operatorname{sn}\left(z+\frac{2K(m)(j+k)}{p}\,\middle|\,m\right)=\frac{p}{2K(m)}\int_0^{2K(m)}\prod_{k=0}^{r-1}\operatorname{sn}\left(t+\frac{2kK(m)}{p}\,\middle|\,m\right)dt\,/;\,p-3\in\mathbb{N}\bigwedge\frac{r}{2}\in\mathbb{N}^+\bigwedge r<p-1$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.36.16.0164.01

$$\sum_{j=0}^{p-1}\prod_{k=0}^{r-1}\operatorname{cn}\left(z+\frac{2K(m)(j+k)}{p}\,\middle|\,m\right)=\frac{p}{2K(m)}\int_0^{2K(m)}\prod_{k=0}^{r-1}\operatorname{cn}\left(t+\frac{2kK(m)}{p}\,\middle|\,m\right)dt\,/;\,p-3\in\mathbb{N}\bigwedge\frac{r}{2}\in\mathbb{N}^+\bigwedge r<p-1$$

Khare/Lakshminarayan/Sukhatme_2002

Khare/Lakshminarayan/Sukhatme_JMP_2002

09.36.16.0067.01

$$\sum_{k=0}^{p-1}\prod_{l=0}^{r-1}\operatorname{cn}\left(z+\frac{4(k+n_l)K(m)}{p}\,\middle|\,m\right)=\sum_{k=0}^{p-1}\prod_{l=0}^{r-1}\operatorname{cn}\left(\frac{4(k+n_l)K(m)}{p}\,\middle|\,m\right)/;\,\frac{p-1}{2}\in\mathbb{N}^+\bigwedge\frac{r}{2}\in\mathbb{N}^+\bigwedge n_0==0\bigwedge n_l\in\mathbb{Z}\bigwedge 1\leq n_l<p\bigwedge n_l<n_{l+1}$$

Khare/Sukhatme_2002

Khare/Sukhatme_JMP_2002

Identities

Functional identities

09.36.17.0001.01

$$\left(m\, w(z)^4 - 1\right)^2 w(2\, z)^2 - 4\, w(z)^2 \left(w(z)^2 - 1\right) \left(m\, w(z)^2 - 1\right) = 0 \;/;\; w(z) = \operatorname{sn}(z \mid m)$$

09.36.17.0002.01

$$\operatorname{sn}(z \mid m) = \frac{\left(\sqrt{\mu} + 1\right) \operatorname{sn}\left(\frac{z}{\sqrt{\mu}+1} \middle| \mu\right)}{\sqrt{\mu}\, \operatorname{sn}\left(\frac{z}{\sqrt{\mu}+1} \middle| \mu\right)^2 + 1} \;/;\; \mu = \frac{\left(1 - \sqrt{1-m}\right)^4}{m^2} \bigwedge 0 < m < 1$$

Complex characteristics

Real part

09.36.19.0001.01

$$\operatorname{Re}(\operatorname{sn}(x + i\, y \mid m)) = \frac{\operatorname{dn}(y \mid 1-m) \operatorname{sn}(x \mid m)}{\operatorname{cn}(y \mid 1-m)^2 + m \operatorname{sn}(x \mid m)^2 \operatorname{sn}(y \mid 1-m)^2} \;/;\; \{x, y, m\} \in \mathbb{R}$$

Imaginary part

09.36.19.0002.01

$$\operatorname{Im}(\operatorname{sn}(x + i\, y \mid m)) = \frac{\operatorname{cn}(x \mid m) \operatorname{cn}(y \mid 1-m) \operatorname{dn}(x \mid m) \operatorname{sn}(y \mid 1-m)}{\operatorname{cn}(y \mid 1-m)^2 + m \operatorname{sn}(x \mid m)^2 \operatorname{sn}(y \mid 1-m)^2} \;/;\; \{x, y, m\} \in \mathbb{R}$$

Absolute value

09.36.19.0003.01

$$|\operatorname{sn}(x + i\, y \mid m)| = \frac{\sqrt{\operatorname{dn}(y \mid 1-m)^2 \operatorname{sn}(x \mid m)^2 + \operatorname{cn}(x \mid m)^2 \operatorname{cn}(y \mid 1-m)^2 \operatorname{dn}(x \mid m)^2 \operatorname{sn}(y \mid 1-m)^2}}{\operatorname{cn}(y \mid 1-m)^2 + m \operatorname{sn}(x \mid m)^2 \operatorname{sn}(y \mid 1-m)^2} \;/;\; \{x, y, m\} \in \mathbb{R}$$

Argument

09.36.19.0004.01

$$\arg(\operatorname{sn}(x + i\, y \mid m)) = \tan^{-1}(\operatorname{dn}(y \mid 1-m) \operatorname{sn}(x \mid m),\, \operatorname{cn}(x \mid m) \operatorname{cn}(y \mid 1-m) \operatorname{dn}(x \mid m) \operatorname{sn}(y \mid 1-m)) \;/;\; \{x, y, m\} \in \mathbb{R}$$

Conjugate value

09.36.19.0005.01

$$\overline{\operatorname{sn}(x + i\, y \mid m)} = \frac{\operatorname{sn}(x \mid m) \operatorname{dn}(y \mid 1-m) - i \operatorname{cn}(x \mid m) \operatorname{dn}(x \mid m) \operatorname{sn}(y \mid 1-m) \operatorname{cn}(y \mid 1-m)}{\operatorname{cn}(y \mid 1-m)^2 + m \operatorname{sn}(x \mid m)^2 \operatorname{sn}(y \mid 1-m)^2} \;/;\; \{x, y, m\} \in \mathbb{R}$$

Differentiation

Low-order differentiation

With respect to z

09.36.20.0001.01

$$\frac{\partial \operatorname{sn}(z \mid m)}{\partial z} = \operatorname{cn}(z \mid m) \operatorname{dn}(z \mid m)$$

09.36.20.0002.01

$$\frac{\partial^2 \operatorname{sn}(z \mid m)}{\partial z^2} = 2\, m \operatorname{sn}(z \mid m)^3 - (1+m) \operatorname{sn}(z \mid m)$$

09.36.20.0003.01

$$\frac{\partial^2 \operatorname{sn}(z \mid m)}{\partial z^2} = -\left(m \operatorname{cn}(z \mid m)^2 + \operatorname{dn}(z \mid m)^2\right) \operatorname{sn}(z \mid m)$$

With respect to *m*

09.36.20.0004.01

$$\frac{\partial \operatorname{sn}(z \mid m)}{\partial m} = \frac{1}{2\, m\, (1-m)} (\operatorname{dn}(z \mid m) \operatorname{cn}(z \mid m) ((1-m)\, z - E(\operatorname{am}(z \mid m) \mid m) + m \operatorname{cd}(z \mid m) \operatorname{sn}(z \mid m)))$$

09.36.20.0005.01

$$\frac{\partial^2 \operatorname{sn}(z \mid m)}{\partial m^2} = -\frac{1}{4\, (m-1)^2\, m^2} \Big(m \Big(-m \operatorname{cd}(z \mid m)^2 \operatorname{sn}(z \mid m) \operatorname{dn}(z \mid m)^2 + \operatorname{cd}(z \mid m) \big(((m-1)\, z + E(\operatorname{am}(z \mid m) \mid m)) \operatorname{dn}(z \mid m)^2 + m \operatorname{sc}(z \mid m) \operatorname{sn}(z \mid m)^2 \operatorname{dn}(z \mid m) - m\, ((m-1)\, z + E(\operatorname{am}(z \mid m) \mid m)) \operatorname{sn}(z \mid m)^2\big) + \operatorname{sn}(z \mid m) \Big(((m-1)\, z + E(\operatorname{am}(z \mid m) \mid m))^2 + \operatorname{dn}(z \mid m) \Big(\sqrt{1 - m \operatorname{sn}(z \mid m)^2} - ((m-1)\, z + E(\operatorname{am}(z \mid m) \mid m)) \operatorname{sc}(z \mid m)\Big)\Big)\Big) \operatorname{cn}(z \mid m)^2 - \operatorname{dn}(z \mid m) \Big(-2\, z\, m^2 + 2 \operatorname{cd}(z \mid m) \operatorname{sn}(z \mid m)\, m^2 + 4\, z\, m - 3\, E(\operatorname{am}(z \mid m) \mid m)\, m - F(\operatorname{am}(z \mid m) \mid m)\, m - (m-1)\, ((m-1)\, z + E(\operatorname{am}(z \mid m) \mid m)) \operatorname{nd}(z \mid m) \operatorname{sd}(z \mid m) \operatorname{sn}(z \mid m)\, m + z \operatorname{dn}(z \mid m) \sqrt{1 - m \operatorname{sn}(z \mid m)^2}\, m - 2\, z + E(\operatorname{am}(z \mid m) \mid m) + F(\operatorname{am}(z \mid m) \mid m) - z \operatorname{dn}(z \mid m) \sqrt{1 - m \operatorname{sn}(z \mid m)^2} + E(\operatorname{am}(z \mid m) \mid m) \operatorname{dn}(z \mid m) \sqrt{1 - m \operatorname{sn}(z \mid m)^2}\Big) \operatorname{cn}(z \mid m) + \operatorname{dn}(z \mid m)^2 \operatorname{sn}(z \mid m)\, ((m-1)\, z + E(\operatorname{am}(z \mid m) \mid m) - m \operatorname{cd}(z \mid m) \operatorname{sn}(z \mid m))^2\Big)$$

Symbolic differentiation

With respect to *z*

09.36.20.0008.01

$$\frac{\partial^n \operatorname{sn}(z \mid m)}{\partial z^n} = \operatorname{sn}(z \mid m)\, \delta_n + \sum_{j=0}^{n-1} \binom{n-1}{j} \frac{\partial^j \operatorname{cn}(z \mid m)}{\partial z^j} \frac{\partial^{-j+n-1} \operatorname{dn}(z \mid m)}{\partial z^{-j+n-1}} \,/;\, n \in \mathbb{N}$$

09.36.20.0006.01

$$\frac{\partial^n \operatorname{sn}(z \mid m)}{\partial z^n} = \frac{2^{1-n}\, \pi^{n+1}}{\sqrt{m}\, K(m)^{n+1}} \sum_{k=0}^{\infty} \frac{(2\, k+1)^n\, q(m)^{k+\frac{1}{2}}}{1 - q(m)^{2\, k+1}} \sin\left(\frac{\pi\, n}{2} + \frac{(2\, k+1)\, \pi\, z}{2\, K(m)}\right) /;\, n \in \mathbb{N}^+$$

