LegendreP2General

Notations

Traditional name

Associated Legendre function of the first kind of type 2

Traditional notation

$P_\nu^\mu(z)$

Mathematica StandardForm notation

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LegendreP[ν, μ, 2, z]
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Primary definition

07.08.02.0001.01

$$P_\nu^\mu(z) = \frac{(1+z)^{\mu/2}}{(1-z)^{\mu/2}}\, {}_2\tilde{F}_1\left(-\nu, \nu+1; 1-\mu; \frac{1-z}{2}\right)$$

The function LegendreP[ν,μ,2,z] is the analytic continuation of the function LegendreP[ν, μ, z] from the unit disk $|z| < 1$ to the cut complex plane. Inside the unit disk they coincide by definition. Outside they also coincide because LegendreP[ν, μ, z] is analytically continued and cut in the same manner.

Specific values

Specialized values

For fixed ν, μ

07.08.03.0001.01

$$P_\nu^\mu(0) = \frac{2^\mu \sqrt{\pi}}{\Gamma\left(\frac{1-\mu-\nu}{2}\right)\Gamma\left(1-\frac{\mu-\nu}{2}\right)}$$

07.08.03.0002.01

$$P_\nu^\mu(1) = 0 \;/;\; \mathrm{Re}(\mu) < 0 \lor \mu \in \mathbb{N}^+$$

07.08.03.0003.01

$$P_\nu^\mu(1) = \tilde{\infty} \;/;\; \mathrm{Re}(\mu) > 0 \land \mu \notin \mathbb{N}^+$$

07.08.03.0004.01

$$P_\nu^\mu(1) = ¿ \;/;\; \mathrm{Re}(\mu) = 0 \land \mu \neq 0$$

07.08.03.0005.01

$$P_\nu^\mu(-1) = \tilde{\infty} /; \nu \notin \mathbb{Z}$$

For fixed ν, z

07.08.03.0006.01

$$P_\nu^0(z) = P_\nu(z)$$

07.08.03.0007.01

$$P_\nu^{-\nu-1}(z) = \frac{2^{\nu+1}}{\Gamma(\nu+1)} \left(1-z^2\right)^{-\frac{\nu+1}{2}} \mathrm{B}_{\frac{1-z}{2}}(\nu+1, \nu+1)$$

07.08.03.0008.01

$$P_\nu^{-\nu}(z) = \frac{2^{-\nu}}{\Gamma(\nu+1)} \left(1-z^2\right)^{\nu/2}$$

07.08.03.0009.01

$$P_\nu^{\nu}(z) = \frac{2^{-\nu}}{\Gamma(-\nu)} \left(1-z^2\right)^{\nu/2} \mathrm{B}_{\frac{1-z}{2}}(-\nu, -\nu)$$

07.08.03.0010.01

$$P_\nu^{\nu+1}(z) = \frac{2^{\nu+1}}{\Gamma(-\nu)} \left(1-z^2\right)^{-\frac{\nu+1}{2}}$$

07.08.03.0093.01

$$P_\nu^{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi}} \frac{\cos\left(\left(\nu+\frac{1}{2}\right)\cos^{-1}(z)\right)}{\sqrt[4]{1-z^2}}$$

For fixed μ, z

07.08.03.0011.01

$$P_0^\mu(z) = \frac{1}{\Gamma(1-\mu)} \frac{(1+z)^{\mu/2}}{(1-z)^{\mu/2}}$$

07.08.03.0012.01

$$P_1^\mu(z) = \frac{z-\mu}{\Gamma(2-\mu)} \frac{(1+z)^{\mu/2}}{(1-z)^{\mu/2}}$$

07.08.03.0013.01

$$P_2^\mu(z) = \frac{3z^2 - 3\mu z + \mu^2 - 1}{\Gamma(1-\mu)} \frac{(1+z)^{\mu/2}}{(1-z)^{\mu/2}}$$

07.08.03.0014.01

$$P_3^\mu(z) = \frac{15z^3 - 15\mu z^2 + 3\left(2\mu^2-3\right) z - \mu^3 + 4\mu}{\Gamma(4-\mu)} \frac{(1+z)^{\mu/2}}{(1-z)^{\mu/2}}$$

07.08.03.0015.01

$$P_4^\mu(z) = \frac{9 + 105 z^4 - 105 z^3 \mu - 10\mu^2 + \mu^4 + 45 z^2 \left(\mu^2-2\right) + z\left(55\mu - 10\mu^3\right)}{\Gamma(5-\mu)} \frac{(1+z)^{\mu/2}}{(1-z)^{\mu/2}}$$

07.08.03.0016.01

$$P_5^\mu(z) = \frac{1}{\Gamma(6-\mu)} \left(-945 z^5 + 64 \mu + 945 z^4 \mu - 20 \mu^3 + \mu^5 + 105 z^2 \mu \left(\mu^2 - 7\right) - 210 z^3 \left(2 \mu^2 - 5\right) - 15 z \left(15 - 13 \mu^2 + \mu^4\right)\right) \frac{(1+z)^{\mu/2}}{(1-z)^{\mu/2}}$$

07.08.03.0017.01

$$P_6^\mu(z) = \frac{1}{\Gamma(7-\mu)} \left(10\,395 z^6 - 10\,395 \mu z^5 + 4725 \left(\mu^2 - 3\right) z^4 - 630 \mu \left(2 \mu^2 - 17\right) z^3 + 105 \left(2 \mu^4 - 32 \mu^2 + 45\right) z^2 - 21 \mu \left(\mu^4 - 25 \mu^2 + 99\right) z + (\mu - 5)(\mu - 3)(\mu - 1)(\mu + 1)(\mu + 3)(\mu + 5)\right) \frac{(1+z)^{\mu/2}}{(1-z)^{\mu/2}}$$

07.08.03.0018.01

$$P_7^\mu(z) = \frac{1}{\Gamma(8-\mu)} \left(135\,135 z^7 - 135\,135 \mu z^6 + 31\,185 \left(2 \mu^2 - 7\right) z^5 - 17\,325 \mu \left(\mu^2 - 10\right) z^4 + 1575 \left(2 \mu^4 - 38 \mu^2 + 63\right) z^3 - 189 \mu \left(2 \mu^4 - 60 \mu^2 + 283\right) z^2 + 7 \left(4 \mu^6 - 170 \mu^4 + 1516 \mu^2 - 1575\right) z - \mu \left(\mu^6 - 56 \mu^4 + 784 \mu^2 - 2304\right)\right) \frac{(1+z)^{\mu/2}}{(1-z)^{\mu/2}}$$

07.08.03.0019.01

$$P_8^\mu(z) = \frac{1}{\Gamma(9-\mu)} \left(2\,027\,025 z^8 - 2\,027\,025 \mu z^7 + 945\,945 \left(\mu^2 - 4\right) z^6 - 135\,135 \mu \left(2 \mu^2 - 23\right) z^5 + 51\,975 \left(\mu^4 - 22 \mu^2 + 42\right) z^4 - 3465 \mu \left(2 \mu^4 - 70 \mu^2 + 383\right) z^3 + 315 \left(2 \mu^6 - 100 \mu^4 + 1043 \mu^2 - 1260\right) z^2 - 9 \mu \left(4 \mu^6 - 266 \mu^4 + 4396 \mu^2 - 15\,159\right) z + \mu^8 - 84 \mu^6 + 1974 \mu^4 - 12\,916 \mu^2 + 11\,025\right) \frac{(1+z)^{\mu/2}}{(1-z)^{\mu/2}}$$

07.08.03.0020.01

$$P_9^\mu(z) = \frac{1}{\Gamma(10-\mu)} \left(34\,459\,425 z^9 - 34\,459\,425 \mu z^8 + 8\,108\,100 \left(2 \mu^2 - 9\right) z^7 - 4\,729\,725 \mu \left(\mu^2 - 13\right) z^6 + 945\,945 \left(\mu^4 - 25 \mu^2 + 54\right) z^5 - 135\,135 \mu \left(\mu^4 - 40 \mu^2 + 249\right) z^4 + 6930 \left(2 \mu^6 - 115 \mu^4 + 1373 \mu^2 - 1890\right) z^3 - 495 \mu \left(2 \mu^6 - 154 \mu^4 + 2933 \mu^2 - 11\,601\right) z^2 + 45 \left(\mu^8 - 98 \mu^6 + 2674 \mu^4 - 20\,217 \mu^2 + 19\,845\right) z - \mu \left(\mu^8 - 120 \mu^6 + 4368 \mu^4 - 52\,480 \mu^2 + 147\,456\right)\right) \frac{(1+z)^{\mu/2}}{(1-z)^{\mu/2}}$$

07.08.03.0021.01

$$P_{10}^\mu(z) = \frac{1}{\Gamma(11-\mu)} \left(654\,729\,075 z^{10} - 654\,729\,075 \mu z^9 + 310\,134\,825 \left(\mu^2 - 5\right) z^8 - 45\,945\,900 \mu \left(2 \mu^2 - 29\right) z^7 + 9\,459\,450 \left(2 \mu^4 - 56 \mu^2 + 135\right) z^6 - 2\,837\,835 \mu \left(\mu^4 - 45 \mu^2 + 314\right) z^5 + 315\,315 \left(\mu^6 - 65 \mu^4 + 874 \mu^2 - 1350\right) z^4 - 12\,870 \mu \left(2 \mu^6 - 175 \mu^4 + 3773 \mu^2 - 16\,830\right) z^3 + 1485 \left(\mu^8 - 112 \mu^6 + 3479 \mu^4 - 29\,828 \mu^2 + 33\,075\right) z^2 - 55 \mu \left(\mu^8 - 138 \mu^6 + 5754 \mu^4 - 78\,877 \mu^2 + 251\,865\right) z + \mu^{10} - 165 \mu^8 + 8778 \mu^6 - 172\,810 \mu^4 + 1\,057\,221 \mu^2 - 893\,025\right) \frac{(1+z)^{\mu/2}}{(1-z)^{\mu/2}}$$

07.08.03.0022.01

$$P_n^\mu(z) = \frac{(1+z)^{\mu/2}}{(1-z)^{\mu/2}} \sum_{k=0}^{n} \frac{(-n)_k (n+1)_k}{\Gamma(k-\mu+1)\, k!} \left(\frac{1-z}{2}\right)^k /; n \in \mathbb{N}$$

07.08.03.0023.01

$$P_{-n}^{\mu}(z) = \frac{(1+z)^{\mu/2}}{(1-z)^{\mu/2}} \sum_{k=0}^{n-1} \frac{(1-n)_k\,(n)_k}{\Gamma(k-\mu+1)\,k!} \left(\frac{1-z}{2}\right)^k /;\, n \in \mathbb{N}^+$$

07.08.03.0024.01

$$P_n^{\mu}(z) = \frac{\Gamma(-\mu)\,(z+1)^{\mu/2}}{\Gamma(-n-\mu)\,\Gamma(n-\mu+1)\,(1-z)^{\mu/2}} \sum_{k=0}^{|n|+\theta(n)-1} \frac{(-n)_k\,(n+1)_k}{(\mu+1)_k\,k!} \left(\frac{z+1}{2}\right)^k /;\, n \in \mathbb{Z}$$

07.08.03.0025.01

$$P_n^{\mu}(z) = \frac{(-1)^n\,2^{-n}}{n!} \frac{(1+z)^{\mu/2}}{(1-z)^{\mu/2-n}} \sum_{k=0}^{n} \frac{(2\,n-k)!\,(-n)_k}{k!\,\Gamma(n-k-\mu+1)} \left(\frac{2}{1-z}\right)^k /;\, n \in \mathbb{N}$$

07.08.03.0026.01

$$P_n^m(z) = 0 /;\, n \in \mathbb{N} \bigwedge m \in \mathbb{N}^+ \bigwedge n < m$$

For fixed *z*

07.08.03.0027.01

$$P_0^0(z) = 1$$

07.08.03.0028.01

$$P_1^0(z) = z$$

07.08.03.0029.01

$$P_1^1(z) = -\sqrt{1-z^2}$$

07.08.03.0030.01

$$P_2^0(z) = \frac{1}{2}\left(3\,z^2-1\right)$$

07.08.03.0031.01

$$P_2^1(z) = -3\,z\,\sqrt{1-z^2}$$

07.08.03.0032.01

$$P_2^2(z) = 3-3\,z^2$$

07.08.03.0033.01

$$P_3^0(z) = \frac{1}{2}\,z\left(5\,z^2-3\right)$$

07.08.03.0034.01

$$P_3^1(z) = -\frac{3}{2}\sqrt{1-z^2}\,\left(5\,z^2-1\right)$$

07.08.03.0035.01

$$P_3^2(z) = -15\,(z-1)\,z\,(1+z)$$

07.08.03.0036.01

$$P_3^3(z) = -15\left(1-z^2\right)^{3/2}$$

07.08.03.0037.01

$$P_4^0(z) = \frac{1}{8}\left(3-30\,z^2+35\,z^4\right)$$

07.08.03.0038.01

$$P_4^1(z) == -\frac{5}{2} z \sqrt{1-z^2} \left(7 z^2 - 3\right)$$

07.08.03.0039.01

$$P_4^2(z) == -\frac{15}{2} (z-1)(z+1) \left(7 z^2 - 1\right)$$

07.08.03.0040.01

$$P_4^3(z) == -105 z \left(1-z^2\right)^{3/2}$$

07.08.03.0041.01

$$P_4^4(z) == 105 \left(z^2 - 1\right)^2$$

07.08.03.0042.01

$$P_5^0(z) == \frac{1}{8} z \left(15 - 70 z^2 + 63 z^4\right)$$

07.08.03.0043.01

$$P_5^1(z) == -\frac{15}{8} \sqrt{1-z^2} \left(21 z^4 - 14 z^2 + 1\right)$$

07.08.03.0044.01

$$P_5^2(z) == -\frac{105}{2} (z-1) z (z+1) \left(3 z^2 - 1\right)$$

07.08.03.0045.01

$$P_5^3(z) == -\frac{105}{2} \left(1-z^2\right)^{3/2} \left(9 z^2 - 1\right)$$

07.08.03.0046.01

$$P_5^4(z) == 945 z \left(z^2 - 1\right)^2$$

07.08.03.0047.01

$$P_5^5(z) == -945 \left(1-z^2\right)^{5/2}$$

07.08.03.0048.01

$$P_6^0(z) == \frac{1}{16} \left(231 z^6 - 315 z^4 + 105 z^2 - 5\right)$$

07.08.03.0049.01

$$P_6^1(z) == -\frac{21}{8} z \sqrt{1-z^2} \left(33 z^4 - 30 z^2 + 5\right)$$

07.08.03.0050.01

$$P_6^2(z) == -\frac{105}{8} (z-1)(z+1) \left(33 z^4 - 18 z^2 + 1\right)$$

07.08.03.0051.01

$$P_6^3(z) == -\frac{315}{2} z \left(1-z^2\right)^{3/2} \left(11 z^2 - 3\right)$$

07.08.03.0052.01

$$P_6^4(z) == \frac{945}{2} \left(z^2 - 1\right)^2 \left(11 z^2 - 1\right)$$

07.08.03.0053.01

$$P_6^5(z) = -10\,395\, z\left(1-z^2\right)^{5/2}$$

07.08.03.0054.01

$$P_6^6(z) = 10\,395\left(1-z^2\right)^3$$

07.08.03.0055.01

$$P_7^0(z) = \frac{1}{16} z\left(429 z^6 - 693 z^4 + 315 z^2 - 35\right)$$

07.08.03.0056.01

$$P_7^1(z) = -\frac{7}{16}\sqrt{1-z^2}\left(429 z^6 - 495 z^4 + 135 z^2 - 5\right)$$

07.08.03.0057.01

$$P_7^2(z) = -\frac{63}{8}(z-1) z (z+1)\left(143 z^4 - 110 z^2 + 15\right)$$

07.08.03.0058.01

$$P_7^3(z) = -\frac{315}{8}\left(1-z^2\right)^{3/2}\left(143 z^4 - 66 z^2 + 3\right)$$

07.08.03.0059.01

$$P_7^4(z) = \frac{3465}{2} z\left(z^2-1\right)^2\left(13 z^2 - 3\right)$$

07.08.03.0060.01

$$P_7^5(z) = -\frac{10\,395}{2}\left(1-z^2\right)^{5/2}\left(13 z^2 - 1\right)$$

07.08.03.0061.01

$$P_7^6(z) = 135\,135\, z\left(1-z^2\right)^3$$

07.08.03.0062.01

$$P_7^7(z) = -135\,135\left(1-z^2\right)^{7/2}$$

07.08.03.0063.01

$$P_8^0(z) = \frac{1}{128}\left(6435 z^8 - 12\,012 z^6 + 6930 z^4 - 1260 z^2 + 35\right)$$

07.08.03.0064.01

$$P_8^1(z) = -\frac{9}{16} z\sqrt{1-z^2}\left(715 z^6 - 1001 z^4 + 385 z^2 - 35\right)$$

07.08.03.0065.01

$$P_8^2(z) = -\frac{315}{16}\left(z^2-1\right)\left(143 z^6 - 143 z^4 + 33 z^2 - 1\right)$$

07.08.03.0066.01

$$P_8^3(z) = -\frac{3465}{8} z\left(1-z^2\right)^{3/2}\left(39 z^4 - 26 z^2 + 3\right)$$

07.08.03.0067.01

$$P_8^4(z) = \frac{10\,395}{8}\left(z^2-1\right)^2\left(65 z^4 - 26 z^2 + 1\right)$$

07.08.03.0068.01

$$P_8^5(z) = -\frac{135135}{2} z (1 - z^2)^{5/2} (5 z^2 - 1)$$

07.08.03.0069.01

$$P_8^6(z) = -\frac{135135}{2} (z^2 - 1)^3 (15 z^2 - 1)$$

07.08.03.0070.01

$$P_8^7(z) = -2027025 z (1 - z^2)^{7/2}$$

07.08.03.0071.01

$$P_8^8(z) = 2027025 (z^2 - 1)^4$$

07.08.03.0072.01

$$P_9^0(z) = \frac{1}{128} z (12155 z^8 - 25740 z^6 + 18018 z^4 - 4620 z^2 + 315)$$

