LaguerreL3General

Notations

Traditional name

Generalized Laguerre function

Traditional notation

$L_{\nu}^{\lambda}(z)$

Mathematica StandardForm notation

`LaguerreL[ν, λ, z]`

Primary definition

07.03.02.0001.01

$$L_{\nu}^{\lambda}(z) = \frac{\Gamma(\nu+\lambda+1)}{\Gamma(\nu+1)}\, {}_1\tilde{F}_1(-\nu; \lambda+1; z)$$

Specific values

Specialized values

For fixed ν, λ

07.03.03.0001.01

$$L_{\nu}^{\lambda}(0) = \binom{\nu+\lambda}{\lambda}$$

For fixed ν, z

07.03.03.0002.01

$$L_{\nu}^{0}(z) = L_{\nu}(z)$$

07.03.03.0003.01

$$L_n^{-\frac{1}{2}}(z) = \frac{(-1)^n}{2^{2n}\, n!} H_{2n}\left(\sqrt{z}\right) /;\ n \in \mathbb{N}$$

07.03.03.0004.01

$$L_n^{\frac{1}{2}}(z) = \frac{(-1)^n}{2^{2n+1}\, n!\, \sqrt{z}} H_{2n+1}\left(\sqrt{z}\right) /;\ n \in \mathbb{N}$$

07.03.03.0005.01

$$L_\nu^1(z) = \frac{\nu L_{\nu-1}(z) - (\nu - z) L_\nu(z)}{z}$$

07.03.03.0006.01

$$L_\nu^\lambda(z) = \left(1 \times .- \frac{\partial .}{\partial z}\right)^\lambda L_\nu(z) \;/;\; \lambda \in \mathbb{N}^+$$

07.03.03.0007.01

$$L_\nu^\lambda(z) = \left(-\frac{\partial .}{\partial z}\right)^\lambda L_\nu(z) \;/;\; \lambda \in \mathbb{N}^+$$

07.03.03.0008.01

$$L_\nu^{-\nu-n}(z) = \tilde{\infty} \;/;\; n \in \mathbb{N}^+$$

For fixed λ, z

07.03.03.0009.01

$$L_0^\lambda(z) = 1$$

07.03.03.0010.01

$$L_1^\lambda(z) = -z + \lambda + 1$$

07.03.03.0011.01

$$L_2^\lambda(z) = \frac{1}{2}\left(z^2 - 2z(2+\lambda) + \lambda^2 + 3\lambda + 2\right)$$

07.03.03.0012.01

$$L_3^\lambda(z) = \frac{1}{6}\left(-z^3 + 3(\lambda+3)z^2 - 3\left(\lambda^2 + 5\lambda + 6\right)z + \lambda^3 + 6\lambda^2 + 11\lambda + 6\right)$$

07.03.03.0013.01

$$L_4^\lambda(z) = \frac{1}{24}\left(z^4 - 4(\lambda+4)z^3 + 6\left(\lambda^2 + 7\lambda + 12\right)z^2 - 4\left(\lambda^3 + 9\lambda^2 + 26\lambda + 24\right)z + \lambda^4 + 10\lambda^3 + 35\lambda^2 + 50\lambda + 24\right)$$

07.03.03.0014.01

$$L_5^\lambda(z) = \frac{1}{120}\left(-z^5 + 5(\lambda+5)z^4 - 10\left(\lambda^2 + 9\lambda + 20\right)z^3 + 10\left(\lambda^3 + 12\lambda^2 + 47\lambda + 60\right)z^2 - 5\left(\lambda^4 + 14\lambda^3 + 71\lambda^2 + 154\lambda + 120\right)z + \lambda^5 + 15\lambda^4 + 85\lambda^3 + 225\lambda^2 + 274\lambda + 120\right)$$

07.03.03.0015.01

$$L_6^\lambda(z) = \frac{1}{720}\left(z^6 - 6(\lambda+6)z^5 + 15\left(\lambda^2 + 11\lambda + 30\right)z^4 - 20\left(\lambda^3 + 15\lambda^2 + 74\lambda + 120\right)z^3 + 15\left(\lambda^4 + 18\lambda^3 + 119\lambda^2 + 342\lambda + 360\right)z^2 - 6\left(\lambda^5 + 20\lambda^4 + 155\lambda^3 + 580\lambda^2 + 1044\lambda + 720\right)z + \lambda^6 + 21\lambda^5 + 175\lambda^4 + 735\lambda^3 + 1624\lambda^2 + 1764\lambda + 720\right)$$

07.03.03.0016.01

$$L_7^\lambda(z) = \frac{1}{5040}\left(-z^7 + 7(\lambda+7)z^6 - 21\left(\lambda^2 + 13\lambda + 42\right)z^5 + 35\left(\lambda^3 + 18\lambda^2 + 107\lambda + 210\right)z^4 - 35\left(\lambda^4 + 22\lambda^3 + 179\lambda^2 + 638\lambda + 840\right)z^3 + 21\left(\lambda^5 + 25\lambda^4 + 245\lambda^3 + 1175\lambda^2 + 2754\lambda + 2520\right)z^2 - 7\left(\lambda^6 + 27\lambda^5 + 295\lambda^4 + 1665\lambda^3 + 5104\lambda^2 + 8028\lambda + 5040\right)z + \lambda^7 + 28\lambda^6 + 322\lambda^5 + 1960\lambda^4 + 6769\lambda^3 + 13132\lambda^2 + 13068\lambda + 5040\right)$$

07.03.03.0017.01

$$L_8^\lambda(z) == \frac{1}{40\,320}\left(z^8 - 8(\lambda+8)z^7 + 28\left(\lambda^2+15\lambda+56\right)z^6 - 56\left(\lambda^3+21\lambda^2+146\lambda+336\right)z^5 + 70\left(\lambda^4+26\lambda^3+251\lambda^2+1066\lambda+1680\right)z^4 - 56\left(\lambda^5+30\lambda^4+355\lambda^3+2070\lambda^2+5944\lambda+6720\right)z^3 + 28\left(\lambda^6+33\lambda^5+445\lambda^4+3135\lambda^3+12\,154\lambda^2+24\,552\lambda+20\,160\right)z^2 - 8\left(\lambda^7+35\lambda^6+511\lambda^5+4025\lambda^4+18\,424\lambda^3+48\,860\lambda^2+69\,264\lambda+40\,320\right)z + \lambda^8 + 36\lambda^7+546\lambda^6+4536\lambda^5+22\,449\lambda^4+67\,284\lambda^3+118\,124\lambda^2+109\,584\lambda+40\,320\right)$$

07.03.03.0018.01

$$L_9^\lambda(z) == \frac{1}{362\,880}\left(-z^9 + 9(\lambda+9)z^8 - 36\left(\lambda^2+17\lambda+72\right)z^7 + 84\left(\lambda^3+24\lambda^2+191\lambda+504\right)z^6 - 126\left(\lambda^4+30\lambda^3+335\lambda^2+1650\lambda+3024\right)z^5 + 126\left(\lambda^5+35\lambda^4+485\lambda^3+3325\lambda^2+11\,274\lambda+15\,120\right)z^4 - 84\left(\lambda^6+39\lambda^5+625\lambda^4+5265\lambda^3+24\,574\lambda^2+60\,216\lambda+60\,480\right)z^3 + 36\left(\lambda^7+42\lambda^6+742\lambda^5+7140\lambda^4+40\,369\lambda^3+133\,938\lambda^2+241\,128\lambda+181\,440\right)z^2 - 9\left(\lambda^8+44\lambda^7+826\lambda^6+8624\lambda^5+54\,649\lambda^4+214\,676\lambda^3+509\,004\lambda^2+663\,696\lambda+362\,880\right)z + \lambda^9 + 45\lambda^8+870\lambda^7+9450\lambda^6+63\,273\lambda^5+269\,325\lambda^4+723\,680\lambda^3+1\,172\,700\lambda^2+1\,026\,576\lambda+362\,880\right)$$

07.03.03.0019.01

$$L_{10}^\lambda(z) == \frac{1}{3\,628\,800}\left(z^{10} - 10(\lambda+10)z^9 + 45\left(\lambda^2+19\lambda+90\right)z^8 - 120\left(\lambda^3+27\lambda^2+242\lambda+720\right)z^7 + 210\left(\lambda^4+34\lambda^3+431\lambda^2+2414\lambda+5040\right)z^6 - 252\left(\lambda^5+40\lambda^4+635\lambda^3+5000\lambda^2+19\,524\lambda+30\,240\right)z^5 + 210\left(\lambda^6+45\lambda^5+835\lambda^4+8175\lambda^3+44\,524\lambda^2+127\,860\lambda+151\,200\right)z^4 - 120\left(\lambda^7+49\lambda^6+1015\lambda^5+11\,515\lambda^4+77\,224\lambda^3+305\,956\lambda^2+662\,640\lambda+604\,800\right)z^3 + 45\left(\lambda^8+52\lambda^7+1162\lambda^6+14\,560\lambda^5+111\,769\lambda^4+537\,628\lambda^3+1\,580\,508\lambda^2+2\,592\,720\lambda+1\,814\,400\right)z^2 - 10\left(\lambda^9+54\lambda^8+1266\lambda^7+16\,884\lambda^6+140\,889\lambda^5+761\,166\lambda^4+2\,655\,764\lambda^3+5\,753\,736\lambda^2+6\,999\,840\lambda+3\,628\,800\right) z + \lambda^{10}+55\lambda^9+1320\lambda^8+18\,150\lambda^7+157\,773\lambda^6 + 902\,055\lambda^5+3\,416\,930\lambda^4+8\,409\,500\lambda^3+12\,753\,576\lambda^2+10\,628\,640\lambda+3\,628\,800\right)$$

07.03.03.0020.01

$$L_n^\lambda(z) == \frac{\Gamma(n+\lambda+1)}{n!}\sum_{k=0}^{n}\frac{(-n)_k\,z^k}{\Gamma(k+\lambda+1)\,k!}\ /;\ n\in\mathbb{N}$$

07.03.03.0021.01

$$L_{-n}^\lambda(z) == 0\ /;\ n\in\mathbb{N}^+$$

07.03.03.0022.01

$$L_{-n-\lambda}^\lambda(z) == \tilde{\infty}\ /;\ n\in\mathbb{N}^+$$

General characteristics

Domain and analyticity

$L_\nu^\lambda(z)$ is an analytical function of ν, λ, z which is defined in $\mathbb{C}^3$. For fixed ν, λ, it is an entire function of z. For positive integer ν, $L_\nu^\lambda(z)$ degenerates to a polynomial in z.