Fractional integro-differentiation

With respect to *z*

09.36.20.0007.01

$$\frac{\partial^{\alpha}\operatorname{sn}(z\mid m)}{\partial z^{\alpha}} = \frac{2^{\alpha-1}\pi^{5/2}z^{1-\alpha}}{\sqrt{m}\,K(m)^2}\sum_{k=0}^{\infty}\frac{(2k+1)\,q(m)^{k+\frac{1}{2}}}{1-q(m)^{2k+1}}\,{}_1\tilde{F}_2\left(1;1-\frac{\alpha}{2},\frac{3-\alpha}{2};-\frac{(2k+1)^2\pi^2 z^2}{16K(m)^2}\right)$$

Integration

Indefinite integration

Involving only one direct function

09.36.21.0001.01

$$\int \operatorname{sn}(z\mid m)\,dz = \frac{1}{\sqrt{m}}\log\left(\operatorname{dn}(z\mid m)-\sqrt{m}\,\operatorname{cn}(z\mid m)\right)$$

Involving functions of the direct function

Involving elementary functions of the direct function

Involving powers of the direct function

09.36.21.0002.01

$$\int \operatorname{sn}(z\mid m)^2\,dz = \frac{z}{m}+\frac{E(\operatorname{am}(z\mid m)\mid m)\left(\operatorname{sn}(z\mid m)^2-\frac{1}{m}\right)}{\operatorname{dn}(z\mid m)\sqrt{1-m\operatorname{sn}(z\mid m)^2}}$$

09.36.21.0003.01

$$\int \operatorname{sn}(z\mid m)^2\,dz = \frac{z-\frac{E(\operatorname{am}(z|m)|m)\sqrt{1-m\operatorname{sn}(z|m)^2}}{\operatorname{dn}(z|m)}}{m}$$

09.36.21.0004.01

$$\int \operatorname{sn}(z\mid m)^3\,dz = \frac{\operatorname{cn}(z\mid m)\operatorname{dn}(z\mid m)+\frac{(m+1)\log\left(\operatorname{dn}(z|m)-\sqrt{m}\,\operatorname{cn}(z|m)\right)}{\sqrt{m}}}{2m}$$

09.36.21.0005.01

$$\int \frac{dz}{\operatorname{sn}(z\mid m)} = \log\left(\frac{\operatorname{dn}(z\mid m)-\operatorname{cn}(z\mid m)}{\operatorname{sn}(z\mid m)}\right)$$

09.36.21.0006.01

$$\int \frac{1}{\operatorname{sn}(z\mid m)}\,dz = \log\left(\frac{\operatorname{sn}(z\mid m)}{\operatorname{cn}(z\mid m)+\operatorname{dn}(z\mid m)}\right)$$

09.36.21.0007.01

$$\int \frac{dz}{\operatorname{sn}(z\mid m)^2} = z-\frac{\operatorname{cn}(z\mid m)\operatorname{dn}(z\mid m)}{\operatorname{sn}(z\mid m)}+\frac{\left(m\operatorname{sn}(z\mid m)^2-1\right)E(\operatorname{am}(z\mid m)\mid m)}{\operatorname{dn}(z\mid m)\sqrt{1-m\operatorname{sn}(z\mid m)^2}}$$

09.36.21.0008.01

$$\int \frac{1}{\operatorname{sn}(z\mid m)^3}\,dz = \frac{1}{2}\left((m+1)\log\left(\frac{\operatorname{sn}(z\mid m)}{\operatorname{cn}(z\mid m)+\operatorname{dn}(z\mid m)}\right)-\frac{\operatorname{cn}(z\mid m)\operatorname{dn}(z\mid m)}{\operatorname{sn}(z\mid m)^2}\right)$$

Involving direct function and elliptic functions

Involving Jacobi functions

Involving cn

09.36.21.0009.01

$$\int \mathrm{sn}(z \mid m)\,\mathrm{cn}(z \mid m)\,dz = -\frac{1}{m}\,\mathrm{dn}(z \mid m)$$

09.36.21.0010.01

$$\int \frac{\mathrm{sn}(z \mid m)}{\mathrm{cn}(z \mid m)}\,dz = \frac{1}{\sqrt{1-m}}\log\left(\frac{\mathrm{dn}(z \mid m)+\sqrt{1-m}}{\mathrm{cn}(z \mid m)}\right)$$

09.36.21.0011.01

$$\int \frac{\mathrm{sn}(z \mid m)}{\mathrm{cn}(z \mid m)^2}\,dz = \frac{1}{1-m}\,\frac{\mathrm{dn}(z \mid m)}{\mathrm{cn}(z \mid m)}$$

09.36.21.0012.01

$$\int \frac{\mathrm{cn}(z \mid m)}{\mathrm{sn}(z \mid m)}\,dz = \log\left(\frac{1-\mathrm{dn}(z \mid m)}{\mathrm{sn}(z \mid m)}\right)$$

09.36.21.0013.01

$$\int \frac{\mathrm{cn}(z \mid m)}{\mathrm{sn}(z \mid m)^2}\,dz = -\frac{\mathrm{dn}(z \mid m)}{\mathrm{sn}(z \mid m)}$$

09.36.21.0014.01

$$\int \frac{dz}{\mathrm{sn}(z \mid m)\,\mathrm{cn}(z \mid m)} = \frac{1}{\sqrt{1-m}}\log\left(\frac{\mathrm{dn}(z \mid m)+\sqrt{1-m}}{\mathrm{cn}(z \mid m)}\right)+\log\left(\frac{1-\mathrm{dn}(z \mid m)}{\mathrm{sn}(z \mid m)}\right)$$

09.36.21.0015.01

$$\int \left(\frac{\mathrm{cn}(z \mid m)}{\mathrm{sn}(z \mid m)}+\frac{\mathrm{sn}(z \mid m)}{\mathrm{cn}(z \mid m)}\right)dz = \frac{1}{\sqrt{1-m}}\log\left(\frac{\mathrm{dn}(z \mid m)+\sqrt{1-m}}{\mathrm{cn}(z \mid m)}\right)+\log\left(\frac{1-\mathrm{dn}(z \mid m)}{\mathrm{sn}(z \mid m)}\right)$$

Involving dn

09.36.21.0016.01

$$\int \mathrm{sn}(z \mid m)\,\mathrm{dn}(z \mid m)\,dz = -\mathrm{cn}(z \mid m)$$

09.36.21.0017.01

$$\int \frac{\mathrm{sn}(z \mid m)}{\mathrm{dn}(z \mid m)}\,dz = \frac{i}{\sqrt{m}\,\sqrt{1-m}}\log\left(\frac{i\sqrt{1-m}-\sqrt{m}\,\mathrm{cn}(z \mid m)}{\mathrm{dn}(z \mid m)}\right)$$

09.36.21.0018.01

$$\int \frac{\mathrm{sn}(z \mid m)}{\mathrm{dn}(z \mid m)}\,dz = \frac{\cot^{-1}\left(\frac{\sqrt{m}\,\mathrm{cn}(z|m)}{\sqrt{1-m}}\right)}{\sqrt{-(m-1)\,m}}$$

09.36.21.0019.01

$$\int \frac{\operatorname{sn}(z \mid m)}{\operatorname{dn}(z \mid m)^2} \, dz = -\frac{1}{1-m} \frac{\operatorname{cn}(z \mid m)}{\operatorname{dn}(z \mid m)}$$

09.36.21.0020.01

$$\int \frac{\operatorname{dn}(z \mid m)}{\operatorname{sn}(z \mid m)} \, dz = \log\left(\frac{1-\operatorname{cn}(z \mid m)}{\operatorname{sn}(z \mid m)}\right)$$

09.36.21.0021.01

$$\int \frac{\operatorname{dn}(z \mid m)}{\operatorname{sn}(z \mid m)} \, dz = \frac{1}{2} \log\left(\frac{1-\operatorname{cn}(z \mid m)}{\operatorname{cn}(z \mid m)+1}\right)$$

09.36.21.0022.01

$$\int \frac{\operatorname{dn}(z \mid m)}{\operatorname{sn}(z \mid m)^2} \, dz = -\frac{\operatorname{cn}(z \mid m)}{\operatorname{sn}(z \mid m)}$$

09.36.21.0023.01

$$\int \frac{dz}{\operatorname{sn}(z \mid m) \operatorname{dn}(z \mid m)} = \frac{\sqrt{m}}{\sqrt{1-m}} \cot^{-1}\left(\frac{\sqrt{m}}{\sqrt{1-m}} \operatorname{cn}(z \mid m)\right) + \frac{1}{2} \log\left(\frac{1-\operatorname{cn}(z \mid m)}{1+\operatorname{cn}(z \mid m)}\right)$$

09.36.21.0024.01

$$\int \left(\frac{\operatorname{sn}(z \mid m)\, m}{\operatorname{dn}(z \mid m)} + \frac{\operatorname{dn}(z \mid m)}{\operatorname{sn}(z \mid m)}\right) dz = \frac{\sqrt{m}}{\sqrt{1-m}} \cot^{-1}\left(\frac{\sqrt{m}}{\sqrt{1-m}} \operatorname{cn}(z \mid m)\right) + \frac{1}{2} \log\left(\frac{1-\operatorname{cn}(z \mid m)}{1+\operatorname{cn}(z \mid m)}\right)$$

Involving cnand dn

09.36.21.0025.01

$$\int \frac{\operatorname{sn}(z \mid m) \operatorname{cn}(z \mid m)}{\operatorname{dn}(z \mid m)} \, dz = -\frac{1}{m} \log(\operatorname{dn}(z \mid m))$$

09.36.21.0026.01

$$\int \frac{\operatorname{sn}(z \mid m)}{\operatorname{cn}(z \mid m) \operatorname{dn}(z \mid m)} \, dz = \frac{1}{1-m} \log\left(\frac{\operatorname{dn}(z \mid m)}{\operatorname{cn}(z \mid m)}\right)$$

09.36.21.0027.01

$$\int \frac{\operatorname{sn}(z \mid m) \operatorname{dn}(z \mid m)}{\operatorname{cn}(z \mid m)} \, dz = -\log(\operatorname{cn}(z \mid m))$$

09.36.21.0028.01

$$\int \frac{\operatorname{cn}(z \mid m) \operatorname{dn}(z \mid m)}{\operatorname{sn}(z \mid m)} \, dz = \log(\operatorname{sn}(z \mid m))$$

