KleinInvariantJ

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Notations

Traditional name

Klein invariant modular function

Traditional notation

$J(z)$

Mathematica StandardForm notation

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KleinInvariantJ[z]
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Primary definition

09.50.02.0001.01

$$J(z) = \frac{\left(\vartheta_2\left(0, e^{\pi i z}\right)^8 + \vartheta_3\left(0, e^{\pi i z}\right)^8 + \vartheta_4\left(0, e^{\pi i z}\right)^8\right)^3}{54\left(\vartheta_2\left(0, e^{\pi i z}\right)\vartheta_3\left(0, e^{\pi i z}\right)\vartheta_4\left(0, e^{\pi i z}\right)\right)^8} \;/;\; \operatorname{Im}(z) > 0$$

This definition can be continued to selected points on the real axis.

Specific values

Specialized values

09.50.03.0001.01

$$J(i + m) = 1 \;/;\; m \in \mathbb{Z}$$

Values at fixed points

Values at complex z

09.50.03.0002.01

$$J(i) = 1$$

Values at quadratic irrationalities (Heegner Numbers)

09.50.03.0013.01

$$J\left(\frac{-1 + i\sqrt{3}}{2}\right) = 1$$

09.50.03.0004.01

$$J\left(\frac{1+i\sqrt{3}}{2}\right) = 0$$

09.50.03.0005.01

$$J\left(\frac{1+i\sqrt{7}}{2}\right) = -\frac{125}{64}$$

09.50.03.0006.01

$$J\left(\frac{1+i\sqrt{11}}{2}\right) = -\frac{512}{27}$$

09.50.03.0007.01

$$J\left(\frac{1+i\sqrt{19}}{2}\right) = -512$$

09.50.03.0008.01

$$J\left(\frac{1+i\,3\sqrt{3}}{2}\right) = -\frac{64\,000}{9}$$

09.50.03.0009.01

$$J\left(\frac{1+i\sqrt{43}}{2}\right) = -512\,000$$

09.50.03.0010.01

$$J\left(\frac{1+i\sqrt{67}}{2}\right) = -85\,184\,000$$

09.50.03.0011.01

$$J\left(\frac{1+i\sqrt{163}}{2}\right) = -151\,931\,373\,056\,000$$

Values at infinities

09.50.03.0012.01

$$J(i\,\infty) = \infty$$

General characteristics

Domain and analyticity

$J(z)$ is an analytical function of z which is defined over the upper half of the complex z-plane.

09.50.04.0001.01

$$z \longrightarrow J(z) :: \mathbb{C} \longrightarrow \mathbb{C}$$

Symmetries and periodicities

Periodicity

09.50.04.0002.01

$J(m+z) == J(z)\ /;\ m \in \mathbb{Z}$

Poles and essential singularities

On the boundary of analyticity the function $J(z)$ has a dense set of poles.

09.50.04.0003.01

$\mathcal{S}ing_z(J(z)) == \{\}\ /;\ \mathrm{Im}(z) > 0$

Branch points

The function $J(z)$ does not have branch points.

09.50.04.0004.01

$\mathcal{BP}_z(J(z)) == \{\}$

Branch cuts

The function $J(z)$ does not have branch cuts.

09.50.04.0005.01

$\mathcal{BC}_z(J(z)) == \{\}$

Natural boundary of analyticity

The real axis $\mathrm{Im}(z) == 0$ is the natural boundary of the region of analyticity.