07.03.04.0001.01

$$(\nu*\lambda*z)\longrightarrow L_\nu^\lambda(z) :: (\mathbb{C}\otimes\mathbb{C}\otimes\mathbb{C})\longrightarrow\mathbb{C}$$

Symmetries and periodicities

Mirror symmetry

07.03.04.0002.01

$$L_{\bar{\nu}}^{\bar{\lambda}}(\bar{z}) == \overline{L_{\nu}^{\lambda}(z)}$$

Periodicity

No periodicity

Poles and essential singularities

With respect to *z*

For fixed ν /; $\nu \notin \mathbb{N}$, λ, the function $L_{\nu}^{\lambda}(z)$ has only one singular point at $z = \tilde{\infty}$. It is an essential singular point.

07.03.04.0003.01

$$\mathcal{S}ing_z\left(L_{\nu}^{\lambda}(z)\right) == \{\{\tilde{\infty}, \infty\}\} /; \nu \notin \mathbb{N}$$

For positive integer ν, the function $L_{\nu}^{\lambda}(z)$ is polynomial and has pole of order ν at $z = \tilde{\infty}$.

07.03.04.0004.01

$$\mathcal{S}ing_z\left(L_{\nu}^{\lambda}(z)\right) == \{\{\tilde{\infty}, \nu\}\} /; \nu \in \mathbb{N}^+$$

With respect to λ

For fixed ν, z, the function $L_{\nu}^{\lambda}(z)$ has an infinite set of singular points:

a) $\lambda == -\nu - k$ /; $k \in \mathbb{N}^+$, are the simple poles with residues $\frac{(-1)^{k-1}}{\Gamma(\nu+1)\,(k-1)!}\, {}_1\tilde{F}_1(-\nu; -k-\nu+1; z)$ /; $k \in \mathbb{N}^+$;

b) $\lambda == \tilde{\infty}$ is an essential singular point.

07.03.04.0005.01

$$\mathcal{S}ing_{\lambda}\left(L_{\nu}^{\lambda}(z)\right) == \left\{\left\{\{-\nu - k, 1\} /; k \in \mathbb{N}^+\right\}, \{\tilde{\infty}, \infty\}\right\}$$

07.03.04.0006.01

$$\mathrm{res}_{\lambda}\left(L_{\nu}^{\lambda}(z)\right)(-\nu - k) == \frac{(-1)^{k-1}}{\Gamma(\nu+1)\,(k-1)!}\, {}_1\tilde{F}_1(-\nu; -k-\nu+1; z) /; k \in \mathbb{N}^+$$

With respect to ν

For fixed λ, z, the function $L_{\nu}^{\lambda}(z)$ has an infinite set of singular points:

a) $\nu == -\lambda - k$ /; $k \in \mathbb{N}^+$, are the simple poles with residues $\frac{(-1)^{k-1}}{\Gamma(1-k-\lambda)\,(k-1)!}\, {}_1\tilde{F}_1(\lambda + k; \lambda + 1; z)$ /; $k \in \mathbb{N}^+$;

b) $\nu == \tilde{\infty}$ is an essential singular point.

07.03.04.0007.01

$$\mathcal{S}ing_{\nu}\left(L_{\nu}^{\lambda}(z)\right) == \left\{\left\{\{-\lambda - k, 1\} /; k \in \mathbb{N}^+\right\}, \{\tilde{\infty}, \infty\}\right\}$$

07.03.04.0008.01

$$\mathrm{res}_{\lambda}\left(L_{\nu}^{\lambda}(z)\right)(-\nu - k) == \frac{(-1)^{k-1}}{\Gamma(1-k-\lambda)\,(k-1)!}\, {}_1\tilde{F}_1(\lambda + k; \lambda + 1; z) /; k \in \mathbb{N}^+$$

Branch points

With respect to z

For fixed ν, λ, the function $L_\nu^\lambda(z)$ does not have branch points.

07.03.04.0009.01

$$\mathcal{BP}_z\left(L_\nu^\lambda(z)\right) == \{\}$$

With respect to λ

For fixed ν, z, the function $L_\nu^\lambda(z)$ does not have branch points.

07.03.04.0010.01

$$\mathcal{BP}_\lambda\left(L_\nu^\lambda(z)\right) == \{\}$$

With respect to ν

For fixed λ, z, the function $L_\nu^\lambda(z)$ does not have branch points.

07.03.04.0011.01

$$\mathcal{BP}_\nu\left(L_\nu^\lambda(z)\right) == \{\}$$

Branch cuts

With respect to z

The function $L_\nu^\lambda(z)$ does not have branch cuts with respect to z.

07.03.04.0012.01

$$\mathcal{BC}_z\left(L_\nu^\lambda(z)\right) == \{\}$$

With respect to λ

The function $L_\nu^\lambda(z)$ does not have branch cuts with respect to λ.

07.03.04.0013.01

$$\mathcal{BC}_\lambda\left(L_\nu^\lambda(z)\right) == \{\}$$

With respect to ν

The function $L_\nu^\lambda(z)$ does not have branch cuts with respect to ν.

07.03.04.0014.01

$$\mathcal{BC}_\nu\left(L_\nu^\lambda(z)\right) == \{\}$$

Series representations

Generalized power series

Expansions at generic point $\lambda == \lambda_0$

For the function itself

07.03.06.0008.01

$$L_n^{\lambda}(z) \propto \frac{(-1)^n z^n}{n!} + \frac{1}{n!} \sum_{s=0}^{n} \sum_{j=1}^{n-s} \frac{(-1)^{j+n-s} z^s}{s!} (-n)_s S_{n-s}^{(j)} (s+\lambda_0+1)^j \left(1 + \frac{j}{s+\lambda_0+1} (\lambda-\lambda_0) + \frac{(j-1) j}{2 (s+\lambda_0+1)^2} (\lambda-\lambda_0)^2 + \ldots\right) /; (\lambda \to \lambda_0) \bigwedge n \in \mathbb{N}$$

07.03.06.0009.01

$$L_n^{\lambda}(z) \propto \frac{(-1)^n z^n}{n!} + \frac{1}{n!} \sum_{s=0}^{n} \sum_{j=1}^{n-s} \frac{(-1)^{j+n-s} z^s}{s!} (-n)_s S_{n-s}^{(j)} (s+\lambda_0+1)^j \left(1 + \frac{j}{s+\lambda_0+1} (\lambda-\lambda_0) + \frac{(j-1) j}{2 (s+\lambda_0+1)^2} (\lambda-\lambda_0)^2 + O\big((\lambda-\lambda_0)^3\big)\right) /; n \in \mathbb{N}$$

07.03.06.0010.01

$$L_n^{\lambda}(z) == \frac{(-1)^n z^n}{n!} + \sum_{k=0}^{\infty} \frac{1}{k!\, n!} \sum_{s=0}^{n} \frac{(-n)_s z^s}{s!} \sum_{j=1}^{n-s} (-1)^{j+n-s} S_{n-s}^{(j)} (j-k+1)_k (s+\lambda_0+1)^{j-k} (\lambda-\lambda_0)^k /; n \in \mathbb{N}$$

07.03.06.0011.01

$$L_n^{\lambda}(z) \propto L_n^{\lambda}(z_0) (1 + O(\lambda-\lambda_0)) /; n \in \mathbb{N}$$

Expansions at generic point $z == z_0$

For the function itself

07.03.06.0012.01

$$L_\nu^{\lambda}(z) \propto L_\nu^{\lambda}(z_0) - L_{\nu-1}^{\lambda+1}(z_0) (z-z_0) + \frac{1}{2} L_{\nu-2}^{\lambda+2}(z_0) (z-z_0)^2 + \ldots /; (z \to z_0)$$

07.03.06.0013.01

$$L_\nu^{\lambda}(z) \propto L_\nu^{\lambda}(z_0) - L_{\nu-1}^{\lambda+1}(z_0) (z-z_0) + \frac{1}{2} L_{\nu-2}^{\lambda+2}(z_0) (z-z_0)^2 + O\big((z-z_0)^3\big)$$

07.03.06.0014.01

$$L_\nu^{\lambda}(z) == \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} L_{\nu-k}^{k+\lambda}(z_0) (z-z_0)^k$$

07.03.06.0015.01

$$L_\nu^{\lambda}(z) == \Gamma(\lambda+\nu+1) \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\, \Gamma(\nu-k+1)} \, {}_1\tilde{F}_1(k-\nu; k+\lambda+1; z_0) (z-z_0)^k$$

07.03.06.0016.01

$$L_\nu^{\lambda}(z) == \frac{\Gamma(\lambda+\nu+1)}{\Gamma(\nu+1)} \tilde{F}_{1\times0\times0}^{1\times0\times0}\left(\begin{matrix} -\nu;;; \\ \lambda+1;;; \end{matrix} z_0, z-z_0 \right)$$

07.03.06.0017.01

$$L_\nu^{\lambda}(z) \propto L_\nu^{\lambda}(z_0) (1 + O(z-z_0))$$

Expansions at $z == 0$

For the function itself

General case

07.03.06.0001.02

$$L_\nu^\lambda(z) \propto \frac{\Gamma(\nu+\lambda+1)}{\Gamma(\nu+1)}\left(\frac{1}{\Gamma(\lambda+1)} - \frac{\nu z}{\Gamma(\lambda+2)} - \frac{(1-\nu)\,\nu\,z^2}{2\,\Gamma(\lambda+3)} - \ldots\right) /;\ (z \to 0)$$