09.36.21.0029.01

$$\int \frac{\operatorname{cn}(z \mid m)}{\operatorname{sn}(z \mid m) \operatorname{dn}(z \mid m)} \, dz = \log\left(\frac{\operatorname{sn}(z \mid m)}{\operatorname{dn}(z \mid m)}\right)$$

09.36.21.0030.01

$$\int \frac{\operatorname{dn}(z \mid m)}{\operatorname{sn}(z \mid m) \operatorname{cn}(z \mid m)} \, dz = \log\left(\frac{\operatorname{sn}(z \mid m)}{\operatorname{cn}(z \mid m)}\right)$$

Definite integration

Involving functions of the direct function

Involving elementary functions of the direct function

Involving products of the direct function

09.36.21.0031.01

$$\int_0^{2K(m)} \operatorname{sn}(t \mid m)^3 \operatorname{sn}(a+t \mid m)\, dt = \frac{2 \operatorname{cs}(a \mid m) \operatorname{ds}(a \mid m) E(m) - 2 K(m) \left(\operatorname{cs}(a \mid m) \operatorname{ds}(a \mid m) - \operatorname{ns}(a \mid m)^3 Z(\operatorname{am}(a \mid m) \mid m)\right)}{m^2}$$

Khare/Lakshminarayan/Sukhatme_2003

09.36.21.0032.01

$$\int_0^{2K(m)} \operatorname{sn}(t \mid m) \operatorname{sn}(a+t \mid m) \operatorname{sn}(b+t \mid m) \operatorname{sn}(c+t \mid m)\, dt = \frac{1}{m^2} ((2 K(m)) (\operatorname{ns}(b-a \mid m) \operatorname{ns}(c-a \mid m) \operatorname{ns}(a \mid m) Z(\operatorname{am}(a \mid m) \mid m) - \operatorname{ns}(b \mid m) \operatorname{ns}(b-a \mid m) \operatorname{ns}(c-b \mid m) Z(\operatorname{am}(b \mid m) \mid m) + \operatorname{ns}(c-a \mid m) \operatorname{ns}(c-b \mid m) \operatorname{ns}(c \mid m) Z(\operatorname{am}(c \mid m) \mid m)))$$

Khare/Lakshminarayan/Sukhatme_2003

Involving direct function and elliptic functions

Involving Jacobi functions

Involving dn

09.36.21.0033.01

$$\int_0^{2K(m)} \operatorname{dn}(t \mid m) \operatorname{sn}(t \mid m) \operatorname{dn}(a+t \mid m) \operatorname{sn}(a+t \mid m)\, dt = \frac{1}{m} \left(4 \operatorname{cs}(a \mid m) \operatorname{ns}(a \mid m) E(m) - 2 K(m) \operatorname{ns}(a \mid m) \left(\operatorname{cs}(a \mid m) \left(\operatorname{dn}(a \mid m)^2 + 1\right) - \left(\operatorname{cn}(a \mid m)^2 + 1\right) \operatorname{ds}(a \mid m) \operatorname{ns}(a \mid m) Z(\operatorname{am}(a \mid m) \mid m)\right)\right)$$

Khare/Lakshminarayan/Sukhatme_2003

Representations through equivalent functions

With inverse function

09.36.27.0001.01

$$\operatorname{sn}\left(\operatorname{sn}^{-1}(z \mid m) \mid m\right) = z$$

With related functions

Involving am

09.36.27.0002.01

$$\operatorname{sn}(z \mid m) = \sin(\operatorname{am}(z \mid m))$$

Involving one other Jacobi elliptic function

Involving cd

09.36.27.0003.01

$$\operatorname{sn}(z \mid m)^2 = \frac{\operatorname{cd}(z \mid m)^2 - 1}{m \operatorname{cd}(z \mid m)^2 - 1}$$

Involving cn

09.36.27.0006.01

$$\operatorname{sn}(z \mid m)^2 = 1 - \operatorname{cn}(z \mid m)^2$$

Involving cs

09.36.27.0008.01

$$\operatorname{sn}(z \mid m) = -\frac{i}{\operatorname{cs}(i z \mid 1 - m)}$$

09.36.27.0030.01

$$\operatorname{sn}(z \mid m)^2 = \frac{1}{\operatorname{cs}(z \mid m)^2 + 1}$$

Involving dc

09.36.27.0009.01

$$\operatorname{sn}(z \mid m)^2 = \frac{1 - \operatorname{dc}(z \mid m)^2}{m - \operatorname{dc}(z \mid m)^2}$$

Involving dn

09.36.27.0012.01

$$\operatorname{sn}(z \mid m)^2 = \frac{1 - \operatorname{dn}(z \mid m)^2}{m}$$

Involving ds

09.36.27.0014.01

$$\operatorname{sn}(z \mid m)^2 = \frac{1}{\operatorname{ds}(z \mid m)^2 + m}$$

Involving nc

09.36.27.0016.01

$$\operatorname{sn}(z \mid m)^2 = 1 - \frac{1}{\operatorname{nc}(z \mid m)^2}$$

Involving nd

09.36.27.0018.01

$$\operatorname{sn}(z \mid m)^2 = \frac{\operatorname{nd}(z \mid m)^2 - 1}{m \operatorname{nd}(z \mid m)^2}$$

Involving ns

09.36.27.0019.01

$$\operatorname{sn}(z \mid m) = \frac{1}{\operatorname{ns}(z \mid m)}$$

Involving sc

09.36.27.0020.01

$$\operatorname{sn}(z \mid m) = -i \operatorname{sc}(i z \mid 1 - m)$$

09.36.27.0021.01

$$\operatorname{sn}(z \mid m)^2 = \frac{\operatorname{sc}(z \mid m)^2}{\operatorname{sc}(z \mid m)^2 + 1}$$

Involving sd

09.36.27.0022.01

$$\operatorname{sn}(z \mid m)^2 = \frac{\operatorname{sd}(z \mid m)^2}{m \operatorname{sd}(z \mid m)^2 + 1}$$

Involving two other Jacobi elliptic functions

Involving cn and cs

09.36.27.0004.01

$$\operatorname{sn}(z \mid m) = \frac{\operatorname{cn}(z \mid m)}{\operatorname{cs}(z \mid m)}$$

Involving cn and sc

09.36.27.0005.01

$$\operatorname{sn}(z \mid m) = \operatorname{sc}(z \mid m) \operatorname{cn}(z \mid m)$$

Involving cs and nc

09.36.27.0007.01

$$\operatorname{sn}(z \mid m) = \frac{1}{\operatorname{cs}(z \mid m) \operatorname{nc}(z \mid m)}$$

09.36.27.0031.01

$$\operatorname{sn}(z \mid m) = \frac{\operatorname{cs}(z \mid m) \operatorname{nc}(z \mid m)}{\operatorname{cs}(z \mid m)^2 + 1}$$

09.36.27.0032.01

$$\operatorname{sn}(z \mid m) = \frac{\operatorname{cs}(z \mid m) (\operatorname{nc}(z \mid m) - 1) (\operatorname{nc}(z \mid m) + 1)}{\operatorname{nc}(z \mid m)}$$

Involving cs and ns

09.36.27.0033.01

$$\operatorname{sn}(z \mid m) = \frac{\operatorname{ns}(z \mid m)}{\operatorname{cs}(z \mid m)^2 + 1}$$

Involving dn and ds

09.36.27.0010.01

$$\operatorname{sn}(z \mid m) == \frac{\operatorname{dn}(z \mid m)}{\operatorname{ds}(z \mid m)}$$

09.36.27.0034.01

$$\operatorname{sn}(z \mid m) = \frac{\operatorname{ds}(z \mid m)}{\operatorname{dn}(z \mid m) \left(\operatorname{ds}(z \mid m)^2 + m\right)}$$

09.36.27.0035.01

$$\operatorname{sn}(z \mid m) = -\frac{(\operatorname{dn}(z \mid m) - 1)\,(\operatorname{dn}(z \mid m) + 1)\,\operatorname{ds}(z \mid m)}{m\,\operatorname{dn}(z \mid m)}$$

Involving dn and ns

09.36.27.0036.01

$$\operatorname{sn}(z \mid m) = -\frac{(\operatorname{dn}(z \mid m) - 1)\,(\operatorname{dn}(z \mid m) + 1)\,\operatorname{ns}(z \mid m)}{m}$$

Involving dn and sd

09.36.27.0011.01

$$\operatorname{sn}(z \mid m) == \operatorname{sd}(z \mid m)\,\operatorname{dn}(z \mid m)$$

Involving ds and nd

09.36.27.0013.01

$$\operatorname{sn}(z \mid m) == \frac{1}{\operatorname{ds}(z \mid m)\,\operatorname{nd}(z \mid m)}$$

Involving nc and ns

09.36.27.0037.01

$$\operatorname{sn}(z \mid m) = \frac{(\operatorname{nc}(z \mid m) - 1)\,(\operatorname{nc}(z \mid m) + 1)\,\operatorname{ns}(z \mid m)}{\operatorname{nc}(z \mid m)^2}$$

Involving nc and sc

09.36.27.0015.01

$$\operatorname{sn}(z \mid m) == \frac{\operatorname{sc}(z \mid m)}{\operatorname{nc}(z \mid m)}$$

09.36.27.0038.01

$$\operatorname{sn}(z \mid m) = \frac{\operatorname{nc}(z \mid m)\,\operatorname{sc}(z \mid m)}{\operatorname{sc}(z \mid m)^2 + 1}$$

Involving nd and sd

09.36.27.0017.01

$$\operatorname{sn}(z \mid m) == \frac{\operatorname{sd}(z \mid m)}{\operatorname{nd}(z \mid m)}$$

Involving ns and sc

09.36.27.0039.01

$$\operatorname{sn}(z \mid m) = \frac{\operatorname{ns}(z \mid m)\operatorname{sc}(z \mid m)^2}{\operatorname{sc}(z \mid m)^2 + 1}$$

Involving three other Jacobi elliptic functions

09.36.27.0040.01

$$\operatorname{sn}(z \mid m) = \frac{\operatorname{ds}(z \mid m)\left(1 - \operatorname{nc}(z \mid m)^2\right)}{\operatorname{cd}(z \mid m)\operatorname{nc}(z \mid m)\left((m-1)\operatorname{nc}(z \mid m)^2 - m\right)}$$

09.36.27.0041.01

$$\operatorname{sn}(z \mid m) = \frac{\operatorname{cs}(z \mid m)\operatorname{dc}(z \mid m)}{\left(\operatorname{cs}(z \mid m)^2 + 1\right)\operatorname{dn}(z \mid m)}$$

