PolyLog

Notations

Traditional name

Polylogarithm

Traditional notation

$\text{Li}_\nu(z)$

Mathematica StandardForm notation

`PolyLog[ν, z]`

Primary definition

10.08.02.0001.01

$$\text{Li}_\nu(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^\nu} \;/;\; |z| < 1$$

Specific values

Specialized values

For fixed ν

General cases

10.08.03.0001.01

$$\text{Li}_\nu(0) = 0$$

10.08.03.0002.01

$$\text{Li}_\nu(1) = \zeta(\nu) \;/;\; \text{Re}(\nu) > 1$$

10.08.03.0034.01

$$\text{Li}_\nu(1) = \tilde{\infty} \;/;\; \text{Re}(\nu) < 1$$

10.08.03.0035.01

$$\text{Li}_\nu(1) = ¿ \;/;\; \text{Re}(\nu) = 1$$

10.08.03.0003.01

$$\text{Li}_\nu(-1) = \left(2^{1-\nu} - 1\right) \zeta(\nu)$$

10.08.03.0004.01

$$\operatorname{Li}_\nu\left(e^{\frac{2\pi i p}{q}}\right) = \frac{1}{q^\nu}\sum_{k=1}^{q} e^{\frac{2\pi i p k}{q}} \zeta\left(\nu, \frac{k}{q}\right) /; p \in \mathbb{N}^+ \bigwedge q \in \mathbb{N}^+ \bigwedge p \leq q$$

For fixed integer ν

10.08.03.0005.01

$$\operatorname{Li}_{2n}(1) = \frac{(-1)^{n+1}(2\pi)^{2n}}{2(2n)!} B_{2n} /; n \in \mathbb{N}$$

10.08.03.0006.01

$$\operatorname{Li}_{2n}(-1) = \frac{(-1)^n(4^n-2)\pi^{2n}}{2(2n)!} B_{2n} /; n \in \mathbb{N}$$

10.08.03.0007.01

$$\operatorname{Li}_n\left(\exp\left(\frac{2\pi i p}{q}\right)\right) = \frac{(-1)^n}{q^n (n-1)!}\sum_{k=1}^{q-1} \exp\left(\frac{2\pi i k p}{q}\right) \psi^{(n-1)}\left(\frac{k}{q}\right) + \frac{\zeta(n)}{q^n} /; p \in \mathbb{N}^+ \bigwedge q \in \mathbb{N}^+ \bigwedge n-1 \in \mathbb{N}^+$$

10.08.03.0008.01

$$\operatorname{Li}_{-n}(z) = z\left(\sum_{j=0}^{n} \binom{n}{j} \operatorname{Li}_{-j}(z) + 1\right) /; n \in \mathbb{N}$$

10.08.03.0033.01

$$\operatorname{Li}_{-n}(z) = (1-z)^{-(n+1)} \sum_{m=1}^{n} \left(\sum_{k=1}^{m} (-1)^{k+1} \binom{n+1}{k-1} (-k+m+1)^n\right) z^m /; n \in \mathbb{N}^+$$

10.08.03.0009.01

$$\operatorname{Li}_{-3}(z) = \frac{z\left(1+4z+z^2\right)}{(z-1)^4}$$

10.08.03.0010.01

$$\operatorname{Li}_{-2}(z) = -\frac{z(1+z)}{(z-1)^3}$$

10.08.03.0011.01

$$\operatorname{Li}_{-1}(z) = \frac{z}{(z-1)^2}$$

10.08.03.0012.01

$$\operatorname{Li}_0(z) = \frac{z}{1-z}$$

10.08.03.0013.01

$$\operatorname{Li}_1(z) = -\log(1-z)$$

Values at fixed points

Dilogarithmical case

10.08.03.0014.01

$$\operatorname{Li}_2(-1) = -\frac{\pi^2}{12}$$

10.08.03.0015.01

$$\mathrm{Li}_2(1) = \frac{\pi^2}{6}$$

10.08.03.0016.01

$$\mathrm{Li}_2\left(\frac{1}{2}\right) = \frac{\pi^2}{12} - \frac{\log^2(2)}{2}$$

(L.Euler)

10.08.03.0017.01

$$\mathrm{Li}_2(2) = \frac{\pi^2}{4} - \pi\, i \log(2)$$

10.08.03.0018.01

$$\mathrm{Li}_2(i) = i\, C - \frac{\pi^2}{48}$$

10.08.03.0019.01

$$\mathrm{Li}_2(-i) = -i\, C - \frac{\pi^2}{48}$$

10.08.03.0020.01

$$\mathrm{Li}_2(1-i) = \frac{\pi^2}{16} - i\, C - \frac{\pi\, i}{4} \log(2)$$

10.08.03.0021.01

$$\mathrm{Li}_2(1+i) = \frac{\pi^2}{16} + i\, C + \frac{\pi\, i}{4} \log(2)$$

10.08.03.0022.01

$$\mathrm{Li}_2\left(\frac{1-\sqrt{5}}{2}\right) = \frac{1}{2} \log^2\left(\frac{\sqrt{5}-1}{2}\right) - \frac{\pi^2}{15}$$

10.08.03.0023.01

$$\mathrm{Li}_2\left(-\frac{1+\sqrt{5}}{2}\right) = -\log^2\left(\frac{1+\sqrt{5}}{2}\right) - \frac{\pi^2}{10}$$

10.08.03.0024.01

$$\mathrm{Li}_2\left(\frac{3-\sqrt{5}}{2}\right) = \frac{\pi^2}{15} - \log^2\left(\frac{\sqrt{5}-1}{2}\right)$$

10.08.03.0025.01

$$\mathrm{Li}_2\left(\frac{\sqrt{5}-1}{2}\right) = \frac{\pi^2}{10} - \log^2\left(\frac{\sqrt{5}-1}{2}\right)$$

10.08.03.0026.01

$$\mathrm{Li}_2\left(\frac{1-i}{2}\right) = -\frac{1}{8} \log^2(2) + \frac{\pi\, i}{8} \log(2) + \frac{5\,\pi^2}{96} - i\, C$$

Trilogarithmical case

10.08.03.0027.01

$$\mathrm{Li}_3\left(\frac{1}{2}\right) = \frac{7}{8}\zeta(3) - \frac{\pi^2}{12}\log(2) + \frac{1}{6}\log^3(2)$$

10.08.03.0028.01

$$\mathrm{Li}_3\left(\frac{3-\sqrt{5}}{2}\right) = \frac{4}{5}\zeta(3) - \frac{1}{12}\log^3\left(\frac{3-\sqrt{5}}{2}\right) + \frac{1}{15}\pi^2\log\left(\frac{3-\sqrt{5}}{2}\right)$$

10.08.03.0029.01

$$\mathrm{Li}_3\left(e^{\frac{\pi i}{3}}\right) = \frac{1}{3}\zeta(3) + \frac{5 i \pi^3}{162}$$

10.08.03.0030.01

$$\mathrm{Li}_3\left(e^{-\frac{\pi i}{3}}\right) = \frac{1}{3}\zeta(3) - \frac{5 i \pi^3}{162}$$

10.08.03.0031.01

$$\mathrm{Li}_3\left(e^{\frac{2\pi i}{3}}\right) = -\frac{4}{9}\zeta(3) + \frac{2 i \pi^3}{81}$$

10.08.03.0032.01

$$\mathrm{Li}_3\left(e^{-\frac{2\pi i}{3}}\right) = -\frac{4}{9}\zeta(3) - \frac{2 i \pi^3}{81}$$

General characteristics

Domain and analyticity

$\mathrm{Li}_\nu(z)$ is an analytical function of ν, z which is defined in $\mathbb{C}^2$.

