# LaguerreL3



## Notations

### Traditional name

Generalized Laguerre polynomials

### Traditional notation

$L_n^\lambda(z)$

### *Mathematica* StandardForm notation

`LaguerreL[n, λ, z]`

## Primary definition

05.08.02.0001.01

$$L_n^\lambda(z) = \frac{\Gamma(\lambda+n+1)}{n!} \sum_{k=0}^{n} \frac{(-n)_k\, z^k}{\Gamma(\lambda+k+1)\, k!} \;/;\; n \in \mathbb{N}$$

## Specific values

### Specialized values

#### For fixed *n*, *λ*

05.08.03.0001.01

$$L_n^\lambda(0) = \binom{n+\lambda}{\lambda}$$

#### For fixed *n*, *z*

05.08.03.0002.01

$$L_n^0(z) = L_n(z)$$

05.08.03.0003.01

$$L_n^{-\frac{1}{2}}(z) = \frac{(-1)^n}{2^{2n}\, n!} H_{2n}\left(\sqrt{z}\right)$$

05.08.03.0004.01

$$L_n^{\frac{1}{2}}(z) = \frac{(-1)^n}{2^{2n+1}\, n!\, \sqrt{z}} H_{2n+1}\left(\sqrt{z}\right)$$

05.08.03.0005.01

$$L_n^1(z) == \frac{n\, L_{n-1}(z) - (n-z)\, L_n(z)}{z}$$

05.08.03.0006.01

$$L_n^\lambda(z) == \left(1 \times .- \frac{\partial .}{\partial z}\right)^\lambda L_n(z) \;/;\; \lambda \in \mathbb{N}^+$$

05.08.03.0007.01

$$L_n^\lambda(z) == \left(-\frac{\partial .}{\partial z}\right)^\lambda L_{n+\lambda}(z) \;/;\; \lambda \in \mathbb{N}^+$$

05.08.03.0008.01

$$L_n^{-n-m}(z) == \tilde{\infty} \;/;\; m \in \mathbb{N}^+$$

## For fixed $\lambda$, $z$

05.08.03.0009.01

$$L_0^\lambda(z) == 1$$

05.08.03.0010.01

$$L_1^\lambda(z) == -z + \lambda + 1$$

05.08.03.0011.01

$$L_2^\lambda(z) == \frac{1}{2}\left(z^2 - 2\, z\, (2+\lambda) + \lambda^2 + 3\, \lambda + 2\right)$$

05.08.03.0012.01

$$L_3^\lambda(z) == \frac{1}{6}\left(-z^3 + 3\, (\lambda+3)\, z^2 - 3\left(\lambda^2 + 5\, \lambda + 6\right) z + \lambda^3 + 6\, \lambda^2 + 11\, \lambda + 6\right)$$

05.08.03.0013.01

$$L_4^\lambda(z) == \frac{1}{24}\left(z^4 - 4\, (\lambda+4)\, z^3 + 6\left(\lambda^2 + 7\, \lambda + 12\right) z^2 - 4\left(\lambda^3 + 9\, \lambda^2 + 26\, \lambda + 24\right) z + \lambda^4 + 10\, \lambda^3 + 35\, \lambda^2 + 50\, \lambda + 24\right)$$

05.08.03.0014.01

$$L_5^\lambda(z) == \frac{1}{120}\left(-z^5 + 5\, (\lambda+5)\, z^4 - 10\left(\lambda^2 + 9\, \lambda + 20\right) z^3 + 10\left(\lambda^3 + 12\, \lambda^2 + 47\, \lambda + 60\right) z^2 - 5\left(\lambda^4 + 14\, \lambda^3 + 71\, \lambda^2 + 154\, \lambda + 120\right) z + \lambda^5 + 15\, \lambda^4 + 85\, \lambda^3 + 225\, \lambda^2 + 274\, \lambda + 120\right)$$

05.08.03.0015.01

$$L_6^\lambda(z) == \frac{1}{720}\left(z^6 - 6\, (\lambda+6)\, z^5 + 15\left(\lambda^2 + 11\, \lambda + 30\right) z^4 - 20\left(\lambda^3 + 15\, \lambda^2 + 74\, \lambda + 120\right) z^3 + 15\left(\lambda^4 + 18\, \lambda^3 + 119\, \lambda^2 + 342\, \lambda + 360\right) z^2 - 6\left(\lambda^5 + 20\, \lambda^4 + 155\, \lambda^3 + 580\, \lambda^2 + 1044\, \lambda + 720\right) z + \lambda^6 + 21\, \lambda^5 + 175\, \lambda^4 + 735\, \lambda^3 + 1624\, \lambda^2 + 1764\, \lambda + 720\right)$$

05.08.03.0016.01

$$L_7^\lambda(z) == \frac{1}{5040}\left(-z^7 + 7\, (\lambda+7)\, z^6 - 21\left(\lambda^2 + 13\, \lambda + 42\right) z^5 + 35\left(\lambda^3 + 18\, \lambda^2 + 107\, \lambda + 210\right) z^4 - 35\left(\lambda^4 + 22\, \lambda^3 + 179\, \lambda^2 + 638\, \lambda + 840\right) z^3 + 21\left(\lambda^5 + 25\, \lambda^4 + 245\, \lambda^3 + 1175\, \lambda^2 + 2754\, \lambda + 2520\right) z^2 - 7\left(\lambda^6 + 27\, \lambda^5 + 295\, \lambda^4 + 1665\, \lambda^3 + 5104\, \lambda^2 + 8028\, \lambda + 5040\right) z + \lambda^7 + 28\, \lambda^6 + 322\, \lambda^5 + 1960\, \lambda^4 + 6769\, \lambda^3 + 13\,132\, \lambda^2 + 13\,068\, \lambda + 5040\right)$$

05.08.03.0017.01

$$L_8^{\lambda}(z) = \frac{1}{40\,320} \left(z^8 - 8\,(\lambda+8)\,z^7 + 28\left(\lambda^2+15\,\lambda+56\right)z^6 - 56\left(\lambda^3+21\,\lambda^2+146\,\lambda+336\right)z^5 + 70\left(\lambda^4+26\,\lambda^3+251\,\lambda^2+1066\,\lambda+1680\right)z^4 - 56\left(\lambda^5+30\,\lambda^4+355\,\lambda^3+2070\,\lambda^2+5944\,\lambda+6720\right)z^3 + 28\left(\lambda^6+33\,\lambda^5+445\,\lambda^4+3135\,\lambda^3+12\,154\,\lambda^2+24\,552\,\lambda+20\,160\right)z^2 - 8\left(\lambda^7+35\,\lambda^6+511\,\lambda^5+4025\,\lambda^4+18\,424\,\lambda^3+48\,860\,\lambda^2+69\,264\,\lambda+40\,320\right)z + \lambda^8 + 36\,\lambda^7+546\,\lambda^6+4536\,\lambda^5+22\,449\,\lambda^4+67\,284\,\lambda^3+118\,124\,\lambda^2+109\,584\,\lambda+40\,320\right)$$

05.08.03.0018.01

$$L_9^{\lambda}(z) = \frac{1}{362\,880} \left(-z^9 + 9\,(\lambda+9)\,z^8 - 36\left(\lambda^2+17\,\lambda+72\right)z^7 + 84\left(\lambda^3+24\,\lambda^2+191\,\lambda+504\right)z^6 - 126\left(\lambda^4+30\,\lambda^3+335\,\lambda^2+1650\,\lambda+3024\right)z^5 + 126\left(\lambda^5+35\,\lambda^4+485\,\lambda^3+3325\,\lambda^2+11\,274\,\lambda+15\,120\right)z^4 - 84\left(\lambda^6+39\,\lambda^5+625\,\lambda^4+5265\,\lambda^3+24\,574\,\lambda^2+60\,216\,\lambda+60\,480\right)z^3 + 36\left(\lambda^7+42\,\lambda^6+742\,\lambda^5+7140\,\lambda^4+40\,369\,\lambda^3+133\,938\,\lambda^2+241\,128\,\lambda+181\,440\right)z^2 - 9\left(\lambda^8+44\,\lambda^7+826\,\lambda^6+8624\,\lambda^5+54\,649\,\lambda^4+214\,676\,\lambda^3+509\,004\,\lambda^2+663\,696\,\lambda+362\,880\right)z + \lambda^9 + 45\,\lambda^8+870\,\lambda^7+9450\,\lambda^6+63\,273\,\lambda^5+269\,325\,\lambda^4+723\,680\,\lambda^3+1\,172\,700\,\lambda^2+1\,026\,576\,\lambda+362\,880\right)$$