09.50.04.0006.01

$\mathcal{AB}_z(J(z)) == \{(-\infty,\ \infty)\}$

Series representations

Generalized power series

Expansions at generic point $z == z_0$

For the function itself

09.50.06.0003.01

$$J(z) \propto J(z_0) + \frac{8\,i\,K(w)^2\,(w-1)\,w\left(\lambda(z_0)^2-\lambda(z_0)+1\right)^2\left(2\,\lambda(z_0)^3-3\,\lambda(z_0)^2-3\,\lambda(z_0)+2\right)}{27\,(K(1-w)\,(K(w)-E(w))-E(1-w)\,K(w))\,(\lambda(z_0)-1)^3\,\lambda(z_0)^3}\,(z-z_0) - \frac{8\,(1-w)\,w\left(\lambda(z_0)^2-\lambda(z_0)+1\right)}{27\,(K(1-w)\,(E(w)-K(w))+E(1-w)\,K(w))^2\,(\lambda(z_0)-1)^4\,\lambda(z_0)^4} \Big(E(w)\,\lambda(z_0)\left(-2\,\lambda(z_0)^6+7\,\lambda(z_0)^5-7\,\lambda(z_0)^4+7\,\lambda(z_0)^2-7\,\lambda(z_0)+2\right) + w\,K(w)\Big(2\,\lambda(z_0)^7-9\,\lambda(z_0)^6+13\,\lambda(z_0)^5-4\,\lambda(z_0)^4-9\,\lambda(z_0)^3-11\,\lambda(z_0)^2+18\,\lambda(z_0)+2\,w\left(\lambda(z_0)^6-3\,\lambda(z_0)^5+2\,\lambda(z_0)^4+\lambda(z_0)^3+9\,\lambda(z_0)^2-10\,\lambda(z_0)+3\right)-6\Big)\Big)\,K(w)^3\,(z-z_0)^2 + \ldots\ /;\ (z\to z_0)\land w = q^{-1}\left(e^{i\,\pi\,z_0}\right)$$

09.50.06.0004.01

$$J(z) \propto J(z_0)\,(1 + O(z - z_0))$$

Exponential Fourier series

09.50.06.0001.01

$$J(z) = \frac{1}{1728}\left(e^{-2 i \pi z} + 744 + \sum_{k=1}^{\infty} a_k\, e^{2 k i \pi z}\right) /;$$

$$a_k = \frac{2\pi}{\sqrt{k}} \sum_{j=1}^{\infty} \frac{1}{j} A_j(k)\, I_1\!\left(\frac{(4\pi)\sqrt{k}}{j}\right) \bigwedge \left(A_j(k) = \sum_{h=0}^{j-1} \delta_{1,\gcd[h,j]} \exp\!\left(-\frac{(2\pi i)\,(h k + H(j, h))}{j}\right) /; (h\, H(j, h)) \bmod j = -1\right)$$

09.50.06.0002.01

$$J(z) = \frac{1}{1728}\left(e^{-2 i \pi z} + 744 + \sum_{k=1}^{\infty} a_k\, e^{2 k i \pi z}\right) /;$$

$$\Bigg(a_1 = 196\,884 \wedge a_2 = 21\,493\,760 \wedge a_3 = 864\,299\,970 \wedge a_4 = 20\,245\,856\,256,\ a_5 = 333\,202\,640\,600 \bigwedge$$

$$\left(a_\nu = a_{2n+1} + \frac{1}{2}\left(a_n^2 - a_n\right) + \sum_{k=1}^{n-1} a_k\, a_{2n-k} /; n = \frac{\nu}{4} \bigwedge \nu \bmod 4 = 0\right) \bigwedge$$

$$\left(a_\nu = a_{2n+3} - a_2\, a_{2n} + \frac{1}{2}\left(a_{n+1}^2 - a_{n+1}\right) + \frac{1}{2}\left(a_{2n}^2 + a_{2n}\right) + \sum_{k=1}^{n} a_k\, a_{2n-k+2} - \sum_{k=1}^{2n-1} (-1)^{k-1} a_k\, a_{4n-k} + \sum_{k=1}^{n-1} a_k\, a_{4n-4k} /; n = \frac{\nu - 1}{4} \bigwedge \nu \bmod 4 = 1\right) \bigwedge$$

$$\left(a_\nu = a_{2n+2} + \sum_{k=1}^{n} a_k\, a_{2n-k+1} /; n = \frac{\nu - 2}{4} \bigwedge \nu \bmod 4 = 2\right) \bigwedge$$

$$\left(a_\nu = a_{2n+4} - a_2\, a_{2n+1} - \frac{1}{2}\left(a_{2n+1}^2 - a_{2n+1}\right) + \sum_{k=1}^{n+1} a_k\, a_{2n-k+3} - \sum_{k=1}^{2n} (-1)^{k-1} a_k\, a_{4n-k+2} + \sum_{k=1}^{n} a_k\, a_{4n-4k+2} /; n = \frac{\nu - 3}{4} \bigwedge \nu \bmod 4 = 3\right)\Bigg)$$