10.08.04.0001.01

$$(\nu * z) \longrightarrow \mathrm{Li}_\nu(z) :: (\mathbb{C} \otimes \mathbb{C}) \longrightarrow \mathbb{C}$$

Symmetries and periodicities

Mirror symmetry

10.08.04.0002.01

$$\mathrm{Li}_{\bar{\nu}}(\bar{z}) = \overline{\mathrm{Li}_\nu(z)} \;/;\; z \notin (-\infty, 0)$$

Periodicity

No periodicity

Poles and essential singularities

With respect to z

For fixed ν, the function $\mathrm{Li}_\nu(z)$ does not have poles and essential singularities.

10.08.04.0003.01

$$\mathcal{Sing}_z(\mathrm{Li}_\nu(z)) = \{\}$$

With respect to ν

For fixed z, the function $\mathrm{Li}_\nu(z)$ has only one singular point at $\nu = \tilde{\infty}$. It is an essential singular point.

10.08.04.0004.01

$$\mathcal{Sing}_\nu(\mathrm{Li}_\nu(z)) = \{\{\tilde{\infty}, \infty\}\}$$

Branch points

With respect to z

For fixed ν, the function $\mathrm{Li}_\nu(z)$ has two branch points: $z = 1$, $z = \tilde{\infty}$.

10.08.04.0005.01

$$\mathcal{BP}_z(\mathrm{Li}_\nu(z)) = \{1, \tilde{\infty}\}$$

10.08.04.0006.01

$$\mathcal{R}_z(\mathrm{Li}_\nu(z), 1) = \log \;/;\; \nu \in \mathbb{Z} \bigvee \nu \notin \mathbb{Q}$$

10.08.04.0007.01

$$\mathcal{R}_z(\mathrm{Li}_\nu(z), 1) = s \;/;\; \nu = \frac{r}{s} \bigwedge r \in \mathbb{Z} \bigwedge s - 1 \in \mathbb{N}^+ \bigwedge \gcd(r, s) = 1$$

With respect to ν

For fixed z, the function $\mathrm{Li}_\nu(z)$ does not have branch points.

10.08.04.0008.01

$$\mathcal{BP}_\nu(\mathrm{Li}_\nu(z)) = \{\}$$

Branch cuts

With respect to z

For fixed ν, the function $\mathrm{Li}_\nu(z)$ is a single-valued function on the z-plane cut along the interval $\{1, \infty\}$, where it is continuous from below.

10.08.04.0009.01

$$\mathcal{BC}_z(\mathrm{Li}_\nu(z)) = \{\{\{1, \infty\}, i\}\}$$

10.08.04.0010.01

$$\lim_{\epsilon \to +0} \mathrm{Li}_\nu(x - i\,\epsilon) = \mathrm{Li}_\nu(x) \;/;\; x > 1$$

10.08.04.0011.01

$$\lim_{\epsilon \to +0} \mathrm{Li}_\nu(x + i\,\epsilon) = \mathrm{Li}_\nu(x) + \frac{2\,i\,\pi}{\Gamma(\nu)} \log^{\nu-1}(x) \;/;\; x > 1$$

With respect to ν

For fixed z, the function $\mathrm{Li}_\nu(z)$ does not have branch cuts.

10.08.04.0012.01

$$\mathcal{BC}_\nu(\mathrm{Li}_\nu(z)) = \{\}$$

Series representations

Generalized power series

Expansions at $z == 0$

For the function itself

General case

10.08.06.0001.02

$$\mathrm{Li}_\nu(z) \propto z + 2^{-\nu} z^2 + 3^{-\nu} z^3 + \ldots \;/;\; (z \to 0)$$

10.08.06.0027.01

$$\mathrm{Li}_\nu(z) \propto z + 2^{-\nu} z^2 + 3^{-\nu} z^3 + O\left(z^4\right)$$

10.08.06.0002.01

$$\mathrm{Li}_\nu(z) == \sum_{k=1}^{\infty} \frac{z^k}{k^\nu} \;/;\; |z| < 1$$

10.08.06.0003.02

$$\mathrm{Li}_\nu(z) \propto z\,(1 + O(z))$$

10.08.06.0028.01

$$\mathrm{Li}_\nu(z) == F_\infty(z, \nu) \;/;\; \left(\left(F_n(z, \nu) == \sum_{k=1}^{n} \frac{z^k}{k^\nu} == \mathrm{Li}_\nu(z) - z^{n+1}\, \Phi(z, \nu, n+1)\right) \bigwedge n \in \mathbb{N}\right)$$

Summed form of the truncated series expansion.

Special cases

10.08.06.0029.01

$$\mathrm{Li}_n(z) == z\; {}_{n+1}F_n(1, a_1, a_2, \ldots, a_n; a_1 + 1, a_2 + 1, \ldots, a_n + 1; z) \;/;\; a_1 == a_2 == \ldots == a_n == 1 \bigwedge n \in \mathbb{N}^+$$

Expansions at $z == 1$

For the function itself

General case

10.08.06.0030.01

$$\mathrm{Li}_\nu(z) = \Gamma(1-\nu)\,(-(z-1))^{\nu-1} \left(1 + \frac{1}{2}(1-\nu)(z-1)\, \frac{1}{24}\left(3\nu^2 - \nu - 2\right)(z-1)^2 + \ldots\right) + \zeta(\nu) + \zeta(\nu-1)(z-1) + \frac{1}{2}(\zeta(\nu-2) - \zeta(\nu-1))(z-1)^2 + \ldots \;/;\; \neg(\nu \in \mathbb{N})$$