05.08.03.0019.01

$$L_{10}^{\lambda}(z) = \frac{1}{3\,628\,800} \left(z^{10} - 10\,(\lambda+10)\,z^9 + 45\left(\lambda^2+19\,\lambda+90\right)z^8 - 120\left(\lambda^3+27\,\lambda^2+242\,\lambda+720\right)z^7 + 210\left(\lambda^4+34\,\lambda^3+431\,\lambda^2+2414\,\lambda+5040\right)z^6 - 252\left(\lambda^5+40\,\lambda^4+635\,\lambda^3+5000\,\lambda^2+19\,524\,\lambda+30\,240\right)z^5 + 210\left(\lambda^6+45\,\lambda^5+835\,\lambda^4+8175\,\lambda^3+44\,524\,\lambda^2+127\,860\,\lambda+151\,200\right)z^4 - 120\left(\lambda^7+49\,\lambda^6+1015\,\lambda^5+11\,515\,\lambda^4+77\,224\,\lambda^3+305\,956\,\lambda^2+662\,640\,\lambda+604\,800\right)z^3 + 45\left(\lambda^8+52\,\lambda^7+1162\,\lambda^6+14\,560\,\lambda^5+111\,769\,\lambda^4+537\,628\,\lambda^3+1\,580\,508\,\lambda^2+2\,592\,720\,\lambda+1\,814\,400\right)z^2 - 10\left(\lambda^9+54\,\lambda^8+1266\,\lambda^7+16\,884\,\lambda^6+140\,889\,\lambda^5+761\,166\,\lambda^4+2\,655\,764\,\lambda^3+5\,753\,736\,\lambda^2+6\,999\,840\,\lambda+3\,628\,800\right) z + \lambda^{10} + 55\,\lambda^9+1320\,\lambda^8+18\,150\,\lambda^7+157\,773\,\lambda^6 + 902\,055\,\lambda^5+3\,416\,930\,\lambda^4+8\,409\,500\,\lambda^3+12\,753\,576\,\lambda^2+10\,628\,640\,\lambda+3\,628\,800\right)$$

05.08.03.0020.01

$$L_n^{\lambda}(z) = \frac{\Gamma(n+\lambda+1)}{n!} \sum_{k=0}^{n} \frac{(-n)_k\, z^k}{\Gamma(k+\lambda+1)\, k!}$$

05.08.03.0021.01

$$L_n^{\lambda}(z) = \tilde{\infty} \;/;\; \lambda \in \mathbb{Z} \bigwedge \lambda < -n$$

### Values at infinities

05.08.03.0022.01

$$L_n^{\lambda}(\infty) = (-1)^n\, \infty \;/;\; n > 0$$

05.08.03.0023.01

$$L_n^{\lambda}(-\infty) = \infty \;/;\; n > 0$$

## General characteristics

### Domain and analyticity

The function $L_n^\lambda(z)$is defined over $\mathbb{N} \otimes \mathbb{C} \otimes \mathbb{C}$. For fixed $n$, $\lambda$, the function $L_n^\lambda(z)$ is a polynomial in $z$ of degree $n$. For fixed $n$, $z$, the function $L_n^\lambda(z)$ is a polynomial in $\lambda$ of degree $n$.

05.08.04.0001.01

$$(n * \lambda * z) \longrightarrow L_n^\lambda(z) :: (\mathbb{N} \otimes \mathbb{C} \otimes \mathbb{C}) \longrightarrow \mathbb{C}$$

## Symmetries and periodicities

### Mirror symmetry

05.08.04.0002.01

$$L_n^{\bar{\lambda}}(\bar{z}) = \overline{L_n^\lambda(z)}$$

### Periodicity

No periodicity

## Poles and essential singularities

### With respect to $z$

For fixed $\lambda$ the function $L_n^\lambda(z)$ is polynomial and has pole of order $n$ at $z = \tilde{\infty}$.

05.08.04.0003.01

$$\mathcal{S}ing_z\left(L_n^\lambda(z)\right) = \{\{\tilde{\infty}, n\}\}$$

### With respect to $\lambda$

For fixed $n$, $z$, the function $L_n^\lambda(z)$ has an infinite set of singular points:

a) $\lambda = -n-k \,/;\, k \in \mathbb{N}^+$, are the simple poles with residues $\frac{(-1)^{k-1}}{\Gamma(n+1)\,(k-1)!}\,{}_1\tilde{F}_1(-n; -k-n+1; z) \,/;\, k \in \mathbb{N}^+$;

b) $\lambda = \tilde{\infty}$ is an essential singular point.

05.08.04.0004.01

$$\mathcal{S}ing_\lambda\left(L_n^\lambda(z)\right) = \left\{\left\{\{-n-k,\, 1\} \,/;\, k \in \mathbb{N}^+\right\},\, \{\tilde{\infty},\, \infty\}\right\}$$

05.08.04.0005.01

$$\text{res}_\lambda\left(L_n^\lambda(z)\right)(-n-k) = \frac{(-1)^{k-1}}{\Gamma(n+1)\,(k-1)!}\,{}_1\tilde{F}_1(-n; -k-n+1; z) \,/;\, k \in \mathbb{N}^+$$

## Branch points

### With respect to $z$

For fixed $n$, $\lambda$, the function $L_n^\lambda(z)$ does not have branch points.

05.08.04.0006.01

$$\mathcal{BP}_z\left(L_n^\lambda(z)\right) = \{\}$$

### With respect to $\lambda$

For fixed $n$, $z$, the function $L_n^\lambda(z)$ does not have branch points.

05.08.04.0007.01

$$\mathcal{BP}_\lambda\left(L_n^\lambda(z)\right) == \{\}$$

## Branch cuts

### With respect to $z$

For fixed $n$, $\lambda$, the function $L_n^\lambda(z)$ does not have branch cuts.

05.08.04.0008.01

$$\mathcal{BC}_z\left(L_n^\lambda(z)\right) == \{\}$$

### With respect to $\lambda$

For fixed $n$, $z$, the function $L_n^\lambda(z)$ does not have branch cuts.

05.08.04.0009.01

$$\mathcal{BC}_\lambda\left(L_n^\lambda(z)\right) == \{\}$$

# Series representations

## Generalized power series

### Expansions at generic point $\lambda == \lambda_0$

## For the function itself

05.08.06.0011.01

$$L_n^\lambda(z) \propto \frac{(-1)^n z^n}{n!} + \frac{1}{n!}\sum_{s=0}^{n}\sum_{j=1}^{n-s}\frac{(-1)^{j+n-s} z^s}{s!}(-n)_s\, S_{n-s}^{(j)}\,(s+\lambda_0+1)^j\left(1+\frac{j}{s+\lambda_0+1}(\lambda-\lambda_0)+\frac{(j-1)\,j}{2\,(s+\lambda_0+1)^2}(\lambda-\lambda_0)^2+\ldots\right) /;$$
$$(\lambda \to \lambda_0)$$

05.08.06.0012.01

$$L_n^\lambda(z) \propto \frac{(-1)^n z^n}{n!} + \frac{1}{n!}\sum_{s=0}^{n}\sum_{j=1}^{n-s}\frac{(-1)^{j+n-s} z^s}{s!}(-n)_s\, S_{n-s}^{(j)}\,(s+\lambda_0+1)^j\left(1+\frac{j}{s+\lambda_0+1}(\lambda-\lambda_0)+\frac{(j-1)\,j}{2\,(s+\lambda_0+1)^2}(\lambda-\lambda_0)^2+O\left((\lambda-\lambda_0)^3\right)\right)$$

05.08.06.0013.01

$$L_n^\lambda(z) == \frac{(-1)^n z^n}{n!} + \sum_{k=0}^{\infty}\frac{1}{k!\,n!}\sum_{s=0}^{n}\frac{(-n)_s z^s}{s!}\sum_{j=1}^{n-s}(-1)^{j+n-s} S_{n-s}^{(j)}\,(j-k+1)_k\,(s+\lambda_0+1)^{j-k}\,(\lambda-\lambda_0)^k$$

05.08.06.0014.01

$$L_n^\lambda(z) \propto L_n^\lambda(z_0)\,(1+O(\lambda-\lambda_0))$$

### Expansions at generic point $z == z_0$

## For the function itself

05.08.06.0015.01

$$L_n^{\lambda}(z) \propto L_n^{\lambda}(z_0) - L_{n-1}^{\lambda+1}(z_0)\,(z-z_0) + \frac{1}{2}\,L_{n-2}^{\lambda+2}(z_0)\,(z-z_0)^2 + \ldots \,/;\, (z \to z_0)$$

05.08.06.0016.01

$$L_n^{\lambda}(z) \propto L_n^{\lambda}(z_0) - L_{n-1}^{\lambda+1}(z_0)\,(z-z_0) + \frac{1}{2}\,L_{n-2}^{\lambda+2}(z_0)\,(z-z_0)^2 + O\!\left((z-z_0)^3\right)$$

05.08.06.0017.01

$$L_n^{\lambda}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}\,L_{n-k}^{k+\lambda}(z_0)\,(z-z_0)^k$$

05.08.06.0018.01

$$L_n^{\lambda}(z) = \Gamma(\lambda+n+1) \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(n-k+1)}\,{}_1\tilde{F}_1(k-n;\,k+\lambda+1;\,z_0)\,(z-z_0)^k$$