07.08.03.0073.01

$$P_9^1(z) = -\frac{45}{128} \sqrt{1 - z^2} (2431 z^8 - 4004 z^6 + 2002 z^4 - 308 z^2 + 7)$$

07.08.03.0074.01

$$P_9^2(z) = -\frac{495}{16} (z - 1) z (z + 1) (221 z^6 - 273 z^4 + 91 z^2 - 7)$$

07.08.03.0075.01

$$P_9^3(z) = -\frac{3465}{16} (1 - z^2)^{3/2} (221 z^6 - 195 z^4 + 39 z^2 - 1)$$

07.08.03.0076.01

$$P_9^4(z) = \frac{135135}{8} (z^2 - 1)^2 (17 z^5 - 10 z^3 + z)$$

07.08.03.0077.01

$$P_9^5(z) = -\frac{135135}{8} (1 - z^2)^{5/2} (85 z^4 - 30 z^2 + 1)$$

07.08.03.0078.01

$$P_9^6(z) = -\frac{675675}{2} z (z^2 - 1)^3 (17 z^2 - 3)$$

07.08.03.0079.01

$$P_9^7(z) = -\frac{2027025}{2} (1 - z^2)^{7/2} (17 z^2 - 1)$$

07.08.03.0080.01

$$P_9^8(z) = 34459425 z (z^2 - 1)^4$$

07.08.03.0081.01

$$P_9^9(z) = -34459425 (1 - z^2)^{9/2}$$

07.08.03.0082.01

$$P_{10}^0(z) = \frac{1}{256} (46189 z^{10} - 109395 z^8 + 90090 z^6 - 30030 z^4 + 3465 z^2 - 63)$$

07.08.03.0083.01

$$P_{10}^{1}(z) = -\frac{55}{128} z \sqrt{1-z^2} \left(4199 z^8 - 7956 z^6 + 4914 z^4 - 1092 z^2 + 63\right)$$

07.08.03.0084.01

$$P_{10}^{2}(z) = -\frac{495}{128} \left(z^2 - 1\right) \left(4199 z^8 - 6188 z^6 + 2730 z^4 - 364 z^2 + 7\right)$$

07.08.03.0085.01

$$P_{10}^{3}(z) = -\frac{6435}{16} z \left(1 - z^2\right)^{3/2} \left(323 z^6 - 357 z^4 + 105 z^2 - 7\right)$$

07.08.03.0086.01

$$P_{10}^{4}(z) = \frac{45\,045}{16} \left(z^2 - 1\right)^2 \left(323 z^6 - 255 z^4 + 45 z^2 - 1\right)$$

07.08.03.0087.01

$$P_{10}^{5}(z) = -\frac{135\,135}{8} z \left(1 - z^2\right)^{5/2} \left(323 z^4 - 170 z^2 + 15\right)$$

07.08.03.0088.01

$$P_{10}^{6}(z) = -\frac{675\,675}{8} \left(z^2 - 1\right)^3 \left(323 z^4 - 102 z^2 + 3\right)$$

07.08.03.0089.01

$$P_{10}^{7}(z) = -\frac{11\,486\,475}{2} z \left(1 - z^2\right)^{7/2} \left(19 z^2 - 3\right)$$

07.08.03.0090.01

$$P_{10}^{8}(z) = \frac{34\,459\,425}{2} \left(z^2 - 1\right)^4 \left(19 z^2 - 1\right)$$

07.08.03.0091.01

$$P_{10}^{9}(z) = -654\,729\,075 z \left(1 - z^2\right)^{9/2}$$

07.08.03.0092.01

$$P_{10}^{10}(z) = 654\,729\,075 \left(1 - z^2\right)^5$$

General characteristics

Domain and analyticity

$P_\nu^\mu(z)$ is an analytical function of ν, μ and z which is defined over $\mathbb{C}^3$. For integer ν, $P_\nu^\mu(z)$ degenerates to a polynomial in z multiplied on function $\frac{(1+z)^{\mu/2}}{(1-z)^{\mu/2}}$.

07.08.04.0001.01

$$(\nu * \mu * 2 * z) \longrightarrow P_\nu^\mu(z) :: (\mathbb{C} \otimes \mathbb{C} \otimes \{2\} \otimes \mathbb{C}) \longrightarrow \mathbb{C}$$

Symmetries and periodicities

Parity

07.08.04.0002.01

$$P_{-\nu}^{\mu}(z) == P_{\nu-1}^{\mu}(z)$$

Mirror symmetry

07.08.04.0003.01

$$P_{\bar{\nu}}^{\bar{\mu}}(\bar{z}) == \overline{P_{\nu}^{\mu}(z)} \;/;\; z \notin (-\infty, -1) \bigwedge z \notin (1, \infty)$$

Periodicity

No periodicity

Poles and essential singularities

With respect to *z*

For fixed ν, μ /; $\frac{\mu}{2} \notin \mathbb{Z}$, the function $P_{\nu}^{\mu}(z)$ does not have poles and essential singularities.

07.08.04.0004.01

$$\mathcal{S}ing_z\left(P_{\nu}^{\mu}(z)\right) == \{\} \;/;\; \frac{\mu}{2} \notin \mathbb{Z}$$

For integer ν and integer $\frac{\mu}{2}$, the function $P_{\nu}^{\mu}(z)$ is polynomial and has pole of order ν at $z = \tilde{\infty}$ (for $\nu \in \mathbb{N}^+$) or order $-\nu - 1$ at $z = \tilde{\infty}$ (for $-\nu \in \mathbb{N}^+$).

07.08.04.0005.01

$$\mathcal{S}ing_z\left(P_{\nu}^{\mu}(z)\right) == \{\{\tilde{\infty}, \nu\}\} \;/;\; \frac{\mu}{2} \in \mathbb{Z} \bigwedge \nu \in \mathbb{N}^+$$

07.08.04.0006.01

$$\mathcal{S}ing_z\left(P_{\nu}^{\mu}(z)\right) == \{\{\tilde{\infty}, -\nu - 1\}\} \;/;\; \frac{\mu}{2} \in \mathbb{Z} \bigwedge -\nu \in \mathbb{N}^+$$

With respect to μ

For fixed ν, z, the function $P_{\nu}^{\mu}(z)$ has only one singular point at $\mu = \tilde{\infty}$. It is an essential singular point. .

07.08.04.0007.01

$$\mathcal{S}ing_{\mu}\left(P_{\nu}^{\mu}(z)\right) == \{\{\tilde{\infty}, \infty\}\}$$

With respect to ν

For fixed μ, z, the function $P_{\nu}^{\mu}(z)$ has only one singular point at $\nu = \tilde{\infty}$. It is an essential singular point. .

07.08.04.0008.01

$$\mathcal{S}ing_{\nu}\left(P_{\nu}^{\mu}(z)\right) == \{\{\tilde{\infty}, \infty\}\}$$

Branch points

With respect to *z*

For fixed generic ν, μ /; $\frac{\mu}{2} \notin \mathbb{Z}$, the function $P_\nu^\mu(z)$ has three branch points: $z == -1$, $z == 1$and $z == \tilde{\infty}$. For fixed noninteger ν and integer $\frac{\mu}{2}$, the function $P_\nu^\mu(z)$ has two branch points: $z == -1$and $z == \tilde{\infty}$. For fixed integers ν and integers $\frac{\mu}{2}$, the function $P_\nu^\mu(z)$does not have branch points.

07.08.04.0009.01

$$\mathcal{BP}_z\left(P_\nu^\mu(z)\right) == \{-1, 1, \tilde{\infty}\} /; \frac{\mu}{2} \notin \mathbb{Z}$$

07.08.04.0010.01

$$\mathcal{BP}_z\left(P_\nu^\mu(z)\right) == \{-1, \tilde{\infty}\} /; \nu \notin \mathbb{Z} \bigwedge \frac{\mu}{2} \in \mathbb{Z}$$

07.08.04.0011.01

$$\mathcal{BP}_z\left(P_\nu^\mu(z)\right) == \{\} /; \nu \in \mathbb{Z} \bigwedge \frac{\mu}{2} \in \mathbb{Z}$$

07.08.04.0012.01

$$\mathcal{R}_z\left(P_\nu^\mu(z), -1\right) == \log /; \mu \in \mathbb{Z} \vee \mu \notin \mathbb{Q}$$

07.08.04.0013.01

$$\mathcal{R}_z\left(P_\nu^\mu(z), -1\right) == s /; \mu == \frac{r}{s} \bigwedge r \in \mathbb{Z} \bigwedge s - 1 \in \mathbb{N}^+ \bigwedge \gcd(r, s) == 1$$

07.08.04.0014.01

$$\mathcal{R}_z\left(P_\nu^\mu(z), 1\right) == \log /; \frac{\mu}{2} \notin \mathbb{Z} \bigwedge \frac{\mu}{2} \notin \mathbb{Q}$$

07.08.04.0015.01

$$\mathcal{R}_z\left(P_\nu^\mu(z), \tilde{\infty}\right) == \log /; \nu + \frac{1}{2} \in \mathbb{Z} \bigvee \nu \notin \mathbb{Q}$$

07.08.04.0016.01

$$\mathcal{R}_z\left(P_\nu^\mu(z), \tilde{\infty}\right) == s /; \nu == \frac{r}{s} \bigwedge \{r, s\} \in \mathbb{Z} \bigwedge s > 2 \bigwedge \gcd(r, s) == 1$$

With respect to μ

For fixed ν, z, the function $P_\nu^\mu(z)$ does not have branch points.

07.08.04.0017.01

$$\mathcal{BP}_\mu\left(P_\nu^\mu(z)\right) == \{\}$$

With respect to ν

For fixed μ, z, the function $P_\nu^\mu(z)$ does not have branch points.

07.08.04.0018.01

$$\mathcal{BP}_\nu\left(P_\nu^\mu(z)\right) == \{\}$$

Branch cuts

With respect to z

For fixed generic ν, μ /; $\frac{\mu}{2} \notin \mathbb{Z}$, the function $P_\nu^\mu(z)$ is a single-valued function on the z-plane cut along the intervals $(-\infty, -1)$ and $(1, \infty)$. The function $P_\nu^\mu(z)$ is continuous from above on the interval $(-\infty, -1]$ and from below on the interval $[1, \infty)$.

For fixed noninteger ν and integer $\frac{\mu}{2}$, the function $P_\nu^\mu(z)$ is a single-valued function on the z-plane cut along the interval $(-\infty, -1)$, where it is continuous from above.

For fixed integers ν and $\frac{\mu}{2}$, the function $P_\nu^\mu(z)$ is a polynomial and does not have branch cuts.

07.08.04.0019.01

$$\mathcal{BC}_z\left(P_\nu^\mu(z)\right) == \{\{(-\infty, -1), -i\}, \{(1, \infty), i\}\} /; \frac{\mu}{2} \notin \mathbb{Z}$$

07.08.04.0020.01

$$\mathcal{BC}_z\left(P_\nu^\mu(z)\right) == \{\{(-\infty, -1), -i\}\} /; \nu \notin \mathbb{Z} \bigwedge \frac{\mu}{2} \in \mathbb{Z}$$

07.08.04.0021.01

$$\mathcal{BC}_z\left(P_\nu^\mu(z)\right) == \{\} /; \nu \in \mathbb{Z} \bigwedge \frac{\mu}{2} \in \mathbb{Z}$$

07.08.04.0022.01

$$\lim_{\epsilon \to +0} P_\nu^\mu(x + i\,\epsilon) == P_\nu^\mu(x) /; x < -1$$

07.08.04.0023.01

$$\lim_{\epsilon \to +0} P_\nu^\mu(x - i\,\epsilon) == e^{i\,\mu\,\pi}\, P_\nu^\mu(x) + \frac{2\,i\,\pi}{\Gamma(1-\mu+\nu)\,\Gamma(-\mu-\nu)}\, P_\nu^{-\mu}(-x) /; x < -1$$

07.08.04.0024.01

$$\lim_{\epsilon \to +0} P_\nu^\mu(x + i\,\epsilon) == e^{i\,\pi\,\mu}\, P_\nu^\mu(x) /; x > 1$$

07.08.04.0025.01

$$\lim_{\epsilon \to +0} P_\nu^\mu(x - i\,\epsilon) == P_\nu^\mu(x) /; x > 1$$

With respect to μ

For fixed ν, z, the function $P_\nu^\mu(z)$ does not have branch cuts.

07.08.04.0026.01

$$\mathcal{BC}_\mu\left(P_\nu^\mu(z)\right) == \{\}$$

With respect to ν

For fixed μ, z, the function $P_\nu^\mu(z)$ does not have branch cuts.

07.08.04.0027.01

$$\mathcal{BC}_\nu\left(P_\nu^\mu(z)\right) == \{\}$$

Series representations

Generalized power series

Expansions at generic point $z == z_0$

For the function itself

07.08.06.0043.01

$$P_\nu^\mu(z) == \frac{\sin(\pi\nu)}{\pi\,\Gamma(-\mu-\nu)\,\Gamma(1-\mu+\nu)}\left(\frac{1}{1-z_0}\right)^{-\frac{1}{2}\mu\left\lfloor\frac{\arg(z_0-z)}{2\pi}\right\rfloor}(1-z_0)^{-\frac{1}{2}\mu\left(\left\lfloor\frac{\arg(z_0-z)}{2\pi}\right\rfloor+1\right)}\left(\frac{1}{z_0+1}\right)^{\frac{1}{2}\mu\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor}(z_0+1)^{\frac{1}{2}\mu\left(\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor+1\right)}$$
$$\left(2\,i\,\pi\,e^{-i\pi\mu\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor}\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor\left\lfloor\frac{\arg(z_0+1)+\pi}{2\pi}\right\rfloor\Gamma(-\nu)\,\Gamma(\nu+1)\,{}_2\tilde{F}_1\left(-\nu,\nu+1;\mu+1;\frac{z_0+1}{2}\right)-\right.$$
$$\left(\frac{1}{z_0+1}\right)^{-\mu\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor}(z_0+1)^{-\mu\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor}G_{2,2}^{2,2}\left(\frac{1}{2}(z_0+1)\,\middle|\,\begin{matrix}\nu+1,-\nu\\0,-\mu\end{matrix}\right)+\frac{1}{2\left(z_0^2-1\right)}\,e^{-i\pi\mu\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor}\left(\frac{1}{z_0+1}\right)^{-\mu\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor}$$
$$(z_0+1)^{-\mu\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor}\left(2\,i\,\pi\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor\left\lfloor\frac{\arg(z_0+1)+\pi}{2\pi}\right\rfloor\left(\frac{1}{z_0+1}\right)^{\mu\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor}(z_0+1)^{\mu\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor}\left(\Gamma(1-\nu)\,\Gamma(\nu+1)\left(z_0^2-1\right)\right.\right.$$
$$\left.{}_2\tilde{F}_1\left(1-\nu,\nu+2;\mu+2;\frac{z_0+1}{2}\right)-2\,\mu\,\Gamma(-\nu)\,\Gamma(\nu+2)\,{}_2\tilde{F}_1\left(-\nu,\nu+1;\mu+1;\frac{z_0+1}{2}\right)\right)+2\,e^{i\pi\mu\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor}\mu$$
$$\left.\left.G_{2,2}^{2,2}\left(\frac{z_0+1}{2}\,\middle|\,\begin{matrix}\nu+1,-\nu\\0,-\mu\end{matrix}\right)+\left(z_0^2-1\right)e^{i\pi\mu\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor}G_{2,2}^{2,2}\left(\frac{z_0+1}{2}\,\middle|\,\begin{matrix}\nu,-\nu-1\\0,-\mu-1\end{matrix}\right)\right)(z-z_0)+\ldots\right)/;\,(z\to z_0)$$

07.08.06.0044.01

$$P_\nu^\mu(z) == \frac{\sin(\pi\nu)}{\pi\,\Gamma(-\mu-\nu)\,\Gamma(1-\mu+\nu)}\left(\frac{1}{1-z_0}\right)^{-\frac{1}{2}\mu\left\lfloor\frac{\arg(z_0-z)}{2\pi}\right\rfloor}(1-z_0)^{-\frac{1}{2}\mu\left(\left\lfloor\frac{\arg(z_0-z)}{2\pi}\right\rfloor+1\right)}\left(\frac{1}{z_0+1}\right)^{\frac{1}{2}\mu\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor}(z_0+1)^{\frac{1}{2}\mu\left(\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor+1\right)}$$
$$\sum_{k=0}^{\infty}\sum_{i=0}^{k}\sum_{j=0}^{i}\frac{(-1)^{k-j}\,2^{i-k}}{(i-j)!\,j!\,(k-i)!}\left(-\frac{\mu}{2}\right)_j\left(\frac{\mu}{2}\right)_{i-j}(1-z_0)^{j-i}\,(z_0+1)^{-j}\left(2\,\pi\,i\,(-1)^k\,e^{-i\pi\mu\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor}\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor\right.$$
$$\left\lfloor\frac{\arg(z_0+1)+\pi}{2\pi}\right\rfloor\Gamma(-i+k-\nu)\,\Gamma(i+\nu+1)\,{}_2\tilde{F}_1\left(-i+k-\nu,-i+k+\nu+1;-i+k+\mu+1;\frac{z_0+1}{2}\right)-$$
$$\left.(-1)^i\left(\frac{1}{z_0+1}\right)^{-\mu\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor}(z_0+1)^{-\mu\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor}G_{2,2}^{2,2}\left(\frac{z_0+1}{2}\,\middle|\,\begin{matrix}i-k+\nu+1,\,i-k-\nu\\0,\,i-k-\mu\end{matrix}\right)\right)(z-z_0)^k$$