10.08.06.0004.02

$$\mathrm{Li}_\nu(z) \propto \zeta(\nu) + \zeta(\nu-1)(z-1) + \frac{1}{2}(\zeta(\nu-2) - \zeta(\nu-1))(z-1)^2 + \ldots \;/;\; (z \to 1) \bigwedge \mathrm{Re}(\nu) > 1 \bigwedge \nu \notin \mathbb{Z}$$

10.08.06.0031.01

$$\mathrm{Li}_\nu(z) \propto \zeta(\nu) + \zeta(\nu-1)(z-1) + \frac{1}{2}(\zeta(\nu-2) - \zeta(\nu-1))(z-1)^2 + O\left((z-1)^3\right) \;/;\; \mathrm{Re}(\nu) > 1 \bigwedge \nu \notin \mathbb{Z}$$

10.08.06.0009.01

$$\mathrm{Li}_\nu(z) \propto \Gamma(1-\nu)\,(1-z)^{\nu-1}\,(1+O(z-1))\;/;\;(z\to 1)\land \mathrm{Re}(\nu)<1$$

10.08.06.0025.02

$$\mathrm{Li}_\nu(z) = \Gamma(1-\nu)\,(-\log(z))^{\nu-1} + \sum_{k=1}^{\infty}\left(\sum_{j=1}^{k}\frac{S_k^{(j)}\,\zeta(\nu-j)}{k!}\right)(z-1)^k + \zeta(\nu)\;/;\;\neg\,(\nu\in\mathbb{N})$$

10.08.06.0032.01

$$\mathrm{Li}_\nu(z) = \Gamma(1-\nu)\left(\sum_{k=0}^{\infty}\frac{(-1)^k\,(z-1)^k}{k+1}\right)^{\nu-1}(-(z-1))^{\nu-1} + \zeta(\nu) + \sum_{k=1}^{\infty}\left(\sum_{j=1}^{k}\frac{S_k^{(j)}\,\zeta(\nu-j)}{k!}\right)(z-1)^k\;/;\;\neg\,(\nu\in\mathbb{N})$$

10.08.06.0033.01

$$\mathrm{Li}_\nu(z) \propto \Gamma(1-\nu)\,(-(z-1))^{\nu-1}\,(1+O(z-1)) + \zeta(\nu)\,(1+O(z-1))\;/;\;\nu\neq 1$$

Special cases

10.08.06.0034.01

$$\mathrm{Li}_3(z) \propto \zeta(3) + \frac{\pi^2}{6}(z-1) + \frac{9-\pi^2}{12}(z-1)^2 + \frac{2\pi^2-21}{36}(z-1)^3 + \ldots - \frac{(z-1)^2}{2}\left(2-z+\frac{11}{12}(z-1)^2-\frac{5}{6}(z-1)^3+\ldots\right)\log(1-z)\;/;\;(z\to 1)$$

10.08.06.0035.01

$$\mathrm{Li}_n(z) \propto \zeta(n) + \frac{(z-1)^{n-1}}{(n-1)!}\left(\psi(n)+\gamma-\frac{\psi(n)\,(n-1)+\gamma\,(n-1)-1}{2}(z-1) - \frac{6\,n+\left(-3\,n^2+n+2\right)(\psi(n)+\gamma)-1}{24}(z-1)^2 - \frac{-3\,n^2-4\,n+(n-1)\,(n+1)\,(n+2)\,(\psi(n)+\gamma)+1}{48}(z-1)^3+\ldots\right) - \frac{(z-1)^{n-1}}{(n-1)!}\left(1-\frac{n-1}{2}(z-1)+\frac{(n-1)\,(3\,n+2)}{24}(z-1)^2-\frac{(n-1)\,(n+1)\,(n+2)}{48}(z-1)^3+\ldots\right)\log(1-z) + \sum_{j=1}^{n-2}\frac{\zeta(n-j)\,(z-1)^j}{j!}\left(1-\frac{j}{2}(z-1)+\frac{j\,(3\,j+5)}{24}(z-1)^2-\frac{j\,(j+2)\,(j+3)}{48}(z-1)^3+\ldots\right)\;/;\;(z\to 1)\land n\in\mathbb{Z}\land n>3$$

10.08.06.0036.01

$$\mathrm{Li}_n(z) = -\frac{(z-1)^{n-1}}{(n-1)!}\left(i\,\pi+2\,i\,\pi\left\lfloor-\frac{1}{2\,\pi}\arg\left(\frac{\log(z)}{z-1}\right)-\frac{\arg(z-1)}{2\,\pi}\right\rfloor+\log(z-1)\right)\sum_{k=0}^{\infty}p_{n-1,k}\,(z-1)^k - \frac{(z-1)^{n-1}}{(n-1)!}\sum_{k=0}^{\infty}p_{n-1,k}\,(z-1)^k\left(\log\left(\frac{\log(z)}{z-1}\right)-\psi(n)-\gamma\right) + \sum_{j=1}^{n-2}\frac{\zeta(n-j)\,(z-1)^j}{j!}\sum_{k=0}^{\infty}p_{j,k}\,(z-1)^k + \sum_{j=n}^{\infty}\frac{\zeta(n-j)\,(z-1)^j}{j!}\sum_{k=0}^{\infty}p_{j,k}\,(z-1)^k + \zeta(n)\;/;\;p_{j,0}=1\bigwedge p_{j,k}=\frac{1}{k}\sum_{i=1}^{k}\frac{(-1)^i\,(j\,i+i-k)}{i+1}\,p_{j,k-i}\bigwedge k\in\mathbb{N}^+\bigwedge n\in\mathbb{N}^+$$

10.08.06.0037.01

$$\mathrm{Li}_n(z) = \frac{(-\log(-\log(z))+\psi(n)+\gamma)}{(n-1)!}\log^{n-1}(z) + \sum_{j=1}^{n-2}\frac{\zeta(n-j)\log^j(z)}{j!} + \sum_{j=n}^{\infty}\frac{\zeta(n-j)\log^j(z)}{j!} + \zeta(n)\;/;\;n\in\mathbb{N}^+$$

Expansions at $z = \infty$

For the function itself

General case

10.08.06.0038.01

$$\mathrm{Li}_\nu(z) \propto \frac{(2\pi)^\nu}{\Gamma(\nu)} e^{\frac{\pi i \nu}{2}} \sum_{k=0}^{\infty} \frac{1}{\left(k + \frac{\log(-z)}{2\pi i} + \frac{1}{2}\right)^{1-\nu}} - \frac{e^{\pi i \nu}}{z} \left(1 + \frac{1}{4z} + \ldots\right) /; (|z| \to \infty) \land \mathrm{Re}(\nu) < 0$$