05.08.06.0019.01

$$L_n^{\lambda}(z) = \frac{\Gamma(\lambda+n+1)}{n!}\,\tilde{F}_{1\times0\times0}^{1\times0\times0}\!\left(\begin{matrix} -\vee;;; \\ \lambda+1;;; \end{matrix}\; z_0,\, z-z_0\right)$$

05.08.06.0020.01

$$L_n^{\lambda}(z) \propto L_n^{\lambda}(z_0)\,(1+O(z-z_0))$$

**Expansions at $z = 0$**

## For the function itself

05.08.06.0001.02

$$L_n^{\lambda}(z) \propto \frac{\Gamma(n+\lambda+1)}{n!}\left(\frac{1}{\Gamma(\lambda+1)} - \frac{n\,z}{\Gamma(\lambda+2)} - \frac{(1-n)\,n\,z^2}{2\,\Gamma(\lambda+3)} - \ldots\right) /;\, (z \to 0)$$

05.08.06.0021.01

$$L_n^{\lambda}(z) \propto \frac{\Gamma(n+\lambda+1)}{\Gamma(n+1)}\left(\frac{1}{\Gamma(\lambda+1)} - \frac{n\,z}{\Gamma(\lambda+2)} - \frac{(1-n)\,n\,z^2}{2\,\Gamma(\lambda+3)} + O\!\left(z^3\right)\right)$$

05.08.06.0002.01

$$L_n^{\lambda}(z) = \frac{\Gamma(\lambda+n+1)}{n!} \sum_{k=0}^{n} \frac{(-n)_k\,z^k}{\Gamma(\lambda+k+1)\,k!}$$

05.08.06.0005.01

$$L_n^{\lambda}(z) = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \binom{n+\lambda}{n-k} z^k$$

05.08.06.0006.01

$$L_n^{\lambda}(z) = \frac{1}{n!} \sum_{k=0}^{n} \frac{(-n)_k\,(k+\lambda+1)_{n-k}\,z^k}{k!}$$

05.08.06.0003.01

$$L_n^{\lambda}(z) = \frac{\Gamma(n+\lambda+1)}{n!}\,{}_1\tilde{F}_1(-n;\,\lambda+1;\,z)$$

05.08.06.0004.02

$$L_n^\lambda(z) \propto \frac{(\lambda+1)_n}{n!} (1 + O(z))$$

**Expansions at $z == \infty$**

## For the function itself

05.08.06.0007.02

$$L_n^\lambda(z) \propto \frac{(-z)^n}{n!} \left(1 + \frac{n(-n-\lambda)}{z} + \frac{(n-1)\,n\,(-n-\lambda)\,(1-n-\lambda)}{2\,z^2} + \ldots\right) /;\ (|z| \to \infty)$$

05.08.06.0022.01

$$L_n^\lambda(z) \propto \frac{(-z)^n}{n!} \left(1 + \frac{n(-n-\lambda)}{z} + \frac{(n-1)\,n\,(-n-\lambda)\,(1-n-\lambda)}{2\,z^2} + O\left(\frac{1}{z^3}\right)\right)$$

05.08.06.0008.01

$$L_n^\lambda(z) == \frac{(-z)^n}{n!} \sum_{k=0}^{n} \frac{(-1)^k\,(-n)_k\,(-n-\lambda)_k\,z^{-k}}{k!}$$

05.08.06.0009.01

$$L_n^\lambda(z) == \frac{(-z)^n}{n!}\, {}_2F_0\left(-n, -n-\lambda; ; -\frac{1}{z}\right)$$

05.08.06.0010.02

$$L_n^\lambda(z) \propto \frac{1}{n!} (-z)^n \left(1 + O\left(\frac{1}{z}\right)\right)$$

**Expansions at $\lambda == 0$**

05.08.06.0023.01

$$L_n^\lambda(z) \propto L_n(z) + \sum_{k=0}^{n-1} \frac{L_k(z)}{n-k}\,\lambda + \sum_{j=0}^{n-1} \frac{H_{-j+n-1}\,L_j(z)}{n-j}\,\lambda^2 + \ldots /;\ (\lambda \to 0)$$

05.08.06.0024.01

$$L_n^\lambda(z) = \sum_{k=0}^{n} \sum_{j=0}^{n-k} \frac{(-1)^{n+k}\,z^{n-j-k}\,(-j-k)_j}{j!\,(n-j-k)!\,(j+k)!}\, B_j^{(j+k+1)}(n+1)\,\lambda^k$$

05.08.06.0025.01

$$L_n^\lambda(z) \propto L_n(z)\,(1 + O(\lambda))$$

**Expansions at $\lambda == \infty$**

05.08.06.0026.01

$$L_n^\lambda(z) \propto \frac{\lambda^n}{n!} \left(1 + \frac{n\,(n-2\,z+1)}{2\,\lambda} + \frac{(n-1)\,n\left(3\,n^2 + (5-12\,z)\,n + 12\,(z-2)\,z + 2\right)}{24\,\lambda^2} + \ldots\right) /;\ (|\lambda| \to \infty)$$

05.08.06.0027.01

$$L_n^\lambda(z) = \lambda^n \sum_{k=0}^{n} \sum_{j=0}^{k} \frac{(-1)^k\,z^{k-j}\,(k-j-n)_j}{j!\,(k-j)!\,(j-k+n)!}\, B_j^{(j-k+n+1)}(n+1)\,\lambda^{-k}$$

05.08.06.0028.01

$$L_n^{\lambda}(z) \propto \frac{\lambda^n}{n!}\left(1 + O\left(\frac{1}{\lambda}\right)\right)$$

## Expansions at $n == \infty$

05.08.06.0029.01

$$L_n^{\lambda}(z) \propto \frac{e^{z/2} z^{-\frac{2\lambda+1}{4}} n^{\frac{2\lambda-1}{4}}}{\sqrt{\pi}} \left(\cos\left(2\sqrt{z\left(\frac{\lambda+1}{2}+n\right)} - \frac{\pi(2\lambda+1)}{4}\right) + \frac{1-4\lambda^2}{16\sqrt{n}\sqrt{z}} \sin\left(2\sqrt{z\left(\frac{\lambda+1}{2}+n\right)} - \frac{\pi(2\lambda+1)}{4}\right) + \frac{1}{512\, n\, z}\left(64(\lambda+1) z^2 \cos\left(2\sqrt{z\left(\frac{\lambda+1}{2}+n\right)} - \frac{\pi(2\lambda+5)}{4}\right) + (2\lambda-1)\left(-8\lambda^3 - 4\lambda^2 + 18\lambda + 64 z(\lambda+1) + 9\right) \cos\left(2\sqrt{z\left(\frac{\lambda+1}{2}+n\right)} - \frac{\pi(2\lambda+1)}{4}\right)\right) - \frac{1}{24576\, n^{3/2} z^{3/2}}\left(3072(2n+\lambda+1) z^3 \cos\left(2\sqrt{z\left(\frac{\lambda+1}{2}+n\right)} - \frac{\pi(2\lambda+7)}{4}\right) + 192\left(4\lambda^3 + 20\lambda^2 + 31\lambda + 15\right) z^2 \sin\left(2\sqrt{z\left(\frac{\lambda+1}{2}+n\right)} - \frac{\pi(2\lambda+5)}{4}\right) + \left(8\lambda^3 - 12\lambda^2 - 2\lambda + 3\right)\left(-8\lambda^3 - 12\lambda^2 + 50\lambda + 192 z(\lambda+1) + 75\right) \sin\left(2\sqrt{z\left(\frac{\lambda+1}{2}+n\right)} - \frac{\pi(2\lambda+1)}{4}\right)\right) + \ldots\right) /; (n \to \infty)$$