09.36.27.0042.01

$$\operatorname{sn}(z \mid m) = \frac{(\operatorname{cd}(z \mid m) - 1)(\operatorname{cd}(z \mid m) + 1)\operatorname{cs}(z \mid m)\operatorname{dn}(z \mid m)}{(m-1)\operatorname{cd}(z \mid m)}$$

09.36.27.0043.01

$$\operatorname{sn}(z \mid m) = \frac{\operatorname{cs}(z \mid m)(\operatorname{dc}(z \mid m) - \operatorname{dn}(z \mid m))(\operatorname{dc}(z \mid m) + \operatorname{dn}(z \mid m))}{\operatorname{dc}(z \mid m)\operatorname{dn}(z \mid m)}$$

09.36.27.0044.01

$$\operatorname{sn}(z \mid m) = -\frac{\operatorname{cs}(z \mid m)(\operatorname{cd}(z \mid m)\operatorname{dn}(z \mid m) - 1)(\operatorname{cd}(z \mid m)\operatorname{dn}(z \mid m) + 1)}{\operatorname{cd}(z \mid m)\operatorname{dn}(z \mid m)}$$

09.36.27.0045.01

$$\operatorname{sn}(z \mid m) = -\frac{\operatorname{cd}(z \mid m)\operatorname{cs}(z \mid m)(\operatorname{dn}(z \mid m) - 1)\operatorname{dn}(z \mid m)(\operatorname{dn}(z \mid m) + 1)}{\operatorname{dn}(z \mid m)^2 + m - 1}$$

09.36.27.0046.01

$$\operatorname{sn}(z \mid m) = \frac{\operatorname{ds}(z \mid m)}{\left(\operatorname{cs}(z \mid m)^2 + 1\right)\operatorname{dn}(z \mid m)}$$

09.36.27.0047.01

$$\operatorname{sn}(z \mid m) = \frac{(\operatorname{dc}(z \mid m) - \operatorname{dn}(z \mid m))(\operatorname{dc}(z \mid m) + \operatorname{dn}(z \mid m))\operatorname{ds}(z \mid m)}{\operatorname{dc}(z \mid m)^2\operatorname{dn}(z \mid m)}$$

09.36.27.0048.01

$$\operatorname{sn}(z \mid m) = -\frac{(\operatorname{cd}(z \mid m)\operatorname{dn}(z \mid m) - 1)(\operatorname{cd}(z \mid m)\operatorname{dn}(z \mid m) + 1)\operatorname{ds}(z \mid m)}{\operatorname{dn}(z \mid m)}$$

09.36.27.0049.01

$$\operatorname{sn}(z \mid m) = \frac{\operatorname{cn}(z \mid m)\operatorname{ds}(z \mid m)}{\operatorname{dc}(z \mid m)\left(\operatorname{ds}(z \mid m)^2 + m - 1\right)}$$

09.36.27.0050.01

$$\operatorname{sn}(z \mid m) = -\frac{\operatorname{cs}(z \mid m)(\operatorname{dn}(z \mid m) - 1)(\operatorname{dn}(z \mid m) + 1)\operatorname{nc}(z \mid m)}{m}$$

09.36.27.0051.01

$$\operatorname{sn}(z \mid m) = \frac{\operatorname{ds}(z \mid m)\operatorname{nc}(z \mid m)}{\operatorname{dc}(z \mid m)\left(\operatorname{ds}(z \mid m)^2 + m\right)}$$

09.36.27.0052.01

$$\operatorname{sn}(z \mid m) = \frac{\operatorname{dc}(z \mid m)\left(\operatorname{ds}(z \mid m)^2 + m - 1\right)\operatorname{nc}(z \mid m)}{\operatorname{ds}(z \mid m)\left(\operatorname{ds}(z \mid m)^2 + m\right)}$$

09.36.27.0053.01

$$\operatorname{sn}(z \mid m) = \frac{\operatorname{cd}(z \mid m)\operatorname{ds}(z \mid m)\operatorname{nc}(z \mid m)}{\operatorname{cd}(z \mid m)^2\operatorname{ds}(z \mid m)^2 + 1}$$

09.36.27.0054.01

$$\operatorname{sn}(z \mid m) = \frac{\operatorname{ds}(z \mid m)(\operatorname{nc}(z \mid m) - 1)(\operatorname{nc}(z \mid m) + 1)}{\operatorname{dn}(z \mid m)\operatorname{nc}(z \mid m)^2}$$

09.36.27.0055.01

$$\operatorname{sn}(z \mid m) = \frac{\operatorname{cd}(z \mid m)\operatorname{ds}(z \mid m)(\operatorname{nc}(z \mid m) - 1)(\operatorname{nc}(z \mid m) + 1)}{\operatorname{nc}(z \mid m)}$$

09.36.27.0056.01

$$\operatorname{sn}(z \mid m) = \frac{\operatorname{ds}(z \mid m)(\operatorname{nc}(z \mid m) - 1)(\operatorname{nc}(z \mid m) + 1)}{\operatorname{dc}(z \mid m)\operatorname{nc}(z \mid m)}$$

09.36.27.0057.01

$$\operatorname{sn}(z \mid m) = \frac{\operatorname{cs}(z \mid m)\operatorname{nd}(z \mid m)}{\operatorname{cd}(z \mid m)\left(\operatorname{cs}(z \mid m)^2 + 1\right)}$$

09.36.27.0058.01

$$\operatorname{sn}(z \mid m) = \frac{\operatorname{cd}(z \mid m)\left(\operatorname{cs}(z \mid m)^2 - m + 1\right)\operatorname{nd}(z \mid m)}{\operatorname{cs}(z \mid m)\left(\operatorname{cs}(z \mid m)^2 + 1\right)}$$

09.36.27.0059.01

$$\operatorname{sn}(z \mid m) = \frac{\operatorname{cs}(z \mid m)\operatorname{dc}(z \mid m)\operatorname{nd}(z \mid m)}{\operatorname{cs}(z \mid m)^2 + 1}$$

09.36.27.0060.01

$$\operatorname{sn}(z \mid m) = \frac{\operatorname{ds}(z \mid m)\operatorname{nd}(z \mid m)}{\operatorname{cs}(z \mid m)^2 + 1}$$

09.36.27.0061.01

$$\operatorname{sn}(z \mid m) = \frac{\operatorname{dc}(z \mid m)^2\operatorname{ds}(z \mid m)\operatorname{nd}(z \mid m)}{\operatorname{dc}(z \mid m)^2 + \operatorname{ds}(z \mid m)^2}$$

09.36.27.0062.01

$$\operatorname{sn}(z \mid m) = \frac{\operatorname{ds}(z \mid m)\,(\operatorname{nc}(z \mid m) - 1)\,(\operatorname{nc}(z \mid m) + 1)\,\operatorname{nd}(z \mid m)}{\operatorname{nc}(z \mid m)^2}$$

09.36.27.0063.01

$$\operatorname{sn}(z \mid m) = -\frac{\operatorname{cs}(z \mid m)\,(\operatorname{cd}(z \mid m) - \operatorname{nd}(z \mid m))\,(\operatorname{cd}(z \mid m) + \operatorname{nd}(z \mid m))}{\operatorname{cd}(z \mid m)\,\operatorname{nd}(z \mid m)}$$

09.36.27.0064.01

$$\operatorname{sn}(z \mid m) = -\frac{\operatorname{ds}(z \mid m)\,(\operatorname{cd}(z \mid m) - \operatorname{nd}(z \mid m))\,(\operatorname{cd}(z \mid m) + \operatorname{nd}(z \mid m))}{\operatorname{nd}(z \mid m)}$$

09.36.27.0065.01

$$\operatorname{sn}(z \mid m) = \frac{\operatorname{cs}(z \mid m)\,(\operatorname{dc}(z \mid m)\,\operatorname{nd}(z \mid m) - 1)\,(\operatorname{dc}(z \mid m)\,\operatorname{nd}(z \mid m) + 1)}{\operatorname{dc}(z \mid m)\,\operatorname{nd}(z \mid m)}$$

09.36.27.0066.01

$$\operatorname{sn}(z \mid m) = \frac{(\operatorname{dc}(z \mid m) - \operatorname{dn}(z \mid m))\,(\operatorname{dc}(z \mid m) + \operatorname{dn}(z \mid m))\,\operatorname{ns}(z \mid m)}{\operatorname{dc}(z \mid m)^2}$$

09.36.27.0067.01

$$\operatorname{sn}(z \mid m) = -(\operatorname{cd}(z \mid m)\,\operatorname{dn}(z \mid m) - 1)\,(\operatorname{cd}(z \mid m)\,\operatorname{dn}(z \mid m) + 1)\,\operatorname{ns}(z \mid m)$$

09.36.27.0068.01

$$\operatorname{sn}(z \mid m) = \frac{\operatorname{dc}(z \mid m)^2\,\operatorname{ns}(z \mid m)}{\operatorname{dc}(z \mid m)^2 + \operatorname{ds}(z \mid m)^2}$$

09.36.27.0069.01

$$\operatorname{sn}(z \mid m) = -\frac{(\operatorname{cd}(z \mid m) - \operatorname{nd}(z \mid m))\,(\operatorname{cd}(z \mid m) + \operatorname{nd}(z \mid m))\,\operatorname{ns}(z \mid m)}{\operatorname{nd}(z \mid m)^2}$$

09.36.27.0070.01

$$\operatorname{sn}(z \mid m) = \frac{(\operatorname{dc}(z \mid m)\,\operatorname{nd}(z \mid m) - 1)\,(\operatorname{dc}(z \mid m)\,\operatorname{nd}(z \mid m) + 1)\,\operatorname{ns}(z \mid m)}{\operatorname{dc}(z \mid m)^2\,\operatorname{nd}(z \mid m)^2}$$

09.36.27.0071.01

$$\operatorname{sn}(z \mid m) = \frac{(\operatorname{cd}(z \mid m) - 1)\,(\operatorname{cd}(z \mid m) + 1)\,\operatorname{cn}(z \mid m)}{(m - 1)\,\operatorname{cd}(z \mid m)^2\,\operatorname{sc}(z \mid m)}$$

09.36.27.0072.01

$$\operatorname{sn}(z \mid m) = -\frac{\operatorname{cn}(z \mid m)\,(\operatorname{dc}(z \mid m) - 1)\,(\operatorname{dc}(z \mid m) + 1)}{(m - 1)\,\operatorname{sc}(z \mid m)}$$

