NorlundB3

Notations

Traditional name

Norlund polynomial

Traditional notation

$B_n^{(\alpha)}(z)$

Mathematica StandardForm notation

`NorlundB[n, α, z]`

Primary definition

05.17.02.0001.01

$$B_n^{(\alpha)}(z) == n! \left([t^n] \left(\frac{t}{e^t - 1} \right)^\alpha e^{z t} \right) /; n \in \mathbb{N}$$

Specific values

Specialized values

For fixed *n*, *α*

05.17.03.0001.01

$$B_n^{(\alpha)}(0) = B_n^{(\alpha)}$$

05.17.03.0002.01

$$B_n^{(\alpha)}(1) = \frac{\alpha - n - 1}{\alpha - 1} B_n^{(\alpha-1)}$$

For fixed *n*, *z*

05.17.03.0003.01

$$B_n^{(0)}(z) = z^n$$

05.17.03.0004.01

$$B_n^{(1)}(z) = B_n(z)$$

05.17.03.0005.01

$$B_n^{(-1)}(z) = \frac{(z+1)^{n+1}}{n+1} - \frac{z^{n+1}}{n+1}$$

For fixed α, z

05.17.03.0006.01

$$B_0^{(\alpha)}(z) = 1$$

05.17.03.0007.01

$$B_1^{(\alpha)}(z) = z - \frac{\alpha}{2}$$

05.17.03.0008.01

$$B_2^{(\alpha)}(z) = z^2 - \alpha\, z + \frac{1}{12}\, \alpha\, (3\, \alpha - 1)$$

05.17.03.0009.01

$$B_3^{(\alpha)}(z) = \frac{1}{8} \left(8\, z^3 - 12\, \alpha\, z^2 + 2\, \alpha\, (3\, \alpha - 1)\, z - (\alpha - 1)\, \alpha^2\right)$$

05.17.03.0010.01

$$B_4^{(\alpha)}(z) = z^4 - 2\, \alpha\, z^3 + \frac{1}{2}\, \alpha\, (3\, \alpha - 1)\, z^2 - \frac{1}{2}\, (\alpha - 1)\, \alpha^2\, z + \frac{1}{240}\, \alpha \left(15\, \alpha^3 - 30\, \alpha^2 + 5\, \alpha + 2\right)$$

05.17.03.0011.01

$$B_5^{(\alpha)}(z) = \frac{1}{96} \left(96\, z^5 - 240\, \alpha\, z^4 + 80\, \alpha\, (3\, \alpha - 1)\, z^3 - 120\, (\alpha - 1)\, \alpha^2\, z^2 + 2\, \alpha \left(15\, \alpha^3 - 30\, \alpha^2 + 5\, \alpha + 2\right) z - \alpha^2 \left(3\, \alpha^3 - 10\, \alpha^2 + 5\, \alpha + 2\right)\right)$$

05.17.03.0012.01

$$B_6^{(\alpha)}(z) = z^6 - 3\, \alpha\, z^5 + \frac{5}{4}\, \alpha\, (3\, \alpha - 1)\, z^4 - \frac{5}{2}\, (\alpha - 1)\, \alpha^2\, z^3 + \frac{1}{16}\, \alpha \left(15\, \alpha^3 - 30\, \alpha^2 + 5\, \alpha + 2\right) z^2 - \frac{1}{16}\, \alpha^2 \left(3\, \alpha^3 - 10\, \alpha^2 + 5\, \alpha + 2\right) z + \frac{\alpha \left(63\, \alpha^5 - 315\, \alpha^4 + 315\, \alpha^3 + 91\, \alpha^2 - 42\, \alpha - 16\right)}{4032}$$

05.17.03.0013.01

$$B_7^{(\alpha)}(z) = \frac{1}{1152} \left(1152\, z^7 - 4032\, \alpha\, z^6 + 2016\, \alpha\, (3\, \alpha - 1)\, z^5 - 5040\, (\alpha - 1)\, \alpha^2\, z^4 + 168\, \alpha \left(15\, \alpha^3 - 30\, \alpha^2 + 5\, \alpha + 2\right) z^3 - 252\, \alpha^2 \left(3\, \alpha^3 - 10\, \alpha^2 + 5\, \alpha + 2\right) z^2 + 2\, \alpha \left(63\, \alpha^5 - 315\, \alpha^4 + 315\, \alpha^3 + 91\, \alpha^2 - 42\, \alpha - 16\right) z + \alpha^2 \left(-9\, \alpha^5 + 63\, \alpha^4 - 105\, \alpha^3 - 7\, \alpha^2 + 42\, \alpha + 16\right)\right)$$