05.08.06.0030.01

$$L_n^{\lambda}(z) \propto \frac{e^{z/2} z^{-\frac{2\lambda+1}{4}} n^{\frac{2\lambda-1}{4}}}{\sqrt{\pi}} \Bigg( \cos\left(2\sqrt{z\left(\frac{\lambda+1}{2}+n\right)} - \frac{\pi(2\lambda+1)}{4}\right) + \sum_{k=1}^{\infty} \Bigg( \sum_{j=0}^{k}\sum_{r=0}^{k-j}\sum_{s=0}^{k-j-r} \frac{(-1)^{j+r+s} 2^{2j-2k+s} z^{k-j-2r-s} (\lambda+1)^s}{j!\, s! \left(\frac{1}{2}\right)_r} A_{2(k-j-r-s)} \cos\Bigg( \pi\left(j-k+r+s-\frac{\lambda}{2}-\frac{1}{4}\right) + 2\sqrt{z\left(n+\frac{\lambda+1}{2}\right)} \Bigg) B_j^{(\lambda+1)}(\lambda+1) \left(j-k+r+s-\frac{\lambda}{2}+\frac{1}{4}\right)_r \left(j-k+r+s-\frac{\lambda}{2}+\frac{3}{4}\right)_r \left(k-j-s+\frac{\lambda}{2}+\frac{1}{4}\right)_s \left(k-j-r-s+\frac{\lambda}{2}+\frac{1}{4}\right)_r \left(k-j-r-s+\frac{\lambda}{2}+\frac{3}{4}\right)_r (-\lambda)_j - \frac{2}{z} \sum_{j=0}^{k-1}\sum_{r=0}^{k-j-1}\sum_{s=0}^{k-j-r-1} \frac{(-1)^{j+r+s} 2^{2j-2k+s} z^{k-j-2r-s} (\lambda+1)^s}{j!\, s! \left(\frac{3}{2}\right)_r} A_{2(k-j-r-s)-1} \sin\Bigg( \pi\left(j-k+r+s-\frac{\lambda}{2}+\frac{1}{4}\right) + 2\sqrt{z\left(n+\frac{\lambda+1}{2}\right)} \Bigg) B_j^{(\lambda+1)}(\lambda+1) \left(j-k+r+s-\frac{\lambda}{2}+\frac{1}{4}\right)_{r+1} \left(j-k+r+s-\frac{\lambda}{2}+\frac{3}{4}\right)_{r+1} \left(k-j-s+\frac{\lambda}{2}+\frac{1}{4}\right)_s \left(k-j-r-s+\frac{\lambda}{2}+\frac{1}{4}\right)_r \left(k-j-r-s+\frac{\lambda}{2}+\frac{3}{4}\right)_r (-\lambda)_j \Bigg) n^{-k} + \frac{\sqrt{z}}{2\sqrt{n}} \sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{r=0}^{k-j}\sum_{s=0}^{k-j-r} \frac{(-1)^{j+r+s} 2^{2j-2k+s} z^{k-j-2r-s} (\lambda+1)^s n^{-k}}{j!\, s! \left(\frac{3}{2}\right)_r} B_j^{(\lambda+1)}(\lambda+1) \left(j-k+r+s-\frac{\lambda}{2}-\frac{1}{4}\right)_r \left(j-k+r+s-\frac{\lambda}{2}+\frac{1}{4}\right)_r \left(k-j-s+\frac{\lambda}{2}+\frac{3}{4}\right)_s \left(k-j-r-s+\frac{\lambda}{2}+\frac{3}{4}\right)_r \left(k-j-r-s+\frac{\lambda}{2}+\frac{5}{4}\right)_r (-\lambda)_j \Bigg( (2r+1) A_{2(k-j-r-s)+1} \cos\Bigg( \pi\left(j-k+r+s-\frac{\lambda}{2}-\frac{3}{4}\right) + 2\sqrt{z\left(n+\frac{\lambda+1}{2}\right)} \Bigg) - \frac{(4j-4k+8r+4s-2\lambda-1)(4j-4k+8r+4s-2\lambda+1)}{8z} A_{2(k-j-r-s)} \sin\Bigg( \pi\left(j-k+r+s-\frac{\lambda}{2}-\frac{1}{4}\right) + 2\sqrt{z\left(n+\frac{\lambda+1}{2}\right)} \Bigg) \Bigg) \Bigg) /;$$

$$(n \to \infty) \bigwedge A_0 = 1 \bigwedge A_1 = 0 \bigwedge A_2 = \frac{\lambda+1}{2} \bigwedge A_m = \frac{m+\lambda-1}{m} A_{m-2} - (2n+\lambda+1) A_{m-3} \bigwedge m \in \mathbb{N}^+$$

05.08.06.0031.01

$$L_n^\lambda(z) \propto \frac{\Gamma(n+\lambda+1)}{n!} e^{z/2} \sum_{k=0}^{\infty} A_k\, 2^{-k} z^k\, {}_0\tilde{F}_1\left(; k+\lambda+1; -\frac{z(2n+\lambda+1)}{2}\right) /;$$
$$(n \to \infty) \bigwedge A_0 = 1 \bigwedge A_1 = 0 \bigwedge A_2 = \frac{\lambda+1}{2} \bigwedge A_m = \frac{m+\lambda-1}{m} A_{m-2} - (2n+\lambda+1) A_{m-3} \bigwedge m \in \mathbb{N}^+$$

05.08.06.0032.01

$$L_n^\lambda(z) \propto \frac{\Gamma(n+\lambda+1)}{n!} \left(\frac{2n+\lambda+1}{2}\right)^{-\frac{\lambda}{2}} z^{-\frac{\lambda}{2}} e^{z/2} \sum_{k=0}^{\infty} A_k \left(\frac{z}{2(2n+\lambda+1)}\right)^{k/2} J_{k+\lambda}\left(\sqrt{2(2n+\lambda+1)z}\right) /;$$
$$(n \to \infty) \bigwedge A_0 = 1 \bigwedge A_1 = 0 \bigwedge A_2 = \frac{\lambda+1}{2} \bigwedge A_m = \frac{m+\lambda-1}{m} A_{m-2} - (2n+\lambda+1) A_{m-3} \bigwedge m \in \mathbb{N}^+$$

05.08.06.0033.01

$$L_n^\lambda(z) \propto \frac{1}{\sqrt{\pi}} e^{z/2} z^{-\frac{2\lambda+1}{4}} n^{\frac{2\lambda-1}{4}} \cos\left(2\sqrt{n z} - \frac{\pi(2\lambda+1)}{4}\right)(1+\ldots) /; (n \to \infty)$$

# Integral representations

## On the real axis

### Of the direct function

05.08.07.0001.01

$$L_n^\lambda(z) = \frac{1}{\Gamma(n+1)} z^{-\frac{\lambda}{2}} e^z \int_0^\infty e^{-t} t^{n+\frac{\lambda}{2}} J_\lambda\left(2\sqrt{t z}\right) dt /; \operatorname{Re}(n+\lambda) > -1$$

## Integral representations of negative integer order

Rodrigues-type formula.

05.08.07.0002.01

$$L_n^\lambda(z) = \frac{e^z}{z^\lambda\, n!} \frac{\partial^n \left(z^{n+\lambda} e^{-z}\right)}{\partial z^n}$$

# Limit representations

05.08.09.0001.01

$$L_n^\lambda(z) = \lim_{b\to\infty} P_n^{(\lambda,b)}\left(1 - \frac{2z}{b}\right)$$

# Generating functions

05.08.11.0001.01

$$L_n^\lambda(z) = \left([t^n]\,(1-t)^{-\lambda-1} \exp\left(\frac{t z}{t-1}\right)\right)$$

# Differential equations

## Ordinary linear differential equations and wronskians

### For the direct function itself

05.08.13.0006.01

$$z\,w''(z)+(\lambda+1-z)\,w'(z)+n\,w(z)=0\;/;\;w(z)=c_1\,L_n^{\lambda}(z)\;+\;c_2\,G_{1,2}^{2,0}\!\left(-z\,\middle|\,\begin{matrix}n+1\\0,\,-\lambda\end{matrix}\right)$$

05.08.13.0007.01

$$W_z\!\left(L_n^{\lambda}(z),\,G_{1,2}^{2,0}\!\left(-z\,\middle|\,\begin{matrix}n+1\\0,\,-\lambda\end{matrix}\right)\right)=\frac{e^z\,(-z)^{-\lambda-1}}{n!}$$

05.08.13.0003.01

$$z\,w''(z)+(\lambda+1-z)\,w'(z)+n\,w(z)=0\;/;\;w(z)=c_1\,L_n^{\lambda}(z)\;+\;c_2\,z^{-\lambda}\,{}_1\tilde{F}_1(-n-\lambda;\,1-\lambda;\,z)\;\;/;\;\lambda\notin\mathbb{Z}$$

05.08.13.0008.01

$$W_z\!\left(L_n^{\lambda}(z),\,z^{-\lambda}\,{}_1\tilde{F}_1(-\lambda-n;\,1-\lambda;\,z)\right)=-\frac{\Gamma(\lambda+n+1)\sin(\pi\,\lambda)}{\pi\,n!}\,e^z\,z^{-\lambda-1}$$

05.08.13.0009.01

$$z\,w''(z)+(\lambda+1-z)\,w'(z)+n\,w(z)=0\;/;\;w(z)=c_1\,L_n^{\lambda}(z)\;+\;c_2\,z^{-\lambda}\,{}_1F_1(-\lambda-n;\,1-\lambda;\,z)\;\bigwedge\;\lambda\notin\mathbb{Z}$$

05.08.13.0004.02

$$W_z\!\left(L_n^{\lambda}(z),\,z^{-\lambda}\,{}_1F_1(-n-\lambda;\,1-\lambda;\,z)\right)=-\frac{e^z\,z^{-\lambda-1}\,\Gamma(n+\lambda+1)}{\Gamma(\lambda)\,n!}$$

05.08.13.0010.01

$$w''(z)+\left(\frac{(\lambda+1)\,g'(z)}{g(z)}-g'(z)-\frac{g''(z)}{g'(z)}\right)w'(z)+\frac{n\,g'(z)^2}{g(z)}\,w(z)=0\;/;\;w(z)=c_1\,L_n^{\lambda}(g(z))\;+c_2\,G_{1,2}^{2,0}\!\left(-g(z)\,\middle|\,\begin{matrix}n+1\\0,\,-\lambda\end{matrix}\right)$$

