Multinomial

Notations

Traditional name

Multinomial coefficient

Traditional notation

$(n_1 + n_2 + \ldots + n_m; n_1, n_2, \ldots, n_m)$

Mathematica StandardForm notation

`Multinomial[`n_1`, `n_2`, …, `n_m`]`

Primary definition

06.04.02.0001.01

$$(n_1 + n_2 + \ldots + n_m; n_1, n_2, \ldots, n_m) = \frac{\Gamma(n+1)}{\prod_{k=1}^{m} \Gamma(n_k + 1)} \;/;\; -n \notin \mathbb{N}^+ \bigwedge n = \sum_{k=1}^{m} n_k$$

06.04.02.0002.01

$$(n_1 + n_2 + \ldots + n_m; n_1, n_2, \ldots, n_m) = 0 \;/;\; -n \in \mathbb{N}^+ \bigwedge n = \sum_{k=1}^{m} n_k$$

$(n; n_1, n_2, \ldots, n_m)$ is the number of ways of putting $n = n_1 + n_2 + \ldots n_m$ different objects into m different boxes with n_k in the kth box, $k = 1, 2, \ldots, m$.

Specific values

Specialized values

06.04.03.0001.01

$$(n; n) = 1$$

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$$(n_1 + n_2; n_1, n_2) = \binom{n_1 + n_2}{n_2}$$

Values at fixed points

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$$(0; 0, 0, \ldots, 0) = 1$$

General characteristics

Domain and analyticity

$(n_1 + n_2 + \ldots + n_m;\, n_1,\, n_2,\, \ldots,\, n_m)$ is an analytical function of $n_1, n_2, \ldots, n_m$ which is defined over $\mathbb{C}^m$.

06.04.04.0001.01

$$(n_1 * n_2 * \ldots * n_m) \longrightarrow (n_1 + n_2 + \ldots + n_m;\, n_1,\, n_2,\, \ldots,\, n_m) :: \mathbb{C}^m \longrightarrow \mathbb{C}$$

Symmetries and periodicities

Mirror symmetry

06.04.04.0002.01

$$(\overline{n_1} + \overline{n_2} + \ldots + \overline{n_m};\, \overline{n_1},\, \overline{n_2},\, \ldots,\, \overline{n_m}) = \overline{(n_1 + n_2 + \ldots + n_m;\, n_1,\, n_2,\, \ldots,\, n_m)}$$

Permutation symmetry

06.04.04.0003.01

$$(n_1 + n_2;\, n_1,\, n_2) = (n_1 + n_2;\, n_2,\, n_1)$$

06.04.04.0004.01

$$\left(n_1 + n_2 + \ldots + n_k + \ldots + n_j + \ldots + n_m;\, n_1,\, n_2,\, \ldots,\, n_k,\, \ldots,\, n_j,\, \ldots,\, n_m\right) = \left(n_1 + n_2 + \ldots + n_j + \ldots + n_k + \ldots + n_m;\, n_1,\, n_2,\, \ldots,\, n_j,\, \ldots,\, n_k,\, \ldots,\, n_m\right) /;\, n_k \neq n_j \bigwedge k \neq j$$

Periodicity

No periodicity

Poles and essential singularities

With respect to n_k

By variable n_k, $1 \leq k \leq m$, (with fixed other variables) the function $(n_1 + n_2 + \ldots + n_m;\, n_1,\, n_2,\, \ldots,\, n_m)$ has an infinite set of singular points:

a) $n_k = -\tilde{N}_k - j /;\, j \in \mathbb{N}^+$, are the simple poles with residues $\frac{(-1)^{j-1}}{\Gamma\left(1-j-\tilde{N}_k\right) \prod_{r=1}^{k-1} \Gamma(n_r+1) \prod_{r=k+1}^{m} \Gamma(n_r+1)\, (j-1)!} /;\, \tilde{N}_k = \sum_{r=1}^{k-1} n_r + \sum_{r=k+1}^{m} n_r \bigwedge j \in \mathbb{N}^+$;

b) $n_k = \tilde{\infty}$ is the point of convergence of poles, which is an essential singular point.