07.08.06.0045.01

$$P_\nu^\mu(z) = \frac{\sin(\pi\nu)}{\Gamma(-\mu-\nu)\,\Gamma(1-\mu+\nu)} \left(\frac{1}{1-z_0}\right)^{-\frac{1}{2}\mu\left\lfloor\frac{\arg(z_0-z)}{2\pi}\right\rfloor} (1-z_0)^{-\frac{1}{2}\mu\left(\left\lfloor\frac{\arg(z_0-z)}{2\pi}\right\rfloor+1\right)} \left(\frac{1}{z_0+1}\right)^{\frac{1}{2}\mu\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor} (z_0+1)^{\frac{1}{2}\mu\left(\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor+1\right)} \sum_{k=0}^{\infty}\sum_{i=0}^{k}\sum_{j=0}^{i} \frac{(-1)^j}{(i-j)!\,j!\,(k-i)!} \left(-\frac{\mu}{2}\right)_j \left(\frac{\mu}{2}\right)_{i-j} (1-z_0)^{j-i}(z_0+1)^{-j} \left(2^{i-k}\,\Gamma(-i+k-\nu)\,(z_0+1)^{k+\mu}\left(\csc(\pi\mu)\,\Gamma(-i+k+\nu+1)\,(z_0+1)^{-k-\mu-\mu\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor}\left(\frac{1}{z_0+1}\right)^{-\mu\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor} + i\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor\left\lfloor\frac{\arg(z_0+1)+\pi}{2\pi}\right\rfloor\Gamma(i+\nu+1)\right)\,{}_2\tilde{F}_1\left(-i+k-\nu,\,-i+k+\nu+1;\,-i+k+\mu+1;\,\frac{z_0+1}{2}\right) - 2^\mu\csc(\pi\mu)\,\Gamma(-\mu-\nu)\,\Gamma(-\mu+\nu+1)\,(z_0+1)^i\left(\frac{1}{z_0+1}\right)^{-\mu\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor}(z_0+1)^{-k-\mu-\mu\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor}\,{}_2\tilde{F}_1\left(-\mu-\nu,\,-\mu+\nu+1;\,i-k-\mu+1;\,\frac{z_0+1}{2}\right)\right)(z-z_0)^k \;/;\; \mu\notin\mathbb{Z}$$

07.08.06.0046.01

$$P_\nu^\mu(z) = \frac{\sin(\pi\nu)}{\pi\,\Gamma(-\mu-\nu)\,\Gamma(-\mu+\nu+1)} \left(\frac{1}{1-z_0}\right)^{-\frac{1}{2}\mu\left\lfloor\frac{\arg(z_0-z)}{2\pi}\right\rfloor} (1-z_0)^{-\frac{1}{2}\mu\left(\left\lfloor\frac{\arg(z_0-z)}{2\pi}\right\rfloor+1\right)} (z_0+1)^{\frac{1}{2}\mu\left(\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor+1\right)} \left(\frac{1}{z_0+1}\right)^{\frac{1}{2}\mu\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor} \left(2\pi i\,e^{-i\pi\mu\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor}\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor\left\lfloor\frac{\arg(z_0+1)+\pi}{2\pi}\right\rfloor\Gamma(-\nu)\,\Gamma(\nu+1)\,{}_2\tilde{F}_1\left(-\nu,\,\nu+1;\,\mu+1;\,\frac{z_0+1}{2}\right) - \left(\frac{1}{z_0+1}\right)^{-\mu\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor}(z_0+1)^{-\mu\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor} G_{2,2}^{2,2}\left(\frac{z_0+1}{2}\,\middle|\,\begin{matrix}\nu+1,\,-\nu\\ 0,\,-\mu\end{matrix}\right)\right) + O(z-z_0)$$

Expansions on branch cuts

For the function itself

Expansion at a point at the right half-plane branch cut

07.08.06.0047.01

$$P_\nu^\mu(z) = -\frac{\sin(\pi\nu)}{\pi\,\Gamma(-\mu-\nu)\,\Gamma(-\mu+\nu+1)}\, e^{-i\pi\mu\left\lfloor\frac{\arg(x-z)}{2\pi}\right\rfloor} (1-x)^{-\frac{\mu}{2}} (x+1)^{\mu/2} \left(G_{2,2}^{2,2}\left(\frac{x+1}{2}\,\middle|\,\begin{matrix}\nu+1,-\nu\\0,-\mu\end{matrix}\right) + \frac{1}{2(1-x^2)}\left((x^2-1)\,G_{2,2}^{2,2}\left(\frac{x+1}{2}\,\middle|\,\begin{matrix}\nu,-\nu-1\\0,-\mu-1\end{matrix}\right) + 2\mu\, G_{2,2}^{2,2}\left(\frac{x+1}{2}\,\middle|\,\begin{matrix}\nu+1,-\nu\\0,-\mu\end{matrix}\right)\right)(z-x) + \frac{1}{8(x^2-1)^2}\left(G_{2,2}^{2,2}\left(\frac{x+1}{2}\,\middle|\,\begin{matrix}\nu-1,-\nu-2\\0,-\mu-2\end{matrix}\right)(x^2-1)^2 + 4\mu\left((x^2-1)\,G_{2,2}^{2,2}\left(\frac{x+1}{2}\,\middle|\,\begin{matrix}\nu,-\nu-1\\0,-\mu-1\end{matrix}\right) + (2x+\mu)\,G_{2,2}^{2,2}\left(\frac{x+1}{2}\,\middle|\,\begin{matrix}\nu+1,-\nu\\0,-\mu\end{matrix}\right)\right)\right)(z-x)^2 + \ldots\right) /; (z\to x)\land x\in\mathbb{R}\land x>1$$

07.08.06.0048.01

$$P_\nu^\mu(z) = -\frac{\sin(\pi\nu)}{\pi\,\Gamma(-\mu-\nu)\,\Gamma(-\mu+\nu+1)}\, e^{-i\pi\mu\left\lfloor\frac{\arg(x-z)}{2\pi}\right\rfloor} (1-x)^{-\frac{\mu}{2}} (x+1)^{\mu/2} \left(G_{2,2}^{2,2}\left(\frac{x+1}{2}\,\middle|\,\begin{matrix}\nu+1,-\nu\\0,-\mu\end{matrix}\right) + \frac{1}{2(1-x^2)}\left((x^2-1)\,G_{2,2}^{2,2}\left(\frac{x+1}{2}\,\middle|\,\begin{matrix}\nu,-\nu-1\\0,-\mu-1\end{matrix}\right) + 2\mu\, G_{2,2}^{2,2}\left(\frac{x+1}{2}\,\middle|\,\begin{matrix}\nu+1,-\nu\\0,-\mu\end{matrix}\right)\right)(z-x) + \frac{1}{8(x^2-1)^2}\left(G_{2,2}^{2,2}\left(\frac{x+1}{2}\,\middle|\,\begin{matrix}\nu-1,-\nu-2\\0,-\mu-2\end{matrix}\right)(x^2-1)^2 + 4\mu\left((x^2-1)\,G_{2,2}^{2,2}\left(\frac{x+1}{2}\,\middle|\,\begin{matrix}\nu,-\nu-1\\0,-\mu-1\end{matrix}\right) + (2x+\mu)\,G_{2,2}^{2,2}\left(\frac{x+1}{2}\,\middle|\,\begin{matrix}\nu+1,-\nu\\0,-\mu\end{matrix}\right)\right)\right)(z-x)^2 + O\left((z-x)^3\right)\right) /; x\in\mathbb{R}\land x>1$$

07.08.06.0049.01

$$P_\nu^\mu(z) = -\frac{\sin(\pi\nu)}{\pi\,\Gamma(-\mu-\nu)\,\Gamma(1-\mu+\nu)}\, e^{-\pi i\mu\left\lfloor\frac{\arg(x-z)}{2\pi}\right\rfloor} (x+1)^{\frac{\mu}{2}} (1-x)^{-\frac{\mu}{2}} \sum_{k=0}^{\infty}\sum_{i=0}^{k}\sum_{j=0}^{i} \frac{(-1)^{i-j+k}\, 2^{i-k} \left(-\frac{\mu}{2}\right)_j \left(\frac{\mu}{2}\right)_{i-j} (1-x)^{j-i} (x+1)^{-j}}{(i-j)!\, j!\, (k-i)!}\, G_{2,2}^{2,2}\left(\frac{x+1}{2}\,\middle|\,\begin{matrix}i-k+\nu+1,\, i-k-\nu\\0,\, i-k-\mu\end{matrix}\right)(z-x)^k /; x\in\mathbb{R}\land x>1$$

07.08.06.0050.01

$$P_\nu^\mu(z) = -\frac{\sin(\pi\nu)\csc(\pi\mu)}{\Gamma(-\mu-\nu)\,\Gamma(1-\mu+\nu)}\, e^{-\pi i\mu\left\lfloor\frac{\arg(x-z)}{2\pi}\right\rfloor} (x+1)^{\frac{\mu}{2}} (1-x)^{-\frac{\mu}{2}} \sum_{k=0}^{\infty}\sum_{i=0}^{k}\sum_{j=0}^{i} \frac{(-1)^j\, 2^{i-k}}{(i-j)!\, j!\, (k-i)!} \left(-\frac{\mu}{2}\right)_j \left(\frac{\mu}{2}\right)_{i-j} (1-x)^{j-i} (x+1)^{-j} \left(2^{-i+k+\mu} (x+1)^{i-k-\mu}\, \Gamma(-\mu-\nu)\,\Gamma(1-\mu+\nu)\, {}_2\tilde{F}_1\left(-\mu-\nu, -\mu+\nu+1; i-k-\mu+1; \frac{x+1}{2}\right) - \Gamma(-i+k-\nu)\, \Gamma(1-i+k+\nu)\, {}_2\tilde{F}_1\left(-i+k-\nu, -i+k+\nu+1; -i+k+\mu+1; \frac{x+1}{2}\right)\right)(z-x)^k /; \mu\notin\mathbb{Z}\bigwedge x\in\mathbb{R}\bigwedge x>1$$

07.08.06.0051.01

$$P_\nu^\mu(z) = -\frac{\sin(\pi\nu)}{\pi\,\Gamma(-\mu-\nu)\,\Gamma(1-\mu+\nu)}\, e^{-i\pi\mu\left\lfloor\frac{\arg(x-z)}{2\pi}\right\rfloor} (1-x)^{-\frac{\mu}{2}} (x+1)^{\mu/2}\, G_{2,2}^{2,2}\left(\frac{x+1}{2}\,\middle|\,\begin{matrix}\nu+1,-\nu\\0,-\mu\end{matrix}\right) + O(z-x) /; x\in\mathbb{R}\land x>1$$

Expansion at a point at the left half-plane branch cut

07.08.06.0052.01

$$P_\nu^\mu(z) = \frac{\sin(\pi\nu)}{\pi\,\Gamma(-\mu-\nu)\,\Gamma(-\mu+\nu+1)}(1-x)^{-\frac{\mu}{2}}(x+1)^{\frac{\mu}{2}} e^{\pi i\mu\left\lfloor\frac{\arg(z-x)}{2\pi}\right\rfloor}\left(2\pi^2 i\, e^{-i\pi\mu\left\lfloor\frac{\arg(z-x)}{2\pi}\right\rfloor}\left\lfloor\frac{\arg(z-x)}{2\pi}\right\rfloor\csc(\pi\nu)\,{}_2\tilde{F}_1\left(-\nu,\nu+1;\mu+1;\frac{x+1}{2}\right)-\right.$$
$$e^{-2i\pi\mu\left\lfloor\frac{\arg(z-x)}{2\pi}\right\rfloor} G_{2,2}^{2,2}\left(\frac{x+1}{2}\,\middle|\,\begin{matrix}\nu+1,-\nu\\0,-\mu\end{matrix}\right)+\frac{1}{2\left(x^2-1\right)}e^{-2i\pi\mu\left\lfloor\frac{\arg(z-x)}{2\pi}\right\rfloor}\left(2e^{i\pi\mu\left\lfloor\frac{\arg(z-x)}{2\pi}\right\rfloor} i\pi^2\csc(\pi\nu)\left\lfloor\frac{\arg(z-x)}{2\pi}\right\rfloor\right.$$
$$\left(\left(x^2-1\right)\nu\,{}_2\tilde{F}_1\left(1-\nu,\nu+2;\mu+2;\frac{x+1}{2}\right)+2\mu(\nu+1)\,{}_2\tilde{F}_1\left(-\nu,\nu+1;\mu+1;\frac{x+1}{2}\right)\right)+$$
$$\left.\left.\left(x^2-1\right)G_{2,2}^{2,2}\left(\frac{x+1}{2}\,\middle|\,\begin{matrix}\nu,-\nu-1\\0,-\mu-1\end{matrix}\right)+2\mu\,G_{2,2}^{2,2}\left(\frac{x+1}{2}\,\middle|\,\begin{matrix}\nu+1,-\nu\\0,-\mu\end{matrix}\right)\right)(z-x)+\ldots\right)/;\ (z\to x)\land x\in\mathbb{R}\land x<-1$$

07.08.06.0053.01

$$P_\nu^\mu(z) = \frac{1}{\pi\,\Gamma(-\mu-\nu)\,\Gamma(-\mu+\nu+1)}\sin(\pi\nu)(1-x)^{-\frac{\mu}{2}}$$
$$(x+1)^{\frac{\mu}{2}} e^{\pi i\mu\left\lfloor\frac{\arg(z-x)}{2\pi}\right\rfloor}\sum_{k=0}^{\infty}\sum_{i=0}^{k}\sum_{j=0}^{i}\frac{(-1)^{k-j}2^{i-k}}{(i-j)!\,j!\,(k-i)!}\left(-\frac{\mu}{2}\right)_j\left(\frac{\mu}{2}\right)_{i-j}(1-x)^{j-i}(x+1)^{-j}$$
$$\left(2\pi i(-1)^k e^{-i\pi\mu\left\lfloor\frac{\arg(z-x)}{2\pi}\right\rfloor}\left\lfloor\frac{\arg(z-x)}{2\pi}\right\rfloor\Gamma(-i+k-\nu)\,\Gamma(i+\nu+1)\,{}_2\tilde{F}_1\left(-i+k-\nu,-i+k+\nu+1;-i+k+\mu+1;\right.\right.$$
$$\left.\left.\frac{x+1}{2}\right)-(-1)^i e^{-2\pi i\mu\left\lfloor\frac{\arg(z-x)}{2\pi}\right\rfloor}G_{2,2}^{2,2}\left(\frac{x+1}{2}\,\middle|\,\begin{matrix}i-k+\nu+1,\ i-k-\nu\\0,\ i-k-\mu\end{matrix}\right)\right)(z-x)^k\ /;\ x\in\mathbb{R}\land x<-1$$

07.08.06.0054.01

$$P_\nu^\mu(z) = \frac{\sin(\pi\nu)}{\Gamma(-\mu-\nu)\,\Gamma(-\mu+\nu+1)}(1-x)^{-\frac{\mu}{2}}(x+1)^{\frac{\mu}{2}} e^{\pi i\mu\left\lfloor\frac{\arg(z-x)}{2\pi}\right\rfloor}\sum_{k=0}^{\infty}\sum_{i=0}^{k}\sum_{j=0}^{i}\frac{(-1)^j 2^{i-k}}{(i-j)!\,j!\,(k-i)!}\left(-\frac{\mu}{2}\right)_j\left(\frac{\mu}{2}\right)_{i-j}(1-x)^{j-i}(x+1)^{-j}$$
$$\left(\left(2i\,e^{-i\pi\mu\left\lfloor\frac{\arg(z-x)}{2\pi}\right\rfloor}\left\lfloor\frac{\arg(z-x)}{2\pi}\right\rfloor\Gamma(i+\nu+1)\,\Gamma(-i+k-\nu)+\csc(\pi\mu)\,e^{-2\pi i\mu\left\lfloor\frac{\arg(z-x)}{2\pi}\right\rfloor}\Gamma(-i+k+\nu+1)\,\Gamma(-i+k-\nu)\right)\right.$$
$${}_2\tilde{F}_1\left(-i+k-\nu,-i+k+\nu+1;-i+k+\mu+1;\frac{x+1}{2}\right)-2^{-i+k+\mu}\csc(\pi\mu)(x+1)^{i-k-\mu}\,\Gamma(-\mu-\nu)\,\Gamma(-\mu+\nu+1)$$
$$\left.e^{-2\pi i\mu\left\lfloor\frac{\arg(z-x)}{2\pi}\right\rfloor}{}_2\tilde{F}_1\left(-\mu-\nu,-\mu+\nu+1;i-k-\mu+1;\frac{x+1}{2}\right)\right)(z-x)^k\ /;\ \mu\notin\mathbb{Z}\bigwedge x\in\mathbb{R}\bigwedge x<-1$$

07.08.06.0055.01

$$P_\nu^\mu(z) =$$
$$-\frac{1}{\pi\,\Gamma(-\mu-\nu)\,\Gamma(-\mu+\nu+1)}(1-x)^{-\frac{\mu}{2}}(x+1)^{\frac{\mu}{2}} e^{\pi i\mu\left\lfloor\frac{\arg(z-x)}{2\pi}\right\rfloor}\left(2\pi^2 i\,e^{-i\pi\mu\left\lfloor\frac{\arg(z-x)}{2\pi}\right\rfloor}\left\lfloor\frac{\arg(z-x)}{2\pi}\right\rfloor{}_2\tilde{F}_1\left(-\nu,\nu+1;\mu+1;\frac{x+1}{2}\right)+\right.$$
$$\left.\sin(\pi\nu)\,e^{-2i\pi\mu\left\lfloor\frac{\arg(z-x)}{2\pi}\right\rfloor}G_{2,2}^{2,2}\left(\frac{x+1}{2}\,\middle|\,\begin{matrix}\nu+1,-\nu\\0,-\mu\end{matrix}\right)\right)+O(z-x)\ /;\ x\in\mathbb{R}\land x<-1$$

Expansions at $z = 0$

07.08.06.0001.01

$$P_\nu^\mu(z) = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-\nu)_k(\nu+1)_k\left(\frac{\mu}{2}-k\right)_m\left(-\frac{\mu}{2}\right)_j(-1)^j z^{j+m}}{\Gamma(k-\mu+1)\,k!\,m!\,j!\,2^k}\ /;\ |z|<1$$