05.08.13.0011.01

$$W_z\!\left(L_n^{\lambda}(g(z)),\,G_{1,2}^{2,0}\!\left(-g(z)\,\middle|\,\begin{matrix}n+1\\0,\,-\lambda\end{matrix}\right)\right)=\frac{g'(z)\,e^{g(z)}\,(-g(z))^{-\lambda-1}}{n!}$$

05.08.13.0012.01

$$h(z)^2\,w''(z)\;+h(z)^2\left(\frac{(\lambda+1)\,g'(z)}{g(z)}-g'(z)-\frac{2\,h'(z)}{h(z)}-\frac{g''(z)}{g'(z)}\right)w'(z)\;+$$
$$\left(2\,h'(z)^2+h(z)\left(g'(z)\,h'(z)+\frac{g''(z)\,h'(z)}{g'(z)}-h''(z)\right)-\frac{h(z)\,g'(z)\,(-n\,h(z)\,g'(z)+(\lambda+1)\,h'(z))}{g(z)}\right)w(z)\;=0\;/;$$
$$w(z)=c_1\,h(z)\,L_n^{\lambda}(z)\;+c_2\,h(z)\,G_{1,2}^{2,0}\!\left(-g(z)\,\middle|\,\begin{matrix}n+1\\0,\,-\lambda\end{matrix}\right)$$

05.08.13.0013.01

$$W_z\!\left(h(z)\,L_n^{\lambda}(g(z)),\,h(z)\,G_{1,2}^{2,0}\!\left(-g(z)\,\middle|\,\begin{matrix}n+1\\0,\,-\lambda\end{matrix}\right)\right)=\frac{h(z)^2\,g'(z)\,e^{g(z)}\,(-g(z))^{-\lambda-1}}{n!}$$

05.08.13.0014.01

$$z^2\,w''(z)\;+z\,(-2\,s+r\,(\lambda-a\,z^r)+1)\,w'(z)+(a\,r\,(s+r\,n)\,z^r+s\,(s-\lambda\,r))\,w(z)=0\;/;$$
$$w(z)=c_1\,z^s\,L_n^{\lambda}(a\,z^r)\;+c_2\,z^s\,G_{1,2}^{2,0}\!\left(-a\,z^r\,\middle|\,\begin{matrix}n+1\\0,\,-\lambda\end{matrix}\right)$$

05.08.13.0015.01

$$W_z\left(z^s L_n^\lambda(a z^r),\, z^s G_{1,2}^{2,0}\left(-a z^r \,\middle|\, \begin{matrix} n+1 \\ 0, -\lambda \end{matrix}\right)\right) = \frac{a r e^{a z^r} z^{r+2s-1} (-a z^r)^{-\lambda-1}}{n!}$$

05.08.13.0016.01

$$w''(z) - ((a r^z - \lambda) \log(r) + 2 \log(s)) w'(z) + \left(a n \log^2(r) r^z + \log^2(s) + (a r^z - \lambda) \log(r) \log(s)\right) w(z) = 0 /;$$
$$w(z) = c_1 s^z L_n^\lambda(a r^z) + c_2 s^z G_{1,2}^{2,0}\left(-a r^z \,\middle|\, \begin{matrix} n+1 \\ 0, -\lambda \end{matrix}\right)$$

05.08.13.0017.01

$$W_z\left(s^z L_n^\lambda(a r^z),\, s^z G_{1,2}^{2,0}\left(-a r^z \,\middle|\, \begin{matrix} n+1 \\ 0, -\lambda \end{matrix}\right)\right) = \frac{a e^{a r^z} r^z (-a r^z)^{-\lambda-1} s^{2z} \log(r)}{n!}$$

### Integral equations whose solutions contain the direct function

05.08.13.0005.01

$$w(z) = \frac{1}{2} (-1)^n \int_0^\infty J_\lambda\left(\sqrt{t z}\right) w(t) \, dt /; w(z) = e^{-\frac{z}{2}} z^{\lambda/2} L_n^\lambda(z)$$

## Transformations

### Transformations and argument simplifications

#### Argument involving basic arithmetic operations

05.08.16.0001.01

$$L_n^\lambda(-z) = e^{-z} (-1)^\lambda L_{-n-\lambda-1}^\lambda(z) /; \lambda \in \mathbb{Z}$$

### Addition formulas

05.08.16.0002.01

$$L_n^\lambda(z_1 + z_2) = \sum_{k=0}^n L_k^\mu(z_1) L_{n-k}^{\lambda-\mu-1}(z_2)$$

05.08.16.0003.01

$$L_n^\lambda(z_1 + z_2) = e^{z_1} \sum_{k=0}^\infty \frac{(-1)^k z_1^k}{k!} L_n^{k+\lambda}(z_2)$$

### Multiple arguments

05.08.16.0004.01

$$L_n^\lambda(z_1 z_2) = \sum_{k=0}^n \binom{n+\lambda}{n-k} z_1^k (1 - z_1)^{n-k} L_k^\lambda(z_2)$$

## Identities

### Recurrence identities

#### Consecutive neighbors

## With respect to *n*

05.08.17.0001.01

$$L_n^{\lambda}(z) = \frac{\lambda + 2n + 3 - z}{\lambda + n + 1} L_{n+1}^{\lambda}(z) - \frac{n+2}{\lambda + n + 1} L_{n+2}^{\lambda}(z)$$

05.08.17.0002.01

$$L_n^{\lambda}(z) = \frac{\lambda + 2n - 1 - z}{n} L_{n-1}^{\lambda}(z) - \frac{\lambda + n - 1}{n} L_{n-2}^{\lambda}(z)$$

## With respect to $\lambda$

05.08.17.0010.01

$$L_n^{\lambda}(z) = \frac{z + \lambda + 1}{\lambda + n + 1} L_n^{\lambda+1}(z) - \frac{z}{\lambda + n + 1} L_n^{\lambda+2}(z)$$

05.08.17.0011.01

$$L_n^{\lambda}(z) = \frac{z + \lambda - 1}{z} L_n^{\lambda-1}(z) - \frac{\lambda + n - 1}{z} L_n^{\lambda-2}(z)$$

### Distant neighbors

## With respect to *n*

05.08.17.0012.01

$$L_n^{\lambda}(z) = C_m(n, \lambda, z) L_{n+m}^{\lambda}(z) - \frac{m+n+1}{m+\lambda+n} C_{m-1}(n, \lambda, z) L_{n+m+1}^{\lambda}(z) \;/;\; C_0(n, \lambda, z) = 1 \bigwedge C_1(n, \lambda, z) = \frac{\lambda + 2n + 3 - z}{\lambda + n + 1} \bigwedge$$
$$C_m(n, \lambda, z) = \frac{2m - z + \lambda + 2n + 1}{m + \lambda + n} C_{m-1}(n, \lambda, z) - \frac{m+n}{m + \lambda + n - 1} C_{m-2}(n, \lambda, z) \bigwedge m \in \mathbb{N}^+$$

05.08.17.0013.01

$$L_n^{\lambda}(z) = C_m(n, \lambda, z) L_{n-m}^{\lambda}(z) - \frac{-m+\lambda+n}{-m+n+1} C_{m-1}(n, \lambda, z) L_{n-m-1}^{\lambda}(z) \;/;\; C_0(n, \lambda, z) = 1 \bigwedge C_1(n, \lambda, z) = \frac{\lambda + 2n - 1 - z}{n} \bigwedge$$
$$C_m(n, \lambda, z) = \frac{\lambda + 2n - 2m + 1 - z}{n - m + 1} C_{m-1}(n, \lambda, z) - \frac{\lambda + n - m + 1}{n - m + 2} C_{m-2}(n, \lambda, z) \bigwedge m \in \mathbb{N}^+$$

## With respect to $\lambda$

05.08.17.0014.01

$$L_n^{\lambda}(z) = C_m(n, \lambda, z) L_n^{\lambda+m}(z) - \frac{z}{\lambda + n + m} C_{m-1}(n, \lambda, z) L_n^{\lambda+m+1}(z) \;/;$$
$$C_0(n, \lambda, z) = 1 \bigwedge C_1(n, \lambda, z) = \frac{z + \lambda + 1}{\lambda + n + 1} \bigwedge C_m(n, \lambda, z) = \frac{m + z + \lambda}{m + \lambda + n} C_{m-1}(n, \lambda, z) - \frac{z}{m + \lambda + n - 1} C_{m-2}(n, \lambda, z) \bigwedge m \in \mathbb{N}^+$$

05.08.17.0015.01

$$L_n^{\lambda}(z) = C_m(n, \lambda, z) L_n^{\lambda-m}(z) - \frac{\lambda + n - m}{z} C_{m-1}(n, \lambda, z) L_n^{\lambda-m-1}(z) \;/;\; C_0(n, \lambda, z) = 1 \bigwedge$$
$$C_1(n, \lambda, z) = \frac{z + \lambda - 1}{z} \bigwedge C_m(n, \lambda, z) = \frac{-m + z + \lambda}{z} C_{m-1}(n, \lambda, z) - \frac{-m + \lambda + n + 1}{z} C_{m-2}(n, \lambda, z) \bigwedge m \in \mathbb{N}^+$$