10.08.06.0010.02

$$\mathrm{Li}_\nu(z) \propto \frac{(2\pi)^\nu}{\Gamma(\nu)} e^{\frac{\pi i \nu}{2}} \sum_{k=0}^{\infty} \frac{1}{\left(k + \frac{\log(-z)}{2\pi i} + \frac{1}{2}\right)^{1-\nu}} - \frac{e^{\pi i \nu}}{z} \left(1 + \frac{1}{4z} + O\left(\frac{1}{z^2}\right)\right) /; \mathrm{Re}(\nu) < 0$$

10.08.06.0011.01

$$\mathrm{Li}_\nu(z) = \frac{(2\pi)^\nu}{\Gamma(\nu)} e^{\frac{\pi i \nu}{2}} \sum_{k=0}^{\infty} \frac{1}{\left(k + \frac{\log(-z)}{2\pi i} + \frac{1}{2}\right)^{1-\nu}} - e^{\pi i \nu} \sum_{k=1}^{\infty} \frac{1}{k^\nu z^k} /; |z| > 1 \land \mathrm{Re}(\nu) < 0$$

10.08.06.0012.01

$$\mathrm{Li}_\nu(z) = \frac{(2\pi)^\nu}{\Gamma(\nu)} e^{\frac{\pi i \nu}{2}} \sum_{k=0}^{\infty} \frac{1}{\left(k + \frac{\log(-z)}{2\pi i} + \frac{1}{2}\right)^{1-\nu}} - e^{\pi i \nu} \mathrm{Li}_\nu\left(\frac{1}{z}\right) /; |z| > 1 \land \mathrm{Re}(\nu) < 0$$

10.08.06.0039.01

$$\mathrm{Li}_\nu(z) \propto \frac{(2\pi)^\nu}{\Gamma(\nu)} e^{\frac{\pi i \nu}{2}} \zeta\left(1-\nu, \frac{\log(-z)}{2\pi i} + \frac{1}{2}\right) - e^{\pi i \nu} \mathrm{Li}_\nu\left(\frac{1}{z}\right)$$

10.08.06.0013.02

$$\mathrm{Li}_\nu(z) \propto \frac{(2\pi)^\nu}{\Gamma(\nu)} e^{\frac{\pi i \nu}{2}} \zeta\left(1-\nu, \frac{\log(-z)}{2\pi i} + \frac{1}{2}\right) - \frac{e^{\pi i \nu}}{z} \left(1 + O\left(\frac{1}{z}\right)\right)$$

Special cases

10.08.06.0014.01

$$\mathrm{Li}_{-n}(z) = (-1)^{n-1} \sum_{k=1}^{\infty} \frac{k^n}{z^k} /; |z| > 1 \bigwedge n \in \mathbb{N}^+$$

10.08.06.0015.02

$$\mathrm{Li}_{-n}(z) \propto \frac{(-1)^{n-1}}{z} \left(1 + O\left(\frac{1}{z}\right)\right) /; n \in \mathbb{N}^+$$

10.08.06.0017.01

$$\mathrm{Li}_n(z) = (-1)^{n-1} \sum_{k=1}^{\infty} \frac{1}{k^n z^k} - \frac{(2\pi i)^n}{n!} B_n\left(\frac{\log(-z)}{2\pi i} + \frac{1}{2}\right) /; |z| > 1 \bigwedge n \in \mathbb{N}^+$$

10.08.06.0018.02

$$\mathrm{Li}_n(z) \propto \frac{(-1)^{n-1}}{z} \left(1 + O\left(\frac{1}{z}\right)\right) - \frac{(2\pi i)^n}{n!} B_n\left(\frac{\log(-z)}{2\pi i} + \frac{1}{2}\right) /; n \in \mathbb{N}^+$$

10.08.06.0019.01

$$\operatorname{Li}_n(z) = (-1)^{n-1} \sum_{k=1}^{\infty} \frac{1}{k^n z^k} - \frac{\log^n(-z)}{n!} + 2 \sum_{k=1}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{\operatorname{Li}_{2k}(-1) \log^{n-2k}(-z)}{(n-2k)!} \,/;\, |z| > 1 \bigwedge n \in \mathbb{N}^+$$

10.08.06.0020.01

$$\operatorname{Li}_n(z) \propto \frac{(-1)^{n-1}}{z} \left(1 + O\left(\frac{1}{z}\right)\right) - \frac{\log^n(-z)}{n!} + 2 \sum_{k=1}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{\operatorname{Li}_{2k}(-1) \log^{n-2k}(-z)}{(n-2k)!} \,/;\, (|z| \to \infty) \bigwedge n \in \mathbb{N}^+$$

10.08.06.0021.01

$$\operatorname{Li}_\nu(-e^z) \propto -\cos(\pi\nu) \sum_{k=1}^{\infty} \frac{(-1)^k e^{-kz}}{k^\nu} - 2 \sum_{k=0}^{\left\lfloor \frac{\nu}{2} \right\rfloor} \frac{\left(1 - 2^{1-2k}\right) z^{\nu-2k} \zeta(2k)}{\Gamma(\nu + 1 - 2k)} + \frac{2\sin(\nu\pi)}{\pi} \sum_{k=\left\lfloor \frac{\nu}{2} \right\rfloor + 1}^{\infty} \frac{\left(1 - 2^{1-2k}\right) z^{\nu-2k} \zeta(2k)}{\Gamma(2k - \nu)} \,/;$$

$$(|z| \to \infty) \bigwedge 2\nu \in \mathbb{N}^+$$

10.08.06.0040.01

$$\operatorname{Li}_n(z) \propto -\frac{\log^n(-z)}{n!} \,/;\, (|z| \to \infty) \bigwedge n \in \mathbb{N}^+$$

Residue representations

10.08.06.0022.02

$$\operatorname{Li}_\nu(z) = \sum_{j=1}^{\infty} \operatorname{res}_s\left(\left(\Gamma(1-s)\left(\frac{\Gamma(-s)}{\Gamma(1-s)}\right)^\nu (-z)^{-s}\right)\Gamma(s)\right)(-j) \,/;\, |z| < 1 \vee (|z| = 1 \wedge \operatorname{Re}(\nu) > 1)$$

10.08.06.0023.02

$$\operatorname{Li}_n(z) = \sum_{j=1}^{\infty} \operatorname{res}_s\left(\frac{\Gamma(s+1)(-z)^{-s}}{(-s)^n} \Gamma(-s)\right)(j) \,/;\, |z| > 1 \bigwedge n \in \mathbb{N}^+$$

10.08.06.0026.02

$$\operatorname{Li}_\nu\left(\frac{1}{z}\right) = \sum_{j=1}^{\infty} \operatorname{res}_s\left(\left(\Gamma(s)\left(\frac{\Gamma(s)}{\Gamma(s+1)}\right)^\nu (-z)^{-s}\right)\Gamma(1-s)\right)(j) \,/;\, |z| > 1 \vee (|z| = 1 \wedge \operatorname{Re}(\nu) > 1)$$