07.08.06.0002.01

$$P_\nu^\mu(z) = \frac{\mu}{2}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-\nu)_k\,(\nu+1)_k\left(\frac{\mu}{2}+1\right)_m\left(1-\frac{\mu}{2}\right)_k\left(-\frac{\mu}{2}\right)_j\,(-1)^{j+k}\,z^{j+k+m+1}}{(2)_{k+m}\,\Gamma(k-\mu+1)\,k!\,j!\,2^k} + \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-\nu)_{k+m}\,(\nu+1)_{k+m}\left(m-\frac{\mu}{2}+1\right)_k\left(-\frac{\mu}{2}\right)_j\,(-z)^{j+k}}{\Gamma(k+m-\mu+1)\,(k+m)!\;k!\,j!\,2^{k+m}} \;/;\; |z|<1$$

07.08.06.0003.01

$$P_\nu^\mu(z) = \frac{\mu\,z}{2}\sum_{k=0}^{\infty}(k+1)!\left(\frac{\mu}{2}+1\right)_k z^k\;{}_1F_0\left(-\frac{\mu}{2};;-z\right){}_3\tilde{F}_2\left(-\nu,\,\nu+1,\,1-\frac{\mu}{2};\,k+2,\,1-\mu;\,-\frac{z}{2}\right) + \Gamma\left(1-\frac{\mu}{2}\right)\sum_{k=0}^{\infty}\left(-\frac{\mu}{2}\right)_k(-z)^k\,\tilde{F}_{2\times1\times0}^{3\times1\times0}\left(\begin{matrix}-\nu,\,\nu+1,\,1-\frac{\mu}{2};\,1;\,; & \frac{1}{2},\,-\frac{z}{2}\\ 1,\,1-\mu;\,1-\frac{\mu}{2};\,; & \end{matrix}\right)$$

07.08.06.0004.01

$$P_\nu^\mu(z) \propto \frac{2^\mu\sqrt{\pi}}{\Gamma\left(\frac{1-\mu-\nu}{2}\right)\Gamma\left(1-\frac{\mu-\nu}{2}\right)}\,(1+O(z)) \;/;\; (z\to 0)$$

07.08.06.0005.01

$$P_n^m(z) = (-1)^m\left(1-z^2\right)^{m/2}\,2^{-n}\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}(-1)^k\binom{n}{k}\binom{2n-2k}{n}(n-m-2k+1)_m\,z^{n-m-2k} \;/;\; m\in\mathbb{N}\land n\in\mathbb{N}$$

07.08.06.0006.01

$$P_n^m(z) \propto (-1)^{m+\left\lfloor\frac{n-m}{2}\right\rfloor}\,2^{-n}\binom{n}{\left\lfloor\frac{n-m}{2}\right\rfloor}\binom{2n-2\left\lfloor\frac{n-m}{2}\right\rfloor}{n}\left(n-m-2\left\lfloor\frac{n-m}{2}\right\rfloor+1\right)_m z^{n-m-2\left\lfloor\frac{n-m}{2}\right\rfloor}(1+O(z)) \;/;\; (z\to 0)\land m\in\mathbb{N}\land n\in\mathbb{N}\land n\geq m$$

Expansions at $z = 1$

07.08.06.0007.01

$$P_\nu^\mu(z) = \frac{(1+z)^{\mu/2}}{(1-z)^{\mu/2}}\left(\frac{1}{\Gamma(1-\mu)} - \frac{(-\nu)(\nu+1)(z-1)}{2\,\Gamma(2-\mu)} + \frac{(-\nu)(1-\nu)(\nu+1)(\nu+2)(z-1)^2}{8\,\Gamma(3-\mu)} - \ldots\right) /;\; \left|\frac{1-z}{2}\right| < 1$$

07.08.06.0008.01

$$P_\nu^\mu(z) = \frac{(1+z)^{\mu/2}}{(1-z)^{\mu/2}}\sum_{k=0}^{\infty}\frac{(-\nu)_k\,(\nu+1)_k}{\Gamma(k-\mu+1)\,k!}\left(\frac{1-z}{2}\right)^k /;\; \left|\frac{1-z}{2}\right| < 1$$

07.08.06.0009.01

$$P_\nu^\mu(z) = \frac{(1+z)^{\mu/2}}{(1-z)^{\mu/2}}\,{}_2\tilde{F}_1\left(-\nu,\,\nu+1;\,1-\mu;\,\frac{1-z}{2}\right)$$

07.08.06.0010.01

$$P_\nu^\mu(z) = \frac{2^{\mu/2}}{(1-z)^{\mu/2}}\sum_{k=0}^{\infty}\sum_{j=0}^{k}\frac{(-1)^k\left(-\frac{\mu}{2}\right)_{k-j}(-\nu)_j\,(\nu+1)_j\,(z-1)^k}{\Gamma(j-\mu+1)\,(k-j)!\,j!\,2^k} /;\; \left|\frac{1-z}{2}\right| < 1$$

07.08.06.0011.01

$$P_\nu^\mu(z) \propto \frac{2^{\mu/2}}{\Gamma(1-\mu)}(1-z)^{-\mu/2}\,(1+O(z-1)) \;/;\; (z\to 1)\bigwedge \mu\notin\mathbb{N}^+$$

07.08.06.0012.01

$$P_\nu^m(z) = \left(1-z^2\right)^{m/2} 2^{-m} (-\nu)_m (\nu+1)_m \sum_{k=0}^{\infty} \frac{(m-\nu)_k (m+\nu+1)_k}{\Gamma(k+m+1)\, k!} \left(\frac{1-z}{2}\right)^k /; m \in \mathbb{N}$$

07.08.06.0013.01

$$P_\nu^m(z) \propto -\frac{2^{-\frac{m}{2}} \Gamma(m-\nu) \Gamma(m+\nu+1) \sin(\pi \nu)}{\pi\, m!} (1-z)^{\frac{m}{2}} (1+O(z-1)) /; (z \to 1) \land m \in \mathbb{N}$$

07.08.06.0014.01

$$P_n^\mu(z) = \frac{(1+z)^{\mu/2}}{(1-z)^{\mu/2}} \sum_{k=0}^{n} \frac{(-n)_k (n+1)_k}{\Gamma(k-\mu+1)\, k!} \left(\frac{1-z}{2}\right)^k /; n \in \mathbb{N}$$

07.08.06.0015.01

$$P_{-n}^\mu(z) = \frac{(1+z)^{\mu/2}}{(1-z)^{\mu/2}} \sum_{k=0}^{n-1} \frac{(1-n)_k (n)_k}{\Gamma(k-\mu+1)\, k!} \left(\frac{1-z}{2}\right)^k /; n \in \mathbb{N}^+$$

Expansions at $z == -1$

07.08.06.0016.01

$$P_\nu^\mu(z) = \frac{2^{-\mu/2} \Gamma(-\mu) (z+1)^{\mu/2}}{\Gamma(-\mu-\nu) \Gamma(1-\mu+\nu)} \left(1 + \frac{\mu(1+\mu) - 2\nu(1+\nu)}{4(1+\mu)} (z+1) + \frac{\mu(1+\mu)(2+\mu)^2 - 4(2+\mu(2+\mu))\nu - 4(1+\mu)^2\nu^2 + 8\nu^3 + 4\nu^4}{32(1+\mu)(2+\mu)} (z+1)^2 + \ldots\right) -$$
$$\frac{2^{\mu/2} \sin(\pi\nu) \Gamma(\mu) (z+1)^{-\frac{\mu}{2}}}{\pi} \left(1 + \frac{\mu(1-\mu) + 2\nu(1+\nu)}{4(-1+\mu)} (z+1) + \frac{-5\mu^3 + \mu^4 + 4(-1+\nu)\nu(1+\nu)(2+\nu) - 4\mu^2\left(-2+\nu+\nu^2\right) + \mu(-4+8\nu(1+\nu))}{32(-2+\mu)(-1+\mu)} (z+1)^2 + \ldots\right) /; \left|\frac{z+1}{2}\right| < 1 \bigwedge \mu \notin \mathbb{Z}$$

07.08.06.0017.01

$$P_\nu^\mu(z) = \frac{\Gamma(-\mu)}{\Gamma(-\mu-\nu)\Gamma(\nu-\mu+1)} \frac{(1+z)^{\mu/2}}{(1-z)^{\mu/2}} \sum_{k=0}^{\infty} \frac{(-\nu)_k (\nu+1)_k}{(\mu+1)_k\, k!} \left(\frac{z+1}{2}\right)^k -$$
$$\frac{2^\mu \sin(\pi\nu)\Gamma(\mu)}{\pi} \left(1-z^2\right)^{-\mu/2} \sum_{k=0}^{\infty} \frac{(\nu-\mu+1)_k (-\mu-\nu)_k}{(1-\mu)_k\, k!} \left(\frac{z+1}{2}\right)^k /; \left|\frac{z+1}{2}\right| < 1 \bigwedge \mu \notin \mathbb{Z}$$

07.08.06.0018.01

$$P_\nu^\mu(z) = \frac{\Gamma(-\mu)}{\Gamma(-\mu-\nu)\Gamma(\nu-\mu+1)} \frac{(1+z)^{\mu/2}}{(1-z)^{\mu/2}} {}_2F_1\left(-\nu, \nu+1; \mu+1; \frac{z+1}{2}\right) -$$
$$\frac{2^\mu \sin(\pi\nu)\Gamma(\mu)}{\pi} \left(1-z^2\right)^{-\mu/2} {}_2F_1\left(\nu-\mu+1, -\mu-\nu; 1-\mu; \frac{z+1}{2}\right) /; \mu \notin \mathbb{Z}$$

07.08.06.0019.01

$$P_\nu^\mu(z) = \frac{\Gamma(-\mu)\, 2^{-\frac{\mu}{2}}}{\Gamma(-\mu-\nu)\,\Gamma(\nu-\mu+1)} (z+1)^{\mu/2} \sum_{k=0}^{\infty}\sum_{j=0}^{k} \frac{\left(\frac{\mu}{2}\right)_{k-j} (-\nu)_j\, (\nu+1)_j\, 2^{-k}\, (z+1)^k}{(\mu+1)_j\, j!\, (k-j)!} - \frac{2^{\mu/2} \sin(\pi\,\nu)\,\Gamma(\mu)}{\pi} (z+1)^{-\frac{\mu}{2}} \sum_{k=0}^{\infty}\sum_{j=0}^{k} \frac{\left(\frac{\mu}{2}\right)_{k-j} (-\mu-\nu)_j\, (\nu-\mu+1)_j\; 2^{-k}\, (z+1)^k}{(1-\mu)_j\, j!\, (k-j)!} \;/;\; \mu \notin \mathbb{Z}$$

07.08.06.0020.01

$$P_\nu^\mu(z) \propto \frac{2^{-\mu/2}\,\Gamma(-\mu)}{\Gamma(-\mu-\nu)\,\Gamma(\nu-\mu+1)} (z+1)^{\mu/2} (1+O(z+1)) - \frac{2^{\mu/2} \sin(\pi\,\nu)\,\Gamma(\mu)}{\pi} (z+1)^{-\mu/2} (1+O(z+1)) \;/;\; (z\to -1) \bigwedge \mu \notin \mathbb{Z}$$

07.08.06.0021.01

$$P_n^\mu(z) = \frac{\Gamma(-\mu)\,(z+1)^{\mu/2}}{\Gamma(-n-\mu)\,\Gamma(n-\mu+1)\,(1-z)^{\mu/2}} \sum_{k=0}^{|n|+\theta(n)-1} \frac{(-n)_k\,(n+1)_k}{(\mu+1)_k\, k!} \left(\frac{z+1}{2}\right)^k \;/;\; n \in \mathbb{Z}$$

07.08.06.0022.01

$$P_\nu^m(z) = \frac{(-1)^{m-1}}{m!\,\Gamma(-m-\nu)\,\Gamma(\nu-m+1)} \log\left(\frac{z+1}{2}\right) \frac{(1+z)^{m/2}}{(1-z)^{m/2}}\, {}_2F_1\left(-\nu,\, \nu+1;\, m+1;\, \frac{z+1}{2}\right) - \frac{2^m \sin(\pi\,\nu)\,(m-1)!}{\pi} \left(1-z^2\right)^{-m/2} \sum_{k=0}^{m-1} \frac{(-m-\nu)_k\,(\nu-m+1)_k}{k!\,(1-m)_k} \left(\frac{z+1}{2}\right)^k + \frac{(-1)^m}{\Gamma(-m-\nu)\,\Gamma(-m+\nu+1)} \frac{(1+z)^{m/2}}{(1-z)^{m/2}} \sum_{k=0}^{\infty} \frac{(-\nu)_k\,(\nu+1)_k}{k!\,(k+m)!} (\psi(k+1)+\psi(k+m+1)-\psi(k+\nu+1)-\psi(k-\nu)) \left(\frac{z+1}{2}\right)^k \;/;\; \left|\frac{z+1}{2}\right| < 1 \bigwedge m \in \mathbb{N}^+ \bigwedge \nu \notin \mathbb{Z}$$

07.08.06.0023.01

$$P_\nu^m(z) \propto \frac{(-1)^{m-1}\, 2^{-\frac{m}{2}}}{m!\,\Gamma(-m-\nu)\,\Gamma(\nu-m+1)} (z+1)^{m/2} \left(\log\left(\frac{z+1}{2}\right) - \psi(m+1) + \psi(-\nu) + \psi(\nu+1) + \gamma\right)(1+O(z+1)) - \frac{2^{m/2} \sin(\pi\,\nu)\,(m-1)!}{\pi} (z+1)^{-\frac{m}{2}} (1+O(z+1)) \;/;\; (z\to -1) \bigwedge m \in \mathbb{N}^+ \bigwedge \nu \notin \mathbb{Z}$$

07.08.06.0024.01

$$P_\nu^0(z) = \frac{\sin(\pi\,\nu)}{\pi} \log\left(\frac{z+1}{2}\right) {}_2F_1\left(-\nu,\, \nu+1;\, 1;\, \frac{z+1}{2}\right) - \frac{\sin(\pi\,\nu)}{\pi} \sum_{k=0}^{\infty} \frac{(-\nu)_k\,(\nu+1)_k\,(2\,\psi(k+1)-\psi(k+\nu+1)-\psi(k-\nu))}{k!^2} \left(\frac{z+1}{2}\right)^k \;/;\; \left|\frac{z+1}{2}\right| < 1 \bigwedge \nu \notin \mathbb{Z}$$

07.08.06.0025.01

$$P_\nu^0(z) \propto \frac{\sin(\pi\,\nu)}{\pi} \log\left(\frac{z+1}{2}\right)(1+O(z+1)) + \frac{\sin(\pi\,\nu)}{\pi} (-\pi\cot(\pi\,\nu) + 2\,\psi(-\nu) + 2\,\gamma)\,(1+O(z+1)) \;/;\; (z\to -1) \bigwedge \nu \notin \mathbb{Z}$$

07.08.06.0026.01

$$P_\nu^{-m}(z) = \frac{2^{-m}(-1)^m \sin(\pi\nu)}{\pi\, m!} \log\left(\frac{z+1}{2}\right)\left(1-z^2\right)^{m/2} {}_2F_1\left(m+\nu+1, m-\nu; m+1; \frac{z+1}{2}\right) + \frac{(m-1)!}{\Gamma(m-\nu)\,\Gamma(m+\nu+1)} \frac{(1-z)^{m/2}}{(1+z)^{m/2}} \sum_{k=0}^{m-1} \frac{(-\nu)_k\,(\nu+1)_k}{k!\,(1-m)_k}\left(\frac{z+1}{2}\right)^k - \frac{2^{-m}(-1)^m \sin(\pi\nu)}{\pi}\left(1-z^2\right)^{m/2} \sum_{k=0}^{\infty} \frac{(m-\nu)_k\,(m+\nu+1)_k}{k!\,(k+m)!} (\psi(k+1)+\psi(k+m+1)-\psi(k+m+\nu+1)-\psi(k+m-\nu))\left(\frac{z+1}{2}\right)^k /;\ m \in \mathbb{N}^+ \bigwedge \nu \notin \mathbb{Z}$$

07.08.06.0027.01

$$P_\nu^{-m}(z) \propto \frac{2^{m/2}(m-1)!}{\Gamma(m-\nu)\,\Gamma(m+\nu+1)} (z+1)^{-\frac{m}{2}} (1+O(z+1)) + \frac{(-1)^m\, 2^{-\frac{m}{2}} \sin(\pi\nu)}{\pi\, m!} (z+1)^{m/2} \left(\log\left(\frac{z+1}{2}\right) - \psi(m+1) + \psi(m-\nu) + \psi(m+\nu+1) + \gamma\right)(1+O(z+1)) /;\ (z \to -1) \bigwedge m \in \mathbb{N}^+ \bigwedge \nu \notin \mathbb{Z}$$

Expansions at $z = \infty$

07.08.06.0028.01

$$P_\nu^\mu(z) = \frac{1}{\sqrt{\pi}} \frac{(1+z)^{\mu/2}}{(1-z)^{\mu/2}} \left(\frac{2^\nu (z-1)^\nu}{\Gamma(\nu-\mu+1)} \Gamma\left(\nu+\frac{1}{2}\right)\left(1+\frac{\mu-\nu}{1-z}+\frac{(1-\nu)(\mu-\nu)(1+\mu-\nu)}{(1-2\nu)(1-z)^2}+\ldots\right) + \frac{2^{-\nu-1}(z-1)^{-\nu-1}}{\Gamma(-\mu-\nu)} \Gamma\left(-\nu-\frac{1}{2}\right)\left(1+\frac{1+\mu+\nu}{1-z}+\frac{(2+\nu)(1+\mu+\nu)(2+\mu+\nu)}{(3+2\nu)(z-1)^2}+\ldots\right)\right) /;\ \left|\frac{1-z}{2}\right| > 1 \bigwedge 2\nu \notin \mathbb{Z}$$

07.08.06.0029.01

$$P_\nu^\mu(z) = \frac{1}{\sqrt{\pi}} \frac{(1+z)^{\mu/2}}{(1-z)^{\mu/2}} \left(\frac{2^\nu (z-1)^\nu}{\Gamma(\nu-\mu+1)} \Gamma\left(\nu+\frac{1}{2}\right) \sum_{k=0}^{\infty} \frac{(-\nu)_k\,(\mu-\nu)_k}{(-2\nu)_k\, k!}\left(\frac{2}{1-z}\right)^k + \frac{2^{-\nu-1}\,(z-1)^{-\nu-1}}{\Gamma(-\mu-\nu)} \Gamma\left(-\nu-\frac{1}{2}\right) \sum_{k=0}^{\infty} \frac{(\nu+1)_k\,(\mu+\nu+1)_k}{(2\nu+2)_k\, k!}\left(\frac{2}{1-z}\right)^k\right) /;\ \left|\frac{1-z}{2}\right| > 1 \bigwedge 2\nu \notin \mathbb{Z}$$