09.36.27.0073.01

$$\operatorname{sn}(z \mid m) = -\frac{\operatorname{cn}(z \mid m)\,(\operatorname{dn}(z \mid m) - 1)\,(\operatorname{dn}(z \mid m) + 1)}{\left(\operatorname{dn}(z \mid m)^2 + m - 1\right)\operatorname{sc}(z \mid m)}$$

09.36.27.0074.01

$$\operatorname{sn}(z \mid m) = \frac{\operatorname{cn}(z \mid m)}{\left(\operatorname{ds}(z \mid m)^2 + m - 1\right)\operatorname{sc}(z \mid m)}$$

09.36.27.0075.01

$$\operatorname{sn}(z \mid m) = -\frac{\operatorname{cn}(z \mid m) - \operatorname{nc}(z \mid m)}{\operatorname{sc}(z \mid m)}$$

09.36.27.0076.01

$$\operatorname{sn}(z \mid m) = \frac{\operatorname{cn}(z \mid m)\,(\operatorname{nd}(z \mid m) - 1)\,(\operatorname{nd}(z \mid m) + 1)}{\left(m \operatorname{nd}(z \mid m)^2 - \operatorname{nd}(z \mid m)^2 + 1\right) \operatorname{sc}(z \mid m)}$$

09.36.27.0077.01

$$\operatorname{sn}(z \mid m) = \frac{\operatorname{dc}(z \mid m) \left(\operatorname{dn}(z \mid m)^2 + m - 1\right) \operatorname{sc}(z \mid m)}{m \operatorname{dn}(z \mid m)}$$

09.36.27.0078.01

$$\operatorname{sn}(z \mid m) = \frac{(m - 1) \operatorname{cd}(z \mid m)^2 \operatorname{nc}(z \mid m) \operatorname{sc}(z \mid m)}{m \operatorname{cd}(z \mid m)^2 - 1}$$

09.36.27.0079.01

$$\operatorname{sn}(z \mid m) = \frac{(m - 1) \operatorname{nc}(z \mid m) \operatorname{sc}(z \mid m)}{m - \operatorname{dc}(z \mid m)^2}$$

09.36.27.0080.01

$$\operatorname{sn}(z \mid m) = \frac{\left(\operatorname{dn}(z \mid m)^2 + m - 1\right) \operatorname{nc}(z \mid m) \operatorname{sc}(z \mid m)}{m}$$

09.36.27.0081.01

$$\operatorname{sn}(z \mid m) = \frac{\left(\operatorname{ds}(z \mid m)^2 + m - 1\right) \operatorname{nc}(z \mid m) \operatorname{sc}(z \mid m)}{\operatorname{ds}(z \mid m)^2 + m}$$

09.36.27.0082.01

$$\operatorname{sn}(z \mid m) = \frac{(m - 1) \operatorname{dc}(z \mid m) \operatorname{nd}(z \mid m) \operatorname{sc}(z \mid m)}{m - \operatorname{dc}(z \mid m)^2}$$

09.36.27.0083.01

$$\operatorname{sn}(z \mid m) = \frac{\operatorname{nc}(z \mid m) \left(m \operatorname{nd}(z \mid m)^2 - \operatorname{nd}(z \mid m)^2 + 1\right) \operatorname{sc}(z \mid m)}{m \operatorname{nd}(z \mid m)^2}$$

09.36.27.0084.01

$$\operatorname{sn}(z \mid m) = \frac{\operatorname{nc}(z \mid m)}{\operatorname{cs}(z \mid m) + \operatorname{sc}(z \mid m)}$$

09.36.27.0085.01

$$\operatorname{sn}(z \mid m) = \frac{\operatorname{ns}(z \mid m) \operatorname{sc}(z \mid m)}{\operatorname{cs}(z \mid m) + \operatorname{sc}(z \mid m)}$$

09.36.27.0086.01

$$\operatorname{sn}(z \mid m) = \frac{\operatorname{ns}(z \mid m) \operatorname{sc}(z \mid m) - \operatorname{cn}(z \mid m)}{\operatorname{sc}(z \mid m)}$$

09.36.27.0087.01

$$\operatorname{sn}(z \mid m) = \frac{\operatorname{dc}(z \mid m) \operatorname{sc}(z \mid m)}{\operatorname{dn}(z \mid m) \left(\operatorname{sc}(z \mid m)^2 + 1\right)}$$

09.36.27.0088.01

$$\operatorname{sn}(z \mid m) = \frac{\operatorname{dc}(z \mid m)\operatorname{nd}(z \mid m)\operatorname{sc}(z \mid m)}{\operatorname{sc}(z \mid m)^2 + 1}$$

09.36.27.0089.01

$$\operatorname{sn}(z \mid m) = \frac{\operatorname{ds}(z \mid m)\operatorname{sc}(z \mid m)^2}{\operatorname{dn}(z \mid m)\left(\operatorname{sc}(z \mid m)^2 + 1\right)}$$

09.36.27.0090.01

$$\operatorname{sn}(z \mid m) = \frac{\operatorname{ds}(z \mid m)\operatorname{nd}(z \mid m)\operatorname{sc}(z \mid m)^2}{\operatorname{sc}(z \mid m)^2 + 1}$$

09.36.27.0091.01

$$\operatorname{sn}(z \mid m) = -\frac{\operatorname{cd}(z \mid m)\operatorname{nd}(z \mid m)\operatorname{sc}(z \mid m)\left(m\operatorname{sc}(z \mid m)^2 - \operatorname{sc}(z \mid m)^2 - 1\right)}{\operatorname{sc}(z \mid m)^2 + 1}$$

09.36.27.0092.01

$$\operatorname{sn}(z \mid m) = \frac{(\operatorname{cd}(z \mid m) - 1)(\operatorname{cd}(z \mid m) + 1)\operatorname{cn}(z \mid m)}{(m-1)\operatorname{cd}(z \mid m)\operatorname{sd}(z \mid m)}$$

09.36.27.0093.01

$$\operatorname{sn}(z \mid m) = -\frac{\operatorname{cn}(z \mid m)(\operatorname{dc}(z \mid m) - 1)(\operatorname{dc}(z \mid m) + 1)}{(m-1)\operatorname{dc}(z \mid m)\operatorname{sd}(z \mid m)}$$

09.36.27.0094.01

$$\operatorname{sn}(z \mid m) = \frac{(m-1)\operatorname{dc}(z \mid m)\operatorname{nc}(z \mid m)\operatorname{sd}(z \mid m)}{m - \operatorname{dc}(z \mid m)^2}$$

09.36.27.0095.01

$$\operatorname{sn}(z \mid m) = -\frac{\operatorname{cd}(z \mid m)\left(m\operatorname{nc}(z \mid m)^2 - \operatorname{nc}(z \mid m)^2 - m\right)\operatorname{sd}(z \mid m)}{\operatorname{nc}(z \mid m)}$$

09.36.27.0096.01

$$\operatorname{sn}(z \mid m) = -\frac{\left(-\operatorname{cs}(z \mid m)^2 + m - 1\right)\operatorname{nd}(z \mid m)\operatorname{sd}(z \mid m)}{\operatorname{cs}(z \mid m)^2 + 1}$$

09.36.27.0097.01

$$\operatorname{sn}(z \mid m) = \frac{(m-1)\operatorname{dc}(z \mid m)^2\operatorname{nd}(z \mid m)\operatorname{sd}(z \mid m)}{m - \operatorname{dc}(z \mid m)^2}$$

09.36.27.0098.01

$$\operatorname{sn}(z \mid m) = -\frac{\left(m\operatorname{nc}(z \mid m)^2 - \operatorname{nc}(z \mid m)^2 - m\right)\operatorname{nd}(z \mid m)\operatorname{sd}(z \mid m)}{\operatorname{nc}(z \mid m)^2}$$

09.36.27.0099.01

$$\operatorname{sn}(z \mid m) = -\frac{\operatorname{nd}(z \mid m)\left(m\operatorname{sc}(z \mid m)^2 - \operatorname{sc}(z \mid m)^2 - 1\right)\operatorname{sd}(z \mid m)}{\operatorname{sc}(z \mid m)^2 + 1}$$

09.36.27.0100.01

$$\operatorname{sn}(z \mid m) = \frac{\operatorname{cd}(z \mid m)\operatorname{nc}(z \mid m)\operatorname{sd}(z \mid m)}{\operatorname{cd}(z \mid m)^2 + \operatorname{sd}(z \mid m)^2}$$

09.36.27.0101.01

$$\operatorname{sn}(z \mid m) = \frac{\operatorname{cn}(z \mid m)\operatorname{sd}(z \mid m)}{\operatorname{dc}(z \mid m)\left(m\operatorname{sd}(z \mid m)^2 - \operatorname{sd}(z \mid m)^2 + 1\right)}$$

09.36.27.0102.01

$$\operatorname{sn}(z \mid m) = \frac{\operatorname{cn}(z \mid m)\operatorname{sd}(z \mid m)^2}{\operatorname{sc}(z \mid m)\left(m\operatorname{sd}(z \mid m)^2 - \operatorname{sd}(z \mid m)^2 + 1\right)}$$

09.36.27.0103.01

$$\operatorname{sn}(z \mid m) = \frac{\operatorname{nc}(z \mid m)\operatorname{sc}(z \mid m)\left(m\operatorname{sd}(z \mid m)^2 - \operatorname{sd}(z \mid m)^2 + 1\right)}{m\operatorname{sd}(z \mid m)^2 + 1}$$

09.36.27.0104.01

$$\operatorname{sn}(z \mid m) = \frac{\operatorname{dc}(z \mid m)^2\operatorname{ns}(z \mid m)\operatorname{sd}(z \mid m)^2}{\operatorname{dc}(z \mid m)^2\operatorname{sd}(z \mid m)^2 + 1}$$

Involving four other Jacobi elliptic functions

09.36.27.0105.01

$$\operatorname{sn}(z \mid m) = -\frac{\operatorname{cs}(z \mid m)\left(\operatorname{cd}(z \mid m)\operatorname{dn}(z \mid m)^2 - \operatorname{dc}(z \mid m)\right)}{\operatorname{dn}(z \mid m)}$$

09.36.27.0106.01

$$\operatorname{sn}(z \mid m) = -\frac{\operatorname{cd}(z \mid m)\operatorname{cs}(z \mid m)\operatorname{dn}(z \mid m)^2 - \operatorname{ds}(z \mid m)}{\operatorname{dn}(z \mid m)}$$