Allan Cortzen

Other series representations

10.08.06.0024.01

$$\operatorname{Li}_\nu(z) = \Gamma(1-\nu)(-\log(z))^{\nu-1} + \sum_{k=0}^{\infty} \frac{\zeta(\nu-k)\log^k(z)}{k!}$$

Integral representations

On the real axis

Of the direct function

10.08.07.0001.01

$$\operatorname{Li}_\nu(z) = \frac{z}{\Gamma(\nu)} \int_0^\infty \frac{t^{\nu-1}}{e^t - z} \, dt \,/;\, \operatorname{Re}(\nu) > 0$$

10.08.07.0002.01

$$\mathrm{Li}_{\nu}(z) = \frac{z}{\Gamma(\nu)} \int_{1}^{\infty} \frac{\log^{\nu-1}(t)}{t\,(t-z)}\,dt \;/;\; \mathrm{Re}(\nu) > 0$$

10.08.07.0003.01

$$\mathrm{Li}_{n}(z) = -\frac{1}{(n-2)!} \int_{0}^{1} \frac{\log(1-t\,z)}{t} \log^{n-2}\left(\frac{1}{t}\right) dt \;/;\; |z| < 1 \bigwedge n \in \mathbb{N}^{+}$$

10.08.07.0004.01

$$\mathrm{Li}_{n}(z) = \int_{0}^{\infty} \frac{e^{t/z}}{t} \left(\sum_{k=1}^{\infty} \frac{(-t)^{k}}{k^{n-1}\,k!} \right) dt \;/;\; \mathrm{Re}(z) < 0 \bigwedge n \in \mathbb{N}^{+}$$

10.08.07.0010.01

$$\mathrm{Li}_{n}(z) = \frac{1}{\Gamma(n-1)} \int_{0}^{z} \frac{\log^{n-1}\left(\frac{z}{t}\right)}{1-t}\,dt \;/;\; \mathrm{Re}(z) < 1 \bigwedge n \in \mathbb{N}^{+}$$

10.08.07.0005.01

$$\mathrm{Li}_{2}(z) = -\int_{0}^{z} \frac{\log(1-t)}{t}\,dt$$

10.08.07.0006.01

$$\mathrm{Li}_{2}(z) = \frac{\pi^{2}}{6} - \int_{1}^{z} \frac{\log(1-t)}{t}\,dt$$

Contour integral representations

10.08.07.0007.01

$$\mathrm{Li}_{n}(z) = -\frac{1}{2\pi i} \int_{\mathcal{L}} \frac{\Gamma(s+1)\,\Gamma(-s)^{n+1}\,(-z)^{-s}}{\Gamma(1-s)^{n}}\,ds \;/;\; |\arg(-z)| < \pi \bigwedge n \in \mathbb{N}^{+}$$

10.08.07.0008.02

$$\mathrm{Li}_{\nu}(z) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{i\infty+\gamma} \Gamma(t)\,\Gamma(-t)\,(-t)^{1-\nu}\,(-z)^{-t}\,dt \;/;\; -1 < \gamma < 0 \bigwedge z \neq 0 \bigwedge (\arg(-z) < \pi \bigvee \mathrm{Re}(\nu) > 1)$$

Multiple integral representations

10.08.07.0009.01

$$\mathrm{Li}_{2}(z) = -\int_{0}^{z} \frac{1}{w} \left(\int_{1}^{1-w} \frac{1}{t}\,dt \right) dw$$

Limit representations

10.08.09.0001.01

$$\mathrm{Li}_{2}(z) = \lim_{\varepsilon \to 0} \frac{{}_2F_1(\varepsilon, \varepsilon; 1; z) - 1}{\varepsilon^{2}}$$

Differential equations

Ordinary linear differential equations and Wronskians

For the direct function itself

10.08.13.0001.01

$$\left(\prod_{k=1}^{n+1}\left(z\frac{d}{dz}+1\right)-\frac{d}{dz}\prod_{l=1}^{n}\left(z\frac{d}{dz}+1\right)\right)w(z)=0\;/;\;w(z)=\frac{\text{Li}_n(z)}{z}\bigwedge n\in\mathbb{N}^+$$

There is no algebraic partial differential equation for $\text{Li}_\nu(z)$ (A. Ostrowski, 1920).

Transformations

Multiple arguments

Dilogarithmical case

10.08.16.0001.01

$$\text{Li}_2(z\,w)=\text{Li}_2(w)+\text{Li}_2(z)-\text{Li}_2\left(\frac{z\,(1-w)}{1-w\,z}\right)-\text{Li}_2\left(\frac{w\,(1-z)}{1-w\,z}\right)-\log\left(\frac{1-w}{1-w\,z}\right)\log\left(\frac{1-z}{1-w\,z}\right)/;$$
$$(|z|<1\land|w|<1)\lor z\,w<1\lor(z>1\land w>1)$$

(L.Rogers, 1906)

10.08.16.0002.01

$$\text{Li}_2(z\,w)=\text{Li}_2(z)+\text{Li}_2(w)+\text{Li}_2\left(\frac{w\,(z-1)}{1-w}\right)+\text{Li}_2\left(\frac{z\,(w-1)}{1-z}\right)+\frac{1}{2}\log^2\left(\frac{1-w}{1-z}\right)/;$$
$$\neg\left(z\in\mathbb{R}\bigwedge w\in\mathbb{R}\bigwedge z>1\bigwedge w>1\bigwedge\frac{z\,(w-1)}{1-z}>1\bigwedge\frac{w\,(z-1)}{1-w}>1\right)\bigvee$$
$$(z\in\mathbb{R}\land w\in\mathbb{R}\land(z>1\land w>1)\lor(z<1\land w<1))$$

(C.J.Hill,1830)

10.08.16.0003.01

$$\text{Li}_2(z\,w)=\text{Li}_2(z)+\text{Li}_2(w)-\text{Li}_2\left(\frac{z\,(1-w)}{1-w\,z}\right)-\text{Li}_2\left(\frac{w\,(1-z)}{1-w\,z}\right)+\frac{1}{2}\left(\log(1-w)\log(w)+\right.$$
$$\left.\log(1-z)\log(z)-\log\left(\frac{1-z}{1-w\,z}\right)\log\left(\frac{z\,(1-w)}{1-w\,z}\right)-\log\left(\frac{1-w}{1-w\,z}\right)\log\left(\frac{w\,(1-z)}{1-w\,z}\right)-\log(w\,z)\log(1-w\,z)\right)/;$$
$$(|z|<1\land 0<w<1)\lor(|w|<1\land 0<z<1)\lor(z\in\mathbb{R}\land w\in\mathbb{R}\land(0<z<1\land w<1)\lor(0<w<1\land z<1)\lor(z>1\land w>1))$$