05.17.03.0014.01

$$B_8^{(\alpha)}(z) = z^8 - 4\, \alpha\, z^7 + \frac{7}{3}\, \alpha\, (3\, \alpha - 1)\, z^6 - 7\, (\alpha - 1)\, \alpha^2\, z^5 + \frac{7}{24}\, \alpha \left(15\, \alpha^3 - 30\, \alpha^2 + 5\, \alpha + 2\right) z^4 - \frac{7}{12}\, \alpha^2 \left(3\, \alpha^3 - 10\, \alpha^2 + 5\, \alpha + 2\right) z^3 + \frac{1}{144}\, \alpha \left(63\, \alpha^5 - 315\, \alpha^4 + 315\, \alpha^3 + 91\, \alpha^2 - 42\, \alpha - 16\right) z^2 - \frac{1}{144}\, \alpha^2 \left(9\, \alpha^5 - 63\, \alpha^4 + 105\, \alpha^3 + 7\, \alpha^2 - 42\, \alpha - 16\right) z + \frac{\alpha \left(135\, \alpha^7 - 1260\, \alpha^6 + 3150\, \alpha^5 - 840\, \alpha^4 - 2345\, \alpha^3 - 540\, \alpha^2 + 404\, \alpha + 144\right)}{34\,560}$$

05.17.03.0015.01

$$B_9^{(\alpha)}(z) = \frac{1}{7680}\left(7680\, z^9 - 34\,560\, \alpha\, z^8 + 23\,040\, \alpha\, (3\,\alpha - 1)\, z^7 - 80\,640\, (\alpha - 1)\, \alpha^2\, z^6 + 4032\, \alpha\left(15\,\alpha^3 - 30\,\alpha^2 + 5\,\alpha + 2\right) z^5 - 10\,080\, \alpha^2\left(3\,\alpha^3 - 10\,\alpha^2 + 5\,\alpha + 2\right) z^4 + 160\,\alpha\left(63\,\alpha^5 - 315\,\alpha^4 + 315\,\alpha^3 + 91\,\alpha^2 - 42\,\alpha - 16\right) z^3 - 240\,\alpha^2\left(9\,\alpha^5 - 63\,\alpha^4 + 105\,\alpha^3 + 7\,\alpha^2 - 42\,\alpha - 16\right) z^2 + 2\,\alpha\left(135\,\alpha^7 - 1260\,\alpha^6 + 3150\,\alpha^5 - 840\,\alpha^4 - 2345\,\alpha^3 - 540\,\alpha^2 + 404\,\alpha + 144\right) z - \alpha^2\left(15\,\alpha^7 - 180\,\alpha^6 + 630\,\alpha^5 - 448\,\alpha^4 - 665\,\alpha^3 + 100\,\alpha^2 + 404\,\alpha + 144\right)\right)$$

05.17.03.0016.01

$$B_{10}^{(\alpha)}(z) = z^{10} - 5\,\alpha\, z^9 + \frac{15}{4}\,\alpha\,(3\,\alpha - 1)\, z^8 - 15\,(\alpha - 1)\,\alpha^2\, z^7 + \frac{7}{8}\,\alpha\left(15\,\alpha^3 - 30\,\alpha^2 + 5\,\alpha + 2\right) z^6 - \frac{21}{8}\,\alpha^2\left(3\,\alpha^3 - 10\,\alpha^2 + 5\,\alpha + 2\right) z^5 + \frac{5}{96}\,\alpha\left(63\,\alpha^5 - 315\,\alpha^4 + 315\,\alpha^3 + 91\,\alpha^2 - 42\,\alpha - 16\right) z^4 - \frac{5}{48}\,\alpha^2\left(9\,\alpha^5 - 63\,\alpha^4 + 105\,\alpha^3 + 7\,\alpha^2 - 42\,\alpha - 16\right) z^3 + \frac{1}{768}\,\alpha\left(135\,\alpha^7 - 1260\,\alpha^6 + 3150\,\alpha^5 - 840\,\alpha^4 - 2345\,\alpha^3 - 540\,\alpha^2 + 404\,\alpha + 144\right) z^2 - \frac{1}{768}\,\alpha^2\left(15\,\alpha^7 - 180\,\alpha^6 + 630\,\alpha^5 - 448\,\alpha^4 - 665\,\alpha^3 + 100\,\alpha^2 + 404\,\alpha + 144\right) z + \frac{1}{101\,376}\,\alpha\left(99\,\alpha^9 - 1485\,\alpha^8 + 6930\,\alpha^7 - 8778\,\alpha^6 - 8085\,\alpha^5 + 8195\,\alpha^4 + 11\,792\,\alpha^3 + 2068\,\alpha^2 - 2288\,\alpha - 768\right)$$