Differential equations

Ordinary nonlinear differential equations

09.50.13.0001.01

$$\left(-36\, w(z)^2 + 41\, w(z) - 32\right) w'(z)^4 - 72\, (w(z) - 1)^2\, w(z)^2\, w^{(3)}(z)\, w'(z) + 108\, (w(z) - 1)^2\, w(z)^2\, w''(z)^2 = 0 /; w(z) = J(z)$$

Transformations

Transformations and argument simplifications

Argument involving basic arithmetic operations

09.50.16.0001.01

$$J(z+1) == J(z)$$

09.50.16.0002.01

$$J\left(-\frac{1}{z}\right) == J(z)$$

Identities

Functional identities

09.50.17.0001.01

$$J(z) == J(z+1)$$

09.50.17.0002.01

$$J(z) == J\left(-\frac{1}{z}\right)$$

09.50.17.0003.01

$$64\,J(z)^3 - 110\,592\,J(2\,z)^2\,J(z)^2 + 95\,232\,J(2\,z)\,J(z)^2 - 6000\,J(z)^2 + 95\,232\,J(2\,z)^2\,J(z) + 1\,510\,125\,J(2\,z)\,J(z) + 187\,500\,J(z) + 64\,J(2\,z)^3 - 6000\,J(2\,z)^2 + 187\,500\,J(2\,z) - 1\,953\,125 == 0$$

Differentiation

Low-order differentiation

09.50.20.0005.01

$$J'(i) = 0$$

09.50.20.0001.02

$$\frac{\partial J(z)}{\partial z} = -\frac{16\,i\,K(\lambda(z))^2\,(\lambda(z)-2)\,(\lambda(z)+1)\,(2\,\lambda(z)-1)\,\left(\lambda(z)^2-\lambda(z)+1\right)^2}{27\,\pi\,(\lambda(z)-1)^2\,\lambda(z)^2}$$

09.50.20.0002.01

$$\frac{\partial J(z)}{\partial z} == \frac{\pi\,i}{864}\left(-e^{-2\,i\,\pi\,z} + \sum_{k=1}^{\infty} k\,a_k\,e^{2\,k\,i\,\pi\,z}\right) /;$$

$$a_k == \frac{2\,\pi}{\sqrt{k}} \sum_{j=1}^{\infty} \frac{1}{j}\,A_j(k)\,I_1\left(\frac{(4\,\pi)\,\sqrt{k}}{j}\right) \bigwedge \left(A_j(k) == \sum_{h=0}^{j-1} \delta_{1,\gcd[h,j]}\,\exp\left(-\frac{(2\,\pi\,i)\,(h\,k + H(j,\,h))}{j}\right) /; (h\,H(j,\,h)) \bmod j == -1\right)$$

09.50.20.0003.01

$$\frac{\partial J(z)}{\partial z} == \frac{\pi\,i}{864}\left(-e^{-2\,i\,\pi\,z} + \sum_{k=1}^{\infty} k\,a_k\,e^{2\,k\,i\,\pi\,z}\right) /;$$

$$a_k == \frac{2\,\pi}{\sqrt{k}} \sum_{j=1}^{\infty} \frac{1}{j}\,A_j(k)\,I_1\left(\frac{(4\,\pi)\,\sqrt{k}}{j}\right) \bigwedge \left(A_j(k) == \sum_{h=0}^{j-1} \delta_{1,\gcd[h,j]}\,\exp\left(-\frac{(2\,\pi\,i)\,(h\,k + H(j,\,h))}{j}\right) /; (h\,H(j,\,h)) \bmod j == -1\right)$$