06.04.04.0005.01

$$\mathcal{Sing}_{n_k}((n_1 + n_2 + \ldots + n_m;\, n_1,\, n_2,\, \ldots,\, n_m)) = \left\{\left\{\left\{-\tilde{N}_k - j,\, 1\right\} /;\, j \in \mathbb{N}^+\right\},\, \{\tilde{\infty},\, \infty\}\right\}$$

06.04.04.0006.01

$$\operatorname{res}_{n_k}((n_1 + n_2 + \ldots + n_m;\, n_1,\, n_2,\, \ldots,\, n_m))\left(-\tilde{N}_k - j\right) = \frac{(-1)^{j-1}}{\Gamma\left(1 - j - \tilde{N}_k\right) \prod_{r=1}^{k-1} \Gamma(n_r + 1) \prod_{r=k+1}^{m} \Gamma(n_r + 1)\, (j-1)!} /;$$

$$\tilde{N}_k = \sum_{r=1}^{k-1} n_r + \sum_{r=k+1}^{m} n_r \bigwedge j \in \mathbb{N}^+$$

Branch points

With respect to n_k

The function $(n_1 + n_2 + \ldots + n_m; n_1, n_2, \ldots, n_m)$ does not have branch points.

06.04.04.0007.01

$$\mathcal{BP}_{n_k}((n_1 + n_2 + \ldots + n_m; n_1, n_2, \ldots, n_m)) = \{\} /; 1 \leq k \leq m$$

Branch cuts

With respect to n_k

The function $(n_1 + n_2 + \ldots + n_m; n_1, n_2, \ldots, n_m)$ does not have branch cuts.

06.04.04.0008.01

$$\mathcal{BC}_{n_k}((n_1 + n_2 + \ldots + n_m; n_1, n_2, \ldots, n_m)) = \{\} /; 1 \leq k \leq m$$

Series representations

Asymptotic series expansions

06.04.06.0001.01

$$(n_1 + n_2 + \ldots + n_m; n_1, n_2, \ldots, n_m) \propto \frac{n_1^{a-1}}{\prod_{k=2}^{m} \Gamma(n_k + 1)} \sum_{k=0}^{\infty} \frac{(-1)^k (1-a)_k B(k, a, a) n_1^{-k}}{k!} /;$$
$$(|n_1| \to \infty) \bigwedge a = \sum_{k=2}^{m} n_k + 1 \bigwedge B(n, \alpha, z) = n! \left([t^n] \frac{t^\alpha e^{t z}}{(e^t - 1)^\alpha}\right) \bigwedge |\arg(a + n_1)| < \pi$$

06.04.06.0002.01

$$(n_1 + n_2 + \ldots + n_m; n_1, n_2, \ldots, n_m) \propto \frac{n_1^{a-1}}{\prod_{k=2}^{m} \Gamma(n_k + 1)} \left(1 + \frac{(a-1) a}{2 n_1} + O\left(\frac{1}{n_1^2}\right)\right) /; (|n_1| \to \infty) \bigwedge a = \sum_{k=2}^{m} n_k + 1 \bigwedge |\arg(a + n_1)| < \pi$$

Other series representations

06.04.06.0003.01

$$(n_1 + n_2 + \ldots + n_m; n_1, n_2, \ldots, n_m) = \left([t_1^{n_1}, t_2^{n_2}, \ldots, t_m^{n_m}] \left(\sum_{k=1}^{m} t_k\right)^n\right) /; n = \sum_{k=1}^{m} n_k$$

Identities

Recurrence identities

Consecutive neighbors

06.04.17.0001.01

$$(n_1 + n_2 + \ldots + n_m; n_1, n_2, \ldots, n_m) = \sum_{l=1}^{m} (n_1 + n_2 + \ldots + n_m - 1; n_1, n_2, \ldots, n_{l-1}, n_l - 1, n_{l+1}, \ldots, n_m)$$

06.04.17.0005.01

$$(n_1 + n_2 + \ldots + n_m; n_1, n_2, \ldots, n_m) = \frac{\sum_{j=1}^{m} n_j}{n_l} (n_1 + n_2 + \ldots + n_m - 1; n_1, n_2, \ldots, n_{l-1}, n_l - 1, n_{l+1}, \ldots, n_m)$$