## Functional identities

### Relations between contiguous functions

## Recurrence relations

05.08.17.0003.01

$$(n+\lambda)\, L_{n-1}^{\lambda}(z) + (n+1)\, L_{n+1}^{\lambda}(z) = (2\,n - z + \lambda + 1)\, L_n^{\lambda}(z)$$

05.08.17.0004.01

$$L_n^{\lambda}(z) = \frac{(n+\lambda)\, L_{n-1}^{\lambda}(z) + (n+1)\, L_{n+1}^{\lambda}(z)}{2\,n - z + \lambda + 1}$$

05.08.17.0005.01

$$L_n^{\lambda}(z) = \frac{(n+\lambda)\, L_{n-1}^{\lambda}(z) - z\, L_n^{\lambda+1}(z)}{n - z}$$

05.08.17.0006.01

$$L_n^{\lambda}(z) = \frac{1}{n}\left((-z+\lambda+1)\, L_{n-1}^{\lambda+1}(z) - z\, L_{n-2}^{\lambda+2}(z)\right)$$

## Normalized recurrence relation

05.08.17.0007.01

$$z\, p(n, z) = p(n+1, z) + n\,(n+\lambda)\, p(n-1, z) + (2\,n + \lambda + 1)\, p(n, z) \;/;\; p(n, z) = (-1)^n\, n!\, L_n^{\lambda}(z)$$

### Additional relations between contiguous functions

05.08.17.0016.01

$$L_n^{\lambda}(z) = L_{n-1}^{\lambda}(z) + L_n^{\lambda-1}(z)$$

05.08.17.0008.01

$$L_n^{\lambda}(z) = \frac{1}{z}\left((n+\lambda)\, L_n^{\lambda-1}(z) - (n+1)\, L_{n+1}^{\lambda-1}(z)\right)$$

### Relations of special kind

05.08.17.0009.01

$$L_n^{-m}(z) = \frac{z^m}{(-n)_m}\, L_{n-m}^{m}(z) \;/;\; m \in \mathbb{N}$$

# Complex characteristics

## Real part

05.08.19.0001.01

$$\operatorname{Re}\left(L_n^{\lambda}(x + i\, y)\right) = \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(-1)^j\, y^{2j}}{(2\,j)!}\, L_{n-2\,j}^{2\,j+\lambda}(x) \;/;\; x \in \mathbb{R} \land y \in \mathbb{R} \land \lambda \in \mathbb{R}$$

## Imaginary part

05.08.19.0002.01

$$\operatorname{Im}\left(L_n^{\lambda}(x+i\,y)\right) = \sum_{j=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor} \frac{(-1)^{j-1}\,y^{2j+1}}{(2\,j+1)!}\,L_{-2\,j+n-1}^{2\,j+\lambda+1}(x)\;/;\;x\in\mathbb{R}\bigwedge y\in\mathbb{R}\bigwedge\lambda\in\mathbb{R}$$

### Argument

05.08.19.0003.01

$$\arg\left(L_n^{\lambda}(x+i\,y)\right) = \tan^{-1}\left(\sum_{j=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\frac{(-1)^j\,L_{n-2\,j}^{2\,j+\lambda}(x)\,y^{2\,j}}{(2\,j)!},\;\sum_{j=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}\frac{(-1)^{j-1}\,L_{n-2\,j-1}^{2\,j+\lambda+1}(x)\,y^{2\,j+1}}{(2\,j+1)!}\right)/;\;x\in\mathbb{R}\bigwedge y\in\mathbb{R}\bigwedge\lambda\in\mathbb{R}$$

### Conjugate value

05.08.19.0004.01

$$\overline{L_n^{\lambda}(x+i\,y)} = \sum_{j=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\frac{(-1)^j\,L_{n-2\,j}^{2\,j+\lambda}(x)\,y^{2\,j}}{(2\,j)!} - i\sum_{j=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}\frac{(-1)^{j-1}\,L_{n-2\,j-1}^{2\,j+\lambda+1}(x)\,y^{2\,j+1}}{(2\,j+1)!}\;/;\;x\in\mathbb{R}\bigwedge y\in\mathbb{R}\bigwedge\lambda\in\mathbb{R}$$

## Differentiation

### Low-order differentiation

#### With respect to $\lambda$

05.08.20.0013.01

$$\frac{\partial L_n^{\lambda}(z)}{\partial\lambda} = \sum_{k=0}^{n-1}\frac{L_k^{\lambda}(z)}{n-k}$$

05.08.20.0001.01

$$\frac{\partial L_n^{\lambda}(z)}{\partial\lambda} = (\psi(n+\lambda+1)-\psi(\lambda+1))\,L_n^{\lambda}(z) + \frac{z\,\Gamma(n+\lambda+1)}{(\lambda+1)\,\Gamma(\lambda+2)\,\Gamma(n)}\,F_{2\times0\times1}^{1\times1\times2}\left(\begin{matrix}1-n;\,1;\,1,\,\lambda+1;\\ 2,\,\lambda+2;;\,\lambda+2;\end{matrix}\;z,\,z\right)$$

05.08.20.0002.01

$$\frac{\partial L_n^{\lambda}(z)}{\partial\lambda} = \psi(n+\lambda+1)\,L_n^{\lambda}(z) - \frac{\Gamma(n+\lambda+1)}{\Gamma(n+1)}\sum_{k=0}^{n}\frac{(-n)_k\,\psi(k+\lambda+1)\,z^k}{k!\,\Gamma(k+\lambda+1)}$$

05.08.20.0014.01

$$\frac{\partial^2 L_n^{\lambda}(z)}{\partial\lambda^2} = 2\sum_{j=0}^{n-1}\frac{H_{n-j-1}\,L_j^{\lambda}(z)}{n-j}$$

05.08.20.0003.01

$$\frac{\partial^2 L_n^{\lambda}(z)}{\partial\lambda^2} = \left(\psi(n+\lambda+1)^2+\psi^{(1)}(n+\lambda+1)\right)L_n^{\lambda}(z) +$$
$$\frac{\Gamma(n+\lambda+1)}{\Gamma(n+1)}\sum_{k=0}^{n}\frac{(-n)_k\,z^k}{k!\,\Gamma(k+\lambda+1)}\left(\psi(k+\lambda+1)^2-2\,\psi(n+\lambda+1)\,\psi(k+\lambda+1)-\psi^{(1)}(k+\lambda+1)\right)$$

05.08.20.0004.01

$$\frac{\partial^2 L_n^{\lambda}(z)}{\partial \lambda^2} = \frac{1}{n!} \sum_{k=0}^{n} \frac{(-n)_k z^k}{k!} \sum_{j=1}^{n-k} (-1)^{j-k+n} S_{n-k}^{(j)} (j-1) j (k+\lambda+1)^{j-2}$$

### With respect to $z$

Forward shift operator:

05.08.20.0005.01

$$\frac{\partial L_n^{\lambda}(z)}{\partial z} = -L_{n-1}^{\lambda+1}(z)$$

05.08.20.0006.01

$$\frac{\partial^2 L_n^{\lambda}(z)}{\partial z^2} = L_{n-2}^{\lambda+2}(z)$$

Backward shift operator:

05.08.20.0007.01

$$z \frac{\partial L_n^{\lambda}(z)}{\partial z} + (\lambda - z) L_n^{\lambda}(z) = (n+1) L_{n+1}^{\lambda-1}(z)$$

05.08.20.0008.01

$$\frac{\partial\left(e^{-z} z^{\lambda} L_n^{\lambda}(z)\right)}{\partial z} = (n+1) e^{-z} z^{\lambda-1} L_{n+1}^{\lambda-1}(z)$$

## Symbolic differentiation

### With respect to $\lambda$

05.08.20.0009.02

$$\frac{\partial^m L_n^{\lambda}(z)}{\partial \lambda^m} = \frac{(-1)^n z^n \delta_m}{n!} + \frac{1}{n!} \sum_{k=0}^{n} \frac{(-n)_k z^k}{k!} \sum_{j=1}^{n-k} (-1)^{j-k+n} S_{n-k}^{(j)} (j-m+1)_m (k+\lambda+1)^{j-m} \text{ /; } m \in \mathbb{N}$$

### With respect to $z$

05.08.20.0010.02

$$\frac{\partial^m L_n^{\lambda}(z)}{\partial z^m} = (-1)^m L_{n-m}^{m+\lambda}(z) \text{ /; } m \in \mathbb{N}$$

05.08.20.0011.02

$$\frac{\partial^m L_n^{\lambda}(z)}{\partial z^m} = \frac{\Gamma(n+\lambda+1) z^{-m}}{\Gamma(n+1)} {}_2\tilde{F}_2(1, -n; 1-m, \lambda+1; z) \text{ /; } m \in \mathbb{N}$$