07.03.06.0018.01

$$L_\nu^\lambda(z) \propto \frac{\Gamma(\nu+\lambda+1)}{\Gamma(\nu+1)}\left(\frac{1}{\Gamma(\lambda+1)} - \frac{\nu z}{\Gamma(\lambda+2)} - \frac{(1-\nu)\,\nu\,z^2}{2\,\Gamma(\lambda+3)} - O\big(z^3\big)\right)$$

07.03.06.0002.01

$$L_\nu^\lambda(z) = \frac{\Gamma(\nu+\lambda+1)}{\Gamma(\nu+1)} \sum_{k=0}^{\infty} \frac{(-\nu)_k\, z^k}{\Gamma(k+\lambda+1)\, k!}$$

07.03.06.0019.01

$$L_\nu^\lambda(z) = \frac{\Gamma(\lambda+\nu+1)}{\Gamma(\nu+1)}\ {}_1\tilde{F}_1(-\nu;\ \lambda+1;\ z)$$

07.03.06.0003.02

$$L_\nu^\lambda(z) \propto \frac{(\lambda+1)_\nu}{\Gamma(\nu+1)}\,(1+O(z))$$

07.03.06.0020.01

$$L_\nu^\lambda(z) = F_\infty(z,\ \nu,\ \lambda) /; \left(\left(F_m(z,\ \nu,\ \lambda) = \frac{\Gamma(\lambda+\nu+1)}{\Gamma(\nu+1)} \sum_{k=0}^{m} \frac{(-\nu)_k\, z^k}{\Gamma(k+\lambda+1)\, k!} = L_\nu^\lambda(z) + \frac{\Gamma(m-\nu+1)\,\Gamma(\lambda+\nu+1)\sin(\pi\,\nu)}{\pi\,(m+1)!\,\Gamma(m+\lambda+2)}\, z^{m+1}\ {}_2F_2(1,\ m-\nu+1;\ m+2,\ m+\lambda+2;\ z)\right) \bigwedge m \in \mathbb{N}\right)$$

Summed form of the truncated series expansion.

Special cases

07.03.06.0004.01

$$L_n^\lambda(z) = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \binom{n+\lambda}{n-k} z^k /;\ n \in \mathbb{N}$$

07.03.06.0005.01

$$L_n^\lambda(z) = \frac{1}{n!} \sum_{k=0}^{n} \frac{(-n)_k\,(k+\lambda+1)_{n-k}\, z^k}{k!} /;\ n \in \mathbb{N}$$

07.03.06.0021.01

$$L_n^\lambda(z) \propto \frac{(\lambda+1)_n}{n!}\,(1+O(z)) /;\ n \in \mathbb{N}$$

Expansions at $z = \infty$

For the function itself

Special cases

07.03.06.0022.01

$$L_n^\lambda(z) \propto \frac{(-z)^n}{n!}\left(1+\frac{n(-n-\lambda)}{z}+\frac{(n-1)\,n\,(-n-\lambda)\,(1-n-\lambda)}{2z^2}+\ldots\right)/;\ (|z|\to\infty)\land n\in\mathbb{N}$$

07.03.06.0023.01

$$L_n^\lambda(z) \propto \frac{(-z)^n}{n!}\left(1+\frac{n(-n-\lambda)}{z}+\frac{(n-1)\,n\,(-n-\lambda)\,(1-n-\lambda)}{2z^2}+O\left(\frac{1}{z^3}\right)\right)/;\ n\in\mathbb{N}$$

07.03.06.0024.01

$$L_n^\lambda(z) = \frac{(-z)^n}{n!}\sum_{k=0}^{n}\frac{(-1)^k\,(-n)_k\,(-n-\lambda)_k\,z^{-k}}{k!}/;\ n\in\mathbb{N}$$

07.03.06.0025.01

$$L_n^\lambda(z) = \frac{(-z)^n}{n!}\,{}_2F_0\left(-n,\,-n-\lambda;\,;\,-\frac{1}{z}\right)/;\ n\in\mathbb{N}$$

07.03.06.0026.01

$$L_n^\lambda(z) \propto \frac{1}{n!}(-z)^n\left(1+O\left(\frac{1}{z}\right)\right)/;\ n\in\mathbb{N}$$

Asymptotic series expansions

07.03.06.0027.01

$$L_\nu^\lambda(z) \propto \frac{(-z)^\nu}{\Gamma(\nu+1)}\left(1-\frac{(\lambda+\nu)\,\nu}{z}+\frac{(1-\nu)\,(\lambda+\nu)\,(1-\lambda-\nu)\,\nu}{2z^2}+\ldots\right)-$$
$$\frac{\sin(\nu\pi)\,\Gamma(\lambda+\nu+1)\,z^{-\lambda-\nu-1}\,e^z}{\pi}\left(1+\frac{(1+\nu)\,(1+\lambda+\nu)}{z}+\frac{(1+\nu)\,(2+\nu)\,(1+\lambda+\nu)\,(2+\lambda+\nu)}{2z^2}+\ldots\right)/;\ (|z|\to\infty)$$

07.03.06.0028.01

$$L_\nu^\lambda(z) \propto \frac{(-z)^\nu}{\Gamma(\nu+1)}\left(\sum_{k=0}^{n}\frac{(-1)^k\,(-\nu)_k\,(-\lambda-\nu)_k\,z^{-k}}{k!}+O\left(z^{-n-1}\right)\right)-$$
$$\frac{\sin(\nu\pi)\,\Gamma(\lambda+\nu+1)\,z^{-\lambda-\nu-1}\,e^z}{\pi}\left(\sum_{k=0}^{n}\frac{(\nu+1)_k\,(\lambda+\nu+1)_k\,z^{-k}}{k!}+O\left(z^{-n-1}\right)\right)/;\ (|z|\to\infty)$$

07.03.06.0006.01

$$L_\nu^\lambda(z) \propto \frac{(-z)^\nu}{\Gamma(\nu+1)}\,{}_2F_0\left(-\nu,\,-\nu-\lambda;\,;\,-\frac{1}{z}\right)-\frac{\sin(\nu\pi)\,\Gamma(\nu+\lambda+1)}{\pi}\,z^{-\nu-\lambda-1}\,e^z\,{}_2F_0\left(\nu+\lambda+1,\,\nu+1;\,;\,\frac{1}{z}\right)/;\ (|z|\to\infty)$$

07.03.06.0007.01

$$L_\nu^\lambda(z) \propto \frac{1}{\Gamma(\nu+1)}(-z)^\nu\left(1+O\left(\frac{1}{z}\right)\right)-\frac{\sin(\nu\pi)\,\Gamma(\nu+\lambda+1)}{\pi}\,z^{-\nu-\lambda-1}\,e^z\left(1+O\left(\frac{1}{z}\right)\right)/;\ (|z|\to\infty)$$

Integral representations

On the real axis

Of the direct function

07.03.07.0001.01

$$L_\nu^\lambda(z) = -\frac{\sin(\nu \pi)}{\pi} \int_0^1 e^{z t} t^{-\nu-1} (1-t)^{\nu+\lambda} \, dt \,/;\, \mathrm{Re}(\nu+\lambda) > -1 \bigwedge \mathrm{Re}(\nu) < 0$$

07.03.07.0002.01

$$L_\nu^\lambda(z) = \frac{1}{\Gamma(\nu+1)} z^{-\frac{\lambda}{2}} e^z \int_0^\infty e^{-t} t^{\nu+\frac{\lambda}{2}} J_\lambda\left(2\sqrt{t z}\right) dt \,/;\, \mathrm{Re}(\nu+\lambda) > -1$$

Integral representations of negative integer order

Rodrigues-type formula.

07.03.07.0003.01

$$L_n^\lambda(z) = \frac{e^z}{z^\lambda n!} \frac{\partial^n \left(z^{n+\lambda} e^{-z}\right)}{\partial z^n} \,/;\, n \in \mathbb{N}$$

Limit representations

07.03.09.0001.01

$$L_\nu^\lambda(z) = \lim_{b\to\infty} P_\nu^{(\lambda,b)}\left(1 - \frac{2z}{b}\right)$$

Generating functions

07.03.11.0001.01

$$L_n^\lambda(z) = \left([t^n]\,(1-t)^{-\lambda-1} \exp\left(\frac{t z}{t-1}\right)\right) /;\, n \in \mathbb{N}$$

Differential equations

Ordinary linear differential equations and wronskians

For the direct function itself

07.03.13.0005.01

$$z\, w''(z) + (\lambda + 1 - z)\, w'(z) + \nu\, w(z) = 0 \,/;\, w(z) = c_1 L_\nu^\lambda(z) + c_2\left(e^z G_{1,2}^{2,0}\left(z \,\middle|\, \begin{matrix} -\lambda-\nu \\ 0, -\lambda \end{matrix}\right) + G_{1,2}^{2,0}\left(-z \,\middle|\, \begin{matrix} \nu+1 \\ 0, -\lambda \end{matrix}\right)\right)$$

07.03.13.0006.01

$$W_z\left(L_\nu^\lambda(z),\, G_{1,2}^{2,0}\left(-z \,\middle|\, \begin{matrix} \nu+1 \\ 0, -\lambda \end{matrix}\right) + e^z G_{1,2}^{2,0}\left(z \,\middle|\, \begin{matrix} -\lambda-\nu \\ 0, -\lambda \end{matrix}\right)\right) = \frac{e^z (-z)^{-\lambda-1}}{\Gamma(\nu+1)} + \frac{e^z z^{-\lambda-1} \sin(\pi \nu)\, \Gamma(\lambda+\nu+1)}{\pi}$$

07.03.13.0007.01

$$z\, w''(z) + (\lambda + 1 - z)\, w'(z) + \nu\, w(z) = 0 \,/;\, w(z) = c_1 L_\nu^\lambda(z) + c_2 G_{1,2}^{2,0}\left(-z \,\middle|\, \begin{matrix} \nu+1 \\ 0, -\lambda \end{matrix}\right) /;\, -\nu \notin \mathbb{N}^+$$