06.04.17.0006.01

$$(n_1+n_2+\ldots+n_m;\, n_1,\, n_2,\, \ldots,\, n_m) == \frac{n_l+1}{\sum_{j=1}^{m} n_j+1}\,(n_1+n_2+\ldots+n_m+1;\, n_1,\, n_2,\, \ldots,\, n_{l-1},\, n_l+1,\, n_{l+1},\, \ldots,\, n_m)$$

Distant neighbors

06.04.17.0007.01

$$(n_1+n_2+\ldots+n_m;\, n_1,\, n_2,\, \ldots,\, n_m) == \frac{1}{(n_l-p+1)_p}\left(\sum_{j=1}^{m} n_j-p+1\right)_m (n_1+n_2+\ldots+n_m-p;\, n_1,\, n_2,\, \ldots,\, n_{l-1},\, n_l-p\,,\, n_{l+1},\, \ldots,\, n_m)$$

06.04.17.0008.01

$$(n_1+n_2+\ldots+n_m;\, n_1,\, n_2,\, \ldots,\, n_m) == \frac{(n_l+1)_p}{\left(\sum_{j=1}^{m} n_j+1\right)_p}\,(n_1+n_2+\ldots+n_m+p;\, n_1,\, n_2,\, \ldots,\, n_{l-1},\, n_l+p,\, n_{l+1},\, \ldots,\, n_m)$$

Generalized Cauchy summation

06.04.17.0003.01

$$\sum_{j_1=0}^{k_1}\sum_{j_n=0}^{k_1}\ldots\sum_{j_n=0}^{k_1}(p;\, j_1,\, j_2,\, \ldots,\, j_n)\left(q+\sum_{h=1}^{n}k_h;\, k_1-j_1,\, k_2-j_2,\, \ldots,\, k_n-j_n\right) == \left(p+q+\sum_{h=1}^{n}k_h;\, k_1,\, k_2,\, \ldots,\, k_n\right) /;$$
$$k_1\in\mathbb{N}\land k_2\in\mathbb{N}\land\ldots\land k_n\in\mathbb{N}\land p\in\mathbb{N}^+\land q\in\mathbb{N}^+\land n\in\mathbb{N}^+$$

06.04.17.0004.01

$$\sum_{j_1=0}^{k_1}\sum_{j_n=0}^{k_1}\ldots\sum_{j_n=0}^{k_1}(-1)^{\sum_{h=1}^{p} j_{2h}}\,(m;\, j_1,\, j_2,\, \ldots,\, j_n) == 1-\delta_{n \bmod 2} /; m\in\mathbb{N}^+\land n\in\mathbb{N}^+\land p\in\mathbb{N}^+$$

Functional identities

Relations of special kind

06.04.17.0002.01

$$\sum_{k_1=0}^{n}\sum_{k_2=0}^{n}\ldots\sum_{k_m=0}^{n}\delta_{n-\sum_{j=1}^{m}k_j,0}\,(k_1+k_2+\ldots+k_m;\, k_1,\, k_2,\, \ldots,\, k_m)\,(n_1-k_1+n_2-k_2+\ldots+n_m-k_m;\, n_1-k_1,\, n_2-k_2,\, \ldots,\, n_m-k_m)$$

Differentiation

Low-order differentiation

06.04.20.0001.01

$$\frac{\partial(n_1+n_2+\ldots+n_m;\, n_1,\, n_2,\, \ldots,\, n_m)}{\partial n_m} == (n_1+n_2+\ldots+n_m;\, n_1,\, n_2,\, \ldots,\, n_m)\,(\psi(n+1)-\psi(n_m+1)) /; n == \sum_{k=1}^{m} n_k$$

06.04.20.0002.01

$$\frac{\partial^2(n_1+n_2+\ldots+n_m;\, n_1,\, n_2,\, \ldots,\, n_m)}{\partial n_m^2} ==$$
$$(n_1+n_2+\ldots+n_m;\, n_1,\, n_2,\, \ldots,\, n_m)\left(\psi(n+1)^2-2\,\psi(n_3+1)\,\psi(n+1)+\psi(n_m+1)^2+\psi^{(1)}(n+1)-\psi^{(1)}(n_m+1)\right) /; n == \sum_{k=1}^{m} n_k$$