09.36.27.0107.01

$$\operatorname{sn}(z \mid m) = -\frac{(\operatorname{dc}(z \mid m)\operatorname{dn}(z \mid m) - \operatorname{cn}(z \mid m))\operatorname{ds}(z \mid m)}{(m-1)\operatorname{dc}(z \mid m)}$$

09.36.27.0108.01

$$\operatorname{sn}(z \mid m) = \frac{\operatorname{dc}(z \mid m)\operatorname{ds}(z \mid m) - \operatorname{cs}(z \mid m)\operatorname{dn}(z \mid m)^2}{\operatorname{dc}(z \mid m)\operatorname{dn}(z \mid m)}$$

09.36.27.0109.01

$$\operatorname{sn}(z \mid m) = -\operatorname{cs}(z \mid m)\,(\operatorname{cd}(z \mid m)\operatorname{dn}(z \mid m) - \operatorname{nc}(z \mid m))$$

09.36.27.0110.01

$$\operatorname{sn}(z \mid m) = -\operatorname{cd}(z \mid m)\operatorname{ds}(z \mid m)\,(\operatorname{cd}(z \mid m)\operatorname{dn}(z \mid m) - \operatorname{nc}(z \mid m))$$

09.36.27.0111.01

$$\operatorname{sn}(z \mid m) = -\frac{\operatorname{ds}(z \mid m)\,(\operatorname{dc}(z \mid m)\operatorname{dn}(z \mid m) - \operatorname{nc}(z \mid m))}{m\operatorname{dc}(z \mid m)}$$

09.36.27.0112.01

$$\operatorname{sn}(z \mid m) = \frac{\operatorname{ds}(z \mid m)\operatorname{nc}(z \mid m)}{\operatorname{cd}(z \mid m)\operatorname{ds}(z \mid m)^2 + \operatorname{dc}(z \mid m)}$$

09.36.27.0113.01

$$\operatorname{sn}(z \mid m) = \frac{\operatorname{cs}(z \mid m)\,(\operatorname{dc}(z \mid m)\operatorname{nc}(z \mid m) - \operatorname{dn}(z \mid m))}{\operatorname{dc}(z \mid m)}$$

09.36.27.0114.01

$$\operatorname{sn}(z \mid m) = -\frac{\operatorname{cs}(z \mid m)\left(\operatorname{cd}(z \mid m)^2 \operatorname{dn}(z \mid m) - \operatorname{nd}(z \mid m)\right)}{\operatorname{cd}(z \mid m)}$$

09.36.27.0115.01

$$\operatorname{sn}(z \mid m) = -\operatorname{ds}(z \mid m)\left(\operatorname{cd}(z \mid m)^2 \operatorname{dn}(z \mid m) - \operatorname{nd}(z \mid m)\right)$$

09.36.27.0116.01

$$\operatorname{sn}(z \mid m) = \frac{\operatorname{dc}(z \mid m) \operatorname{ds}(z \mid m) \operatorname{nd}(z \mid m)}{\operatorname{dc}(z \mid m) + \operatorname{cs}(z \mid m) \operatorname{ds}(z \mid m)}$$

09.36.27.0117.01

$$\operatorname{sn}(z \mid m) = -\frac{\operatorname{cd}(z \mid m) \operatorname{cs}(z \mid m) (\operatorname{dn}(z \mid m) - 1) (\operatorname{dn}(z \mid m) + 1)}{\operatorname{dn}(z \mid m) + m \operatorname{nd}(z \mid m) - \operatorname{nd}(z \mid m)}$$

09.36.27.0118.01

$$\operatorname{sn}(z \mid m) = \frac{\operatorname{cs}(z \mid m)\left(\operatorname{dc}(z \mid m)^2 \operatorname{nd}(z \mid m) - \operatorname{dn}(z \mid m)\right)}{\operatorname{dc}(z \mid m)}$$

09.36.27.0119.01

$$\operatorname{sn}(z \mid m) = \frac{\operatorname{ds}(z \mid m)\left(\operatorname{dc}(z \mid m)^2 \operatorname{nd}(z \mid m) - \operatorname{dn}(z \mid m)\right)}{\operatorname{dc}(z \mid m)^2}$$

09.36.27.0120.01

$$\operatorname{sn}(z \mid m) = \frac{\operatorname{ds}(z \mid m) (\operatorname{nc}(z \mid m) \operatorname{nd}(z \mid m) - \operatorname{cd}(z \mid m))}{\operatorname{nc}(z \mid m)}$$

09.36.27.0121.01

$$\operatorname{sn}(z \mid m) = \frac{\operatorname{cs}(z \mid m) (\operatorname{nc}(z \mid m) \operatorname{nd}(z \mid m) - \operatorname{cd}(z \mid m))}{\operatorname{nd}(z \mid m)}$$

09.36.27.0122.01

$$\operatorname{sn}(z \mid m) = \frac{\operatorname{ds}(z \mid m) \operatorname{nc}(z \mid m) \operatorname{nd}(z \mid m) - \operatorname{cs}(z \mid m)}{\operatorname{nc}(z \mid m)}$$

09.36.27.0123.01

$$\operatorname{sn}(z \mid m) = \frac{\operatorname{cs}(z \mid m)\left(\operatorname{dc}(z \mid m) \operatorname{nd}(z \mid m)^2 - \operatorname{cd}(z \mid m)\right)}{\operatorname{nd}(z \mid m)}$$

09.36.27.0124.01

$$\operatorname{sn}(z \mid m) = \frac{\operatorname{ds}(z \mid m) \operatorname{nd}(z \mid m)^2 - \operatorname{cd}(z \mid m) \operatorname{cs}(z \mid m)}{\operatorname{nd}(z \mid m)}$$

09.36.27.0125.01

$$\operatorname{sn}(z \mid m) = \frac{\operatorname{dc}(z \mid m) \operatorname{ns}(z \mid m)}{\operatorname{dc}(z \mid m) + \operatorname{cs}(z \mid m) \operatorname{ds}(z \mid m)}$$

09.36.27.0126.01

$$\operatorname{sn}(z \mid m) = \frac{(\operatorname{dc}(z \mid m) \operatorname{nc}(z \mid m) - \operatorname{dn}(z \mid m)) \operatorname{ns}(z \mid m)}{\operatorname{dc}(z \mid m) \operatorname{nc}(z \mid m)}$$

09.36.27.0127.01

$$\operatorname{sn}(z \mid m) = \frac{(\operatorname{nc}(z \mid m) \operatorname{nd}(z \mid m) - \operatorname{cd}(z \mid m)) \operatorname{ns}(z \mid m)}{\operatorname{nc}(z \mid m) \operatorname{nd}(z \mid m)}$$

09.36.27.0128.01

$$\operatorname{sn}(z \mid m) = \operatorname{ns}(z \mid m) - \operatorname{cd}(z \mid m) \operatorname{cs}(z \mid m) \operatorname{dn}(z \mid m)$$

09.36.27.0129.01

$$\operatorname{sn}(z \mid m) = \operatorname{ns}(z \mid m) - \operatorname{cd}(z \mid m)^2 \operatorname{dn}(z \mid m) \operatorname{ds}(z \mid m)$$

09.36.27.0130.01

$$\operatorname{sn}(z \mid m) = \frac{\operatorname{dc}(z \mid m) \operatorname{ns}(z \mid m) - \operatorname{cs}(z \mid m) \operatorname{dn}(z \mid m)}{\operatorname{dc}(z \mid m)}$$

09.36.27.0131.01

$$\operatorname{sn}(z \mid m) = \frac{\operatorname{dc}(z \mid m) \operatorname{ns}(z \mid m) - \operatorname{cn}(z \mid m) \operatorname{ds}(z \mid m)}{\operatorname{dc}(z \mid m)}$$

09.36.27.0132.01

$$\operatorname{sn}(z \mid m) = \frac{\operatorname{dc}(z \mid m)^2 \operatorname{ns}(z \mid m) - \operatorname{dn}(z \mid m) \operatorname{ds}(z \mid m)}{\operatorname{dc}(z \mid m)^2}$$

09.36.27.0133.01

$$\operatorname{sn}(z \mid m) = \frac{\operatorname{nc}(z \mid m) \operatorname{ns}(z \mid m) - \operatorname{cd}(z \mid m) \operatorname{ds}(z \mid m)}{\operatorname{nc}(z \mid m)}$$

09.36.27.0134.01

$$\operatorname{sn}(z \mid m) = \frac{\operatorname{nd}(z \mid m) \operatorname{ns}(z \mid m) - \operatorname{cd}(z \mid m) \operatorname{cs}(z \mid m)}{\operatorname{nd}(z \mid m)}$$

09.36.27.0135.01

$$\operatorname{sn}(z \mid m) = \frac{\operatorname{dc}(z \mid m) \operatorname{nd}(z \mid m) \operatorname{ns}(z \mid m) - \operatorname{cs}(z \mid m)}{\operatorname{dc}(z \mid m) \operatorname{nd}(z \mid m)}$$

09.36.27.0136.01

$$\operatorname{sn}(z \mid m) = \frac{\operatorname{cn}(z \mid m) (\operatorname{cd}(z \mid m) - \operatorname{dc}(z \mid m))}{(m-1) \operatorname{cd}(z \mid m) \operatorname{sc}(z \mid m)}$$

09.36.27.0137.01

$$\operatorname{sn}(z \mid m) = \frac{\operatorname{cn}(z \mid m) - \operatorname{dc}(z \mid m) \operatorname{dn}(z \mid m)}{(m-1) \operatorname{sc}(z \mid m)}$$

09.36.27.0138.01

$$\operatorname{sn}(z \mid m) = -\frac{\operatorname{cn}(z \mid m) (\operatorname{dn}(z \mid m) - \operatorname{nd}(z \mid m))}{(\operatorname{dn}(z \mid m) + m \operatorname{nd}(z \mid m) - \operatorname{nd}(z \mid m)) \operatorname{sc}(z \mid m)}$$

09.36.27.0139.01

$$\operatorname{sn}(z \mid m) = -\frac{\operatorname{cn}(z \mid m) - \operatorname{dc}(z \mid m) \operatorname{nd}(z \mid m)}{\operatorname{sc}(z \mid m)}$$

09.36.27.0140.01

$$\operatorname{sn}(z \mid m) = \frac{(\operatorname{dc}(z \mid m) \operatorname{dn}(z \mid m) + m \operatorname{nc}(z \mid m) - \operatorname{nc}(z \mid m)) \operatorname{sc}(z \mid m)}{m}$$