07.08.06.0030.01

$$P_\nu^\mu(z) = \frac{1}{\sqrt{\pi}} \frac{(1+z)^{\mu/2}}{(1-z)^{\mu/2}} \left(\frac{2^\nu (z-1)^\nu}{\Gamma(\nu-\mu+1)} \Gamma\left(\nu+\frac{1}{2}\right) {}_2F_1\left(\mu-\nu, -\nu; -2\nu; \frac{2}{1-z}\right) + \frac{2^{-\nu-1}(z-1)^{-\nu-1}}{\Gamma(-\mu-\nu)} \Gamma\left(-\nu-\frac{1}{2}\right) {}_2F_1\left(\nu+1, \mu+\nu+1; 2\nu+2; \frac{2}{1-z}\right)\right) /;\ z \notin (-1, 1) \bigwedge 2\nu \notin \mathbb{Z}$$

07.08.06.0031.01

$$P_\nu^\mu(z) \propto \frac{1}{\sqrt{\pi}} \frac{z^{\mu/2}}{(1-z)^{\mu/2}} \left(\frac{2^\nu z^\nu}{\Gamma(\nu-\mu+1)} \Gamma\left(\nu+\frac{1}{2}\right)\left(1+O\left(\frac{1}{z}\right)\right) + \frac{2^{-\nu-1} z^{-\nu-1}}{\Gamma(-\mu-\nu)} \Gamma\left(-\nu-\frac{1}{2}\right)\left(1+O\left(\frac{1}{z}\right)\right)\right) /;\ (|z| \to \infty) \bigwedge 2\nu \notin \mathbb{Z}$$

07.08.06.0032.01

$$P_n^\mu(z) = \frac{(-1)^n\, 2^{-n}}{n!} \frac{(1+z)^{\mu/2}}{(1-z)^{\mu/2-n}} \sum_{k=0}^{n} \frac{(2n-k)!\,(-n)_k}{k!\,\Gamma(n-k-\mu+1)}\left(\frac{2}{1-z}\right)^k /;\ n \in \mathbb{N}$$

07.08.06.0033.01

$$P_\nu^\mu(z) == \frac{2^{\nu+1}\sin(\pi(\mu-\nu))\,\Gamma(\mu+\nu+1)}{\pi\,\Gamma(-\nu)\,\Gamma(2\nu+2)}(z-1)^{-\nu-1}\log\left(\frac{z-1}{2}\right)\frac{(1+z)^{\mu/2}}{(1-z)^{\mu/2}}\,{}_2F_1\left(\nu+1,\,\mu+\nu+1;\,2\nu+2;\,\frac{2}{1-z}\right)+ \frac{2^{-\nu}(z-1)^\nu}{\Gamma(\nu+1)}\frac{(1+z)^{\mu/2}}{(1-z)^{\mu/2}}\sum_{k=0}^{2\nu}\frac{(2\nu-k)!\,(-\nu)_k}{k!\,\Gamma(\nu-\mu-k+1)}\left(\frac{2}{1-z}\right)^k + \frac{2^{-\nu}\sin(\pi(\nu-\mu))\sin(\pi\nu)\,\Gamma(\mu+\nu+1)}{\pi^{3/2}\,\Gamma\left(\nu+\frac{3}{2}\right)}(z-1)^{-\nu-1}\frac{(1+z)^{\mu/2}}{(1-z)^{\mu/2}} \sum_{k=0}^{\infty}\frac{(\nu+1)_k\,(\mu+\nu+1)_k}{k!\,(2\nu+2)_k}(\psi(k+1)-\psi(k+\nu+1)+\psi(k+2\nu+2)-\psi(-k-\mu-\nu))\left(\frac{2}{1-z}\right)^k /;\ 2\nu+1\in\mathbb{N}\bigwedge\nu-\mu\notin\mathbb{Z}$$

07.08.06.0034.01

$$P_\nu^\mu(z)\propto\frac{2^\nu\,\Gamma\left(\nu+\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(\nu-\mu+1)}\frac{z^{\mu/2+\nu}}{(1-z)^{\mu/2}}\left(1+O\left(\frac{1}{z}\right)\right)+\frac{2^{\nu+1}\,\Gamma(\mu+\nu+1)\sin(\pi(\mu-\nu))}{\pi\,\Gamma(-\nu)\,\Gamma(2\nu+2)} \frac{z^{\mu/2-\nu-1}}{(1-z)^{\mu/2}}\left(\log\left(\frac{z}{2}\right)-\psi(-\mu-\nu)-\psi(\nu+1)+\psi(2\nu+2)-\gamma\right)\left(1+O\left(\frac{1}{z}\right)\right) /;\ 2\nu\in\mathbb{N}\bigwedge\nu-\mu\notin\mathbb{Z}$$

07.08.06.0035.01

$$P_{-\frac{1}{2}}^\mu(z)\propto\frac{\sqrt{2}\,\cos(\pi\mu)\,\Gamma\left(\mu+\frac{1}{2}\right)}{\pi^{3/2}}\frac{z^{\frac{\mu-1}{2}}}{(1-z)^{\mu/2}}\left(\log(2z)-\psi\left(\frac{1}{2}-\mu\right)-\gamma\right)\left(1+O\left(\frac{1}{z}\right)\right) /;\ \mu+\frac{1}{2}\notin\mathbb{Z}$$

07.08.06.0036.01

$$P_\nu^\mu(z) == \frac{(-1)^{\mu-\nu-1}\,2^{\nu+1}\,(\mu+\nu)!\,(z-1)^{-\nu-1}}{(2\nu+1)!\,\Gamma(-\nu)}\frac{(1+z)^{\mu/2}}{(1-z)^{\mu/2}}\,{}_2F_1\left(\nu+1,\,\mu+\nu+1;\,2\nu+2;\,\frac{2}{1-z}\right) /;\ 2\nu+1\in\mathbb{N}\bigwedge\mu-\nu\in\mathbb{N}^+$$

07.08.06.0037.01

$$P_\nu^\mu(z)\propto\frac{(-1)^{\mu-\nu-1}\,2^{\nu+1}\,(\mu+\nu)!}{(2\nu+1)!\,\Gamma(-\nu)}\frac{z^{\mu/2-\nu-1}}{(1-z)^{\mu/2}}\left(1+O\left(\frac{1}{z}\right)\right) /;\ (|z|\to\infty)\bigwedge 2\nu+1\in\mathbb{N}\bigwedge\mu-\nu\in\mathbb{N}^+$$

07.08.06.0038.01

$$P_\nu^\mu(z) == \frac{2^\nu\,\Gamma\left(\nu+\frac{1}{2}\right)(z-1)^\nu}{\sqrt{\pi}\,\Gamma(\nu-\mu+1)}\frac{(1+z)^{\mu/2}}{(1-z)^{\mu/2}}\sum_{k=0}^{\nu-\mu}\frac{(\mu-\nu)_k\,(-\nu)_k}{k!\,(-2\nu)_k}\left(\frac{2}{1-z}\right)^k-\frac{(-1)^{\nu-\mu}\,2^{\nu+1}\,\Gamma(\mu+\nu+1)\,(z-1)^{-\nu-1}}{\Gamma(-\nu)\,\Gamma(2\nu+2)} \frac{(1+z)^{\mu/2}}{(1-z)^{\mu/2}}\,{}_2F_1\left(\nu+1,\,\mu+\nu+1;\,2\nu+2;\,\frac{2}{1-z}\right) /;\ 2\nu+1\in\mathbb{N}\bigwedge\nu-\mu\in\mathbb{Z}\bigwedge-\nu\le\mu\le\nu+1$$

07.08.06.0039.01

$$P_\nu^\mu(z)\propto\frac{2^\nu\,\Gamma\left(\nu+\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(\nu-\mu+1)}\frac{z^{\mu/2+\nu}}{(1-z)^{\mu/2}}\left(1+O\left(\frac{1}{z}\right)\right) /;\ (|z|\to\infty)\bigwedge 2\nu+1\in\mathbb{N}\bigwedge\nu-\mu\in\mathbb{Z}\bigwedge-\nu\le\mu\le\nu+1$$

07.08.06.0040.01

$$P_\nu^\mu(z) \equiv \frac{(-1)^{2\nu}\, 2^{1-\mu} \sin(\pi\nu)\, \Gamma(1-\mu)}{\pi\, \Gamma(1-\mu-\nu)\, \Gamma(\nu-\mu+2)} (z-1)^{\mu-1} \frac{(1+z)^{\mu/2}}{(1-z)^{\mu/2}}\, {}_3F_2\left(1, 1, 1-\mu; 1-\mu-\nu, 2-\mu+\nu; \frac{2}{1-z}\right) + \frac{2^\nu\, \Gamma\left(\nu+\frac{1}{2}\right)}{\sqrt{\pi}\, \Gamma(\nu-\mu+1)} (z-1)^\nu \frac{(1+z)^{\mu/2}}{(1-z)^{\mu/2}} \sum_{k=0}^{2\nu} \frac{(\mu-\nu)_k\, (-\nu)_k}{k!\, (-2\nu)_k} \left(\frac{2}{1-z}\right)^k + \frac{(-1)^{2\nu+1}\, 2^{\nu+1}}{(-\mu-\nu-1)!\, \Gamma(-\nu)} (z-1)^{-\nu-1} \frac{(1+z)^{\mu/2}}{(1-z)^{\mu/2}} \sum_{k=0}^{-\mu-\nu-1} \frac{(\nu+1)_k\, (\mu+\nu+1)_k}{k!\, (k+2\nu+1)!} \left(\frac{2}{1-z}\right)^k \left(\log\left(\frac{z-1}{2}\right) + \psi(k+1) - \psi(k+\nu+1) + \psi(k+2\nu+2) - \psi(-k-\mu-\nu)\right) /;\ 2\nu+1 \in \mathbb{N} \land \nu-\mu \in \mathbb{Z} \land \mu+\nu \leq 0$$

07.08.06.0041.01

$$P_\nu^\mu(z) \propto \frac{(-1)^{2\nu+1}\, 2^{\nu+1}}{(-\mu-\nu-1)!\, \Gamma(-\nu)\, \Gamma(2\nu+2)} \left(\log\left(\frac{z}{2}\right) - \psi(-\mu-\nu) - \psi(\nu+1) + \psi(2\nu+2) - \gamma\right) \frac{z^{\mu/2-\nu-1}}{(1-z)^{\mu/2}} \left(1 + O\left(\frac{1}{z}\right)\right) + \frac{2^\nu\, \Gamma\left(\nu+\frac{1}{2}\right)}{\sqrt{\pi}\, \Gamma(\nu-\mu+1)} \frac{z^{\mu/2+\nu}}{(1-z)^{\mu/2}} \left(1 + O\left(\frac{1}{z}\right)\right) + \frac{(-1)^{2\nu}\, 2^{1-\mu} \sin(\pi\nu)\, \Gamma(1-\mu)}{\pi\, \Gamma(1-\mu-\nu)\, \Gamma(2-\mu+\nu)} \frac{z^{3\mu/2-1}}{(1-z)^{\mu/2}} \left(1 + O\left(\frac{1}{z}\right)\right) /;\ (|z| \to \infty) \land 2\nu+1 \in \mathbb{N} \land \nu-\mu \in \mathbb{Z} \land \mu+\nu \leq 0$$

07.08.06.0042.01

$$P_{-\frac{1}{2}}^\mu(z) \propto \sqrt{\frac{2}{\pi}} \frac{\log(2z) - \psi\left(\frac{1}{2}-\mu\right) - \gamma}{\Gamma\left(\frac{1}{2}-\mu\right)} \frac{z^{\frac{\mu-1}{2}}}{(1-z)^{\mu/2}} \left(1 + O\left(\frac{1}{z}\right)\right) /;\ (|z| \to \infty) \land -\mu - \frac{1}{2} \in \mathbb{N}$$

Asymptotic series expansions

Expansions at $\nu == -\frac{1}{2} + i\,\infty$

07.08.06.0056.01

$$P_{i\tau-\frac{1}{2}}^\mu(x) \propto \left(\frac{x+1}{2}\right)^{\mu/2} \tau^\mu\, J_{-\mu}\left(\tau \sqrt{2(x-1)}\right) \left(1 + O\left(\frac{1}{\tau}\right)\right) /;\ (\tau \to \infty) \land x \in \mathbb{R} \land 1 < x < 3$$

07.08.06.0057.01

$$P_{i\tau-\frac{1}{2}}^\mu(x) \propto \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{x-1}} \left(\frac{x+1}{x-1}\right)^{\mu/2} \tau^{\mu-\frac{1}{2}} \cos\left(\frac{1}{2}\pi\left(\mu-\frac{1}{2}\right) + \frac{\tau}{x-1} + \tau \log(2(x-1))\right) \left(1 + O\left(\frac{1}{\tau}\right)\right) /;\ (\tau \to \infty) \land x \in \mathbb{R} \land x > 3$$

07.08.06.0058.01

$$P_{i\tau-\frac{1}{2}}^\mu(3) \propto \sqrt{\frac{2}{\pi}}\, \tau^\mu\, 2^{i\tau-\frac{1}{2}}\, J_{-\mu}\left(\tau\sqrt{2}\right) \left(1 + O\left(\frac{1}{\tau}\right)\right) /;\ (\tau \to \infty) \land \mu < 0$$

Integral representations

On the real axis

Of the direct function

07.08.07.0001.01

$$P_\nu^\mu(z) \equiv \frac{(-1)^{-\nu}\, 2^{\mu-2\nu}}{\Gamma(-\mu-\nu)\, \Gamma(\nu+1)} \frac{(1+z)^{\mu/2}}{(1-z)^{\mu/2-\nu}} \int_{-1}^{1} \frac{(t-1)^\nu}{(t+1)^{\mu+\nu+1}} \left(t - \frac{z+3}{z-1}\right)^\nu dt /;\ -1 < \mathrm{Re}(\nu) < -\mathrm{Re}(\mu)$$

07.08.07.0002.01

$$P_\nu^m(z) = \frac{(-\nu)_m}{\pi}\, e^{\frac{m\pi i}{2}} \int_0^\pi \frac{\cos(m\,t)}{\left(z+\sqrt{z^2-1}\,\cos(t)\right)^{\nu+1}}\,dt \,/;\, m\in\mathbb{N}\land \operatorname{Re}(z)>0$$

07.08.07.0003.01

$$P_\nu^\mu(z) = \frac{1}{\Gamma(-\mu)}\left(1-z^2\right)^{\mu/2}\int_z^1 P_\nu(t)\,(t-z)^{-\mu-1}\,dt \,/;\, \operatorname{Re}(\mu)<0$$

Integral representations of negative integer order

Rodrigues-type formula.

07.08.07.0004.01

$$P_\nu^m(z) = (-1)^m\left(1-z^2\right)^{m/2}\frac{\partial^m P_\nu(z)}{\partial z^m} \,/;\, m\in\mathbb{N}$$

Differential equations

Ordinary linear differential equations and wronskians

For the direct function itself

07.08.13.0001.01

$$\left(1-z^2\right)w''(z)-2\,z\,w'(z)+\left(\nu\,(\nu+1)-\frac{\mu^2}{1-z^2}\right)w(z) = 0 \,/;\, w(z) = c_1\,P_\nu^\mu(z)+c_2\,Q_\nu^\mu(z)$$

07.08.13.0002.02

$$W_z\left(P_\nu^\mu(z),\, Q_\nu^\mu(z)\right) = \frac{\Gamma(\mu+\nu+1)}{\left(1-z^2\right)\Gamma(-\mu+\nu+1)}$$

07.08.13.0003.01

$$g'(z)\,w''(z)-\left(\frac{2\,g(z)\,g'(z)^2}{1-g(z)^2}+g''(z)\right)w'(z)-\frac{\left(\mu^2-\nu\,(\nu+1)\left(1-g(z)^2\right)\right)g'(z)^3}{\left(1-g(z)^2\right)^2}\,w(z)=0 \,/;\, w(z)=c_1\,P_\nu^\mu(g(z))+c_2\,Q_\nu^\mu(g(z))$$

07.08.13.0004.01

$$W_z\left(P_\nu^\mu(g(z)),\, Q_\nu^\mu(g(z))\right) = \frac{\Gamma(\mu+\nu+1)\,g'(z)}{\left(1-g(z)^2\right)\Gamma(1-\mu+\nu)}$$

07.08.13.0005.01

$$g'(z)\,h(z)^2\,w''(z)-\left(\left(\frac{2\,g(z)\,g'(z)^2}{1-g(z)^2}+g''(z)\right)h(z)^2+2\,g'(z)\,h'(z)\,h(z)\right)w'(z)+$$
$$\left(-\frac{\mu^2-\nu\,(\nu+1)\left(1-g(z)^2\right)}{\left(1-g(z)^2\right)^2}\,h(z)^2\,g'(z)^3+2\,h'(z)^2\,g'(z)+h(z)\left(h'(z)\left(\frac{2\,g(z)\,g'(z)^2}{1-g(z)^2}+g''(z)\right)-g'(z)\,h''(z)\right)\right)w(z)=$$
$$0 \,/;\, w(z)=c_1\,h(z)\,P_\nu^\mu(g(z))+c_2\,h(z)\,Q_\nu^\mu(g(z))$$

07.08.13.0006.01

$$W_z\left(h(z)\,P_\nu^\mu(g(z)),\, h(z)\,Q_\nu^\mu(g(z))\right) = \frac{\Gamma(\mu+\nu+1)\,h(z)^2\,g'(z)}{\left(1-g(z)^2\right)\Gamma(1-\mu+\nu)}$$