General characteristics

Domain and analyticity

The function $B_n^{(\alpha)}(z)$ is defined over $\mathbb{N} \otimes \mathbb{C} \otimes \mathbb{C}$. For fixed n, the function $B_n^{(\alpha)}(z)$ is a polynomial in α and z of degree n.

05.17.04.0001.01

$$(n * z * z) \longrightarrow B_n^{(\alpha)}(z) :: (\mathbb{N} \otimes \mathbb{C} \otimes \mathbb{C}) \longrightarrow \mathbb{C}$$

Symmetries and periodicities

Parity

No parity

Mirror symmetry

05.17.04.0002.01

$$B_n^{(\bar{\alpha})}(\bar{z}) = \overline{B_n^{(\alpha)}(z)}$$

Periodicity

No periodicity

Poles and essential singularities

With respect to α

The function $B_n^{(\alpha)}(z)$ is polynomial and has pole of order n at $\alpha = \tilde{\infty}$.

05.17.04.0003.01

$$\mathcal{S}ing_{\alpha}\left(B_n^{(\alpha)}(z)\right) == \{\{\tilde{\infty}, n\}\}$$

With respect to z

The function $B_n^{(\alpha)}(z)$ is polynomial and has pole of order n at $z = \tilde{\infty}$.

05.17.04.0004.01

$$\mathcal{S}ing_{z}\left(B_n^{(\alpha)}(z)\right) == \{\{\tilde{\infty}, n\}\}$$

Branch points

With respect to α

The function $B_n^{(\alpha)}(z)$ does not have branch points.

05.17.04.0005.01

$$\mathcal{BP}_{\alpha}\left(B_n^{(\alpha)}(z)\right) == \{\}$$

With respect to z

The function $B_n^{(\alpha)}(z)$ does not have branch points.

05.17.04.0006.01

$$\mathcal{BP}_{z}\left(B_n^{(\alpha)}(z)\right) == \{\}$$

Branch cuts

With respect to α

The function $B_n^{(\alpha)}(z)$ does not have branch cuts.

05.17.04.0007.01

$$\mathcal{BC}_{\alpha}\left(B_n^{(\alpha)}(z)\right) == \{\}$$

With respect to z

The function $B_n^{(\alpha)}(z)$ does not have branch cuts.

05.17.04.0008.01

$$\mathcal{BC}_{z}\left(B_n^{(\alpha)}(z)\right) == \{\}$$

Series representations

Generalized power series

Expansions at generic point $z == z_0$

05.17.06.0001.01

$$B_n^{(\alpha)}(z) \propto B_n^{(\alpha)}(z_0) + n\, B_{n-1}^{(\alpha)}(z_0)\,(z - z_0) + \frac{n\,(n-1)}{2}\, B_{n-2}^{(\alpha)}(z_0)\,(z - z_0)^2 + \ldots \;/;\; (z \to z_0)$$

05.17.06.0002.01

$$B_n^{(\alpha)}(z) \propto B_n^{(\alpha)}(z_0) + n\, B_{n-1}^{(\alpha)}(z_0)\,(z - z_0) + \frac{n\,(n-1)}{2}\, B_{n-2}^{(\alpha)}(z_0)\,(z - z_0)^2 + O\big((z - z_0)^3\big)$$

05.17.06.0003.01

$$B_n^{(\alpha)}(z) = \sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!}\, B_{n-k}^{(\alpha)}(z_0)\,(z - z_0)^k$$