09.50.20.0004.02

$$\frac{\partial^2 J(z)}{\partial z^2} = \frac{64 K(\lambda(z))^3}{27 \pi^2 (\lambda(z)-1)^2 \lambda(z)^2} \left(E(\lambda(z)) (\lambda(z)-2) (\lambda(z)+1) (2 \lambda(z)-1) ((\lambda(z)-1) \lambda(z)+1)^2 - K(\lambda(z)) \left(\lambda(z)^2-\lambda(z)+1\right) \left(4 \lambda(z)^6 - 11 \lambda(z)^5 + 6 \lambda(z)^4 + 4 \lambda(z)^3 + 13 \lambda(z)^2 - 18 \lambda(z) + 6\right)\right)$$

09.50.20.0006.01

$$\frac{\partial^3 J(z)}{\partial z^3} = \frac{128 i K(\lambda(z))^4}{27 \pi^3 (\lambda(z)-1)^2 \lambda(z)^2} \left((\lambda(z)-2) \left(16 \lambda(z)^8 - 28 \lambda(z)^7 + 25 \lambda(z)^6 + 8 \lambda(z)^5 - 32 \lambda(z)^4 + 98 \lambda(z)^3 - 140 \lambda(z)^2 + 92 \lambda(z) - 23\right) K(\lambda(z))^2 - 6 E(\lambda(z)) \left(\lambda(z)^2-\lambda(z)+1\right) \left(4 \lambda(z)^6 - 11 \lambda(z)^5 + 6 \lambda(z)^4 + 4 \lambda(z)^3 + 13 \lambda(z)^2 - 18 \lambda(z) + 6\right) K(\lambda(z)) + 3 E(\lambda(z))^2 (\lambda(z)-2) (\lambda(z)+1) (2 \lambda(z)-1) \left(\lambda(z)^2-\lambda(z)+1\right)^2\right)$$

09.50.20.0007.01

$$\frac{\partial^4 J(z)}{\partial z^4} = \frac{512 K(\lambda(z))^5}{27 \pi^4 (\lambda(z)-1)^2 \lambda(z)^2} \left(-6 (\lambda(z)-2) (\lambda(z)+1) (2 \lambda(z)-1) \left(\lambda(z)^2-\lambda(z)+1\right)^2 E(\lambda(z))^3 + 18 K(\lambda(z)) \left(\lambda(z)^2-\lambda(z)+1\right) \left(4 \lambda(z)^6 - 11 \lambda(z)^5 + 6 \lambda(z)^4 + 4 \lambda(z)^3 + 13 \lambda(z)^2 - 18 \lambda(z) + 6\right) E(\lambda(z))^2 - 6 K(\lambda(z))^2 (\lambda(z)-2) \left(16 \lambda(z)^8 - 28 \lambda(z)^7 + 25 \lambda(z)^6 + 8 \lambda(z)^5 - 32 \lambda(z)^4 + 98 \lambda(z)^3 - 140 \lambda(z)^2 + 92 \lambda(z) - 23\right) E(\lambda(z)) + K(\lambda(z))^3 \left(32 \lambda(z)^{10} - 112 \lambda(z)^9 + 123 \lambda(z)^8 - 12 \lambda(z)^7 + 36 \lambda(z)^6 - 402 \lambda(z)^5 + 1380 \lambda(z)^4 - 2340 \lambda(z)^3 + 2175 \lambda(z)^2 - 1060 \lambda(z) + 212\right)\right)$$