Power of arguments

10.08.16.0004.01

$$\text{Li}_\nu\left(z^2\right)=2^{\nu-1}\,(\text{Li}_\nu(z)+\text{Li}_\nu(-z))$$

10.08.16.0005.01

$$\text{Li}_2\left(z^2\right)=2\,(\text{Li}_2(z)+\text{Li}_2(-z))$$

10.08.16.0006.01

$$\text{Li}_n(z^m)=m^{n-1}\sum_{k=0}^{m-1}\text{Li}_n\left(e^{\frac{2\pi i k}{m}}\,z\right)/;\;m\in\mathbb{N}^+$$

Identities

Recurrence identities

General cases

Involving two polyilogarithms

10.08.17.0001.01

$$\mathrm{Li}_\nu(z) = \int_0^z \frac{1}{t}\,\mathrm{Li}_{\nu-1}(t)\,dt$$

Involving several polylogarithms

10.08.17.0004.01

$$\mathrm{Li}_n(z) = \frac{(-1)^{n-1}}{(n-2)!}\int_1^z \frac{\log(t)^{n-2}\log(1-t)}{t}\,dt + \mathrm{Li}_n(1) + \sum_{k=1}^{n-2}\frac{(-1)^{k-1}\,\mathrm{Li}_{n-k}(z)\log^k(z)}{k!} \;/;\; n-1\in\mathbb{N}^+$$

Functional identities

General cases

Involving two polyilogarithms

10.08.17.0006.01

$$\mathrm{Li}_\nu(-1) = -\left(1-2^{1-\nu}\right)\mathrm{Li}_\nu(1) \;/;\; \mathrm{Re}(\nu) > 1$$

10.08.17.0007.01

$$\mathrm{Li}_\nu(z) = \frac{(2\pi)^\nu}{\Gamma(\nu)}\,e^{\frac{\pi i \nu}{2}}\,\zeta\!\left(1-\nu,\, \frac{\log(-z)}{2\pi i} + \frac{1}{2}\right) - e^{\pi i \nu}\,\mathrm{Li}_\nu\!\left(\frac{1}{z}\right) \;/;\; z \notin (0,\,1)$$

10.08.17.0057.01

$$\mathrm{Li}_\nu(z) = -e^{\pi i \nu}\,\mathrm{Li}_\nu\!\left(\frac{1}{z}\right) + \left(1 - \sqrt{\frac{z-1}{z}}\,\sqrt{\frac{z}{z-1}}\right)\frac{i\pi}{\Gamma(\nu)}\log^{\nu-1}(z) + \frac{(2\pi)^\nu}{\Gamma(\nu)}\,e^{\frac{\pi i \nu}{2}}\,\zeta\!\left(1-\nu,\, \frac{\log(-z)}{2\pi i} + \frac{1}{2}\right)$$

10.08.17.0008.01

$$\mathrm{Li}_{-n}(z) = (-1)^{n-1}\,\mathrm{Li}_{-n}\!\left(\frac{1}{z}\right) \;/;\; n\in\mathbb{N}^+$$

10.08.17.0009.01

$$\mathrm{Li}_n(z) = (-1)^{n-1}\,\mathrm{Li}_n\!\left(\frac{1}{z}\right) - \frac{(2\pi i)^n}{n!}\,B_n\!\left(\frac{\log(-z)}{2\pi i} + \frac{1}{2}\right) \;/;\; z\notin(0,\,1) \land n\in\mathbb{N}^+$$

10.08.17.0058.01

$$\mathrm{Li}_n(z) = (-1)^{n-1}\,\mathrm{Li}_n\!\left(\frac{1}{z}\right) + \frac{i\pi}{(n-1)!}\left(1 - \sqrt{\frac{z-1}{z}}\,\sqrt{\frac{z}{z-1}}\right)\log^{n-1}(z) - \frac{(2 i \pi)^n}{n!}\,B_n\!\left(\frac{1}{2} - \frac{i\log(-z)}{2\pi}\right) \;/;\; n\in\mathbb{N}^+$$

10.08.17.0059.01

$$\mathrm{Li}_n(z) = (-1)^{n-1}\,\mathrm{Li}_n\!\left(\frac{1}{z}\right) - \frac{1}{n!}\left(\frac{2\pi\sqrt{-(z-1)^2}}{z-1}\right)^n B_n\!\left(1 - \frac{\sqrt{-(z-1)^2}}{(z-1)(2\pi)}\log(z)\right) \;/;\; n\in\mathbb{N}^+$$

10.08.17.0010.01

$$\text{Li}_n(z) = -\frac{\log^n(-z)}{n!} + (-1)^{n-1}\,\text{Li}_n\!\left(\frac{1}{z}\right) + 2\sum_{k=1}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{\text{Li}_{2k}(-1)}{(n-2k)!} \log^{n-2k}(-z) \;/;\; z \notin (0, 1) \bigwedge n \in \mathbb{N}^+$$

10.08.17.0060.01

$$\text{Li}_n(z) = (-1)^{n-1}\,\text{Li}_n\!\left(\frac{1}{z}\right) + \left(1 - \sqrt{\frac{z-1}{z}}\,\sqrt{\frac{z}{z-1}}\right) \frac{i\,\pi}{(n-1)!} \log^{n-1}(z) + 2\sum_{k=1}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{\text{Li}_{2k}(-1)\log^{n-2k}(-z)}{(n-2k)!} - \frac{\log^n(-z)}{n!} \;/;\; n \in \mathbb{N}^+$$

Involving three polylogarithms

10.08.17.0011.01

$$\text{Li}_\nu(z^2) = 2^{\nu-1}\,(\text{Li}_\nu(z) + \text{Li}_\nu(-z))$$

Dilogarithmical cases

Involving two dilogarithms

10.08.17.0012.01

$$\text{Li}_2(z) = -\text{Li}_2(1-z) - \log(1-z)\log(z) + \frac{\pi^2}{6}$$

(L.Euler, 1768)

10.08.17.0013.01

$$\text{Li}_2(z) = -\text{Li}_2\!\left(\frac{1}{z}\right) - \frac{1}{2}\log^2(-z) - \frac{\pi^2}{6} \;/;\; z \notin (0, 1)$$

10.08.17.0061.01

$$\text{Li}_2(z) = -\text{Li}_2\!\left(\frac{1}{z}\right) - \frac{1}{2}\log^2(-z) - i\,\pi\left(\sqrt{\frac{z-1}{z}}\,\sqrt{\frac{z}{z-1}} - 1\right)\log(z) - \frac{\pi^2}{6}$$