09.36.27.0141.01

$$\operatorname{sn}(z \mid m) = \frac{\operatorname{dc}(z \mid m)\,(\operatorname{dn}(z \mid m) + m\,\operatorname{nd}(z \mid m) - \operatorname{nd}(z \mid m))\,\operatorname{sc}(z \mid m)}{m}$$

09.36.27.0142.01

$$\operatorname{sn}(z \mid m) = \frac{\operatorname{dc}(z \mid m)}{\operatorname{dn}(z \mid m)\,(\operatorname{cs}(z \mid m) + \operatorname{sc}(z \mid m))}$$

09.36.27.0143.01

$$\operatorname{sn}(z \mid m) = \frac{\operatorname{dc}(z \mid m)\,\operatorname{nd}(z \mid m)}{\operatorname{cs}(z \mid m) + \operatorname{sc}(z \mid m)}$$

09.36.27.0144.01

$$\operatorname{sn}(z \mid m) = \frac{\operatorname{ds}(z \mid m)\,\operatorname{sc}(z \mid m)}{\operatorname{dn}(z \mid m)\,(\operatorname{cs}(z \mid m) + \operatorname{sc}(z \mid m))}$$

09.36.27.0145.01

$$\operatorname{sn}(z \mid m) = \frac{\operatorname{ds}(z \mid m)\,\operatorname{nd}(z \mid m)\,\operatorname{sc}(z \mid m)}{\operatorname{cs}(z \mid m) + \operatorname{sc}(z \mid m)}$$

09.36.27.0146.01

$$\operatorname{sn}(z \mid m) = \frac{\operatorname{nc}(z \mid m)}{\operatorname{cd}(z \mid m)\,\operatorname{ds}(z \mid m) + \operatorname{sc}(z \mid m)}$$

09.36.27.0147.01

$$\operatorname{sn}(z \mid m) = \frac{\operatorname{cd}(z \mid m)\,\operatorname{nd}(z \mid m)\,(\operatorname{cs}(z \mid m) - m\,\operatorname{sc}(z \mid m) + \operatorname{sc}(z \mid m))}{\operatorname{cs}(z \mid m)^2 + 1}$$

09.36.27.0148.01

$$\operatorname{sn}(z \mid m) = -\frac{\operatorname{dn}(z \mid m)\,\operatorname{ds}(z \mid m)\,\operatorname{sc}(z \mid m) - \operatorname{cn}(z \mid m)}{(m-1)\,\operatorname{sc}(z \mid m)}$$

09.36.27.0149.01

$$\operatorname{sn}(z \mid m) = \frac{1}{m}\left(\operatorname{nc}(z \mid m)\,\operatorname{sc}(z \mid m)\,\operatorname{ds}(z \mid m)^2 - \operatorname{dn}(z \mid m)\,\operatorname{ds}(z \mid m) + m\,\operatorname{nc}(z \mid m)\,\operatorname{sc}(z \mid m) - \operatorname{nc}(z \mid m)\,\operatorname{sc}(z \mid m)\right)$$

09.36.27.0150.01

$$\operatorname{sn}(z \mid m) = \frac{\operatorname{ds}(z \mid m)\,\operatorname{nd}(z \mid m)\,\operatorname{sc}(z \mid m) - \operatorname{cn}(z \mid m)}{\operatorname{sc}(z \mid m)}$$

09.36.27.0151.01

$$\operatorname{sn}(z \mid m) = -\frac{\operatorname{cd}(z \mid m)\,\operatorname{nd}(z \mid m)\left(m\,\operatorname{sc}(z \mid m)^2 - \operatorname{sc}(z \mid m)^2 - 1\right)}{\operatorname{cs}(z \mid m) + \operatorname{sc}(z \mid m)}$$

09.36.27.0152.01

$$\operatorname{sn}(z \mid m) = -\frac{1}{\operatorname{sc}(z \mid m)}\left(m\,\operatorname{cd}(z \mid m)\,\operatorname{nd}(z \mid m)\,\operatorname{sc}(z \mid m)^2 - \operatorname{cd}(z \mid m)\,\operatorname{nd}(z \mid m)\,\operatorname{sc}(z \mid m)^2 + \operatorname{cn}(z \mid m) - \operatorname{cd}(z \mid m)\,\operatorname{nd}(z \mid m)\right)$$

09.36.27.0153.01

$$\operatorname{sn}(z \mid m) = \frac{\operatorname{cn}(z \mid m)\,(\operatorname{cd}(z \mid m) - \operatorname{dc}(z \mid m))}{(m-1)\,\operatorname{sd}(z \mid m)}$$

09.36.27.0154.01

$$\operatorname{sn}(z \mid m) = -\frac{\operatorname{cn}(z \mid m) - \operatorname{nc}(z \mid m)}{\operatorname{dc}(z \mid m)\,\operatorname{sd}(z \mid m)}$$

09.36.27.0155.01

$$\operatorname{sn}(z \mid m) = \frac{\operatorname{cd}(z \mid m)^2 \operatorname{nc}(z \mid m) - \operatorname{cn}(z \mid m)}{m \operatorname{cd}(z \mid m) \operatorname{sd}(z \mid m)}$$

09.36.27.0156.01

$$\operatorname{sn}(z \mid m) = \frac{\operatorname{cd}(z \mid m) \operatorname{nd}(z \mid m) - \operatorname{cn}(z \mid m)}{m \operatorname{cd}(z \mid m) \operatorname{sd}(z \mid m)}$$

09.36.27.0157.01

$$\operatorname{sn}(z \mid m) = \frac{\operatorname{dc}(z \mid m) \operatorname{nd}(z \mid m) - \operatorname{cn}(z \mid m)}{\operatorname{dc}(z \mid m) \operatorname{sd}(z \mid m)}$$

09.36.27.0158.01

$$\operatorname{sn}(z \mid m) = -\frac{(-m \operatorname{cd}(z \mid m) + m \operatorname{nc}(z \mid m) \operatorname{nd}(z \mid m) - \operatorname{nc}(z \mid m) \operatorname{nd}(z \mid m)) \operatorname{sd}(z \mid m)}{\operatorname{nc}(z \mid m)}$$

09.36.27.0159.01

$$\operatorname{sn}(z \mid m) = \frac{\operatorname{nd}(z \mid m) (\operatorname{cs}(z \mid m) - m \operatorname{sc}(z \mid m) + \operatorname{sc}(z \mid m)) \operatorname{sd}(z \mid m)}{\operatorname{cs}(z \mid m) + \operatorname{sc}(z \mid m)}$$

09.36.27.0160.01

$$\operatorname{sn}(z \mid m) = \frac{\operatorname{cn}(z \mid m)}{\operatorname{dc}(z \mid m) (\operatorname{ds}(z \mid m) + m \operatorname{sd}(z \mid m) - \operatorname{sd}(z \mid m))}$$

09.36.27.0161.01

$$\operatorname{sn}(z \mid m) = \frac{\operatorname{dc}(z \mid m) \operatorname{nc}(z \mid m) (\operatorname{ds}(z \mid m) + m \operatorname{sd}(z \mid m) - \operatorname{sd}(z \mid m))}{\operatorname{ds}(z \mid m)^2 + m}$$

09.36.27.0162.01

$$\operatorname{sn}(z \mid m) = \frac{\operatorname{nd}(z \mid m) \operatorname{sc}(z \mid m) - \operatorname{cn}(z \mid m) \operatorname{sd}(z \mid m)}{\operatorname{sc}(z \mid m) \operatorname{sd}(z \mid m)}$$

09.36.27.0163.01

$$\operatorname{sn}(z \mid m) = \frac{\operatorname{dc}(z \mid m) \operatorname{nd}(z \mid m)}{\operatorname{cs}(z \mid m) + \operatorname{dc}(z \mid m) \operatorname{sd}(z \mid m)}$$

09.36.27.0164.01

$$\operatorname{sn}(z \mid m) = \frac{\operatorname{dc}(z \mid m) \operatorname{ns}(z \mid m) \operatorname{sd}(z \mid m)}{\operatorname{cs}(z \mid m) + \operatorname{dc}(z \mid m) \operatorname{sd}(z \mid m)}$$

09.36.27.0165.01

$$\operatorname{sn}(z \mid m) = \frac{\operatorname{cd}(z \mid m) \operatorname{ns}(z \mid m) \operatorname{sd}(z \mid m) - \operatorname{cn}(z \mid m)}{m \operatorname{cd}(z \mid m) \operatorname{sd}(z \mid m)}$$

09.36.27.0166.01

$$\operatorname{sn}(z \mid m) = \frac{\operatorname{dc}(z \mid m) \operatorname{ns}(z \mid m) \operatorname{sd}(z \mid m) - \operatorname{cn}(z \mid m)}{\operatorname{dc}(z \mid m) \operatorname{sd}(z \mid m)}$$

09.36.27.0167.01

$$\operatorname{sn}(z \mid m) = \frac{\operatorname{nd}(z \mid m) \operatorname{sc}(z \mid m)}{\operatorname{cd}(z \mid m) + \operatorname{sc}(z \mid m) \operatorname{sd}(z \mid m)}$$

09.36.27.0168.01

$$\operatorname{sn}(z \mid m) = \frac{\operatorname{nc}(z \mid m) \operatorname{sd}(z \mid m)}{\operatorname{cd}(z \mid m) + \operatorname{sc}(z \mid m) \operatorname{sd}(z \mid m)}$$

09.36.27.0169.01

$$\operatorname{sn}(z \mid m) = \frac{\operatorname{cn}(z \mid m) \operatorname{sd}(z \mid m)}{\operatorname{dc}(z \mid m) + m \operatorname{sc}(z \mid m) \operatorname{sd}(z \mid m) - \operatorname{sc}(z \mid m) \operatorname{sd}(z \mid m)}$$

09.36.27.0170.01

$$\operatorname{sn}(z \mid m) = -\frac{1}{\operatorname{sc}(z \mid m)} \left(-\operatorname{cn}(z \mid m) \operatorname{nd}(z \mid m)^2 + m \operatorname{sc}(z \mid m) \operatorname{sd}(z \mid m) \operatorname{nd}(z \mid m) - \operatorname{sc}(z \mid m) \operatorname{sd}(z \mid m) \operatorname{nd}(z \mid m) + \operatorname{cn}(z \mid m)\right)$$