07.08.13.0007.01

$$z^2 w''(z) - z\left(2s + \frac{r\left(a^2 z^{2r}+1\right)}{1-a^2 z^{2r}} - 1\right)w'(z) + \left(-\frac{a^2 r^2\left(\mu^2 + \left(a^2 z^{2r}-1\right)\nu(\nu+1)\right)z^{2r}}{\left(1-a^2 z^{2r}\right)^2} + s^2 + \frac{r s\left(a^2 z^{2r}+1\right)}{1-a^2 z^{2r}}\right)w(z) = 0 /; w(z) = c_1 z^s P_\nu^\mu(a z^r) + c_2 z^s Q_\nu^\mu(a z^r)$$

07.08.13.0008.01

$$W_z\left(z^s P_\nu^\mu(a z^r), z^s Q_\nu^\mu(a z^r)\right) = \frac{a r z^{r+2s-1}\,\Gamma(\mu+\nu+1)}{\left(1-a^2 z^{2r}\right)\Gamma(-\mu+\nu+1)}$$

07.08.13.0009.01

$$w''(z) - \frac{a^2(\log(r)-2\log(s))\,r^{2z} + \log(r) + 2\log(s)}{1-a^2 r^{2z}}\,w'(z) + \left(-\frac{a^2\left(\mu^2 - \left(1-a^2 r^{2z}\right)\nu(\nu+1)\right)\log^2(r)\,r^{2z}}{\left(1-a^2 r^{2z}\right)^2} + \log^2(s) + \frac{\left(a^2 r^{2z}+1\right)\log(r)\log(s)}{1-a^2 r^{2z}}\right)w(z) = 0 /; w(z) = c_1 s^z P_\nu^\mu\left(a r^z\right) + c_2 s^z Q_\nu^\mu(a r^z)$$

07.08.13.0010.01

$$W_z\left(s^z P_\nu^\mu(a r^z), s^z Q_\nu^\mu(a r^z)\right) = \frac{a r^z s^{2z}\,\Gamma(\mu+\nu+1)\log(r)}{\left(1-a^2 r^{2z}\right)\Gamma(-\mu+\nu+1)}$$

Transformations

Transformations and argument simplifications

Argument involving basic arithmetic operations

07.08.16.0001.01

$$P_{-\nu-1}^\mu(z) = P_\nu^\mu(z)$$

07.08.16.0002.01

$$P_\nu^{-m}(z) = \frac{(-1)^m(\nu-m)!}{(m+\nu)!}\,P_\nu^m(z) /; m \in \mathbb{Z}$$

07.08.16.0004.01

$$P_\nu^{-\mu}(z) = -\frac{1}{\pi}\Gamma(-\mu-\nu)\,\Gamma(\nu-\mu+1)\left(\sin(\nu\pi)\,P_\nu^\mu(z) + \sin(\mu\pi)\,P_\nu^\mu(-z)\right)$$

07.08.16.0003.01

$$P_\nu^\mu(-z) = -\csc(\pi\mu)\left(\sin(\pi\nu)\;P_\nu^\mu(z)\; + \frac{\pi}{\Gamma(-\mu-\nu)\,\Gamma(\nu-\mu+1)}\,P_\nu^{-\mu}(z)\right)$$

07.08.16.0005.01

$$P_\nu^\mu(-z) = \cos((\mu+\nu)\pi)\,P_\nu^\mu(z) - \frac{2}{\pi}\sin((\mu+\nu)\pi)\,Q_\nu^\mu(z)$$

Identities

Recurrence identities

Consecutive neighbors

07.08.17.0001.01

$$P_\nu^\mu(z) == \frac{(2\nu+3)z}{\mu+\nu+1} P_{\nu+1}^\mu(z) + \frac{\mu-\nu-2}{\nu+\mu+1} P_{\nu+2}^\mu(z)$$

07.08.17.0002.01

$$P_\nu^\mu(z) == \frac{(2\nu-1)z}{\nu-\mu} P_{\nu-1}^\mu(z) - \frac{\mu+\nu-1}{\nu-\mu} P_{\nu-2}^\mu(z)$$

07.08.17.0003.01

$$P_\nu^\mu(z) == \frac{2z(1+\mu)}{(\mu(1+\mu)-\nu(1+\nu))\sqrt{1-z^2}} P_\nu^{\mu+1}(z) + \frac{1}{\mu(\mu+1)-\nu(\nu+1)} P_\nu^{\mu+2}(z)$$

07.08.17.0004.01

$$P_\nu^\mu(z) == \frac{2(1-\mu)z}{\sqrt{1-z^2}} P_\nu^{\mu-1}(z) + ((\mu-1)(\mu-2)-\nu(\nu+1)) P_\nu^{\mu-2}(z)$$

Distant neighbors

07.08.17.0009.01

$$P_\nu^\mu(z) == C_n(\nu,\mu,z) P_{\nu+n}^\mu(z) + \frac{\mu-\nu-n-1}{n+\mu+\nu} C_{n-1}(\nu,\mu,z) P_{\nu+n+1}^\mu(z) \;/;\; C_0(\nu,\mu,z) == 1 \bigwedge$$
$$C_1(\nu,\mu,z) == \frac{(2\nu+3)z}{\mu+\nu+1} \bigwedge C_n(\nu,\mu,z) == \frac{z(2n+2\nu+1)}{n+\mu+\nu} C_{n-1}(\nu,\mu,z) + \frac{\mu-\nu-n}{n+\mu+\nu-1} C_{n-2}(\nu,\mu,z) \bigwedge n \in \mathbb{N}^+$$

07.08.17.0010.01

$$P_\nu^\mu(z) == C_n(\nu,\mu,z) P_{\nu-n}^\mu(z) - \frac{\mu+\nu-n}{\nu-\mu-n+1} C_{n-1}(\nu,\mu,z) P_{\nu-n-1}^\mu(z) \;/;\; C_0(\nu,\mu,z) == 1 \bigwedge$$
$$C_1(\nu,\mu,z) == \frac{(2\nu-1)z}{\nu-\mu} \bigwedge C_n(\nu,\mu,z) == \frac{z(2n-2\nu-1)}{n+\mu-\nu-1} C_{n-1}(\nu,\mu,z) - \frac{\mu+\nu-n+1}{\nu-\mu-n+2} C_{n-2}(\nu,\mu,z) \bigwedge n \in \mathbb{N}^+$$

07.08.17.0011.01

$$P_\nu^\mu(z) == C_n(\nu,\mu,z) P_\nu^{\mu+n}(z) + \frac{1}{(n+\mu-1)(n+\mu)-\nu(\nu+1)} C_{n-1}(\nu,\mu,z) P_\nu^{\mu+n+1}(z) \;/;$$
$$C_0(\nu,\mu,z) == 1 \bigwedge C_1(\nu,\mu,z) == \frac{2(\mu+1)z}{(\mu(\mu+1)-\nu(1+\nu))\sqrt{1-z^2}} \bigwedge C_n(\nu,\mu,z) ==$$
$$\frac{2z(n+\mu)}{\sqrt{1-z^2}\,((n-1+\mu)(n+\mu)-\nu(1+\nu))} C_{n-1}(\nu,\mu,z) + \frac{1}{(n-2+\mu)(n-1+\mu)-\nu(1+\nu)} C_{n-2}(\nu,\mu,z) \bigwedge n \in \mathbb{N}^+$$

07.08.17.0012.01

$$P_\nu^\mu(z) == C_n(\nu,\mu,z) P_\nu^{\mu-n}(z) + ((\mu-n-1)(\mu-n)-\nu(\nu+1)) C_{n-1}(\nu,\mu,z) P_\nu^{\mu-n-1}(z) \;/;\; C_0(\nu,\mu,z) == 1 \bigwedge$$
$$C_1(\nu,\mu,z) == \frac{2(1-\mu)z}{\sqrt{1-z^2}} \bigwedge C_n(\nu,\mu,z) == \frac{2z(n-\mu)}{\sqrt{1-z^2}} C_{n-1}(\nu,\mu,z) + ((\mu-n)(\mu-n+1)-\nu(\nu+1)) C_{n-2}(\nu,\mu,z) \bigwedge n \in \mathbb{N}^+$$

Functional identities

Relations between contiguous functions

07.08.17.0005.01

$$(\mu+\nu)\,P_{\nu-1}^{\mu}(z)+(\nu-\mu+1)\,P_{\nu+1}^{\mu}(z)=(2\,\nu+1)\,z\,P_{\nu}^{\mu}(z)$$

07.08.17.0006.01

$$P_{\nu}^{\mu}(z)=\frac{(\mu+\nu)\,P_{\nu-1}^{\mu}(z)+(\nu-\mu+1)\,P_{\nu+1}^{\mu}(z)}{(2\,\nu+1)\,z}$$

07.08.17.0007.01

$$P_{\nu}^{\mu+1}(z)-(\mu\,(\mu-1)-\nu\,(\nu+1))\,P_{\nu}^{\mu-1}(z)+\frac{2\,\mu\,z}{\sqrt{1-z^2}}\,P_{\nu}^{\mu}(z)=0$$

07.08.17.0013.01

$$z\,(\mu+\nu+1)\,P_{\nu}^{\mu}(z)+\sqrt{1-z}\,\sqrt{z+1}\,P_{\nu}^{\mu+1}(z)-(-\mu+\nu+1)\,P_{\nu+1}^{\mu}(z)=0$$

Pavlyk O. (2006)

07.08.17.0014.01

$$P_{\nu}^{\mu+1}(z)-z\,P_{\nu+1}^{\mu+1}(z)-\sqrt{1-z}\,\sqrt{z+1}\,(-\mu+\nu+1)\,P_{\nu+1}^{\mu}(z)=0$$

Pavlyk O. (2006)

Additional relations between contiguous functions

07.08.17.0008.01

$$P_{\nu}^{\mu+1}(z)\,P_{\nu_1}^{\mu_1+1}(z)-P_{\nu-1}^{\mu+1}(z)\,P_{\nu_1-1}^{\mu_1+1}(z)+(\nu-\mu)\,(\nu_1-\mu_1)\,P_{\nu}^{\mu}(z)\,P_{\nu_1}^{\mu_1}(z)\;-(\mu+\nu)\,(\mu_1+\nu_1)\,P_{\nu-1}^{\mu}(z)\,P_{\nu_1-1}^{\mu_1}(z)=0$$

Differentiation

Low-order differentiation

With respect to ν

07.08.20.0001.01

$$\frac{\partial P_{\nu}^{\mu}(z)}{\partial\nu}=\pi\cot(\pi\,\nu)\,P_{\nu}^{\mu}(z)-\frac{(1+z)^{\mu/2}}{(1-z)^{\mu/2}}\sum_{k=0}^{\infty}\frac{(-\nu)_k\,(\nu+1)_k}{\Gamma(k-\mu+1)\,k!}\,(\psi\,(k-\nu)-\psi(k+\nu+1))\left(\frac{1-z}{2}\right)^k\;/;\;\left|\frac{1-z}{2}\right|<1\bigwedge\nu\notin\mathbb{Z}$$

07.08.20.0002.01

$$\frac{\partial P_{\nu}^{\mu}(z)}{\partial\nu}=\frac{(1+z)^{\mu/2}}{(1-z)^{\mu/2}}\sum_{k=0}^{\infty}\frac{1}{\Gamma(k-\mu+1)\,k!}\left(\frac{1-z}{2}\right)^k\sum_{j=1}^{k}S_k^{(j)}\,\nu^j\sum_{r=1}^{k}(-1)^r\,S_k^{(r)}\left(\frac{j}{\nu}+\frac{r}{\nu+1}\right)(\nu+1)^r\;/;\;\left|\frac{1-z}{2}\right|<1$$

07.08.20.0003.01

$$\frac{\partial P_{\nu}^{\mu}(z)}{\partial\nu}=-\frac{2\,\nu+1}{2\,\Gamma(2-\mu)}\,\frac{(1+z)^{\mu/2}}{(1-z)^{\mu/2-1}}\,F_{2\times0\times2}^{2\times1\times3}\left(\begin{matrix}1-\nu,\;\nu+2;\;1;\;1,\;-\nu,\;\nu+1;\\2,\;2-\mu;;\;\nu+2,\;1-\nu;\end{matrix}\;\frac{1-z}{2},\;\frac{1-z}{2}\right)$$

07.08.20.0004.01

$$\frac{\partial^2 P_\nu^\mu(z)}{\partial \nu^2} = \frac{(1+z)^{\mu/2}}{(1-z)^{\mu/2}} \sum_{k=0}^{\infty} \frac{(-\nu)_k (\nu+1)_k}{k!\,\Gamma(k-\mu+1)} \left(\psi(k-\nu)^2 - 2(\pi\cot(\pi\nu) + \psi(k+\nu+1))\,\psi(k-\nu) + \psi(k+\nu+1)^2 + 2\pi\cot(\pi\nu)\,\psi(k+\nu+1) + \psi^{(1)}(k-\nu) + \psi^{(1)}(k+\nu+1)\right)\left(\frac{1-z}{2}\right)^k - \pi^2 P_\nu^\mu(z) \;/;\; \left|\frac{1-z}{2}\right| < 1$$

07.08.20.0005.01

$$\frac{\partial^2 P_\nu^\mu(z)}{\partial \nu^2} = \frac{(1+z)^{\mu/2}}{(1-z)^{\mu/2}} \sum_{k=0}^{\infty} \frac{1}{\Gamma(k-\mu+1)\,k!} \left(\frac{1-z}{2}\right)^k \sum_{i=1}^{k} \nu^{i-2} S_k^{(i)} \sum_{r=1}^{k} (-1)^r (\nu+1)^{r-2} \left((r-1)\,r\,\nu^2 + i^2(\nu+1)^2 + ((2r-1)\,\nu-1)\,i\,(\nu+1)\right) S_k^{(r)} \;/;\; \left|\frac{1-z}{2}\right| < 1$$

With respect to μ

07.08.20.0006.01

$$\frac{\partial P_\nu^\mu(z)}{\partial \mu} = \frac{(1+z)^{\mu/2}}{(1-z)^{\mu/2}} \sum_{k=0}^{\infty} \frac{(-\nu)_k (\nu+1)_k}{\Gamma(k-\mu+1)\,k!} \psi(k-\mu+1) \left(\frac{1-z}{2}\right)^k + \frac{1}{2}(\log(1+z) - \log(1-z))\,P_\nu^\mu(z) \;/;\; \left|\frac{1-z}{2}\right| < 1 \bigwedge \mu \notin \mathbb{N}^+$$

07.08.20.0007.01

$$\frac{\partial P_\nu^\mu(z)}{\partial \mu} = \left(\psi(1-\mu) - \frac{1}{2}(\log(1-z) - \log(1+z))\right) P_\nu^\mu(z) - \frac{\nu(\nu+1)}{2(1-\mu)\,\Gamma(2-\mu)} \frac{(1+z)^{\mu/2}}{(1-z)^{\mu/2-1}} F_{2\times 0\times 1}^{2\times 1\times 2}\left(\begin{matrix} 1-\nu,\, \nu+2;\, 1;\, 1,\, 1-\mu; \\ 2,\, 2-\mu;;\, 2-\mu; \end{matrix} \frac{1-z}{2}, \frac{1-z}{2}\right)$$

07.08.20.0008.01

$$\frac{\partial^2 P_\nu^\mu(z)}{\partial \mu^2} = \frac{(1+z)^{\mu/2}}{(1-z)^{\mu/2}} \sum_{k=0}^{\infty} \frac{(-\nu)_k (\nu+1)_k}{k!\,\Gamma(k-\mu+1)} \left(\frac{1-z}{2}\right)^k \left(\psi(k-\mu+1)^2 + (\log(1+z) - \log(1-z))\,\psi(k-\mu+1) - \psi^{(1)}(k-\mu+1)\right) + \frac{1}{4}(\log(1+z) - \log(1-z))^2 P_\nu^\mu(z) \;/;\; \left|\frac{1-z}{2}\right| < 1 \bigwedge \mu \notin \mathbb{N}^+$$

With respect to z

07.08.20.0009.01

$$\frac{\partial P_\nu^\mu(z)}{\partial z} = \frac{1}{z^2-1}\left(z\,\nu\,P_\nu^\mu(z) - (\mu+\nu)\,P_{\nu-1}^\mu(z)\right)$$

07.08.20.0010.01

$$\frac{\partial^2 P_\nu^\mu(z)}{\partial z^2} = \frac{2z(\mu+\nu)\,P_{\nu-1}^\mu(z) + \left(\mu^2 + \left((\nu-1)\,z^2 - \nu - 1\right)\nu\right) P_\nu^\mu(z)}{\left(z^2-1\right)^2}$$

Symbolic differentiation

With respect to ν

07.08.20.0011.02

$$\frac{\partial^m P_\nu^\mu(z)}{\partial \nu^m} = \frac{(1+z)^{\mu/2}}{(1-z)^{\mu/2}} \sum_{k=0}^{\infty} \frac{1}{\Gamma(k-\mu+1)\,k!} \left(\frac{1-z}{2}\right)^k \sum_{j=0}^{m} \binom{m}{j} \sum_{i=1}^{k} S_k^{(i)}\,(i-j+1)_j\,\nu^{i-j} \sum_{r=1}^{k} (-1)^r\,S_k^{(r)}\,(j-m+r+1)_{m-j}\,(\nu+1)^{j-m+r} \;/;$$
$$\left|\frac{1-z}{2}\right| < 1 \bigwedge m \in \mathbb{N}$$

With respect to z

07.08.20.0012.02

$$\frac{\partial^m P_\nu^\mu(z)}{\partial z^m} =$$
$$\frac{(-1)^m\,(1+z)^{\mu/2}}{(1-z)^{\mu/2+m}}\,\Gamma\!\left(\frac{\mu}{2}+1\right) \sum_{j=0}^{m} \binom{m}{j}\,{}_2\tilde{F}_1\!\left(-j, \frac{\mu}{2}; \frac{\mu}{2}-j+1; \frac{z+1}{z-1}\right) {}_3\tilde{F}_2\!\left(1, -\nu, \nu+1; j-m+1, 1-\mu; \frac{1-z}{2}\right) \left(\frac{z-1}{z+1}\right)^j \;/;\; m \in \mathbb{N}$$