10.08.17.0062.01

$$\text{Li}_2(z) = -\frac{1}{2}\log^2(z) - \frac{\pi\sqrt{-(z-1)^2}}{z-1}\log(z) + \frac{\pi^2}{3} - \text{Li}_2\!\left(\frac{1}{z}\right)$$

10.08.17.0014.01

$$\text{Li}_2(z) = -\text{Li}_2\!\left(\frac{1}{z}\right) - \frac{1}{2}\log^2(z) - \frac{\pi}{z-1}\sqrt{-(z-1)^2}\,\log(z) + \frac{\pi^2}{3}$$

10.08.17.0015.01

$$\text{Li}_2(z) = \text{Li}_2\!\left(\frac{1}{1-z}\right) + \frac{1}{2}\log(1-z)\log\!\left(\frac{1-z}{z^2}\right) - \frac{\pi^2}{6} \;/;\; \text{Re}(z) \le 0 \bigvee z \notin (1, \infty)$$

10.08.17.0016.01

$$\text{Li}_2(z) = \text{Li}_2\!\left(\frac{1}{1-z}\right) + \frac{1}{2}\log^2(1-z) - \log(-z)\log(1-z) - \frac{\pi^2}{6}$$

10.08.17.0017.01

$$\text{Li}_2(z) = \text{Li}_2\!\left(\frac{1}{1-z}\right) + \frac{1}{2}\log^2(z-1) + \frac{\pi^2}{3} - \log(z)\log(1-z) \;/;\; z \notin (0, 1)$$

10.08.17.0018.01

$$\mathrm{Li}_2(z) = -\mathrm{Li}_2\left(\frac{z}{z-1}\right) - \frac{1}{2}\log^2(1-z) \;/;\; z \notin (1, \infty)$$

(J.Landen,1780)

10.08.17.0019.01

$$\mathrm{Li}_2(z) = -\mathrm{Li}_2\left(\frac{z}{z-1}\right) - \frac{1}{2}\log^2(z-1) + \frac{\pi^2}{2} + i\,\pi\log\left(\frac{z-1}{z^2}\right) \;/;\; z > 1$$

10.08.17.0020.01

$$\mathrm{Li}_2(z) = -\mathrm{Li}_2\left(\frac{z}{z-1}\right) - \frac{1}{2}\log^2(z-1) + i\,\pi\log(z-1) + \frac{\pi^2}{2} - 2\,\pi\,i\log(z) \;/;\; z > 1$$

10.08.17.0021.01

$$\mathrm{Li}_2(z) - \mathrm{Li}_2\left(\frac{z-1}{z}\right) = \frac{\log^2(z)}{2} - \log(1-z)\log(z) + \frac{\pi^2}{6} \;/;\; z \notin (-\infty, 0)$$

Involving three dilogarithms

10.08.17.0022.01

$$\mathrm{Li}_2(z) = \mathrm{Li}_2(z+1) - \frac{1}{2}\mathrm{Li}_2\left(\frac{1}{z^2}\right) + \log\left(\frac{z+1}{z}\right)\log(-z) \;/;\; z \notin (0, 1)$$

10.08.17.0023.01

$$\mathrm{Li}_2(z) = \mathrm{Li}_2\left(\frac{z}{z+1}\right) + \frac{1}{2}\left(+\mathrm{Li}_2(z^2) - \log(1-z)\log(z) + \log\left(\frac{1}{z+1}\right)\log\left(\frac{z}{z+1}\right) + \frac{1}{2}\log(z^2)\log(1-z^2)\right) \;/;\; \mathrm{Im}(z) > 0 \lor z > 0$$

Abel's duplication formula

10.08.17.0024.01

$$\mathrm{Li}_2(2z - z^2) = -\log^2(2-z) + \frac{\pi^2}{6} - 2\,\mathrm{Li}_2\left(\frac{1}{2-z}\right) + 2\,\mathrm{Li}_2(z) \;/;\; |z| < 1$$

10.08.17.0025.01

$$\mathrm{Li}_2\left(\frac{z}{z-1}\right) + \mathrm{Li}_2\left(\frac{z}{z+1}\right) - \frac{1}{2}\mathrm{Li}_2\left(\frac{z^2}{z^2-1}\right) = -\frac{1}{4}\log^2\left(\frac{1+z}{1-z}\right) \;/;\; |z| < 1 \lor \mathrm{Re}(z) \le 0 \lor \mathrm{Im}(z) \ge 0$$

10.08.17.0026.01

$$\mathrm{Li}_2\left(\frac{z}{z-1}\right) + \mathrm{Li}_2\left(\frac{z}{z+1}\right) - \frac{1}{2}\mathrm{Li}_2\left(\frac{z^2}{z^2-1}\right) = -\frac{1}{4}\log^2\left(\frac{1+z}{1-z}\right) \;/;\; |z| < 1 \lor \mathrm{Re}(z) \le 0 \lor \mathrm{Im}(z) \ge 0$$

10.08.17.0027.01

$$\mathrm{Li}_2(z^2) - 2\,(\mathrm{Li}_2(z) + \mathrm{Li}_2(-z)) = 0$$

10.08.17.0028.01

$$\mathrm{Li}_2(z^2) - 2\,\mathrm{Li}_2(z) + 2\,\mathrm{Li}_2\left(\frac{z}{z+1}\right) = \log(1-z)\log(z) - \log\left(\frac{1}{z+1}\right)\log\left(\frac{z}{z+1}\right) - \frac{1}{2}\log(z^2)\log(1-z^2) \;/;$$
$$\mathrm{Re}(z) > 0 \lor (\mathrm{Re}(z) = 0 \land \mathrm{Im}(z) > 0)$$

Involving four dilogarithms

10.08.17.0029.01

$$\operatorname{Li}_2(z)-\operatorname{Li}_2(-z)+\operatorname{Li}_2\left(\frac{1-z}{1+z}\right)-\operatorname{Li}_2\left(-\frac{1-z}{1+z}\right)=\log\left(\frac{1+z}{1-z}\right)\log(z)+\frac{\pi^2}{4}\ /;\ z\notin(-\infty,-1)\bigwedge z\notin(1,\infty)$$

Involving five dilogarithms

10.08.17.0030.01

$$L(z)+L(w)-L(z\,w)-L\left(\frac{z\,(1-w)}{1-z\,w}\right)-L\left(\frac{w\,(1-z)}{1-z\,w}\right)=0\ /;$$
$$\left(L(z)=\operatorname{Li}_2(z)+\frac{1}{2}\log(1-z)\log(z)\ /;\ (|z|<1\land 0<w<1)\lor(|w|<1\land 0<z<1)\lor\right.$$
$$\left.(z<1\land 0<w<1)\lor(w<1\land 0<z<1)\lor(z\,w<1\land z>0\land w>0)\lor(z>1\land w>1)\right)$$