09.36.27.0171.01

$$\operatorname{sn}(z \mid m) = \frac{\operatorname{nc}(z \mid m) \operatorname{sd}(z \mid m)}{\operatorname{dc}(z \mid m) \operatorname{sd}(z \mid m)^2 + \operatorname{cd}(z \mid m)}$$

09.36.27.0172.01

$$\operatorname{sn}(z \mid m) = -\frac{1}{\operatorname{nc}(z \mid m)} \left(\operatorname{nc}(z \mid m) \operatorname{sd}(z \mid m)^2 \operatorname{ns}(z \mid m)^3 - \operatorname{nc}(z \mid m) \operatorname{sd}(z \mid m)^2 \operatorname{ns}(z \mid m) - \operatorname{nc}(z \mid m) \operatorname{ns}(z \mid m) - m \operatorname{cd}(z \mid m) \operatorname{sd}(z \mid m)\right)$$

Involving five other Jacobi elliptic functions

09.36.27.0173.01

$$\operatorname{sn}(z \mid m) = -\operatorname{cs}(z \mid m) (\operatorname{cd}(z \mid m) \operatorname{dn}(z \mid m) - \operatorname{dc}(z \mid m) \operatorname{nd}(z \mid m))$$

09.36.27.0174.01

$$\operatorname{sn}(z \mid m) = \operatorname{ds}(z \mid m) \operatorname{nd}(z \mid m) - \operatorname{cd}(z \mid m) \operatorname{cs}(z \mid m) \operatorname{dn}(z \mid m)$$

09.36.27.0175.01

$$\operatorname{sn}(z \mid m) = \frac{\operatorname{dc}(z \mid m) \operatorname{ds}(z \mid m) \operatorname{nd}(z \mid m) - \operatorname{cs}(z \mid m) \operatorname{dn}(z \mid m)}{\operatorname{dc}(z \mid m)}$$

09.36.27.0176.01

$$\operatorname{sn}(z \mid m) = -\operatorname{cd}(z \mid m) (\operatorname{cs}(z \mid m) \operatorname{dn}(z \mid m) - \operatorname{cs}(z \mid m) \operatorname{nd}(z \mid m) + m \operatorname{nd}(z \mid m) \operatorname{sc}(z \mid m) - \operatorname{nd}(z \mid m) \operatorname{sc}(z \mid m))$$

09.36.27.0177.01

$$\operatorname{sn}(z \mid m) = -\frac{1}{\operatorname{nc}(z \mid m)} \left(\operatorname{nc}(z \mid m) \operatorname{ns}(z \mid m) \operatorname{nd}(z \mid m)^2 - \operatorname{nc}(z \mid m) \operatorname{sd}(z \mid m) \operatorname{nd}(z \mid m) - \operatorname{nc}(z \mid m) \operatorname{ns}(z \mid m) - m \operatorname{cd}(z \mid m) \operatorname{sd}(z \mid m)\right)$$

09.36.27.0178.01

$$\operatorname{sn}(z \mid m) = -\frac{1}{\operatorname{sc}(z \mid m)} (\operatorname{cn}(z \mid m) - \operatorname{cd}(z \mid m) \operatorname{nd}(z \mid m) + m \operatorname{nd}(z \mid m) \operatorname{sc}(z \mid m) \operatorname{sd}(z \mid m) - \operatorname{nd}(z \mid m) \operatorname{sc}(z \mid m) \operatorname{sd}(z \mid m))$$

09.36.27.0179.01

$$\operatorname{sn}(z \mid m) = -\frac{-\operatorname{nc}(z \mid m) \operatorname{nd}(z \mid m)^2 + \operatorname{nc}(z \mid m) \operatorname{sd}(z \mid m)^2 + \operatorname{cn}(z \mid m)}{m \operatorname{cd}(z \mid m) \operatorname{sd}(z \mid m)}$$

09.36.27.0180.01

$$\operatorname{sn}(z \mid m) = \frac{\operatorname{nc}(z \mid m) \operatorname{ns}(z \mid m)^2 \operatorname{sd}(z \mid m)^2 - \operatorname{nc}(z \mid m) \operatorname{sd}(z \mid m)^2 - \operatorname{cn}(z \mid m)}{m \operatorname{cd}(z \mid m) \operatorname{sd}(z \mid m)}$$

Involving Weierstrass functions

09.36.27.0023.01

$$\operatorname{sn}(z \mid m) = \frac{\sqrt{e_1 - e_3}\ \sigma\left(\frac{z}{\sqrt{e_1-e_3}}; g_2, g_3\right)}{\sigma_3\left(\frac{z}{\sqrt{e_1-e_3}}; g_2, g_3\right)} /;$$

$$\{\omega_1, \omega_2, \omega_3\} = \{\omega_1(g_2, g_3), -\omega_1(g_2, g_3) - \omega_3(g_2, g_3), \omega_3(g_2, g_3)\} \bigwedge m = \lambda\left(\frac{\omega_3}{\omega_1}\right) \bigwedge e_n = \wp(\omega_n; g_2, g_3) \bigwedge n \in \{1, 2, 3\}$$

09.36.27.0024.01

$$\operatorname{sn}(z \mid m) = \frac{\zeta(z + \omega_1 + \omega_3; g_2, g_3) - \zeta(\omega_1; g_2, g_3) - \zeta(z + \omega_3; g_2, g_3)}{\sqrt{m}} /;$$

$$\{\omega_1, \omega_3\} = \{2K(m), -i K(1-m) - 2K(m)\} \land \{g_2, g_3\} = \{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\}$$

09.36.27.0025.01

$$\operatorname{sn}(u \mid m)^2 = \frac{e_1 - e_3}{\wp\left(\frac{z}{\sqrt{e_1-e_3}}; g_2, g_3\right) - e_3} /;$$

$$\{\omega_1, \omega_2, \omega_3\} = \{\omega_1(g_2, g_3), -\omega_1(g_2, g_3) - \omega_3(g_2, g_3), \omega_3(g_2, g_3)\} \bigwedge m = \lambda\left(\frac{\omega_3}{\omega_1}\right) \bigwedge e_n = \wp(\omega_n; g_2, g_3) \bigwedge n \in \{1, 2, 3\}$$

09.36.27.0026.01

$$\operatorname{sn}\left(z \,\middle|\, \frac{e_2 - e_3}{e_1 - e_3}\right) = \sqrt{\frac{e_1 - e_3}{\wp\left(\frac{z}{\sqrt{e_1-e_3}}; g_2, g_3\right) - e_3}} /;$$

$$\{\omega_1, \omega_2, \omega_3\} = \{\omega_1(g_2, g_3), -\omega_1(g_2, g_3) - \omega_3(g_2, g_3), \omega_3(g_2, g_3)\} \land e_n = \wp(\omega_n; g_2, g_3) \land n \in \{1, 2, 3\}$$

Involving theta functions

09.36.27.0027.02

$$\operatorname{sn}(z \mid m) = \frac{1}{\sqrt[4]{m}} \frac{\vartheta_1\left(\frac{\pi z}{2K(m)}, q(m)\right)}{\vartheta_4\left(\frac{\pi z}{2K(m)}, q(m)\right)}$$

09.36.27.0028.01

$$\operatorname{sn}(z \mid m) = \frac{\vartheta_3(0, q(m))}{\vartheta_2(0, q(m))} \frac{\vartheta_1\left(\frac{z}{\vartheta_3(0,q(m))^2}, q(m)\right)}{\vartheta_4\left(\frac{z}{\vartheta_3(0,q(m))^2}, q(m)\right)}$$

09.36.27.0029.01

$$\operatorname{sn}(z \mid m) = \frac{\vartheta_s(z \mid m)}{\vartheta_n(z \mid m)}$$

Inequalities

09.36.29.0001.01

$$\frac{\operatorname{sn}(x \mid m)}{(\operatorname{cn}(x \mid m) + \operatorname{dn}(x \mid m))^2} \le \frac{x}{K(m)(1-m)} /; x \in \mathbb{R} \land 0 \le x \le K(m) \land 0 \le m \le 1$$

09.36.29.0002.02

$$\frac{\mathrm{sn}(x \mid m)}{\left(\sqrt{m}\ \mathrm{cn}(x \mid m) + \mathrm{dn}(x \mid m)\right)^2} \leq \frac{2\,x}{\pi\,(1-m)^2} \ /;\ x \in \mathbb{R} \land 0 \leq x \leq K(m) \land 0 \leq m \leq 1$$

Zeros

09.36.30.0001.01

$$\mathrm{sn}(2\,r\,K(m) + 2\,i\,s\,K(1-m) \mid m) == 0 \ /;\ \{r, s\} \in \mathbb{Z}$$

Theorems

Mapping of the upper half plane to a rectangle

The function $w(z) == \mathrm{sn}(z \mid m)$ maps the upper half w-plane to a rectangle in the z-plane with vertices $\{-K(m) + i\,K(1-m),\ -K(m),\ K(m),\ K(m) + i\,K(1-m)\}$.

The solutions of the Euler equations for the motion of a free rigid body

The solutions of the Euler equations for the motion of a free rigid body:

$$A\,p'(t) - (B-C)\,q(t)\,r(t) == B\,q'(t) - (C-A)\,r(t)\,p(t) == C\,r'(t) - (A-B)\,p(t)\,q(t) == 0;$$
$$p(t) == c_p\ \mathrm{cn}(\alpha\,(t-t_0) \mid m) \bigwedge q(t) == c_q\ \mathrm{sn}(\alpha\,(t-t_0) \mid m) \bigwedge r(t) == c_r\ \mathrm{dn}(\alpha\,(t-t_0) \mid m),$$

where n and m follow algebraically from A, B, C and the initial conditions.

Parametrization of any biquadratic curve

Any biquadratic curve $a\,x^2\,y^2 + b\left(x^2\,y + x\,y^2\right) + c\left(x^2 + y^2\right) + 2\,d\,x\,y + e(x+y) + f == 0$ can be parametrized in the form $x(t) == (\alpha\,\mathrm{sn}(t \mid m) + \beta) / (\gamma\,\mathrm{sn}(t \mid m) + \delta)$, $y(t) == (\alpha\,\mathrm{sn}(t + \eta \mid m) + \beta) / (\gamma\,\mathrm{sn}(t + \eta \mid m) + \delta)$.

The curvature and the torsion

The curvature and the torsion of the knotted and unknotted centerlines of an elastic rod can be expressed in elliptic functions.

History

– K. F. Gauss (1799)
– C. G. J. Jacobi (1827)
– N. H. Abel (1827)
– C. Gudermann (1838) introduced the notation sn.

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