09.50.20.0008.01

$$\frac{\partial^5 J(z)}{\partial z^5} = \frac{2048 i K(\lambda(z))^6}{27 \pi^5 (\lambda(z)-1)^2 \lambda(z)^2} \left(-15 (\lambda(z)-2) (\lambda(z)+1) (2 \lambda(z)-1) \left(\lambda(z)^2-\lambda(z)+1\right)^2 E(\lambda(z))^4 + 60 K(\lambda(z)) \left(\lambda(z)^2-\lambda(z)+1\right) \left(4 \lambda(z)^6 - 11 \lambda(z)^5 + 6 \lambda(z)^4 + 4 \lambda(z)^3 + 13 \lambda(z)^2 - 18 \lambda(z) + 6\right) E(\lambda(z))^3 - 30 K(\lambda(z))^2 (\lambda(z)-2) \left(16 \lambda(z)^8 - 28 \lambda(z)^7 + 25 \lambda(z)^6 + 8 \lambda(z)^5 - 32 \lambda(z)^4 + 98 \lambda(z)^3 - 140 \lambda(z)^2 + 92 \lambda(z) - 23\right) E(\lambda(z))^2 + 10 K(\lambda(z))^3 \left(32 \lambda(z)^{10} - 112 \lambda(z)^9 + 123 \lambda(z)^8 - 12 \lambda(z)^7 + 36 \lambda(z)^6 - 402 \lambda(z)^5 + 1380 \lambda(z)^4 - 2340 \lambda(z)^3 + 2175 \lambda(z)^2 - 1060 \lambda(z) + 212\right) E(\lambda(z)) - K(\lambda(z))^4 (\lambda(z)-2) \left(64 \lambda(z)^{10} - 64 \lambda(z)^9 - 20 \lambda(z)^8 + 149 \lambda(z)^7 - 131 \lambda(z)^6 + 926 \lambda(z)^5 - 3440 \lambda(z)^4 + 6086 \lambda(z)^3 - 5774 \lambda(z)^2 + 2835 \lambda(z) - 567\right)\right)$$

Operations

Limit operation

09.50.25.0001.01

$$\lim_{\epsilon \to +0} J(i \epsilon) = \infty$$

Representations through equivalent functions

With inverse function

09.50.27.0001.01

$$z = \frac{i(r-s)}{r+s} /; \{r, s\} = \left\{\Gamma\left(\frac{5}{12}\right)^2 {}_2F_1\left(\frac{1}{12}, \frac{1}{12}; \frac{1}{2}; 1-\lambda\right), 2\left(\sqrt{3}-2\right)\Gamma\left(\frac{11}{12}\right)^2 \sqrt{\lambda-1}\, {}_2F_1\left(\frac{7}{12}, \frac{7}{12}; \frac{3}{2}; 1-\lambda\right)\right\} \bigwedge |z| \geq 1 \bigwedge -\frac{1}{2} \leq \operatorname{Re}(z) \leq 0 \bigwedge \lambda = J(z)$$

With related functions

Involving Weierstrass functions

09.50.27.0002.01

$$J\left(\frac{\omega_3}{\omega_1}\right) = \frac{g_2^3}{g_2^3 - 27 g_3^2} /; \{g_2, g_3\} = \{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\} \bigwedge \operatorname{Im}\left(\frac{\omega_3}{\omega_1}\right) > 0$$

09.50.27.0003.01

$$J(z) = \frac{g_2^3}{g_2^3 - 27 g_3^2} /; \{g_2, g_3\} = \{g_2(1, z), g_3(1, z)\} \wedge \operatorname{Im}(z) > 0$$

Involving theta functions

09.50.27.0004.01

$$J(z) = \frac{\left(\vartheta_2(0, e^{\pi i z})^8 + \vartheta_3(0, e^{\pi i z})^8 + \vartheta_4(0, e^{\pi i z})^8\right)^3}{54\left(\vartheta_2(0, e^{\pi i z})\,\vartheta_3(0, e^{\pi i z})\,\vartheta_4(0, e^{\pi i z})\right)^8} /; \operatorname{Im}(z) > 0$$

Involving other related functions

09.50.27.0005.01

$$J(z) = \frac{\left(\eta(z)^{24} + 256\,\eta(2z)^{24}\right)^3}{1728\,\eta(z)^{48}\,\eta(2z)^{24}}$$