07.03.13.0008.01

$$W_z\left(L_\nu^\lambda(z),\, G_{1,2}^{2,0}\left(-z \,\middle|\, \begin{matrix} \nu+1 \\ 0, -\lambda \end{matrix}\right)\right) = \frac{e^z (-z)^{-\lambda-1}}{\Gamma(\nu+1)}$$

07.03.13.0001.01

$$z\, w''(z) + (\lambda + 1 - z)\, w'(z) + \nu\, w(z) = 0 \,/;\, w(z) = c_1 L_\nu^\lambda(z) + c_2 U(-\nu, \lambda+1, z) \,/;\, \nu \notin \mathbb{Z}$$

07.03.13.0002.02

$$W_z\left(L_\nu^\lambda(z),\, U(-\nu,\, \lambda+1,\, z)\right) = \frac{\Gamma(\lambda+\nu+1)\sin(\pi\,\nu)}{\pi}\, e^z\, z^{-\lambda-1}$$

07.03.13.0003.01

$$z\, w''(z) + (\lambda+1-z)\, w'(z) + \nu\, w(z) = 0 \,/;\, w(z) = c_1\, L_\nu^\lambda(z) \,+\, c_2\, z^{-\lambda}\, {}_1\tilde{F}_1(-\nu-\lambda;\, 1-\lambda;\, z) \,/;\, \lambda \notin \mathbb{Z}$$

07.03.13.0009.01

$$W_z\left(L_\nu^\lambda(z),\, z^{-\lambda}\, {}_1\tilde{F}_1(-\lambda-\nu;\, 1-\lambda;\, z)\right) = -\frac{\Gamma(\lambda+\nu+1)\sin(\pi\,\lambda)}{\pi\,\Gamma(\nu+1)}\, e^z\, z^{-\lambda-1}$$

07.03.13.0010.01

$$z\, w''(z) + (\lambda+1-z)\, w'(z) + \nu\, w(z) = 0 \,/;\, w(z) = c_1\, L_\nu^\lambda(z) \,+\, c_2\, z^{-\lambda}\, {}_1F_1(-\lambda-\nu;\, 1-\lambda;\, z) \bigwedge \lambda \notin \mathbb{Z}$$

07.03.13.0004.02

$$W_z\left(L_\nu^\lambda(z),\, z^{-\lambda}\, {}_1F_1(-\nu-\lambda;\, 1-\lambda;\, z)\right) = -\frac{\Gamma(\lambda+\nu+1)}{\Gamma(\lambda)\,\Gamma(\nu+1)}\, e^z\, z^{-\lambda-1}$$

07.03.13.0011.01

$$w''(z) + \left(\frac{(\lambda+1)\, g'(z)}{g(z)} - g'(z) - \frac{g''(z)}{g'(z)}\right) w'(z) + \frac{\nu\, g'(z)^2}{g(z)}\, w(z) = 0 \,/;\, w(z) = c_1\, L_\nu^\lambda(g(z)) \,+ c_2\, U(-\nu,\, \lambda+1,\, g(z))$$

07.03.13.0012.01

$$W_z\left(L_\nu^\lambda(g(z)),\, U(-\nu,\, \lambda+1,\, g(z))\right) = \frac{\Gamma(\lambda+\nu+1)\sin(\pi\,\nu)}{\pi}\, g'(z)\, e^{g(z)}\, g(z)^{-\lambda-1}$$

07.03.13.0013.01

$$h(z)^2\, w''(z) \,+ h(z)^2 \left(\frac{(\lambda+1)\, g'(z)}{g(z)} - g'(z) - \frac{2\, h'(z)}{h(z)} - \frac{g''(z)}{g'(z)}\right) w'(z) \,+$$
$$\left(2\, h'(z)^2 + h(z)\left(g'(z)\, h'(z) + \frac{g''(z)\, h'(z)}{g'(z)} - h''(z)\right) - \frac{h(z)\, g'(z)\,(-\nu\, h(z)\, g'(z) + (\lambda+1)\, h'(z))}{g(z)}\right) w(z) \,= 0 \,/;$$
$$w(z) = c_1\, h(z)\, L_\nu^\lambda(g(z)) \,+ c_2\, h(z)\, U(-\nu,\, \lambda+1,\, g(z))$$

07.03.13.0014.01

$$W_z\left(h(z)\, L_\nu^\lambda(g(z)),\, h(z)\, U(-\nu,\, \lambda+1,\, g(z))\right) = \frac{\Gamma(\lambda+\nu+1)\sin(\pi\,\nu)}{\pi}\, h(z)^2\, g'(z)\, e^{g(z)}\, g(z)^{-\lambda-1}$$

07.03.13.0015.01

$$z^2\, w''(z) \,+ z\,(-2\, s + r\,(\lambda - a\, z^r) + 1)\, w'(z) + (a\, r\,(s + r\,\nu)\, z^r + s\,(s - \lambda\, r))\, w(z) = 0 \,/;\, w(z) = c_1\, z^s\, L_\nu^\lambda(a\, z^r) \,+ c_2\, z^s\, U(-\nu,\, \lambda+1,\, a\, z^r)$$

07.03.13.0016.01

$$W_z\left(z^s\, L_\nu^\lambda(a\, z^r),\, z^s\, U(-\nu,\, \lambda+1,\, a\, z^r)\right) = \frac{a\, r\, \Gamma(\lambda+\nu+1)\sin(\pi\,\nu)}{\pi}\, e^{a\, z^r}\, z^{r+2\, s-1}\, (a\, z^r)^{-\lambda-1}$$

07.03.13.0017.01

$$w''(z) - ((a\, r^z - \lambda)\log(r) + 2\log(s))\, w'(z) + \left(a\, \nu\log^2(r)\, r^z + \log^2(s) + (a\, r^z - \lambda)\log(r)\log(s)\right) w(z) = 0 \,/;$$
$$w(z) = c_1\, s^z\, L_\nu^\lambda(a\, r^z) \,+ c_2\, s^z\, U(-\nu,\, \lambda+1,\, a\, r^z)$$

07.03.13.0018.01

$$W_z\left(s^z\, L_\nu^\lambda(a\, r^z),\, s^z\, U(-\nu,\, \lambda+1,\, a\, r^z)\right) = \frac{a\, \Gamma(\lambda+\nu+1)\log(r)\sin(\pi\,\nu)\, e^{a\, r^z}\, r^z\, (a\, r^z)^{-\lambda-1}\, s^{2\, z}}{\pi}$$

Transformations

Transformations and argument simplifications

Argument involving basic arithmetic operations

07.03.16.0001.01

$$L_\nu^\lambda(-z) \equiv \csc(\pi(\nu+\lambda))\sin(\nu\pi)\, e^{-z}\, L_{-\nu-\lambda-1}^\lambda(z)$$

07.03.16.0002.01

$$L_\nu^\lambda(-z) \equiv e^{-z}\,(-1)^\lambda\, L_{-\nu-\lambda-1}^\lambda(z)\ /;\ \lambda\in\mathbb{Z}$$

Addition formulas

07.03.16.0003.01

$$L_n^\lambda(z_1+z_2) \equiv \sum_{k=0}^n L_k^\mu(z_1)\, L_{n-k}^{\lambda-\mu-1}(z_2)\ /;\ n\in\mathbb{N}$$

07.03.16.0004.01

$$L_n^\lambda(z_1+z_2) \equiv e^{z_1}\sum_{k=0}^\infty \frac{(-1)^k z_1^k}{k!}\, L_n^{k+\lambda}(z_2)\ /;\ n\in\mathbb{N}$$

Multiple arguments

07.03.16.0005.01

$$L_n^\lambda(z_1 z_2) \equiv \sum_{k=0}^n \binom{n+\lambda}{n-k} z_1^k\,(1-z_1)^{n-k}\, L_k^\lambda(z_2)\ /;\ n\in\mathbb{N}$$

Identities

Recurrence identities

Consecutive neighbors

With respect to ν

07.03.17.0001.01

$$L_\nu^\lambda(z) \equiv \frac{\lambda+2\nu+3-z}{\lambda+\nu+1}\, L_{\nu+1}^\lambda(z) - \frac{\nu+2}{\lambda+\nu+1}\, L_{\nu+2}^\lambda(z)$$

07.03.17.0002.01

$$L_\nu^\lambda(z) \equiv \frac{\lambda+2\nu-1-z}{\nu}\, L_{\nu-1}^\lambda(z) - \frac{\lambda+\nu-1}{\nu}\, L_{\nu-2}^\lambda(z)$$

With respect to λ

07.03.17.0010.01

$$L_\nu^\lambda(z) \equiv \frac{z+\lambda+1}{\lambda+\nu+1}\, L_\nu^{\lambda+1}(z) - \frac{z}{\lambda+\nu+1}\, L_\nu^{\lambda+2}(z)$$

07.03.17.0011.01

$$L_\nu^\lambda(z) = \frac{z+\lambda-1}{z} L_\nu^{\lambda-1}(z) - \frac{\lambda+\nu-1}{z} L_\nu^{\lambda-2}(z)$$

Distant neighbors

With respect to ν

07.03.17.0012.01

$$L_\nu^\lambda(z) = C_n(\nu, \lambda, z) L_{\nu+n}^\lambda(z) - \frac{n+\nu+1}{n+\lambda+\nu} C_{n-1}(\nu, \lambda, z) L_{\nu+n+1}^\lambda(z) /; C_0(\nu, \lambda, z) = 1 \bigwedge C_1(\nu, \lambda, z) = \frac{\lambda+2\nu+3-z}{\lambda+\nu+1} \bigwedge C_n(\nu, \lambda, z) = \frac{2n-z+\lambda+2\nu+1}{n+\lambda+\nu} C_{n-1}(\nu, \lambda, z) - \frac{n+\nu}{n+\lambda+\nu-1} C_{n-2}(\nu, \lambda, z) \bigwedge n \in \mathbb{N}^+$$