Symbolic differentiation

06.04.20.0003.02

$$\frac{\partial^u (n_1+n_2+\ldots+n_m;\, n_1,\, n_2,\, \ldots,\, n_m)}{\partial n_m^u} = \frac{(-1)^u\, u!\, \Gamma(s+1)^{u+1}}{\prod_{k=1}^{m-1}\Gamma(n_k+1)\,\Gamma(n_m-s)}\, {}_{u+2}\tilde{F}_{u+1}(a_1,\, a_2,\, \ldots,\, a_{u+1},\, s-n_m+1;\, a_1+1,\, a_2+1,\, \ldots,\, a_{u+1}+1;\, 1)\, /;$$

$$a_1 == a_2 == \ldots == a_{u+1} == s+1 \bigwedge s == \sum_{k=1}^{m} n_k \bigwedge u \in \mathbb{N} \bigwedge s - n_m \notin \mathbb{N}$$

Summation

Finite summation

06.04.23.0001.01

$$\sum_{k_{1,1}=0}^{o} \ldots \sum_{k_{m,n}=0}^{o} \prod_{i=1}^{m} (k_{i,1}+\ldots+k_{i,n};\, k_{i,1},\, \ldots,\, k_{i,n}) == (b_1+\ldots+b_n;\, b_1,\, \ldots,\, b_n)\, /;$$

$$\sum_{j=1}^{n} k_{i,j} == a_i \bigwedge \sum_{i=1}^{m} k_{i,j} == b_i \bigwedge \sum_{i=1}^{m} a_i == \sum_{j=1}^{n} b_j \bigwedge o = \mathrm{Max}(k_{1,1},\, \ldots,\, k_{m,n})$$

06.04.23.0002.01

$$\sum_{j_1=\sigma_0}^{\sigma_1} \sum_{j_2=\sigma_1}^{\sigma_2} \ldots \sum_{j_n=\sigma_{n-1}}^{\sigma_n} (-1)^{\sum_{h=10}^{n} k_h - \sum_{h=1}^{n-1}\sigma_h} \left(\sigma_n;\, \sigma_n-k_n,\, k_n-\sigma_{n-1},\, \sigma_{n-1}-k_{n-1},\, k_{n-1}-\sigma_{n-2},\, \ldots,\, \sigma_{q+2}-k_{q+2},\, k_{q+2}-\sigma_{q+1},\, \sigma_{q+1}-k_q,\, \sigma_q\right) \left(\sigma_q;\, \sigma_q-k_q,\, k_q-\sigma_{q-1},\, \sigma_{q-1}-k_{q-1},\, k_{q-1}-\sigma_{q-2},\, \ldots,\, \sigma_2-k_2,\, k_2-\sigma_1,\, \sigma_1-k_0,\, \sigma_0\right) \prod_{j=1}^{n} \delta_{\sigma_{j-1},\sigma_j}\, /;$$

$$\sigma_0 \in \mathbb{N} \bigwedge \sigma_1 \in \mathbb{N} \bigwedge \ldots \bigwedge \sigma_n \in \mathbb{N} \bigwedge \sigma_0 \le \sigma_1 \le \ldots \le \sigma_{n-1} \le \sigma_n \bigwedge q \in \bigwedge \mathbb{N} \bigwedge\ 0 \le q \le n \bigwedge n \in \mathbb{N}^+$$