07.08.20.0014.01

$$\frac{\partial^m P_\nu^\mu(z)}{\partial z^m} = \frac{\Gamma\!\left(1-\frac{\mu}{2}\right)\Gamma(\mu+\nu+1)}{\Gamma(-\mu+\nu+1)} \sum_{k=0}^{m}\sum_{j=0}^{k} \frac{(-1)^j\,2^{2j-k}\,k!\binom{m}{k}\Gamma(-k+m-\mu+\nu+1)}{(k-j)!\,(2j-k)!\,\Gamma\!\left(1-j-\frac{\mu}{2}\right)\Gamma(k-m+\mu+\nu+1)}\, z^{2j-k}\left(1-z^2\right)^{\frac{1}{2}(-2j+k-m)} P_\nu^{k-m+\mu}(z) \;/;$$
$$m \in \mathbb{N}$$

07.08.20.0015.01

$$\frac{\partial^m P_\nu^\mu(z)}{\partial z^m} = \sqrt{\pi}\sum_{k=0}^{m}(-1)^{m-k}\,z^{-k}\left(1-z^2\right)^{\frac{k-m}{2}}\binom{m}{k}(-\mu-\nu)_{m-k}\,{}_3\tilde{F}_2\!\left(1, -k, \frac{\mu}{2}; \frac{1-k}{2}, 1-\frac{k}{2}; \frac{z^2}{z^2-1}\right)(1-\mu+\nu)_{m-k}\,P_\nu^{k-m+\mu}(z) \;/;$$
$$m \in \mathbb{N}$$

Fractional integro-differentiation

With respect to z

07.08.20.0013.01

$$\frac{\partial^\alpha P_\nu^\mu(z)}{\partial z^\alpha} = \frac{\mu}{2}\,z^{1-\alpha}\sum_{k=0}^{\infty}(k+1)!\left(\frac{\mu}{2}+1\right)_k z^k\,\tilde{F}_{1\times0\times2}^{1\times1\times3}\!\left(\begin{matrix}k+2; -\frac{\mu}{2}; -\nu, \nu+1, 1-\frac{\mu}{2}; \\ k-\alpha+2;; k+2, 1-\mu;\end{matrix}\; -z, -\frac{z}{2}\right) +$$
$$\Gamma\!\left(1-\frac{\mu}{2}\right)z^{-\alpha}\sum_{k=0}^{\infty}\left(-\frac{\mu}{2}\right)_k(-z)^k\,\tilde{F}_{2\times1\times1}^{3\times1\times1}\!\left(\begin{matrix}-\nu, \nu+1, 1-\frac{\mu}{2}; 1; k+1; \\ 1, 1-\mu; 1-\frac{\mu}{2}; k-\alpha+1;\end{matrix}\; \frac{1}{2}, -\frac{z}{2}\right)$$

Integration

Indefinite integration

Involving only one direct function

07.08.21.0001.01

$$\int P_\nu^\mu(z)\,dz = \frac{\mu}{2}\,z^2\sum_{k=0}^{\infty}(k+1)!\left(\frac{\mu}{2}+1\right)_k z^k\,\tilde{F}_{1\times0\times2}^{1\times1\times3}\!\left(\begin{matrix}k+2; -\frac{\mu}{2}; -\nu, \nu+1, 1-\frac{\mu}{2}; \\ k+3;; k+2, 1-\mu;\end{matrix}\; -z, -\frac{z}{2}\right) +$$
$$\Gamma\!\left(1-\frac{\mu}{2}\right)z\sum_{k=0}^{\infty}\left(-\frac{\mu}{2}\right)_k(-z)^k\,\tilde{F}_{2\times1\times1}^{3\times1\times1}\!\left(\begin{matrix}-\nu, \nu+1, 1-\frac{\mu}{2}; 1; k+1; \\ 1, 1-\mu; 1-\frac{\mu}{2}; k+2;\end{matrix}\; \frac{1}{2}, -\frac{z}{2}\right)$$

Definite integration

Involving the direct function

07.08.21.0002.01

$$\int_{-1}^{1} P_n^m(t)^2 \, dt = \frac{2\,(m+n)!}{(2\,n+1)\,(n-m)!} \;/;\; m \in \mathbb{N} \land n \in \mathbb{N} \land m \leq n$$

Orthogonality:

07.08.21.0003.01

$$\int_{-1}^{1} P_n^m(t)\, P_l^m(t)\, dt = \frac{2\,(m+n)!\,\delta_{n,l}}{(2\,n+1)\,(n-m)!} \;/;\; l \in \mathbb{N} \bigwedge m \in \mathbb{N} \bigwedge n \in \mathbb{N}$$

Summation

Finite summation

07.08.23.0001.01

$$\sum_{m=-n}^{n} (-1)^m\, P_n^m(z)\, P_n^{-m}(z) = 1 \;/;\; n \in \mathbb{N}$$

07.08.23.0002.01

$$\sum_{m=-n}^{n} \frac{(n-m)!\,\cos(m\,(\phi-\phi_1))}{(m+n)!}\, P_n^m(\cos(\theta))\, P_n^m(\cos(\theta_1)) = P_n(\cos(\theta)\cos(\theta_1) + \cos(\phi-\phi_1)\sin(\theta)\sin(\theta_1)) \;/;$$
$$0 < \theta < \frac{\pi}{2} \bigwedge 0 < \theta_1 < \frac{\pi}{2} \bigwedge 0 < \phi < \frac{\pi}{2} \bigwedge 0 < \phi_1 < \frac{\pi}{2} \bigwedge n \in \mathbb{N}$$

Operations

Limit operation

07.08.25.0001.01

$$\lim_{\nu\to\infty} \nu^{-\mu}\, P_\nu^\mu\!\left(\cos\!\left(\frac{z}{\nu}\right)\right) = J_{-\mu}(z)$$

Representations through more general functions

Through hypergeometric functions

Involving ${}_2\tilde{F}_1$

07.08.26.0001.01

$$P_\nu^\mu(z) = \frac{(1+z)^{\mu/2}}{(1-z)^{\mu/2}}\, {}_2\tilde{F}_1\!\left(-\nu,\, \nu+1;\, 1-\mu;\, \frac{1-z}{2}\right)$$

07.08.26.0003.01

$$P_\nu^m(z) = \frac{(1+z)^{m/2}}{(1-z)^{m/2}}\, 2^{-m}\, (-\nu)_m\, (\nu+1)_m\, {}_2\tilde{F}_1\!\left(m-\nu,\, m+\nu+1;\, m+1;\, \frac{1-z}{2}\right) /;\; m \in \mathbb{N}^+$$

07.08.26.0032.01

$$P_\nu^\mu(z) = -\frac{2^\mu \sin(\pi\nu)}{\sin(\pi\mu)} \left(1-z^2\right)^{-\frac{\mu}{2}} {}_2\tilde{F}_1\left(-\mu+\nu+1, -\mu-\nu; 1-\mu; \frac{z+1}{2}\right) - \frac{\pi\,(z+1)^{\mu/2}}{(1-z)^{\mu/2}\sin(\pi\mu)\,\Gamma(-\mu-\nu)\,\Gamma(-\mu+\nu+1)}\, {}_2\tilde{F}_1\left(-\nu, \nu+1; \mu+1; \frac{z+1}{2}\right) /; \mu \notin \mathbb{Z}$$

07.08.26.0004.01

$$P_\nu^\mu(z) = 2^{-\nu} \frac{(1+z)^{\mu/2+\nu}}{(1-z)^{\mu/2}}\, {}_2\tilde{F}_1\left(-\nu, -\mu-\nu; 1-\mu; \frac{z-1}{z+1}\right)$$

07.08.26.0033.01

$$P_\nu^\mu(z) = \frac{2^{-\nu-1}\,\Gamma\left(-\nu-\frac{1}{2}\right)\Gamma(2\nu+2)\,(z-1)^{-\nu-1}\,(z+1)^{\mu/2}}{\sqrt{\pi}\;\Gamma(-\mu-\nu)\,(1-z)^{\mu/2}}\, {}_2\tilde{F}_1\left(\nu+1, \mu+\nu+1; 2\nu+2; \frac{2}{1-z}\right) + \frac{2^\nu\,\Gamma\left(\nu+\frac{1}{2}\right)\Gamma(-2\nu)\,(z+1)^{\mu/2}\,(z-1)^\nu}{\sqrt{\pi}\;\Gamma(\nu-\mu+1)\,(1-z)^{\mu/2}}\, {}_2\tilde{F}_1\left(-\nu, \mu-\nu; -2\nu; \frac{2}{1-z}\right) /; z \notin (-1, 1) \bigwedge 2\nu \notin \mathbb{Z}$$

07.08.26.0034.01

$$P_\nu^\mu(z) = 2^\mu\,\pi\left(1-z^2\right)^{-\frac{\mu}{2}} \left(\frac{1}{\Gamma\left(\frac{1}{2}(-\mu-\nu+1)\right)\Gamma\left(\frac{1}{2}(-\mu+\nu+2)\right)}\, {}_2\tilde{F}_1\left(\frac{1}{2}(-\mu+\nu+1), -\frac{1}{2}(\mu+\nu); \frac{1}{2}; z^2\right) - \frac{z}{\Gamma\left(\frac{1}{2}(-\mu-\nu)\right)\Gamma\left(\frac{1}{2}(-\mu+\nu+1)\right)}\, {}_2\tilde{F}_1\left(\frac{1}{2}(-\mu-\nu+1), \frac{1}{2}(-\mu+\nu+2); \frac{3}{2}; z^2\right)\right)$$

07.08.26.0035.01

$$P_\nu^\mu(z) = -\frac{2^{-\nu-3}\,(\cos(2\pi\mu)-\cos(2\pi\nu))\sec(\pi\nu)}{\sqrt{\pi}} \left(1-z^2\right)^{-\frac{\mu}{2}} \left(\frac{\Gamma(\mu+\nu+1)}{z}\left(-z^2\right)^{\frac{\mu-\nu}{2}} \left(\csc\left(\frac{1}{2}\pi(\mu-\nu)\right) - \frac{\sqrt{-z^2}}{z}\sec\left(\frac{1}{2}\pi(\mu-\nu)\right)\right) {}_2\tilde{F}_1\left(\frac{1}{2}(-\mu+\nu+1), \frac{1}{2}(-\mu+\nu+2); \nu+\frac{3}{2}; \frac{1}{z^2}\right) + 2^{2\nu+1}\,\Gamma(\mu-\nu)\left(\csc\left(\frac{1}{2}\pi(\mu+\nu)\right) - \frac{\sqrt{-z^2}}{z}\sec\left(\frac{1}{2}\pi(\mu+\nu)\right)\right) \left(-z^2\right)^{\frac{\mu+\nu}{2}}\, {}_2\tilde{F}_1\left(\frac{1}{2}(-\mu-\nu), \frac{1}{2}(-\mu-\nu+1); \frac{1}{2}-\nu; \frac{1}{z^2}\right)\right) /; z \notin (-1, 0)$$

Involving ${}_2F_1$

07.08.26.0005.01

$$P_\nu^\mu(z) = \frac{1}{\Gamma(1-\mu)}\,\frac{(1+z)^{\mu/2}}{(1-z)^{\mu/2}}\, {}_2F_1\left(-\nu, \nu+1; 1-\mu; \frac{1-z}{2}\right) /; \mu \notin \mathbb{N}^+$$

07.08.26.0006.01

$$P_\nu^\mu(z) = \frac{\Gamma(-\mu)}{\Gamma(-\mu-\nu)\,\Gamma(\nu-\mu+1)}\,\frac{(1+z)^{\mu/2}}{(1-z)^{\mu/2}}\,{}_2F_1\!\left(-\nu,\,\nu+1;\,\mu+1;\,\frac{z+1}{2}\right) - \frac{2^\mu\sin(\pi\nu)\,\Gamma(\mu)}{\pi}\left(1-z^2\right)^{-\mu/2}{}_2F_1\!\left(\nu-\mu+1,\,-\mu-\nu;\,1-\mu;\,\frac{z+1}{2}\right)\,/;\,\mu\notin\mathbb{Z}$$

07.08.26.0036.01

$$P_\nu^\mu(z) = \frac{2^{-\nu}\,(z+1)^{\frac{\mu}{2}+\nu}}{\Gamma(\mu)\,(1-z)^{\mu/2}}\,{}_2F_1\!\left(-\nu,\,-\mu-\nu;\,1-\mu;\,\frac{z-1}{z+1}\right)\,/;\,\mu\notin\mathbb{Z}$$

07.08.26.0007.01

$$P_\nu^\mu(z) = \frac{2^{-\nu-1}\,\Gamma\!\left(-\nu-\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(-\mu-\nu)}\,(z-1)^{-\nu-1}\,\frac{(1+z)^{\mu/2}}{(1-z)^{\mu/2}}\,{}_2F_1\!\left(\nu+1,\,\mu+\nu+1;\,2\nu+2;\,\frac{2}{1-z}\right) + \frac{2^\nu\,\Gamma\!\left(\nu+\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(\nu-\mu+1)}\,\frac{(1+z)^{\mu/2}\,(z-1)^\nu}{(1-z)^{\mu/2}}\,{}_2F_1\!\left(-\nu,\,\mu-\nu;\,-2\nu;\,\frac{2}{1-z}\right)\,/;\,z\notin(-1,\,1)\bigwedge 2\nu\notin\mathbb{Z}$$

07.08.26.0037.01

$$P_\nu^\mu(z) = 2^\mu\sqrt{\pi}\,\left(1-z^2\right)^{-\frac{\mu}{2}}\left(\frac{1}{\Gamma\!\left(\frac{1}{2}(-\mu-\nu+1)\right)\Gamma\!\left(\frac{1}{2}(-\mu+\nu+2)\right)}\,{}_2F_1\!\left(\frac{1}{2}(-\mu+\nu+1),\,-\frac{1}{2}(\mu+\nu);\,\frac{1}{2};\,z^2\right) - \frac{2z}{\Gamma\!\left(\frac{1}{2}(-\mu-\nu)\right)\Gamma\!\left(\frac{1}{2}(-\mu+\nu+1)\right)}\,{}_2F_1\!\left(\frac{1}{2}(-\mu-\nu+1),\,\frac{1}{2}(-\mu+\nu+2);\,\frac{3}{2};\,z^2\right)\right)$$

07.08.26.0038.01

$$P_\nu^\mu(z) = \frac{2^{-\nu-2}\,(\cos(2\pi\mu)-\cos(2\pi\nu))}{\pi\sqrt{\pi}}\left(1-z^2\right)^{-\frac{\mu}{2}}\left(\frac{\Gamma(\mu+\nu+1)\,\Gamma\!\left(\frac{1}{2}-\nu\right)}{z\,(2\nu+1)}\left(\frac{\sqrt{-z^2}}{z}\sec\!\left(\frac{1}{2}\pi(\mu-\nu)\right)-\csc\!\left(\frac{1}{2}\pi(\mu-\nu)\right)\right)\left(-z^2\right)^{\frac{\mu-\nu}{2}}{}_2F_1\!\left(\frac{1}{2}(-\mu+\nu+1),\,\frac{1}{2}(-\mu+\nu+2);\,\nu+\frac{3}{2};\,\frac{1}{z^2}\right) + 2^{2\nu}\,\Gamma(\mu-\nu)\,\Gamma\!\left(\nu+\frac{1}{2}\right)\left(\frac{\sqrt{-z^2}}{z}\sec\!\left(\frac{1}{2}\pi(\mu+\nu)\right)-\csc\!\left(\frac{1}{2}\pi(\mu+\nu)\right)\right)\left(-z^2\right)^{\frac{\mu+\nu}{2}}{}_2F_1\!\left(\frac{1}{2}(-\mu-\nu),\,\frac{1}{2}(-\mu-\nu+1);\,\frac{1}{2}-\nu;\,\frac{1}{z^2}\right)\right)/;\,z\notin(-1,\,0)$$

Through Meijer *G*

Classical cases for the direct function itself

07.08.26.0008.01

$$P_\nu^\mu(z) = -\frac{\sin(\pi\nu)}{\pi}\,\frac{(1+z)^{\mu/2}}{(1-z)^{\mu/2}}\,G_{2,2}^{1,2}\!\left(\frac{z-1}{2}\,\middle|\,\begin{matrix}\nu+1,\,-\nu\\ 0,\,\mu\end{matrix}\right)/;\,\nu\notin\mathbb{Z}$$

07.08.26.0009.01

$$P_n^\mu(z) = -\frac{1}{\pi}\,\frac{(1+z)^{\mu/2}}{(1-z)^{\mu/2}}\,\lim_{\nu\to n}\sin(\pi\nu)\,G_{2,2}^{1,2}\!\left(\frac{z-1}{2}\,\middle|\,\begin{matrix}\nu+1,\,-\nu\\ 0,\,\mu\end{matrix}\right)/;\,n\in\mathbb{Z}$$

Classical cases involving algebraic functions

07.08.26.0010.01

$$(z+1)^{-\frac{\mu}{2}} P_\nu^\mu(2z+1) = -\frac{\sin(\pi\nu)}{\pi} (-z)^{-\mu/2} G_{2,2}^{1,2}\left(z \,\middle|\, \begin{matrix} -\nu, \nu+1 \\ 0, \mu \end{matrix}\right)$$

07.08.26.0011.01

$$(-z-1)^{-\frac{\mu}{2}} P_\nu^\mu\left(\frac{z+2}{z}\right) = -\frac{\sin(\pi\nu)}{\pi} \left(-\frac{1}{z}\right)^{-\mu/2} (-z)^{-\mu/2} G_{2,2}^{2,1}\left(z \,\middle|\, \begin{matrix} 1, 1-\mu \\ -\nu, \nu+1 \end{matrix}\right)$$

07.08.26.0012.01

$$(z+1)^\nu P_\nu^\mu\left(\frac{1-z}{1+z}\right) = \frac{1}{\Gamma(-\mu-\nu)\,\Gamma(-\nu)} G_{2,2}^{1,2}\left(z \,\middle|\, \begin{matrix} 1-\frac{\mu}{2}+\nu, \frac{\mu}{2}+\nu+1 \\ -\frac{\mu}{2}, \frac{\mu}{2} \end{matrix}\right) /; z \notin (-\infty, -1)$$