## Fractional integro-differentiation

### With respect to $\lambda$

05.08.20.0015.01

$$\frac{\partial^{\alpha} L_n^{\lambda}(z)}{\partial \lambda^{\alpha}} = \sum_{k=0}^{n} \sum_{j=0}^{n-k} \frac{(-1)^{k+n} k! z^{n-j-k} (-j-k)_j B_j^{(j+k+1)}(n+1)}{j! (n-j-k)! (j+k)! \Gamma(k-\alpha+1)} \lambda^{k-\alpha}$$

#### With respect to $z$

05.08.20.0012.01

$$\frac{\partial^{\alpha} L_n^{\lambda}(z)}{\partial z^{\alpha}} = \frac{\Gamma(n+\lambda+1)\, z^{-\alpha}}{\Gamma(n+1)}\, {}_2\tilde{F}_2(1, -n; 1-\alpha, \lambda+1; z)$$

## Integration

### Indefinite integration

#### Involving only one direct function

05.08.21.0001.01

$$\int L_n^{\lambda}(z)\, dz = -L_{n+1}^{\lambda-1}(z)$$

#### Involving one direct function and elementary functions

## Involving power function

05.08.21.0002.01

$$\int z^{\alpha-1} L_n^{\lambda}(z)\, dz = \frac{z^{\alpha}\, \Gamma(n+\lambda+1)}{\alpha\, \Gamma(n+1)\, \Gamma(\lambda+1)}\, {}_2F_2(-n, \alpha; \alpha+1, \lambda+1; z)$$

## Involving exponential function

05.08.21.0003.01

$$\int e^{-z} L_n^{\lambda}(z)\, dz = -\frac{\Gamma(n+\lambda)}{\Gamma(n+1)}\, {}_1\tilde{F}_1(n+\lambda; \lambda; -z)$$

## Involving exponential function and a power function

05.08.21.0004.01

$$\int z^{\alpha-1} e^{-c z} L_n^{\lambda}(c z)\, dz = \frac{z^{\alpha}\, \Gamma(\alpha)\, \Gamma(n+\lambda+1)}{\Gamma(n+1)}\, {}_2\tilde{F}_2(n+\lambda+1, \alpha; \lambda+1, \alpha+1; -c z)$$

05.08.21.0005.01

$$\int z^{\alpha-1} e^{-p z} L_n^{\lambda}(z)\, dz = -\frac{\Gamma(\lambda+n+1)\, z^{\alpha}}{\Gamma(n+1)\, (p z)^{\alpha}} \sum_{k=0}^{\infty} \frac{(-n)_k\, \Gamma(k+\alpha, p z)}{\Gamma(k+\lambda+1)\, k!\, p^k}$$

05.08.21.0006.01

$$\int z^{\lambda} e^{-z} L_n^{\lambda}(z)\, dz = \frac{z^{\lambda+1}\, \Gamma(\lambda+n+1)}{\Gamma(n+1)}\, {}_1\tilde{F}_1(\lambda+n+1; \lambda+2; -z)$$

05.08.21.0007.01

$$\int z^{\lambda+n-1} e^{-z} L_n^{\lambda}(z)\, dz = \frac{z^{\lambda+n}\, \Gamma(\lambda+n)}{\Gamma(n+1)}\, {}_1\tilde{F}_1(\lambda+n; \lambda+1; -z)$$

### Definite integration

#### Involving the direct function

Orthogonality:

05.08.21.0008.01

$$\int_0^\infty t^\lambda e^{-t} L_m^\lambda(t) L_n^\lambda(t) \, dt = \frac{\Gamma(n+\lambda+1)\,\delta_{n,m}}{n!} \,/;\, \mathrm{Re}(\lambda) > -1$$

05.08.21.0009.01

$$\int_0^\infty t^{\alpha-1} e^{-p\,t} L_m^\lambda(a\,t) L_n^\beta(b\,t) \, dt = \frac{\Gamma(\alpha)\,(\lambda+1)_m\,(\beta+1)_n\,p^{-\alpha}}{m!\,n!} \sum_{j=0}^m \frac{(-m)_j\,(\alpha)_j}{(\lambda+1)_j\,j!} \left(\frac{a}{p}\right)^j \sum_{k=0}^n \frac{(-n)_k\,(j+\alpha)_k}{(\beta+1)_k\,k!} \left(\frac{b}{p}\right)^k \,/;\, \mathrm{Re}(\alpha) > 0 \land \mathrm{Re}(p) > 0 \land m \in \mathbb{N} \land n \in \mathbb{N}$$

05.08.21.0010.01

$$\int_0^\infty t^{\alpha-1} e^{-p\,t} L_m^\lambda(p\,t) L_n^\beta(p\,t) \, dt = \frac{p^{-\alpha}\,\Gamma(\alpha)\,\Gamma(n-\alpha+\beta+1)\,\Gamma(m+\lambda+1)}{m!\,n!\,\Gamma(1-\alpha+\beta)\,\Gamma(\lambda+1)} \, {}_3F_2(-m, \alpha, \alpha-\beta; -n+\alpha-\beta, \lambda+1; 1) \,/;\, \mathrm{Re}(\alpha) > 0 \land \mathrm{Re}(p) > 0 \land m \in \mathbb{N} \land n \in \mathbb{N}$$

## Summation

### Finite summation

05.08.23.0001.01

$$\sum_{k=0}^n \frac{(\lambda-\beta)_{n-k}\,L_k^\beta(z)}{(n-k)!} = L_n^\lambda(z)$$

05.08.23.0002.01

$$\sum_{k=0}^n \binom{k-\beta+\lambda-1}{k} L_{n-k}^\beta(z) = L_n^\lambda(z)$$

05.08.23.0004.01

$$\sum_{k=0}^n \binom{n+\lambda}{n-k} L_k^\lambda(z)\,w^k\,(1-w)^{n-k} = L_n^\lambda(z\,w)$$

05.08.23.0005.01

$$\sum_{k=0}^n L_k^\mu(z_1)\,L_{n-k}^\lambda(z_2) = L_n^{\lambda+\mu+1}(z_1+z_2)$$

### Infinite summation

05.08.23.0006.01

$$\sum_{n=0}^\infty L_n^\lambda(z)\,w^n = (1-w)^{-\lambda-1}\,e^{\frac{w\,z}{w-1}} \,/;\, |w| < 1$$

05.08.23.0007.01

$$\sum_{n=0}^\infty \frac{1}{(\lambda+1)_n} L_n^\lambda(z)\,w^n = e^w \, {}_0F_1(;\lambda+1; -z\,w) \,/;\, |w| < 1$$

05.08.23.0008.01

$$\sum_{n=0}^\infty \frac{(c)_n}{(\lambda+1)_n} L_n^\lambda(z)\,w^n = (1-w)^{-c} \, {}_1F_1\!\left(c; \lambda+1; \frac{z\,w}{w-1}\right) \,/;\, |w| < 1$$

05.08.23.0009.01

$$\sum_{n=0}^{\infty} \frac{n!\, L_n^{\lambda}(x)\, L_n^{\lambda}(y)}{\Gamma(n+\lambda+1)} = (x\,y)^{-\frac{\lambda}{2}}\, e^{\frac{x+y}{2}}\, \delta(x-y) \,/;\, \mathrm{Re}(\lambda) > -1 \bigwedge x > 0 \bigwedge y > 0$$

05.08.23.0003.01

$$\sum_{k=0}^{\infty} \frac{L_n^{k+\lambda}(z)\, w^k}{k!} = e^w\, L_n^{\lambda}(z-w)$$

05.08.23.0010.01

$$\sum_{k=0}^{\infty} \left(\frac{t}{(t+1)^{\mu+1}}\right)^k L_k^{\lambda+k\,\mu}(z) = \frac{(t+1)^{\lambda+1}\, e^{-z\,t}}{1-\mu\,t}$$

05.08.23.0011.01

$$\sum_{k=0}^{\infty} \left(t\,(1-t)^{\mu}\, e^{\frac{\alpha\,z\,t}{1-t}}\right)^k L_k^{\lambda+k\,\mu}(z\,(k\,\alpha+1)) = \frac{(1-t)^{1-\lambda}\, e^{\frac{z\,t}{t-1}}}{(\mu+1)\,t^2-(-z\,\alpha+\mu+2)\,t+1}$$

05.08.23.0012.01

$$\sum_{k=0}^{\infty} \frac{\left(t\,(1-t)^{\mu}\, e^{\frac{\alpha\,z\,t}{1-t}}\right)^k L_k^{\lambda+k\,\mu}(z\,(k\,\alpha+1))}{k\,\alpha+1} = (1-t)^{-\lambda}\, e^{\frac{z\,t}{t-1}}\, {}_2F_1\!\left(\frac{-\alpha\,\lambda+\mu+1}{\alpha}, 1; \frac{\alpha+1}{\alpha}; t\right)$$