06.04.23.0003.01

$$\sum_{j_1=\sigma_0}^{\sigma_1} \sum_{j_2=\sigma_1}^{\sigma_2} \ldots \sum_{j_n=\sigma_{n-1}}^{\sigma_n} (-1)^{\sum_{h=10}^{n} k_h - \sum_{h=1}^{n-1}\sigma_h} \left(\sigma_n;\, \sigma_n-k_n,\, k_n-\sigma_{n-1},\, \sigma_{n-1}-k_{n-1},\, k_{n-1}-\sigma_{n-2},\, \ldots,\, \sigma_{q+2}-k_{q+2},\, k_{q+2}-\sigma_{q+1},\, \sigma_{q+1}-k_q,\, \sigma_q\right) \left(\sigma_q;\, \sigma_q-k_q,\, k_q-\sigma_{q-1},\, \sigma_{q-1}-k_{q-1},\, k_{q-1}-\sigma_{q-2},\, \ldots,\, \sigma_2-k_2,\, k_2-\sigma_1,\, \sigma_1-k_0,\, \sigma_0\right) == 2^{\sigma_n-\sigma_0} (\sigma_n;\, \sigma_n-\sigma_{n-1},\, \sigma_{n-1}-\sigma_{n-2},\, \ldots,\, \sigma_2-\sigma_1,\, \sigma_1-\sigma_0,\, \sigma_0)\, /;\, \sigma_0 \in \mathbb{N} \bigwedge \sigma_1 \in \mathbb{N} \bigwedge \ldots \bigwedge \sigma_n \in \mathbb{N} \bigwedge \sigma_0 \le \sigma_1 \le \ldots \le \sigma_{n-1} \le \sigma_n \bigwedge q \in \mathbb{N} \bigwedge 0 \le q \le n \bigwedge n \in \mathbb{N}^+$$

Representations through equivalent functions

With related functions

06.04.27.0001.01

$$(n_1+n_2+\ldots+n_m;\, n_1,\, n_2,\, \ldots,\, n_m) == \frac{n!}{\prod_{k=1}^{m} n_k!}\, /;\, n == \sum_{k=1}^{m} n_k$$

06.04.27.0002.01

$$(n_1 + n_2 + \ldots + n_m;\, n_1,\, n_2,\, \ldots,\, n_m) == \frac{(n_m + 1)_{n-n_m}}{\prod_{k=1}^{m-1} \Gamma(n_k + 1)} \text{ /; } n == \sum_{k=1}^{m} n_k$$

Theorems

The multinomial expansion

$$(a_1 + a_2 + \ldots\ + a_m)^n == \sum_{n_1, n_2, \ldots, n_m = 0}^{n} \delta_{n,\, n_1 + n_2 + \ldots + n_m} (n;\, n_1,\, n_2,\, \ldots\, n_m) \prod_{k=1}^{m} a_k^{n_k}.$$

The derivative of product

$$\frac{\partial^n}{\partial z^n} \prod_{k=1}^{m} f_k(z) == \sum_{n_1, n_2, \ldots, n_m = 0}^{n} \delta_{n,\, n_1 + n_2 + \ldots + n_m} (n;\, n_1,\, n_2,\, \ldots,\, n_m) \prod_{k=1}^{m} \frac{\partial^{n_k}}{\partial z^{n_k}} f_k(z)$$

The expected value of the number of real roots of a system of *n* sparse polynomial equations in *n* variables

The expected value of the number of real roots of a system of n sparse polynomial equations in n variables can be expressed in multinomials and the volume of the corresponding Newton polytopes.

Generalized multinomial theorem

$$A_n(x_1,\, x_2,\, \ldots,\, x_m,\, p_1,\, p_2,\, \ldots,\, p_m) == \sum_{k=0}^{n} \binom{n}{k} k!\, (k + x)\, A_{n-k}(x_1 + k,\, x_2,\, \ldots,\, x_m,\, p_1 - 1,\, p_2,\, \ldots,\, p_m) \text{ /;}$$

$$A_n(x_1,\, x_2,\, \ldots,\, x_m,\, p_1,\, p_2,\, \ldots,\, p_m) == \sum_{k_1, k_2, \ldots,\, k_m = 0}^{n} \delta_{n,\, k_1 + k_2 + \ldots + k_m} (k_1 + k_2 + \ldots + k_m;$$

$$k_1,\, k_2,\, \ldots,\, k_m) \prod_{j=1}^{m} \left(x_j + k_j\right)^{k_j + p_j} \text{ /; } n \in \mathbb{N} \bigwedge p_1,\, p_2,\, \ldots,\, p_m \in \mathbb{Z}$$

The volume of the *d*-dimensional region

The volume V of the d-dimensional region $\sum_{k=1}^{d} |x_k|^{p_k} < 1$ is $V = 2^d \big/ s\left(\sum_{k=1}^{d} p_k^{-1};\, p_1^{-1},\, p_2^{-1},\, \ldots,\, p_d^{-1}\right)$.

History

– C. F. Hindenburg (1779)

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