07.08.26.0013.01

$$(z+1)^\nu P_\nu^\mu\left(\frac{z-1}{z+1}\right) = \frac{1}{\Gamma(-\mu-\nu)\,\Gamma(-\nu)} G_{2,2}^{2,1}\left(z \,\middle|\, \begin{matrix} \frac{\mu}{2}+\nu+1, 1-\frac{\mu}{2}+\nu \\ \frac{\mu}{2}, -\frac{\mu}{2} \end{matrix}\right) /; z \notin (-\infty, 0)$$

07.08.26.0014.01

$$(z+1)^{\nu/2} P_\nu^\mu\left(\frac{1}{\sqrt{z+1}}\right) = \frac{2^{-\nu-1}}{\sqrt{\pi}\,\Gamma(-\mu-\nu)} G_{2,2}^{1,2}\left(z \,\middle|\, \begin{matrix} \frac{\nu+1}{2}, \frac{\nu}{2}+1 \\ -\frac{\mu}{2}, \frac{\mu}{2} \end{matrix}\right)$$

07.08.26.0015.01

$$(z+1)^{\nu/2} P_\nu^\mu\left(\sqrt{\frac{z}{z+1}}\right) = \frac{2^{-\nu-1}}{\sqrt{\pi}\,\Gamma(-\mu-\nu)} G_{2,2}^{2,1}\left(z \,\middle|\, \begin{matrix} \frac{\mu+\nu}{2}+1, \frac{\nu-\mu}{2}+1 \\ 0, \frac{1}{2} \end{matrix}\right) /; z \notin (-\infty, 0)$$

Classical cases involving unit step θ

07.08.26.0016.01

$$\theta(1-|z|)\,(1-z)^{-\frac{\mu}{2}} P_\nu^\mu(2z-1) = G_{2,2}^{2,0}\left(z \,\middle|\, \begin{matrix} -\frac{\mu}{2}-\nu, 1-\frac{\mu}{2}+\nu \\ -\frac{\mu}{2}, \frac{\mu}{2} \end{matrix}\right) /; z \notin (-1, 0)$$

07.08.26.0017.01

$$\theta(|z|-1)\,(z-1)^{-\frac{\mu}{2}} P_\nu^\mu\left(\frac{2}{z}-1\right) = G_{2,2}^{0,2}\left(z \,\middle|\, \begin{matrix} 1, 1-\mu \\ -\nu, \nu+1 \end{matrix}\right) /; z \notin (-\infty, -1)$$

07.08.26.0018.01

$$\theta(1-|z|)\,(1-z)^{-\frac{\mu}{2}} P_\nu^\mu\left(\sqrt{z}\right) = 2^\mu G_{2,2}^{2,0}\left(z \,\middle|\, \begin{matrix} \frac{\nu-\mu}{2}+1, \frac{1-\mu-\nu}{2} \\ 0, \frac{1}{2} \end{matrix}\right) /; z \notin (-1, 0)$$

07.08.26.0019.01

$$\theta(|z|-1)\,(z-1)^{-\frac{\mu}{2}} P_\nu^\mu\left(\frac{1}{\sqrt{z}}\right) = 2^\mu G_{2,2}^{0,2}\left(z \,\middle|\, \begin{matrix} \frac{1-\mu}{2}, 1-\frac{\mu}{2} \\ -\frac{\nu}{2}, \frac{\nu+1}{2} \end{matrix}\right)$$

07.08.26.0020.01

$$\frac{\theta(1-|z|)}{\sqrt{1-z}} \left(P_\nu^\mu\left(-\sqrt{1-z}\right) + P_\nu^\mu\left(\sqrt{1-z}\right)\right) = \frac{2^{\mu+1}\pi}{\Gamma\left(\frac{1}{2}(1-\mu-\nu)\right)\Gamma\left(\frac{1}{2}(2-\mu+\nu)\right)} G_{2,2}^{2,0}\left(z \,\middle|\, \begin{matrix} -\frac{\nu}{2}, \frac{\nu+1}{2} \\ -\frac{\mu}{2}, \frac{\mu}{2} \end{matrix}\right) /; z \notin (-1, 0)$$

07.08.26.0021.01

$$\frac{\theta(|z|-1)}{\sqrt{z-1}} \left(P_\nu^\mu\left(-\sqrt{\frac{z-1}{z}}\right) + P_\nu^\mu\left(\sqrt{\frac{z-1}{z}}\right)\right) = \frac{2^{\mu+1}\pi}{\Gamma\left(\frac{1}{2}(-\mu-\nu+1)\right)\Gamma\left(\frac{1}{2}(-\mu+\nu+2)\right)} G_{2,2}^{0,2}\left(z \,\middle|\, \begin{matrix} \frac{1-\mu}{2}, \frac{\mu+1}{2} \\ -\frac{\nu}{2}, \frac{\nu+1}{2} \end{matrix}\right) /; z \notin (-\infty, -1)$$

Classical cases involving sgn

07.08.26.0022.01

$$\frac{((z-1)\,\text{sgn}(|z|-1))^{\nu+\frac{1}{2}}}{\sqrt{\sqrt{z}+1}}\,P_\nu^{\nu+\frac{1}{2}}\left(\frac{2\sqrt[4]{z}}{\sqrt{z}+1}\right) = \frac{2^{-\nu-\frac{3}{2}}}{\pi\,\Gamma\left(-2\nu-\frac{1}{2}\right)\Gamma(-\nu)}\,G_{2,2}^{2,2}\left(z\,\middle|\,\begin{matrix}\nu+1,\ \nu+\frac{5}{4}\\ 0,\ \frac{1}{4}\end{matrix}\right)$$

07.08.26.0023.01

$$\frac{((1-z)\,\text{sgn}(1-|z|))^{-\mu}}{\sqrt{z+1}}\,P_{-\frac{1}{4}}^{\mu}\left(\frac{z^2-6z+1}{(z+1)^2}\right) = \frac{\Gamma\left(\frac{1}{2}-\mu\right)}{2^\mu\sqrt{\pi}}\,G_{2,2}^{1,1}\left(z\,\middle|\,\begin{matrix}\frac{1-\mu}{2},\ \frac{1-3\mu}{2}\\ -\frac{\mu}{2},\ \frac{\mu}{2}\end{matrix}\right)\ /;\ z\notin(-\infty,0)$$

07.08.26.0024.01

$$\frac{((1-z)\,\text{sgn}(1-|z|))^{\mu}}{\sqrt{z+1}}\,P_{-\frac{1}{4}}^{\mu}\left(-\frac{z^2-6z+1}{(z+1)^2}\right) = \frac{2^{-\mu-\frac{1}{2}}\,\Gamma\left(\mu+\frac{1}{2}\right)\cos(\pi\mu)}{\pi^2\,\Gamma\left(\frac{1}{2}-2\mu\right)}\,G_{2,2}^{2,2}\left(z\,\middle|\,\begin{matrix}\frac{\mu+1}{2},\ \frac{1+3\mu}{2}\\ -\frac{\mu}{2},\ \frac{\mu}{2}\end{matrix}\right)\ /;\ z\notin(-\infty,-1)$$

Generalized cases involving algebraic functions

07.08.26.0025.01

$$\left(z^2+1\right)^{\nu/2}P_\nu^\mu\left(\frac{z}{\sqrt{z^2+1}}\right) = \frac{2^{-\nu-1}}{\sqrt{\pi}\,\Gamma(-\mu-\nu)}\,G_{2,2}^{2,1}\left(z,\ \frac{1}{2}\,\middle|\,\begin{matrix}\frac{\mu+\nu}{2}+1,\ \frac{\nu-\mu}{2}+1\\ 0,\ \frac{1}{2}\end{matrix}\right)\ /;\ \text{Re}(z)>0$$

Generalized cases involving unit step θ

07.08.26.0026.01

$$\theta(1-|z|)\left(1-z^2\right)^{-\frac{\mu}{2}}P_\nu^\mu(z) = 2^\mu\,G_{2,2}^{2,0}\left(z,\ \frac{1}{2}\,\middle|\,\begin{matrix}\frac{\nu-\mu}{2}+1,\ \frac{1-\mu-\nu}{2}\\ 0,\ \frac{1}{2}\end{matrix}\right)$$

07.08.26.0027.01

$$\theta(|z|-1)\left(z^2-1\right)^{-\frac{\mu}{2}}P_\nu^\mu\left(\frac{1}{z}\right) = 2^\mu\,G_{2,2}^{0,2}\left(z,\ \frac{1}{2}\,\middle|\,\begin{matrix}\frac{1-\mu}{2},\ 1-\frac{\mu}{2}\\ -\frac{\nu}{2},\ \frac{\nu+1}{2}\end{matrix}\right)\ /;\ |z|<0\lor\text{Re}(z)>0$$

07.08.26.0028.01

$$\frac{\theta(|z|-1)}{\sqrt{z^2-1}}\left(P_\nu^\mu\left(\frac{\sqrt{z^2-1}}{z}\right)+P_\nu^\mu\left(-\frac{\sqrt{z^2-1}}{z}\right)\right) = \frac{2^{\mu+1}\pi}{\Gamma\left(\frac{1-\mu-\nu}{2}\right)\Gamma\left(\frac{\nu-\mu}{2}+1\right)}\,G_{2,2}^{0,2}\left(z,\ \frac{1}{2}\,\middle|\,\begin{matrix}\frac{1-\mu}{2},\ \frac{\mu+1}{2}\\ -\frac{\nu}{2},\ \frac{\nu+1}{2}\end{matrix}\right)\ /;\ |z|<0\lor\text{Re}(z)>0$$

Generalized cases involving sgn

07.08.26.0029.01

$$\frac{\left(\text{sgn}(|z|-1)\left(z^2-1\right)\right)^{\nu+\frac{1}{2}}}{\sqrt{z+1}}\,P_\nu^{\nu+\frac{1}{2}}\left(\frac{2\sqrt{z}}{z+1}\right) = \frac{2^{-\nu-\frac{3}{2}}}{\pi\,\Gamma\left(-2\nu-\frac{1}{2}\right)\Gamma(-\nu)}\,G_{2,2}^{2,2}\left(z,\ \frac{1}{2}\,\middle|\,\begin{matrix}\nu+1,\ \nu+\frac{5}{4}\\ 0,\ \frac{1}{4}\end{matrix}\right)\ /;\ |z|<0\lor\text{Re}(z)>0$$

07.08.26.0030.01

$$\frac{\left(\operatorname{sgn}(|z|-1)\left(z^2-1\right)\right)^{\nu+\frac{1}{2}}}{\sqrt{z+1}} P_\nu^{-\nu-\frac{1}{2}}\left(\frac{2\sqrt{z}}{z+1}\right) = -\frac{\Gamma(\nu+1)}{2^{\nu+\frac{1}{2}}\sqrt{\pi}}\left(\cos\left(\pi\left(\nu+\frac{1}{4}\right)\right) G_{2,2}^{1,1}\left(z, \frac{1}{2} \,\middle|\, \begin{matrix} \nu+1, \nu+\frac{5}{4} \\ \frac{1}{4}, 0 \end{matrix}\right) + \sin\left(\pi\left(\nu+\frac{1}{4}\right)\right) G_{2,2}^{1,1}\left(z, \frac{1}{2} \,\middle|\, \begin{matrix} \nu+\frac{5}{4}, \nu+1 \\ 0, \frac{1}{4} \end{matrix}\right)\right) /; \operatorname{Re}(z) > 0$$

Through other functions

Involving some hypergeometric-type functions

07.08.26.0031.01

$$P_\nu^\mu(z) = \frac{\Gamma(\nu+1)}{\Gamma(\nu-\mu+1)} \frac{(1+z)^{\mu/2}}{(1-z)^{\mu/2}} P_\nu^{(-\mu,\mu)}(z)$$

Involving spheroidal functions

07.08.26.0039.01

$$P_\nu^\mu(z) = PS_{\nu,\mu}(0, z)$$

Representations through equivalent functions

With related functions

Involving Gegenbauer functions

07.08.27.0001.01

$$P_\nu^\mu(z) = \frac{2^{-\mu}\,\Gamma\left(\frac{1}{2}-\mu\right)\Gamma(\mu+\nu+1)}{\sqrt{\pi}\,\Gamma(1-\mu+\nu)}\left(1-z^2\right)^{-\mu/2} C_{\mu+\nu}^{\frac{1}{2}-\mu}(z)$$

Involving Legendre functions

07.08.27.0002.01

$$P_\nu^\mu(z) = \frac{(z-1)^{\mu/2}}{(1-z)^{\mu/2}} \mathfrak{P}_\nu^\mu(z)$$

07.08.27.0013.01

$$P_\nu^\mu(z) = -\csc(\mu\pi)\left(\sin(\nu\pi)\frac{(-z-1)^{\mu/2}}{(z+1)^{\mu/2}} \mathfrak{P}_\nu^\mu(-z) + \frac{\pi}{\Gamma(-\mu-\nu)\,\Gamma(\nu-\mu+1)} \frac{(z+1)^{\mu/2}}{(-z-1)^{\mu/2}} \mathfrak{P}_\nu^{-\mu}(-z)\right)$$

07.08.27.0003.01

$$P_\nu^\mu(z) = -\frac{2\csc(\mu\pi)}{\pi}\left(\cos(\mu\pi)\, Q_\nu^\mu(z) - (\nu-\mu+1)_{2\mu}\, Q_\nu^{-\mu}(z)\right) /; \mu \notin \mathbb{Z}$$

07.08.27.0004.01

$$P_\nu^\mu(z) = \frac{2\csc(\pi\mu)}{\pi^2}\left(\cos(\pi(\mu+\nu))\sin(\pi(\mu-\nu))\,\Gamma(\mu-\nu)\,\Gamma(\mu+\nu+1)\, Q_\nu^{-\mu}(-z) - \pi\cos(\pi\nu)\, Q_\nu^\mu(-z)\right) /; \mu \notin \mathbb{Z}$$

07.08.27.0005.01

$$P_\nu^\mu(x) = e^{\frac{\pi i \mu}{2}} \lim_{z \to x+i0} \mathfrak{P}_\nu^\mu(z) /; x < -1$$

07.08.27.0006.01

$$P_\nu^\mu(x) = e^{-\frac{\pi i \mu}{2}} \lim_{z \to x + i 0} \mathfrak{P}_\nu^\mu(z) /; x > 1$$

07.08.27.0007.01

$$P_\nu^\mu(x) = e^{\frac{\pi i \mu}{2}} \mathfrak{P}_\nu^\mu(x) /; x < -1$$

07.08.27.0008.01

$$P_\nu^\mu(x) = e^{-\frac{\pi i \mu}{2}} \mathfrak{P}_\nu^\mu(x) /; x > 1$$

07.08.27.0009.01

$$P_\nu^\mu(x) = e^{-\frac{\pi i \mu}{2}} \lim_{z \to x - i 0} \mathfrak{P}_\nu^\mu(z) /; -1 < x < 1$$

07.08.27.0010.01

$$P_\nu^\mu(x) = e^{\frac{\pi i \mu}{2}} \mathfrak{P}_\nu^\mu(x) /; -1 < x < 1$$

07.08.27.0011.01

$$P_\nu^\mu(x) = \frac{1}{\pi i} e^{-\pi i \mu} \left(e^{\frac{\pi i \mu}{2}} \lim_{z \to x - i 0} \mathfrak{Q}_\nu^\mu(z) - e^{-\frac{\pi i \mu}{2}} \lim_{z \to x + i 0} \mathfrak{Q}_\nu^\mu(z) \right) /; -1 < x < 1$$

Involving spherical harmonic functions

07.08.27.0012.01

$$P_\nu^\mu(z) = \frac{2 \sqrt{\pi} \sqrt{\Gamma(\mu + \nu + 1)}}{\sqrt{2 \nu + 1} \sqrt{\Gamma(1 - \mu + \nu)}} Y_\nu^\mu\left(\cos^{-1}(z), 0\right)$$

Theorems

One infinite sum

$$\sum_{n=0}^{\infty} (2 n + 1) i^n \sum_{m=0}^{n} (2 - \delta_m) \frac{(n - m)!}{(n + m)!} \cos(\varphi - \alpha) P_n^m(\cos(\beta)) P_n^m(\cos(\vartheta)) J_{n+\frac{1}{2}}(|\boldsymbol{k}| |\boldsymbol{r}|) = e^{i \boldsymbol{k}.\boldsymbol{r}} /;$$
$$r = |\boldsymbol{r}| \{\cos(\varphi) \sin(\vartheta), \sin(\varphi) \sin(\vartheta), \cos(\vartheta)\} \bigwedge \boldsymbol{k} = |\boldsymbol{k}| \{\cos(\alpha) \sin(\beta), \sin(\alpha) \sin(\beta), \cos(\beta)\}$$

Eigenfunctions of the Schrödinger equation

Legendre functions P_n^m are eigenfunctions of the Schrödinger equation with reflectionless, shape invariant, supersymmetric potential:$-\frac{\partial^2 \psi(x)}{\partial x^2} - \left(n(n + 1) \operatorname{sech}^2(x)\right) \psi(x) = -m^2 \psi(x)$.

The solution of Dirichlet problem for Laplace equation in spherical coordinates

The solution of the Dirichlet problem $\psi(1, \phi, \vartheta) = \psi_0(\phi, \vartheta)$ for the $\Delta\psi(\phi, \vartheta) = 0$ on unit sphere in spherical coordinates is given by:

$$\psi(r, \phi, \vartheta) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{2 n + 1}{2 \pi (1 + \delta_m)} \frac{(n - m)!}{(n + m)!} r^n P_n^m(\cos(\vartheta)) \int_0^\pi \left(\int_{-\pi}^{\pi} \psi_0\left(\tilde{\phi}, \tilde{\vartheta}\right) \cos\left(m\left(\phi - \tilde{\phi}\right)\right) d\tilde{\phi} \right) P_n^m\left(\cos\left(\tilde{\vartheta}\right)\right) \sin\left(\tilde{\vartheta}\right) d\tilde{\vartheta}$$

History

– D. Bernoulli (1748)
– A. M. Legendre (1782)
– E. Heine (1842)
– F. Neumann (1848)
– L. Schläfli (1881) used complex μ, ν
– E. Hobson (1896)

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