05.17.06.0004.01

$$B_n^{(\alpha)}(z) \propto B_n^{(\alpha)}(z_0)\,(1 + O(z - z_0))$$

Expansions at $z == 0$

05.17.06.0005.01

$$B_n^{(\alpha)}(z) \propto B_n^{(\alpha)} + n\, B_{n-1}^{(\alpha)}\, z + \frac{n\,(n-1)}{2}\, B_{n-2}^{(\alpha)}\, z^2 + \ldots \;/;\; (z \to 0)$$

05.17.06.0006.01

$$B_n^{(\alpha)}(z) \propto B_n^{(\alpha)} + n\, B_{n-1}^{(\alpha)}\, z + \frac{n\,(n-1)}{2}\, B_{n-2}^{(\alpha)}\, z^2 + O\big(z^3\big)$$

05.17.06.0007.01

$$B_n^{(\alpha)}(z) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k}^{(\alpha)}\, z^k$$

05.17.06.0008.01

$$B_n^{(\alpha)}(z) = n! \sum_{k=0}^{n} \frac{c_{n-k}\, z^k}{k!\,(n-k)!} \;/;$$

$$c_k = (\alpha)_{k+1} \sum_{j=0}^{k} \frac{(-1)^j}{j+\alpha} \binom{k}{j} p_{j,k} \bigwedge p_{j,0} = 1 \bigwedge p_{j,k} = \frac{1}{k} \sum_{m=1}^{k} (j\, m + m - k)\, a_m\, p_{j,k-m} \bigwedge a_k = \frac{1}{(k+1)!} \bigwedge k \in \mathbb{N}$$

05.17.06.0009.01

$$B_n^{(\alpha)}(z) \propto B_n^{(\alpha)}\,(1 + O(z))$$

Expansions at $z == \infty$

05.17.06.0010.01

$$B_n^{(\alpha)}(z) = z^n \left(1 - \frac{\alpha\, n}{2\, z} + \frac{(3\,\alpha - 1)\,(n-1)\,\alpha\, n}{24\, z^2} + \ldots\right) /;\; (|z| \to \infty) \land n > 0$$

05.17.06.0011.01

$$B_n^{(\alpha)}(z) = z^n \left(1 - \frac{\alpha\, n}{2\, z} + \frac{(3\,\alpha - 1)\,(n-1)\,\alpha\, n}{24\, z^2} + O\!\left(\frac{1}{z^3}\right)\right) /;\; n > 0$$

05.17.06.0012.01

$$B_n^{(\alpha)}(z) = z^n \sum_{k=0}^{n} \binom{n}{k} B_k^{(\alpha)}\, z^{-k}$$

05.17.06.0013.01

$$B_n^{(\alpha)}(z) = z^n \left(1 + O\left(\frac{1}{z}\right)\right) /; n > 0$$

Expansions at generic point $\alpha == \alpha_0$

05.17.06.0014.01

$$B_n^{(\alpha)}(z) \propto B_n^{(\alpha_0)}(z) + \sum_{k=0}^{n} \binom{n}{k} z^k \left(\frac{\partial B_{n-k}^{(\alpha)}}{\partial \alpha} |_{\alpha==\alpha_0}\right)(\alpha - \alpha_0) + \frac{1}{2} \sum_{k=0}^{n} \binom{n}{k} z^k \left(\frac{\partial^2 B_{n-k}^{(\alpha)}}{\partial \alpha^2} |_{\alpha==\alpha_0}\right)(\alpha - \alpha_0)^2 + \ldots /; (\alpha \to \alpha_0)$$

05.17.06.0015.01

$$B_n^{(\alpha)}(z) = \sum_{k=0}^{n} \binom{n}{k} z^k \sum_{j=0}^{n-k} \frac{1}{j!} \left(\frac{\partial^j B_{n-k}^{(\alpha)}}{\partial \alpha^j} |_{\alpha==\alpha_0}\right)(\alpha - \alpha_0)^j$$

05.17.06.0016.01

$$B_n^{(\alpha)}(z) \propto B_n^{(\alpha_0)}(z) \, (1 + O(\alpha - \alpha_0))$$

Expansions at $\alpha == 0$

05.17.06.0017.01

$$B_n^{(\alpha)}(z) = z^n \, \theta(n) - \sum_{i=0}^{n-1} \sum_{k=0}^{n-i-1} (-1)^k \binom{n}{i+k+1} z^{n-i-k-1} \sum_{r=0}^{i} S_{i+k+2}^{(r+1)} \sum_{j=1}^{i+k+1} (-1)^j \, j^{r-i-1} \binom{i+k+1}{j} p_{j,i+k+1} \, \alpha^{i+1} /;$$

$$p_{j,0} = 1 \bigwedge p_{j,k} = \frac{1}{k} \sum_{m=1}^{k} (j\, m + m - k) \, a_m \, p_{j,k-m} \bigwedge a_k = \frac{1}{(k+1)!} \bigwedge k \in \mathbb{N}$$