07.03.17.0013.01

$$L_\nu^\lambda(z) = C_n(\nu, \lambda, z) L_{\nu-n}^\lambda(z) - \frac{-n+\lambda+\nu}{-n+\nu+1} C_{n-1}(\nu, \lambda, z) L_{\nu-n-1}^\lambda(z) /; C_0(\nu, \lambda, z) = 1 \bigwedge C_1(\nu, \lambda, z) = \frac{\lambda+2\nu-1-z}{\nu} \bigwedge C_n(\nu, \lambda, z) = \frac{\lambda+2\nu-2n+1-z}{\nu-n+1} C_{n-1}(\nu, \lambda, z) - \frac{\lambda+\nu-n+1}{\nu-n+2} C_{n-2}(\nu, \lambda, z) \bigwedge n \in \mathbb{N}^+$$

With respect to λ

07.03.17.0014.01

$$L_\nu^\lambda(z) = C_n(\nu, \lambda, z) L_\nu^{\lambda+n}(z) - \frac{z}{\lambda+\nu+n} C_{n-1}(\nu, \lambda, z) L_\nu^{\lambda+n+1}(z) /; C_0(\nu, \lambda, z) = 1 \bigwedge C_1(\nu, \lambda, z) = \frac{z+\lambda+1}{\lambda+\nu+1} \bigwedge C_n(\nu, \lambda, z) = \frac{n+z+\lambda}{n+\lambda+\nu} C_{n-1}(\nu, \lambda, z) - \frac{z}{n+\lambda+\nu-1} C_{n-2}(\nu, \lambda, z) \bigwedge n \in \mathbb{N}^+$$

07.03.17.0015.01

$$L_\nu^\lambda(z) = C_n(\nu, \lambda, z) L_\nu^{\lambda-n}(z) - \frac{\lambda+\nu-n}{z} C_{n-1}(\nu, \lambda, z) L_\nu^{\lambda-n-1}(z) /; C_0(\nu, \lambda, z) = 1 \bigwedge C_1(\nu, \lambda, z) = \frac{z+\lambda-1}{z} \bigwedge C_n(\nu, \lambda, z) = \frac{-n+z+\lambda}{z} C_{n-1}(\nu, \lambda, z) - \frac{-n+\lambda+\nu+1}{z} C_{n-2}(\nu, \lambda, z) \bigwedge n \in \mathbb{N}^+$$

Functional identities

Relations between contiguous functions

Recurrence relations

07.03.17.0003.01

$$(\nu+\lambda) L_{\nu-1}^\lambda(z) + (\nu+1) L_{\nu+1}^\lambda(z) = (2\nu-z+\lambda+1) L_\nu^\lambda(z)$$

07.03.17.0004.01

$$L_\nu^\lambda(z) = \frac{(\nu+\lambda) L_{\nu-1}^\lambda(z) + (\nu+1) L_{\nu+1}^\lambda(z)}{2\nu-z+\lambda+1}$$

07.03.17.0005.01

$$L_\nu^\lambda(z) = \frac{(\nu+\lambda) L_{\nu-1}^\lambda(z) - z L_\nu^{\lambda+1}(z)}{\nu-z}$$

Normalized recurrence relation

07.03.17.0006.01

$$z\,p(n,z) == p(n+1,z) + n\,(n+\lambda)\,p(n-1,z) + (2\,n+\lambda+1)\,p(n,z)\;/;\; p(n,z) == (-1)^n\,n!\,L_n^{\lambda}(z) \bigwedge n \in \mathbb{N}$$

Additional relations between contiguous functions

07.03.17.0016.01

$$L_\nu^{\lambda}(z) = L_{\nu-1}^{\lambda}(z) + L_\nu^{\lambda-1}(z)$$

07.03.17.0007.01

$$L_\nu^{\lambda}(z) == \frac{1}{z}\left((\nu+\lambda)\,L_\nu^{\lambda-1}(z) - (\nu+1)\,L_{\nu+1}^{\lambda-1}(z)\right)$$

Relations of special kind

07.03.17.0008.01

$$L_\nu^{-m}(z) == \frac{z^m}{(-\nu)_m}\,L_{\nu-m}^{m}(z)\;/;\; m \in \mathbb{N}$$

07.03.17.0009.01

$$w(z) == \frac{1}{2}\,(-1)^n \int_0^\infty J_\lambda\left(\sqrt{t\,z}\right) w(t)\,dt\;/;\; w(z) == e^{-\frac{z}{2}}\,z^{\lambda/2}\,L_n^{\lambda}(z) \bigwedge n \in \mathbb{N}$$

Complex characteristics

Real part

07.03.19.0001.01

$$\operatorname{Re}\left(L_n^{\lambda}(x+i\,y)\right) == \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(-1)^j\,y^{2j}}{(2\,j)!}\,L_{n-2j}^{2j+\lambda}(x)\;/;\; x \in \mathbb{R} \bigwedge y \in \mathbb{R} \bigwedge \lambda \in \mathbb{R} \bigwedge n \in \mathbb{N}$$

Imaginary part

07.03.19.0002.01

$$\operatorname{Im}\left(L_n^{\lambda}(x+i\,y)\right) == \sum_{j=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \frac{(-1)^{j-1}\,y^{2j+1}}{(2\,j+1)!}\,L_{-2j+n-1}^{2j+\lambda+1}(x)\;/;\; x \in \mathbb{R} \bigwedge y \in \mathbb{R} \bigwedge \lambda \in \mathbb{R} \bigwedge n \in \mathbb{N}$$

Differentiation

Low-order differentiation

With respect to ν

07.03.20.0001.01

$$\frac{\partial L_\nu^{\lambda}(z)}{\partial \nu} == (\psi(\nu+\lambda+1) - \psi(\nu+1))\,L_\nu^{\lambda}(z) - \frac{z\,\Gamma(\nu+\lambda+1)}{\nu!\,\Gamma(\lambda+2)}\,F_{2\times 0\times 1}^{1\times 1\times 2}\left(\begin{matrix} 1-\nu;\; 1;\; 1,\,-\nu; \\ 2,\,\lambda+2;;\; 1-\nu; \end{matrix}\; z,\, z\right)$$

07.03.20.0002.01

$$\frac{\partial L_\nu^\lambda(z)}{\partial \nu} = (\pi \cot(\nu \pi) + \psi(\nu+\lambda+1)) L_\nu^\lambda(z) - \frac{\Gamma(\nu+\lambda+1)}{\nu!} \sum_{k=0}^{\infty} \frac{(-\nu)_k \psi(k-\nu) z^k}{k! \Gamma(k+\lambda+1)}$$

07.03.20.0003.01

$$\frac{\partial^2 L_\nu^\lambda(z)}{\partial \nu^2} = L_\nu^\lambda(z) \left(\psi(\nu+\lambda+1)^2 + 2\pi \cot(\nu\pi) \psi(\nu+\lambda+1) - \pi^2 + \psi^{(1)}(\nu+\lambda+1)\right) + \frac{\Gamma(\nu+\lambda+1)}{\nu!^2} \sum_{k=0}^{\infty} \frac{(-\nu)_k z^k}{k! \Gamma(k+\lambda+1)} \left(\Gamma(\nu+1) \psi(k-\nu) \psi(\nu+1) + \nu! \left(\psi(k-\nu)^2 - (\pi \cot(\nu\pi) + \psi(-\nu) + 2\psi(\nu+\lambda+1)) \psi(k-\nu) + \psi^{(1)}(k-\nu)\right)\right)$$

With respect to λ

07.03.20.0004.01

$$\frac{\partial L_\nu^\lambda(z)}{\partial \lambda} = (\psi(\nu+\lambda+1) - \psi(\lambda+1)) L_\nu^\lambda(z) + \frac{z\, \Gamma(\nu+\lambda+1)}{(\lambda+1)\Gamma(\lambda+2)\Gamma(\nu)} F_{2\times 0\times 1}^{1\times 1\times 2}\left(\begin{matrix} 1-\nu; 1; 1, \lambda+1; \\ 2, \lambda+2;; \lambda+2; \end{matrix} z, z\right)$$

07.03.20.0005.01

$$\frac{\partial L_\nu^\lambda(z)}{\partial \lambda} = \psi(\nu+\lambda+1) L_\nu^\lambda(z) - \frac{\Gamma(\nu+\lambda+1)}{\Gamma(\nu+1)} \sum_{k=0}^{\infty} \frac{(-\nu)_k \psi(k+\lambda+1) z^k}{k! \Gamma(k+\lambda+1)}$$

07.03.20.0006.01

$$\frac{\partial^2 L_\nu^\lambda(z)}{\partial \lambda^2} = \left(\psi(\nu+\lambda+1)^2 + \psi^{(1)}(\nu+\lambda+1)\right) L_\nu^\lambda(z) + \frac{\Gamma(\nu+\lambda+1)}{\Gamma(\nu+1)} \sum_{k=0}^{\infty} \frac{(-\nu)_k z^k}{k! \Gamma(k+\lambda+1)} \left(\psi(k+\lambda+1)^2 - 2\psi(\nu+\lambda+1)\psi(k+\lambda+1) - \psi^{(1)}(k+\lambda+1)\right)$$

07.03.20.0007.01

$$\frac{\partial^2 L_\nu^\lambda(z)}{\partial \lambda^2} = -\frac{2\sin(\nu\pi)}{\pi} \Gamma(\nu+\lambda+1)^3 \sum_{k=0}^{\infty} \frac{z^k}{k!}\, {}_4\tilde{F}_3(\nu+\lambda+1, \nu+\lambda+1, \nu+\lambda+1, \nu-k+1; \nu+\lambda+2, \nu+\lambda+2, \nu+\lambda+2; 1) \;/;\; \nu \notin \mathbb{N}$$

07.03.20.0008.01

$$\frac{\partial^2 L_n^\lambda(z)}{\partial \lambda^2} = \frac{1}{n!} \sum_{k=0}^{n} \frac{(-n)_k z^k}{k!} \sum_{j=1}^{n-k} (-1)^{j-k+n} S_{n-k}^{(j)} (j-1)\, j\, (k+\lambda+1)^{j-2} \;/;\; n \in \mathbb{N}$$

With respect to z

Forward shift operator:

07.03.20.0009.01

$$\frac{\partial L_\nu^\lambda(z)}{\partial z} = -L_{\nu-1}^{\lambda+1}(z)$$

07.03.20.0010.01

$$\frac{\partial^2 L_\nu^\lambda(z)}{\partial z^2} = L_{\nu-2}^{\lambda+2}(z)$$

Backward shift operator:

07.03.20.0011.01

$$z\,\frac{\partial L_{\nu}^{\lambda}(z)}{\partial z} + (\lambda - z)\,L_{\nu}^{\lambda}(z) = (\nu + 1)\,L_{\nu+1}^{\lambda-1}(z)$$

07.03.20.0012.01

$$\frac{\partial\left(e^{-z}\,z^{\lambda}\,L_{\nu}^{\lambda}(z)\right)}{\partial z} = (\nu + 1)\,e^{-z}\,z^{\lambda-1}\,L_{\nu+1}^{\lambda-1}(z)$$

Symbolic differentiation

With respect to ν

07.03.20.0013.02

$$\frac{\partial^m L_{\nu}^{\lambda}(z)}{\partial \nu^m} = \frac{m!}{\Gamma(-\lambda)} \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(j+\lambda+1)\,j!} \sum_{k=0}^{m} \frac{\Gamma(\nu+\lambda+1)^{k+1}}{(m-k)!}\,{}_{k+2}\tilde{F}_{k+1}(a_1, a_2, \ldots, a_{k+1}, \lambda+1; a_1+1, a_2+1, \ldots, a_{k+1}+1; 1) \sum_{p=0}^{j} \frac{(-1)^{j+k}\,\nu^{k-m+p}\,p!\,S_j^{(p)}}{(k-m+p)!} \;/;$$
$$a_1 = a_2 = \ldots = a_{k+1} = \nu+\lambda+1 \bigwedge k \in \mathbb{N}^+ \bigwedge m \in \mathbb{N} \bigwedge \lambda \notin \mathbb{N}$$

With respect to λ

07.03.20.0014.02

$$\frac{\partial^m L_{n}^{\lambda}(z)}{\partial \lambda^m} = \frac{(-1)^n\,z^n\,\delta_m}{n!} + \frac{1}{n!} \sum_{k=0}^{n} \frac{(-n)_k\,z^k}{k!} \sum_{j=1}^{n-k} (-1)^{j-k+n}\,S_{n-k}^{(j)}\,(j-m+1)_m\,(k+\lambda+1)^{j-m} \;/;\; m \in \mathbb{N} \land n \in \mathbb{N}$$

07.03.20.0015.02

$$\frac{\partial^m L_{\nu}^{\lambda}(z)}{\partial \lambda^m} = \frac{(-1)^{m-1}\,m!}{\pi} \sin(\nu\,\pi)\,\Gamma(\nu+\lambda+1)^{m+1} \sum_{k=0}^{\infty} \frac{z^k}{k!}\,{}_{m+2}\tilde{F}_{m+1}(a_1, a_2, \ldots, a_{m+1}, \nu-k+1; a_1+1, a_2+1, \ldots, a_{m+1}+1; 1) \;/;$$
$$a_1 = a_2 = \ldots = a_{m+1} = \nu+\lambda+1 \bigwedge m \in \mathbb{N} \bigwedge \nu \notin \mathbb{N}$$

With respect to z

07.03.20.0016.02

$$\frac{\partial^m L_{\nu}^{\lambda}(z)}{\partial z^m} = (-1)^m\,L_{\nu-m}^{m+\lambda}(z) \;/;\; m \in \mathbb{N}$$

07.03.20.0017.02

$$\frac{\partial^m L_{\nu}^{\lambda}(z)}{\partial z^m} = \frac{\Gamma(\nu+\lambda+1)\,z^{-m}}{\Gamma(\nu+1)}\,{}_2\tilde{F}_2(1, -\nu; 1-m, \lambda+1; z) \;/;\; m \in \mathbb{N}$$

Fractional integro-differentiation

With respect to z

07.03.20.0018.01

$$\frac{\partial^\alpha L_{\nu}^{\lambda}(z)}{\partial z^\alpha} = \frac{\Gamma(\nu+\lambda+1)\,z^{-\alpha}}{\Gamma(\nu+1)}\,{}_2\tilde{F}_2(1, -\nu; 1-\alpha, \lambda+1; z)$$

Integration

Indefinite integration

Involving only one direct function

07.03.21.0001.01

$$\int L_\nu^\lambda(z)\, dz = -L_{\nu+1}^{\lambda-1}(z)$$

Involving one direct function and elementary functions

Involving power function

07.03.21.0002.01

$$\int z^{\alpha-1} L_\nu^\lambda(c\, z)\, dz = \frac{z^\alpha\, \Gamma(\alpha)\, \Gamma(\lambda+\nu+1)}{\Gamma(\nu+1)}\, {}_2\tilde{F}_2(-\nu, \alpha; \lambda+1, \alpha+1; c\, z)$$

07.03.21.0003.01

$$\int z^{\alpha-1} L_\nu^\lambda(z)\, dz = \frac{z^\alpha\, \Gamma(\nu+\lambda+1)}{\alpha\, \Gamma(\nu+1)\, \Gamma(\lambda+1)}\, {}_2F_2(-\nu, \alpha; \alpha+1, \lambda+1; z)$$

07.03.21.0004.01

$$\int z^{\nu-2} L_\nu^\lambda(z)\, dz = \frac{z^{\nu-1}\, \Gamma(\lambda+\nu+1)}{\nu^2-\nu}\, {}_2\tilde{F}_2(-\nu, \nu-1; \lambda+1, \nu; z)$$

Involving exponential function

07.03.21.0005.01

$$\int e^{-z} L_\nu^\lambda(z)\, dz = -\frac{\Gamma(\nu+\lambda)}{\Gamma(\nu+1)}\, {}_1\tilde{F}_1(\nu+\lambda; \lambda; -z)$$

Involving exponential function and a power function

07.03.21.0006.01

$$\int z^{\alpha-1} e^{-p\, z} L_\nu^\lambda(z)\, dz = -\frac{\Gamma(\lambda+\nu+1)\, z^\alpha}{\Gamma(\nu+1)\, (p\, z)^\alpha} \sum_{k=0}^{\infty} \frac{(-\nu)_k\, \Gamma(k+\alpha,\, p\, z)}{\Gamma(k+\lambda+1)\, k!\, p^k}$$

07.03.21.0007.01

$$\int z^{\alpha-1} e^{-c\, z} L_\nu^\lambda(c\, z)\, dz = \frac{z^\alpha\, \Gamma(\alpha)\, \Gamma(\nu+\lambda+1)}{\Gamma(\nu+1)}\, {}_2\tilde{F}_2(\nu+\lambda+1, \alpha; \lambda+1, \alpha+1; -c\, z)$$

07.03.21.0008.01

$$\int z^\lambda\, e^{-z} L_\nu^\lambda(z)\, dz = \frac{z^{\lambda+1}\, \Gamma(\lambda+\nu+1)}{\Gamma(\nu+1)}\, {}_1\tilde{F}_1(\lambda+\nu+1; \lambda+2; -z)$$

07.03.21.0009.01

$$\int z^{\lambda+\nu-1}\, e^{-z} L_\nu^\lambda(z)\, dz = \frac{z^{\lambda+\nu}\, \Gamma(\lambda+\nu)}{\Gamma(\nu+1)}\, {}_1\tilde{F}_1(\lambda+\nu; \lambda+1; -z)$$

Definite integration

Involving the direct function

Orthogonality:

07.03.21.0010.01

$$\int_0^\infty t^\lambda e^{-t} L_m^\lambda(t) L_n^\lambda(t) \, dt = \frac{\Gamma(\lambda+n+1)\,\delta_{n,m}}{n!} \;/;\; \operatorname{Re}(\lambda) > -1 \land n \in \mathbb{N} \land m \in \mathbb{N}$$

Summation

Finite summation

07.03.23.0001.01

$$\sum_{k=0}^{n} \frac{(\lambda-\beta)_{n-k} L_k^\beta(z)}{(n-k)!} = L_n^\lambda(z) \;/;\; n \in \mathbb{N}$$

07.03.23.0002.01

$$\sum_{k=0}^{n} \binom{k-\beta+\lambda-1}{k} L_{n-k}^\beta(z) = L_n^\lambda(z) \;/;\; n \in \mathbb{N}$$

07.03.23.0004.01

$$\sum_{k=0}^{n} \binom{n+\lambda}{n-k} L_k^\lambda(z)\, w^k (1-w)^{n-k} = L_n^\lambda(z\,w) \;/;\; n \in \mathbb{N}$$

07.03.23.0005.01

$$\sum_{k=0}^{n} L_k^\mu(z_1)\, L_{n-k}^\lambda(z_2) = L_n^{\lambda+\mu+1}(z_1+z_2) \;/;\; n \in \mathbb{N}$$

Infinite summation

07.03.23.0003.01

$$\sum_{k=0}^{\infty} \frac{L_n^{k+\lambda}(z)\, w^k}{k!} = e^w L_n^\lambda(z-w) \;/;\; n \in \mathbb{N}$$

07.03.23.0006.01

$$\sum_{n=0}^{\infty} L_n^\lambda(z)\, w^n = (1-w)^{-\lambda-1} e^{\frac{w z}{w-1}} \;/;\; |w| < 1$$

07.03.23.0007.01

$$\sum_{n=0}^{\infty} \frac{1}{(\lambda+1)_n} L_n^\lambda(z)\, w^n = e^w \, {}_0F_1(;\lambda+1;-z\,w) \;/;\; |w| < 1$$

07.03.23.0008.01

$$\sum_{n=0}^{\infty} \frac{(c)_n}{(\lambda+1)_n} L_n^\lambda(z)\, w^n = (1-w)^{-c} \, {}_1F_1\!\left(c;\lambda+1;\frac{z\,w}{w-1}\right) \;/;\; |w| < 1$$

07.03.23.0009.01

$$\sum_{n=0}^{\infty} \frac{n!\, L_n^\lambda(x)\, L_n^\lambda(y)}{\Gamma(n+\lambda+1)} = (x\,y)^{-\frac{\lambda}{2}} e^{\frac{x+y}{2}} \delta(x-y) \;/;\; \operatorname{Re}(\lambda) > -1 \land x > 0 \land y > 0$$

Operations

Limit operation

07.03.25.0001.01

$$\lim_{\nu\to\infty} \frac{1}{\nu^\lambda} L_\nu^\lambda\left(\frac{z^2}{4\nu}\right) = 2^\lambda z^{-\lambda} J_\lambda(z)$$

07.03.25.0002.01

$$\lim_{\nu\to\infty} \frac{1}{\nu^\lambda} L_\nu^\lambda\left(-\frac{z^2}{4\nu}\right) = 2^\lambda z^{-\lambda} I_\lambda(z)$$

07.03.25.0003.01

$$\lim_{\lambda\to\infty} 2^{\nu/2} \Gamma(\nu+1) \lambda^{-\frac{\nu}{2}} L_\nu^\lambda\left(\lambda - \sqrt{2\lambda}\, z\right) = H_\nu(z)$$

Orthogonality, completeness, and Fourier expansions

The set of functions $L_n^\lambda(x)$, $n = 0, 1, \ldots$, forms a complete, orthogonal (with weight $\frac{n!}{\Gamma(n+\lambda+1)} x^\lambda e^{-x}$) system on the interval $(0, \infty)$.