09.50.27.0006.01

$$J(z) = \frac{1}{1728}\left(\frac{256\,\eta(2z)^{16}}{\eta(z)^{16}} + \frac{\eta(z)^8}{\eta(2z)^8}\right)^3 /; \operatorname{Im}(z) > 0$$

09.50.27.0008.01

$$J(2z) = \frac{1}{1728}\left(\frac{\eta(z)^{16}}{\eta(2z)^{16}} + \frac{16\,\eta(2z)^8}{\eta(z)^8}\right)^3$$

09.50.27.0007.01

$$J(z) = \frac{4\left(\lambda(z)^2 - \lambda(z) + 1\right)^3}{27\,\lambda(z)^2\,(1-\lambda(z))^2}$$

09.50.27.0009.01

$$J(z) = \frac{\left(\eta(z)^{12} + 27\,\eta(3z)^{12}\right)\left(\eta(z)^{12} + 243\,\eta(3z)^{12}\right)^3}{1728\,\eta(z)^{36}\,\eta(3z)^{12}}$$

09.50.27.0010.01

$$J(3z) = \frac{\left(\eta(z)^{12} + 3\,\eta(3z)^{12}\right)^3 \left(\eta(z)^{12} + 27\,\eta(3z)^{12}\right)}{1728\,\eta(z)^{12}\,\eta(3z)^{36}}$$

09.50.27.0011.01

$$J(z) = \frac{\left(\eta(z)^{12} + 250\,\eta(5z)^6\,\eta(z)^6 + 3125\,\eta(5z)^{12}\right)^3}{1728\,\eta(z)^{30}\,\eta(5z)^6}$$

09.50.27.0012.01

$$J(5z) = \frac{\left(\eta(z)^{12} + 10\,\eta(5z)^6\,\eta(z)^6 + 5\,\eta(5z)^{12}\right)^3}{1728\,\eta(z)^6\,\eta(5z)^{30}}$$

Zeros

09.50.30.0001.01

$$J(z) = 0 \;/;\; z = e^{\frac{\pi i}{3}} \bigvee z = e^{\frac{2\pi i}{3}}$$

Theorems

Property of modular functions

Every modular function $m(\tau)$ (meaning a function that is invarant under all argument substitutions of the form $z \to (a\,z + b)/(c\,z + d)$ where a, b, c, and d are integers and $a\,d - b\,c = 1$) is a rational function of $J(z)$.

Transcendentality of Klein invariant J for algebraic argument

The value $J(\alpha)$ is transcendental for any algebraic α where $\operatorname{Im}(\alpha) > 0$ and α not being a quadratic irrational.

Moonshine conjecture

The dimensions of the representations of the monster group (the largest simple sporadic group) of order $2^{46} \times 3^{20} \times 5^9 \times 11^2 \times 13^3 \times 17 \times 19 \times 23 \times 29 \times 31 \times 41 \times 47 \times 59 \times 71$ are simple combinations the Fourier coefficients of $1728\,J(z)$.

Solution of Chazy equation

The function $w(z) = \frac{(4-7\,J(\tau))\,J'(\tau)^2 + 6\,(J(\tau)-1)\,J(\tau)\,J''(\tau)}{2\,(J(\tau)-1)\,J(\tau)\,J'(\tau)}$ is a solution of Chazy equation $w'''(z) = 2\,w''(z)\,w(z) - 3\,w(z)^2$.

Replicability of $1728\,J(\tau) - 744$

The function $1728\,J(\tau) - 744$ is completely replicable:

$$J(\sigma) - J(\tau) = \frac{e^{-2\pi i \sigma}}{1728} \prod_{n=-1}^{\infty} \prod_{m=1}^{\infty} \left(1 - e^{2\pi i (m\sigma + n\tau)}\right)^{c(n m)} \;/;\; 1728\,J(\tau) - 744 = \sum_{n=-1}^{\infty} c(n)\,e^{2\pi i \tau}.$$

History

- R. Dedekind (1877)
- F. Klein (1878)
- R. Fricke (1890–1892)

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