05.17.06.0018.01

$$B_n^{(\alpha)}(z) = z^n \, \theta(n) - \sum_{i=0}^{n} \sum_{k=0}^{n-i} (-1)^k \binom{n}{i+k} z^{n-i-k} \sum_{r=1}^{i} S_{i+k+1}^{(r)} \sum_{j=1}^{i+k} (-1)^j \, j^{r-i-1} \binom{i+k}{j} p_{j,k+i} \, \alpha^i /;$$

$$p_{j,0} = 1 \bigwedge p_{j,k} = \frac{1}{k} \sum_{m=1}^{k} (j\, m + m - k) \, a_m \, p_{j,k-m} \bigwedge a_k = \frac{1}{(k+1)!} \bigwedge k \in \mathbb{N}$$

05.17.06.0019.01

$$B_n^{(\alpha)}(z) = z^n \, \theta(n) - \sum_{k=0}^{n} (-1)^k \binom{n}{k} z^{n-k} \sum_{i=0}^{k} (-1)^i \sum_{r=1}^{i} S_{k+1}^{(r)} \sum_{j=1}^{k} (-1)^j \, j^{r-i-1} \binom{k}{j} p_{j,k} \, \alpha^i /;$$

$$p_{j,0} = 1 \bigwedge p_{j,k} = \frac{1}{k} \sum_{m=1}^{k} (j\, m + m - k) \, a_m \, p_{j,k-m} \bigwedge a_k = \frac{1}{(k+1)!} \bigwedge k \in \mathbb{N}$$

05.17.06.0020.01

$$B_n^{(\alpha)}(z) \propto \theta(n) \, z^n + \alpha \left(\frac{n}{2} \theta(n-1) \, z^{n-1} + \sum_{k=0}^{n} \binom{n}{k} z^k \left(n - k - 2 \left\lfloor \frac{n-k}{2} \right\rfloor + 1\right)(n-k)! \sum_{j=1}^{n-k} \frac{(-1)^j}{j} \binom{n-k}{j} p_{j,n-k} - \right.$$

$$\left. \sum_{k=0}^{n} \binom{n}{k} z^k \left(n - k - 2 \left\lfloor \frac{n-k}{2} \right\rfloor\right)(n-k)! \sum_{j=1}^{n-k} \frac{(-1)^j}{j} \binom{n-k}{j} \left(\frac{1}{j} - H_{n-k}\right) p_{j,n-k} \, \alpha\right)(1 + O(\alpha)) /;$$

$$p_{j,0} = 1 \bigwedge p_{j,k} = \frac{1}{k} \sum_{m=1}^{k} (j\, m + m - k) \, a_m \, p_{j,k-m} \bigwedge a_k = \frac{1}{(k+1)!} \bigwedge k \in \mathbb{N}$$

05.17.06.0021.01

$$B_n^{(\alpha)}(z) \propto z^n\,(1+O(\alpha))$$

Expansions at $\alpha == \infty$

05.17.06.0022.01

$$B_n^{(\alpha)}(z) \propto \theta(n)\,z^n + z^n \sum_{k=0}^{n} \binom{n}{k} (-2\,z)^{-k}\,\alpha^k \left(1 + 2^k \sum_{r=1}^{k-1} S_{k+1}^{(r)} \sum_{j=1}^{k} \frac{(-1)^{j+k}\,j^{r-k}}{\alpha} \binom{k}{j} p_{j,k} - 2^k \sum_{r=1}^{k-2} S_{k+1}^{(r)} \sum_{j=1}^{k} \frac{(-1)^{j+k}\,j^{r-k+1}}{\alpha^2} \binom{k}{j} p_{j,k} + \ldots\right) /;$$

$$(|\alpha| \to \infty) \bigwedge p_{j,0} = 1 \bigwedge p_{j,k} = \frac{1}{k} \sum_{m=1}^{k} (j\,m + m - k)\,a_m\,p_{j,k-m} \bigwedge a_k = \frac{1}{(k+1)!} \bigwedge k \in \mathbb{N} \bigwedge n > 0$$