07.03.25.0004.01

$$\sum_{n=0}^{\infty} \left(\sqrt{\frac{n!}{\Gamma(n+\lambda+1)}}\, x^{\lambda/2} e^{-\frac{x}{2}} L_n^\lambda(x)\right)\left(\sqrt{\frac{n!}{\Gamma(n+\lambda+1)}}\, y^{\lambda/2} e^{-\frac{y}{2}} L_n^\lambda(y)\right) = \delta(x-y) /; \operatorname{Re}(\lambda) > -1 \land x > 0 \land y > 0$$

07.03.25.0005.01

$$\int_0^\infty \left(\sqrt{\frac{m!}{\Gamma(m+\lambda+1)}}\, t^{\lambda/2} e^{-\frac{t}{2}} L_m^\lambda(t)\right)\left(\sqrt{\frac{n!}{\Gamma(n+\lambda+1)}}\, t^{\lambda/2} e^{-\frac{t}{2}} L_n^\lambda(t)\right) dt = \delta_{n,m} /; \operatorname{Re}(\lambda) > -1$$

Any sufficiently smooth function $f(x)$ can be expanded in the system $\{L_n^\lambda(x)\}_{n=0,1,\ldots}$ as a generalized Fourier series, with its sum converging to $f(x)$ almost everywhere.

07.03.25.0006.01

$$f(x) = \sum_{n=0}^{\infty} c_n \psi_n(x) /; c_n = \int_0^\infty \psi_n(t) f(t)\, dt \bigwedge \psi_n(x) = \sqrt{\frac{n!}{\Gamma(n+\lambda+1)}}\, x^{\lambda/2} e^{-\frac{x}{2}} L_n^\lambda(x) /; \operatorname{Re}(\lambda) > -1 \land x > 0$$

Representations through more general functions

Through hypergeometric functions

Involving ${}_1\tilde{F}_1$

07.03.26.0001.01

$$L_\nu^\lambda(z) = \frac{\Gamma(\nu+\lambda+1)}{\Gamma(\nu+1)}\, {}_1\tilde{F}_1(-\nu; \lambda+1; z)$$

Involving ${}_1F_1$

07.03.26.0002.01

$$L_{\nu}^{\lambda}(z) = \frac{(\lambda+1)_{\nu}}{\Gamma(\nu+1)}\, {}_1F_1(-\nu;\lambda+1;z) \,/;\, -\lambda \notin \mathbb{N}^+$$

Through Meijer *G*

Classical cases for the direct function itself

07.03.26.0003.01

$$L_{\nu}^{\lambda}(z) = \Gamma(\nu+\lambda+1)\, G_{1,2}^{1,0}\!\left(z \,\middle|\, \begin{matrix} \nu+1 \\ 0, -\lambda \end{matrix}\right) /;\, \nu \notin \mathbb{Z}$$

07.03.26.0006.01

$$L_{\nu}^{\lambda}(z) + L_{\nu}^{\lambda}(-z) = -2^{-\lambda-\nu}\sqrt{\pi}\,\Gamma(\lambda+\nu+1)\sin(\pi\nu)\, G_{3,5}^{1,2}\!\left(\frac{z}{2}, \frac{1}{2} \,\middle|\, \begin{matrix} \frac{\nu+1}{2}, \frac{\nu}{2}+1, \frac{1}{2} \\ 0, \frac{1}{2}, \frac{1}{2}, -\frac{\lambda}{2}, \frac{1-\lambda}{2} \end{matrix}\right)$$

07.03.26.0007.01

$$L_{\nu}^{\lambda}(z) - L_{\nu}^{\lambda}(-z) = -2^{-\lambda-\nu}\sqrt{\pi}\,\Gamma(\lambda+\nu+1)\sin(\pi\nu)\, G_{3,5}^{1,2}\!\left(\frac{z}{2}, \frac{1}{2} \,\middle|\, \begin{matrix} \frac{\nu+1}{2}, \frac{\nu+2}{2}, 1 \\ \frac{1}{2}, 0, 1, -\frac{\lambda}{2}, \frac{1-\lambda}{2} \end{matrix}\right)$$

Classical cases involving exp

07.03.26.0004.01

$$e^{-z} L_{\nu}^{\lambda}(z) = \frac{1}{\Gamma(\nu+1)}\, G_{1,2}^{1,1}\!\left(z \,\middle|\, \begin{matrix} -\nu-\lambda \\ 0, -\lambda \end{matrix}\right)$$

Classical cases involving exp and cosh

07.03.26.0008.01

$$e^{-\frac{z}{2}}\cosh\!\left(\frac{z}{2}\right) L_{\nu}^{\lambda}(z) = \frac{1}{2}\Gamma(\lambda+\nu+1)\, G_{1,2}^{1,0}\!\left(z \,\middle|\, \begin{matrix} \nu+1 \\ 0, -\lambda \end{matrix}\right) + \frac{1}{2\,\Gamma(\nu+1)}\, G_{1,2}^{1,1}\!\left(z \,\middle|\, \begin{matrix} -\lambda-\nu \\ 0, -\lambda \end{matrix}\right)$$

Classical cases involving exp and sinh

07.03.26.0009.01

$$e^{-\frac{z}{2}}\sinh\!\left(\frac{z}{2}\right) L_{\nu}^{\lambda}(z) = \frac{1}{2}\Gamma(\lambda+\nu+1)\, G_{1,2}^{1,0}\!\left(z \,\middle|\, \begin{matrix} \nu+1 \\ 0, -\lambda \end{matrix}\right) - \frac{1}{2\,\Gamma(\nu+1)}\, G_{1,2}^{1,1}\!\left(z \,\middle|\, \begin{matrix} -\lambda-\nu \\ 0, -\lambda \end{matrix}\right)$$

Classical cases for products of Laguerre *L*

07.03.26.0010.01

$$L_{\nu}^{\lambda}(-z)\, L_{\nu}^{\lambda}(z) = -\frac{2^{-\lambda}\,\Gamma(\lambda+\nu+1)\sin(\pi\nu)}{\sqrt{\pi}\,\Gamma(\nu+1)}\, G_{2,4}^{1,2}\!\left(-\frac{z^2}{4} \,\middle|\, \begin{matrix} \nu+1, -\lambda-\nu \\ 0, -\frac{\lambda}{2}, \frac{1-\lambda}{2}, -\lambda \end{matrix}\right)$$

Classical cases involving exp and products of Laguerre *L*

07.03.26.0011.01

$$e^{-z} L_{-\lambda-\nu-1}^{\lambda}(z)\, L_{\nu}^{\lambda}(z) = -\frac{2^{-\lambda}\sqrt{\pi}\,\Gamma(-\nu)\sin(\pi\nu)}{\Gamma(-\lambda-\nu)}\, G_{3,5}^{1,2}\!\left(\frac{z^2}{4} \,\middle|\, \begin{matrix} \nu+1, -\lambda-\nu, \frac{1}{2} \\ 0, -\frac{\lambda}{2}, \frac{1-\lambda}{2}, -\lambda, \frac{1}{2} \end{matrix}\right)$$

Classical cases involving ${}_1F_1$

07.03.26.0012.01

$${}_1F_1(-\nu;\,\lambda+1;\,-z)\,L_\nu^\lambda(z) = -\frac{2^{-\lambda}\,\Gamma(\lambda+1)\sin(\pi\,\nu)}{\sqrt{\pi}}\,G_{2,4}^{1,2}\!\left(-\frac{z^2}{4}\,\middle|\,\begin{matrix}\nu+1,\,-\lambda-\nu\\ 0,\,-\frac{\lambda}{2},\,\frac{1-\lambda}{2},\,-\lambda\end{matrix}\right)$$

07.03.26.0013.01

$${}_1F_1(-\nu;\,\lambda+1;\,-z)\,L_\nu^\lambda(z) = -2^{-\lambda}\sqrt{\pi}\;\Gamma(\lambda+1)\sin(\pi\,\nu)\,G_{3,5}^{1,2}\!\left(\frac{z^2}{4}\,\middle|\,\begin{matrix}\nu+1,\,-\lambda-\nu,\,\frac{1}{2}\\ 0,\,-\frac{\lambda}{2},\,\frac{1-\lambda}{2},\,-\lambda,\,\frac{1}{2}\end{matrix}\right)$$

Classical cases involving exp and ${}_1F_1$

07.03.26.0014.01

$$e^{-z}\,{}_1F_1(\lambda+\nu+1;\,\lambda+1;\,z)\,L_\nu^\lambda(z) = -2^{-\lambda}\sqrt{\pi}\;\Gamma(\lambda+1)\sin(\pi\,\nu)\,G_{3,5}^{1,2}\!\left(\frac{z^2}{4}\,\middle|\,\begin{matrix}\nu+1,\,-\lambda-\nu,\,\frac{1}{2}\\ 0,\,-\frac{\lambda}{2},\,\frac{1}{2}-\frac{\lambda}{2},\,-\lambda,\,\frac{1}{2}\end{matrix}\right)$$