05.08.23.0013.01

$$\sum_{k=0}^{\infty} \frac{\left(t\,(1-t)^{\mu}\, e^{\frac{a\,z\,t}{1-t}}\right)^k}{k+1} L_k^{\lambda+k\,\mu}(z\,(a\,k+1)) = \frac{1}{t\,(\lambda-\mu)}$$

$$\left((1-t)^{-\lambda}\, e^{\frac{-a\,z+z+t\,(-z)}{1-t}} \left({}_1F_1\!\left(\lambda-\mu; \lambda-\mu+1; \frac{z\,(a-1)}{1-t}\right) - (1-t)^{\lambda-\mu}\, {}_1F_1(\lambda-\mu; \lambda-\mu+1; z\,(a-1))\right)\right) /;\, \left|t\,(1-t)^{\mu}\, e^{\frac{a\,z\,t}{1-t}}\right| < 1$$

05.08.23.0014.01

$$\sum_{k=0}^{\infty} \frac{\left(t\,(1-t)^{b}\, e^{\frac{a\,z\,t}{1-t}}\right)^k L_k^{c+b\,k}(z\,(a\,k+1))}{(-c-b\,k-k+m)\left(\frac{-c-b\,k-k+m+q}{q}\right)_p} = \frac{(1-t)^{m-c}\, e^{\frac{z\,t}{t-1}}}{p!}$$

$$\sum_{j=0}^{p} \sum_{k=0}^{m+j\,q} \frac{(-t)^k\,(1-t)^{j\,q-k}\,(-p)_j\,(-m-j\,q)_k}{k!\,j!\,(-c-b\,k-k+m+j\,q)}\, {}_1F_1\!\left(1; \frac{k\,b+b+c+k-m-j\,q+1}{b+1}; \frac{z\,t\,(b-a\,c+a\,m+a\,j\,q+1)}{(1-t)\,(b+1)}\right) /;$$

$$\left|t\,(1-t)^{b}\, e^{\frac{a\,z\,t}{1-t}}\right| < 1 \bigwedge m \in \mathbb{N} \bigwedge p \in \mathbb{N} \bigwedge q \in \mathbb{N}$$

# Operations

## Limit operation

05.08.25.0001.01

$$\lim_{n\to\infty} \frac{1}{n^{\lambda}}\, L_n^{\lambda}\!\left(\frac{z^2}{4\,n}\right) = 2^{\lambda}\, z^{-\lambda}\, J_{\lambda}(z)$$

05.08.25.0002.01

$$\lim_{n\to\infty} \frac{1}{n^{\lambda}} L_n^{\lambda}\left(-\frac{z^2}{4\,n}\right) == 2^{\lambda} z^{-\lambda} I_{\lambda}(z)$$

05.08.25.0003.01

$$\lim_{\lambda\to\infty} 2^{n/2}\, n!\, \lambda^{-\frac{n}{2}} L_n^{\lambda}\left(\lambda - \sqrt{2\,\lambda}\; z\right) == H_n(z)$$

### Orthogonality, completeness, and Fourier expansions

The set of functions $L_n^{\lambda}(x)$, $n = 0, 1, \ldots$, forms a complete, orthogonal (with weight $\frac{n!}{\Gamma(n+\lambda+1)} x^{\lambda} e^{-x}$) system on the interval $(0, \infty)$.

05.08.25.0004.01

$$\sum_{n=0}^{\infty} \left(\sqrt{\frac{n!}{\Gamma(n+\lambda+1)}}\; x^{\lambda/2} e^{-\frac{x}{2}} L_n^{\lambda}(x)\right) \left(\sqrt{\frac{n!}{\Gamma(n+\lambda+1)}}\; y^{\lambda/2} e^{-\frac{y}{2}} L_n^{\lambda}(y)\right) == \delta(x-y) \;/;\; \mathrm{Re}(\lambda) > -1 \bigwedge x > 0 \bigwedge y > 0$$

05.08.25.0005.01

$$\int_0^{\infty} \left(\sqrt{\frac{m!}{\Gamma(m+\lambda+1)}}\; t^{\lambda/2} e^{-\frac{t}{2}} L_m^{\lambda}(t)\right) \left(\sqrt{\frac{n!}{\Gamma(n+\lambda+1)}}\; t^{\lambda/2} e^{-\frac{t}{2}} L_n^{\lambda}(t)\right) dt == \delta_{n,m} \;/;\; \mathrm{Re}(\lambda) > -1$$

Any sufficiently smooth function $f(x)$ can be expanded in the system $\left\{L_n^{\lambda}(x)\right\}_{n=0,1,\ldots}$ as a generalized Fourier series, with its sum converging to $f(x)$ almost everywhere.

05.08.25.0006.01

$$f(x) == \sum_{n=0}^{\infty} c_n \psi_n(x) \;/;\; c_n == \int_0^{\infty} \psi_n(t)\, f(t)\, dt \bigwedge \psi_n(x) == \sqrt{\frac{n!}{\Gamma(n+\lambda+1)}}\; x^{\lambda/2} e^{-\frac{x}{2}} L_n^{\lambda}(x) \bigwedge \mathrm{Re}(\lambda) > -1 \bigwedge x > 0$$

## Representations through more general functions

### Through hypergeometric functions

**Involving $_1\tilde{F}_1$**

05.08.26.0001.01

$$L_n^{\lambda}(z) == \frac{\Gamma(n+\lambda+1)}{\Gamma(n+1)}\, {}_1\tilde{F}_1(-n; \lambda+1; z)$$

**Involving $_1F_1$**

05.08.26.0002.01

$$L_n^{\lambda}(z) == \frac{(\lambda+1)_n}{\Gamma(n+1)}\, {}_1F_1(-n; \lambda+1; z) \;/;\; -\lambda \notin \mathbb{N}^+$$

### Through Meijer *G*

**Classical cases involving exp**

05.08.26.0003.01

$$e^{-z} L_n^{\lambda}(z) = \frac{1}{\Gamma(n+1)} G_{1,2}^{1,1}\left(z \,\middle|\, \begin{array}{c} -n-\lambda \\ 0, -\lambda \end{array}\right)$$

### Through other functions

#### Involving some hypergeometric-type functions

05.08.26.0004.01

$$L_n^{\lambda}(z) = \lim_{b\to\infty} P_n^{(\lambda,b)}\left(1 - \frac{2z}{b}\right)$$

## Theorems

### Expansions in generalized Fourier series

$$f(x) = \sum_{k=0}^{\infty} c_k \psi_k(x) \;/;\; c_k = \int_0^{\infty} f(t)\,\psi_k(t)\,dt, \quad \psi_k(x) = \frac{\sqrt{k!}}{\sqrt{\Gamma(\lambda+k+1)}} x^{\lambda/2} e^{-x/2} L_k^{\lambda}(x), \quad k \in \mathbb{N}.$$

### Quantum mechanical eigenfunctions of the hydrogen atom

Quantum mechanical eigenfunctions $\psi_{n l m}(r, \theta, \phi)$ of the hydrogen atom are:

$$\psi_{n l m}(r, \theta, \phi) = (2\tau)^3 \frac{(n-l-1)!}{2n\,(n+l)!^3} e^{-\tau r} (2\tau r)^l L_{n-l-1}^{2l+1}(2\tau r)\, Y_l^m(\theta, \phi) \;/;\; \tau > 0,$$
$$n \in \mathbb{N}^+, l \in \mathbb{N}, l \le n-1, m \in \mathbb{Z}, |m| \le l$$

### The intensity of the Laguerre-Gauss modes

The intensity of the Laguerre-Gauss modes $\mathcal{L}_n^m(x, y) = e^{-(x^2+y^2)}(x + i y)^m L_n^m\left(2\left(x^2+y^2\right)\right)$ are shape invariant under a Fresnel transformation:

$$\left| \frac{k}{2\pi i} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i k \left((x-\xi)^2+(y-\eta)^2\right)} \mathcal{L}_n^m(x, y)\,dx\,dy \right|^2 = \frac{1}{|\alpha|^2} \left| \mathcal{L}_n^m\left(\frac{\xi}{|\alpha|}, \frac{\eta}{|\alpha|}\right) \right|^2 \;/;\; \alpha = 1 + \frac{2i}{k}$$

### The eigenfunctions of the Schrödinger equation of the one-dimensional hydrogen atom

The eigenfunctions of the Schrödinger equation of the one-dimensional hydrogen atom (the simplest example for a system with dynamical superselection rules) $-\psi_n''(x) - \frac{1}{x}\psi_n''(x) = -\frac{1}{2n^2}\psi_n(x)$ are given by

$$\psi_n(x) = \alpha_1\,\theta(x) \sqrt{\frac{4}{n^5\,n!^2}} (-1)^{n-1} x L_{n-1}^1\left(\frac{2x}{n}\right) e^{-\frac{x}{n}} + \alpha_2\,\theta(-x) \sqrt{\frac{4}{n^5\,n!^2}} (-1)^{n-1} x L_{n-1}^1\left(-\frac{2x}{n}\right) e^{\frac{x}{n}}.$$

## History

– E. N. Laguerre (1879)
– N. J. Sonin (1880)

## Copyright