05.17.06.0023.01

$$B_n^{(\alpha)}(z) = z^n\,\theta(n) - z^n \sum_{k=0}^{n} \binom{n}{k} z^{-k}\,\alpha^k \sum_{i=0}^{k} \sum_{r=1}^{k-i} S_{k+1}^{(r)} \sum_{j=1}^{k} (-1)^{i+j}\,j^{i-k+r-1} \binom{k}{j} p_{j,k}\,\alpha^{-i} /;$$

$$p_{j,0} = 1 \bigwedge p_{j,k} = \frac{1}{k} \sum_{m=1}^{k} (j\,m + m - k)\,a_m\,p_{j,k-m} \bigwedge a_k = \frac{1}{(k+1)!} \bigwedge k \in \mathbb{N}$$

05.17.06.0024.01

$$B_n^{(\alpha)}(z) \propto z^n \sum_{k=0}^{n} \binom{n}{k} (-2\,z)^{-k}\,\alpha^k \left(1 + O\!\left(\frac{1}{\alpha}\right)\right)$$

Generating functions

05.17.11.0001.01

$$B_n^{(\alpha)}(z) == n!\left([t^n]\left(\frac{t}{e^t - 1}\right)^{\alpha} e^{z\,t}\right) /; n \in \mathbb{N}$$

Identities

Functional identities

Relations between contiguous functions

Recurrence relations

05.17.17.0001.01

$$B_n^{(\alpha)}(z) = \frac{1}{n} \sum_{k=1}^{n} \binom{n}{k} ((\alpha+1)\,k - n)\,B_k\!\left(\frac{z}{\alpha}\right) B_{n-k}^{(\alpha)}(z) /; n \in \mathbb{N}^+$$

05.17.17.0002.01

$$B_n^{(m)}(z) = n\,B_{n-1}^{(m-1)}(z-1) + B_n^{(m)}(z-1) /; n \in \mathbb{N} \land m \in \mathbb{N}$$

Differentiation

Low-order differentiation

With respect to α

05.17.20.0001.01

$$\frac{\partial B_n^{(\alpha)}(z)}{\partial \alpha} = n! \sum_{k=0}^{n} \frac{c_{n-k}\, z^k}{k!\,(n-k)!} \,/;\, c_k = (\alpha)_{k+1} \sum_{j=0}^{k} \frac{(-1)^j}{(j+\alpha)^2} \binom{k}{j} (-(j+\alpha)\,\psi(\alpha) + (j+\alpha)\,\psi(k+\alpha+1) - 1)\, p_{j,k} \bigwedge$$

$$p_{j,0} = 1 \bigwedge p_{j,k} = \frac{1}{k} \sum_{m=1}^{k} (j\,m + m - k)\, a_m\, p_{j,k-m} \bigwedge a_k = \frac{1}{(k+1)!} \bigwedge k \in \mathbb{N}$$

With respect to z

05.17.20.0002.01

$$\frac{\partial B_n^{(\alpha)}(z)}{\partial z} = n\, B_{n-1}^{(\alpha)}(z)$$

05.17.20.0003.01

$$\frac{\partial^2 B_n^{(\alpha)}(z)}{\partial z^2} = n\,(n-1)\, B_{n-2}^{(\alpha)}(z)$$

Symbolic differentiation

With respect to α

05.17.20.0004.01

$$\frac{\partial^m B_n^{(\alpha)}(z)}{\partial \alpha^m} = n! \sum_{k=0}^{n} \frac{c_{n-k,m}\, z^k}{k!\,(n-k)!} \,/;$$

$$c_{k,m} = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \sum_{r=0}^{m} \binom{m}{r} (m-r)!\, \alpha^{1-r}\, (j+\alpha)^{r-m-1} \sum_{i=0}^{k} (-1)^{i+k+m-r}\, S_{k+1}^{(i+1)}\, (i-r+2)_r\, \alpha^i\, p_{j,k} \bigwedge$$

$$p_{j,0} = 1 \bigwedge p_{j,k} = \frac{1}{k} \sum_{m=1}^{k} (j\,m + m - k)\, a_m\, p_{j,k-m} \bigwedge a_k = \frac{1}{(k+1)!} \bigwedge k \in \mathbb{N}$$