Classical cases involving ${}_1\tilde{F}_1$

07.03.26.0015.01

$${}_1\tilde{F}_1(-\nu;\,\lambda+1;\,-z)\,L_\nu^\lambda(z) = -\frac{2^{-\lambda}\sin(\pi\,\nu)}{\sqrt{\pi}}\,G_{2,4}^{1,2}\!\left(-\frac{z^2}{4}\,\middle|\,\begin{matrix}\nu+1,\,-\lambda-\nu\\ 0,\,-\frac{\lambda}{2},\,\frac{1-\lambda}{2},\,-\lambda\end{matrix}\right)$$

07.03.26.0016.01

$${}_1\tilde{F}_1(-\nu;\,\lambda+1;\,-z)\,L_\nu^\lambda(z) = -2^{-\lambda}\sqrt{\pi}\;\sin(\pi\,\nu)\,G_{3,5}^{1,2}\!\left(\frac{z^2}{4}\,\middle|\,\begin{matrix}\nu+1,\,-\lambda-\nu,\,\frac{1}{2}\\ 0,\,-\frac{\lambda}{2},\,\frac{1-\lambda}{2},\,-\lambda,\,\frac{1}{2}\end{matrix}\right)$$

Classical cases involving exp and ${}_1\tilde{F}_1$

07.03.26.0017.01

$$e^{-z}\,{}_1\tilde{F}_1(\lambda+\nu+1;\,\lambda+1;\,z)\,L_\nu^\lambda(z) = -2^{-\lambda}\sqrt{\pi}\;\sin(\pi\,\nu)\,G_{3,5}^{1,2}\!\left(\frac{z^2}{4}\,\middle|\,\begin{matrix}\nu+1,\,-\lambda-\nu,\,\frac{1}{2}\\ 0,\,-\frac{\lambda}{2},\,\frac{1}{2}-\frac{\lambda}{2},\,-\lambda,\,\frac{1}{2}\end{matrix}\right)$$

Classical cases involving hypergeometric U

07.03.26.0018.01

$$U(-\nu,\,\lambda+1,\,-z)\,L_\nu^\lambda(z) = -\frac{2^{-\lambda-1}\,\Gamma(\lambda+\nu+1)\sin(\pi\,\nu)}{\pi^{3/2}}\,G_{2,4}^{3,1}\!\left(-\frac{z^2}{4}\,\middle|\,\begin{matrix}\nu+1,\,-\lambda-\nu\\ 0,\,-\frac{\lambda}{2},\,\frac{1}{2}-\frac{\lambda}{2},\,-\lambda\end{matrix}\right)/;\ \mathrm{Re}(z)<0\bigvee\arg(z)=-\frac{\pi}{2}$$

Classical cases involving exp and hypergeometric U

07.03.26.0019.01

$$e^{-z}\,U(\lambda+\nu+1,\,\lambda+1,\,z)\,L_\nu^\lambda(z) = \frac{2^{-\lambda-1}}{\sqrt{\pi}\;\Gamma(\nu+1)}\,G_{2,4}^{3,1}\!\left(\frac{z^2}{4}\,\middle|\,\begin{matrix}-\lambda-\nu,\,\nu+1\\ 0,\,-\frac{\lambda}{2},\,\frac{1}{2}-\frac{\lambda}{2},\,-\lambda\end{matrix}\right)/;\ -\frac{\pi}{2}<\arg(z)\leq\frac{\pi}{2}$$

Generalized cases for products of Laguerre L

07.03.26.0020.01

$$L_\nu^\lambda(-z)\,L_\nu^\lambda(z) = -\frac{2^{-\lambda}\,\Gamma(\lambda+\nu+1)\sin(\pi\,\nu)}{\sqrt{\pi}\;\Gamma(\nu+1)}\,G_{2,4}^{1,2}\!\left(\frac{i\,z}{2},\,\frac{1}{2}\,\middle|\,\begin{matrix}\nu+1,\,-\lambda-\nu\\ 0,\,-\frac{\lambda}{2},\,\frac{1-\lambda}{2},\,-\lambda\end{matrix}\right)$$

Generalized cases involving exp and products of Laguerre L

07.03.26.0021.01

$$e^{-z} L_{-\lambda-\nu-1}^{\lambda}(z) L_{\nu}^{\lambda}(z) = -\frac{2^{-\lambda} \sqrt{\pi}\ \Gamma(-\nu) \sin(\pi \nu)}{\Gamma(-\lambda-\nu)} G_{3,5}^{1,2}\left(\frac{z}{2}, \frac{1}{2} \middle| \begin{array}{c} \nu+1, -\lambda-\nu, \frac{1}{2} \\ 0, -\frac{\lambda}{2}, \frac{1-\lambda}{2}, -\lambda, \frac{1}{2} \end{array}\right)$$

Generalized cases involving ${}_1F_1$

07.03.26.0022.01

$$ {}_1F_1(-\nu; \lambda+1; -z) L_{\nu}^{\lambda}(z) = -\frac{2^{-\lambda} \Gamma(\lambda+1) \sin(\pi \nu)}{\sqrt{\pi}} G_{2,4}^{1,2}\left(\frac{i z}{2}, \frac{1}{2} \middle| \begin{array}{c} \nu+1, -\lambda-\nu \\ 0, -\frac{\lambda}{2}, \frac{1-\lambda}{2}, -\lambda \end{array}\right)$$

07.03.26.0023.01

$$ {}_1F_1(-\nu; \lambda+1; -z) L_{\nu}^{\lambda}(z) = -2^{-\lambda} \sqrt{\pi}\ \Gamma(\lambda+1) \sin(\pi \nu) G_{3,5}^{1,2}\left(\frac{z}{2}, \frac{1}{2} \middle| \begin{array}{c} \nu+1, -\lambda-\nu, \frac{1}{2} \\ 0, -\frac{\lambda}{2}, \frac{1-\lambda}{2}, -\lambda, \frac{1}{2} \end{array}\right)$$

Generalized cases involving exp and ${}_1F_1$

07.03.26.0024.01

$$e^{-z} {}_1F_1(\lambda+\nu+1; \lambda+1; z) L_{\nu}^{\lambda}(z) = -2^{-\lambda} \sqrt{\pi}\ \Gamma(\lambda+1) \sin(\pi \nu) G_{3,5}^{1,2}\left(\frac{z}{2}, \frac{1}{2} \middle| \begin{array}{c} \nu+1, -\lambda-\nu, \frac{1}{2} \\ 0, -\frac{\lambda}{2}, \frac{1}{2}-\frac{\lambda}{2}, -\lambda, \frac{1}{2} \end{array}\right)$$

Generalized cases involving ${}_1\tilde{F}_1$

07.03.26.0025.01

$$ {}_1\tilde{F}_1(-\nu; \lambda+1; -z) L_{\nu}^{\lambda}(z) = -\frac{2^{-\lambda} \sin(\pi \nu)}{\sqrt{\pi}} G_{2,4}^{1,2}\left(\frac{i z}{2}, \frac{1}{2} \middle| \begin{array}{c} \nu+1, -\lambda-\nu \\ 0, -\frac{\lambda}{2}, \frac{1-\lambda}{2}, -\lambda \end{array}\right)$$

07.03.26.0026.01

$$ {}_1\tilde{F}_1(-\nu; \lambda+1; -z) L_{\nu}^{\lambda}(z) = -2^{-\lambda} \sqrt{\pi}\ \sin(\pi \nu) G_{3,5}^{1,2}\left(\frac{z}{2}, \frac{1}{2} \middle| \begin{array}{c} \nu+1, -\lambda-\nu, \frac{1}{2} \\ 0, -\frac{\lambda}{2}, \frac{1-\lambda}{2}, -\lambda, \frac{1}{2} \end{array}\right)$$

Generalized cases involving exp and ${}_1\tilde{F}_1$

07.03.26.0027.01

$$e^{-z} {}_1\tilde{F}_1(\lambda+\nu+1; \lambda+1; z) L_{\nu}^{\lambda}(z) = -2^{-\lambda} \sqrt{\pi}\ \sin(\pi \nu) G_{3,5}^{1,2}\left(\frac{z}{2}, \frac{1}{2} \middle| \begin{array}{c} \nu+1, -\lambda-\nu, \frac{1}{2} \\ 0, -\frac{\lambda}{2}, \frac{1}{2}-\frac{\lambda}{2}, -\lambda, \frac{1}{2} \end{array}\right)$$

Generalized cases involving hypergeometric U

07.03.26.0028.01

$$U(-\nu, \lambda+1, -z) L_{\nu}^{\lambda}(z) = -\frac{2^{-\lambda-1} \Gamma(\lambda+\nu+1) \sin(\pi \nu)}{\pi^{3/2}} G_{2,4}^{3,1}\left(-\frac{z}{2}, \frac{1}{2} \middle| \begin{array}{c} \nu+1, -\lambda-\nu \\ 0, -\frac{\lambda}{2}, \frac{1}{2}-\frac{\lambda}{2}, -\lambda \end{array}\right)$$

Generalized cases involving exp and hypergeometric U

07.03.26.0029.01

$$e^{-z} U(\lambda+\nu+1, \lambda+1, z) L_{\nu}^{\lambda}(z) = \frac{2^{-\lambda-1}}{\sqrt{\pi}\ \Gamma(\nu+1)} G_{2,4}^{3,1}\left(\frac{z}{2}, \frac{1}{2} \middle| \begin{array}{c} -\lambda-\nu, \nu+1 \\ 0, -\frac{\lambda}{2}, \frac{1}{2}-\frac{\lambda}{2}, -\lambda \end{array}\right)$$

Through other functions

Involving some hypergeometric-type functions

07.03.26.0005.01

$$L_\nu^\lambda(z) = \lim_{b\to\infty} P_\nu^{(\lambda,b)}\left(1 - \frac{2\,z}{b}\right)$$

History

– E. N. Laguerre (1879)

– N. J. Sonin (1880)

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