With respect to z

05.17.20.0005.01

$$\frac{\partial^m B_n^{(\alpha)}(z)}{\partial z^m} = \frac{n!}{(n-m)!} B_{n-m}^{(\alpha)}(z) \,/;\, n \in \mathbb{N} \land m \in \mathbb{N} \land m \leq n$$

05.17.20.0006.01

$$\frac{\partial^m B_n^{(\alpha)}(z)}{\partial z^m} = n! \sum_{k=0}^{n-m} \frac{c_{n-m-k}\, z^k}{k!\,(n-m-k)!} \,/;$$

$$c_k = (\alpha)_{k+1} \sum_{j=0}^{k} \frac{(-1)^j}{j+\alpha} \binom{k}{j} p_{j,k} \bigwedge p_{j,0} = 1 \bigwedge p_{j,k} = \frac{1}{k} \sum_{m=1}^{k} (j\,m + m - k)\, a_m\, p_{j,k-m} \bigwedge a_k = \frac{1}{(k+1)!} \bigwedge k \in \mathbb{N}$$

Fractional integro-differentiation

With respect to α

05.17.20.0007.01

$$\frac{\partial^\beta B_n^{(\alpha)}(z)}{\partial \alpha^\beta} = \frac{z^n \alpha^{-\beta} \theta(n)}{\Gamma(1-\beta)} - \sum_{k=0}^{n} (-1)^k \binom{n}{k} z^{n-k} \sum_{i=0}^{k} \frac{i!}{\Gamma(i-\beta+1)} \sum_{r=1}^{i} S_{k+1}^{(r)} \sum_{j=1}^{k} (-1)^{i+j} j^{r-i-1} \binom{k}{j} p(j,k) \alpha^{i-\beta} /;$$

$$p_{j,0} = 1 \bigwedge p_{j,k} = \frac{1}{k} \sum_{m=1}^{k} (j\, m + m - k)\, a_m\, p_{j,k-m} \bigwedge a_k = \frac{1}{(k+1)!} \bigwedge k \in \mathbb{N}$$

With respect to z

05.17.20.0008.01

$$\frac{\partial^\beta B_n^{(\alpha)}(z)}{\partial z^\beta} = n! \sum_{k=0}^{n} \frac{B_{n-k}^{(\alpha)} z^{k-\beta}}{(n-k)!\, \Gamma(k-\beta+1)}$$

Integration

Indefinite integration

For the direct function with respect to α

05.17.21.0001.01

$$\int B_n^{(\alpha)}(z)\, d\alpha = z^n \alpha\, \theta(n) - \sum_{k=0}^{n} (-1)^k \binom{n}{k} z^{n-k} \sum_{i=0}^{k} \sum_{r=1}^{i} S_{k+1}^{(r)} \sum_{j=1}^{k} \frac{(-1)^{i+j} j^{r-i-1}}{i+1} \binom{k}{j} p(j,k)\, \alpha^{i+1}\, p_{j,k}\, \alpha^{i+1} /;$$

$$p_{j,0} = 1 \bigwedge p_{j,k} = \frac{1}{k} \sum_{m=1}^{k} (j\, m + m - k)\, a_m\, p_{j,k-m} \bigwedge a_k = \frac{1}{(k+1)!} \bigwedge k \in \mathbb{N}$$

For the direct function with respect to z

05.17.21.0002.01

$$\int B_n^{(\alpha)}(z)\, dz = \sum_{k=0}^{n} \frac{1}{k+1} \binom{n}{k} B_{n-k}^{(\alpha)} z^{k+1}$$

Summation

Infinite summation

05.17.23.0001.01

$$\sum_{k=0}^{\infty} \frac{B_k^{(\alpha)}(z)\, w^k}{k!} = \left(\frac{w}{e^w - 1}\right)^{\alpha} e^{z w} /; |w| < 2\pi$$

Theorems

Asymptotical expansions of ratios of the gamma functions

Norlund polynomials are used in the asymptotical expansions of ratios of the gamma functions:

$$\frac{\Gamma(a+z)}{\Gamma(b+z)} \propto z^{a-b} \sum_{k=0}^{\infty} \frac{(-1)^k \, (b-a)_k}{k!} \, B_k^{(a-b+1)}(a) \, z^{-k} \; /; \; |\arg(a+z)| < \pi \bigwedge (|z| \to \infty)$$

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