10.08.17.0031.01

$$\operatorname{Li}_2\left(\frac{z\,w}{(1-z)\,(1-w)}\right)=$$
$$+\operatorname{Li}_2\left(\frac{z}{1-w}\right)+\operatorname{Li}_2\left(\frac{w}{1-z}\right)-\operatorname{Li}_2(z)-\operatorname{Li}_2(w)+\frac{1}{2}\left(\log\left(\frac{z}{1-w}\right)\log\left(\frac{w+z-1}{w-1}\right)+\log\left(\frac{w}{1-z}\right)\log\left(\frac{w+z-1}{z-1}\right)-\right.$$
$$\left.\log\left(\frac{w\,z}{(w-1)\,(z-1)}\right)\log\left(-\frac{w+z-1}{(w-1)\,(z-1)}\right)-\log(1-z)\log(z)-\log(1-w)\log(w)\right)\ /;$$
$$(|z|<1\land 0<w<1)\lor|w|<1\land 0<z<1\lor z<1\land 0<w<1\lor w<1\land 0<z<1\lor w+z>1\land w<0\lor w+z>1\land z<0$$

10.08.17.0032.01

$$\operatorname{Li}_2\left(\frac{z\,w}{(1-z)\,(1-w)}\right)=\operatorname{Li}_2\left(\frac{w}{w-1}\right)+\operatorname{Li}_2\left(\frac{z}{z-1}\right)+\operatorname{Li}_2\left(\frac{z}{1-w}\right)+\operatorname{Li}_2\left(\frac{w}{1-z}\right)+\frac{1}{2}\log^2\left(\frac{1-z}{1-w}\right)\ /;$$
$$(|z|<1\land|w|<1)\lor(w<1\land z<1)\lor(w<1\land z<1)\lor 0<w<-z$$

(W.Spence, 1809)

10.08.17.0033.01

$$\operatorname{Li}_2\left(\frac{z\,w}{(1-z)\,(1-w)}\right)=\operatorname{Li}_2\left(\frac{z}{1-w}\right)+\operatorname{Li}_2\left(\frac{w}{1-z}\right)-\operatorname{Li}_2(z)-\operatorname{Li}_2(w)-\log(1-z)\log(1-w)\ /;$$
$$|z|<1\land|w|<1\lor(w\in\mathbb{R}\land w<1\land z\in\mathbb{R}\land z<1)$$

(N.Abel, 1830)

10.08.17.0034.01

$$\operatorname{Li}_2(z\,w)-\operatorname{Li}_2(w)-\operatorname{Li}_2(z)+\operatorname{Li}_2\left(\frac{z\,(1-w)}{1-w\,z}\right)+\operatorname{Li}_2\left(\frac{w\,(1-z)}{1-w\,z}\right)=-\log\left(\frac{1-w}{1-w\,z}\right)\log\left(\frac{1-z}{1-w\,z}\right)\ /;$$
$$(|z|<1\land|w|<1)\lor z\,w<1\lor(z>1\land w>1)$$

(L.Rogers, 1906)

10.08.17.0035.01

$$\operatorname{Li}_2(z\,w)-\operatorname{Li}_2(z)-\operatorname{Li}_2(w)-\operatorname{Li}_2\left(\frac{w\,(z-1)}{1-w}\right)-\operatorname{Li}_2\left(\frac{z\,(w-1)}{1-z}\right)=\frac{1}{2}\log^2\left(\frac{1-w}{1-z}\right)\ /;$$
$$\neg\left(z\in\mathbb{R}\bigwedge w\in\mathbb{R}\bigwedge z>1\bigwedge w>1\bigwedge\frac{z\,(w-1)}{1-z}>1\bigwedge\frac{w\,(z-1)}{1-w}>1\right)\bigvee$$
$$(z\in\mathbb{R}\land w\in\mathbb{R}\land(z>1\land w>1)\lor(z<1\land w<1))$$

(C.J.Hill,1830)

10.08.17.0036.01

$$\mathrm{Li}_2(z\,w)-\mathrm{Li}_2(z)-\mathrm{Li}_2(w)+\mathrm{Li}_2\left(\frac{z\,(1-w)}{1-w\,z}\right)+\mathrm{Li}_2\left(\frac{w\,(1-z)}{1-w\,z}\right)=\frac{1}{2}\left(\log(1-w)\log(w)+\log(1-z)\log(z)-\log\left(\frac{1-z}{1-w\,z}\right)\log\left(\frac{z\,(1-w)}{1-w\,z}\right)-\log\left(\frac{1-w}{1-w\,z}\right)\log\left(\frac{w\,(1-z)}{1-w\,z}\right)-\log(w\,z)\log(1-w\,z)\right)\,/;$$
$$(|z|<1\land 0<w<1)\lor(|w|<1\land 0<z<1)\lor(z\in\mathbb{R}\land w\in\mathbb{R}\land(0<z<1\land w<1)\lor(0<w<1\land z<1)\lor(z>1\land w>1))$$

10.08.17.0037.01

$$\mathrm{Li}_2(z)+\mathrm{Li}_2(w-z)+\mathrm{Li}_2\left(\frac{w-z}{w-1}\right)-\mathrm{Li}_2\left(\frac{z\,(w-z)}{w-1}\right)+\mathrm{Li}_2\left(\frac{z}{w-1}\right)=-\frac{1}{2}\log^2(1-w)\,/;\,|z|<1\land|w|<1$$

10.08.17.0038.01

$$\mathrm{Li}_2\left(\frac{z\,(1-w)}{w\,(1-z)}\right)=\mathrm{Li}_2(w)-\mathrm{Li}_2(z)+\mathrm{Li}_2\left(\frac{z}{w}\right)+\mathrm{Li}_2\left(\frac{1-w}{1-z}\right)+\log(w)\log\left(\frac{1-w}{1-z}\right)-\frac{\pi^2}{6}\,/;$$
$$(0<z<1\land|w|<1)\lor(0<w<1\land|z|<1)\lor(w\in\mathbb{R}\land w>0\land z\in\mathbb{R}\land z<1)$$

(W.Schaeffer, 1846)

10.08.17.0039.01

$$\mathrm{Li}_2\left(\frac{z\,(1-w)^2}{w\,(1-z)^2}\right)=\mathrm{Li}_2\left(\frac{1-w}{1-z}\right)+\mathrm{Li}_2\left(-\frac{z\,(1-w)}{1-z}\right)+\mathrm{Li}_2\left(\frac{z\,(1-w)}{w\,(1-z)}\right)+\mathrm{Li}_2\left(-\frac{1-w}{w\,(1-z)}\right)+\frac{1}{2}\log^2(w)\,/;$$
$$(w>0\land z\in\mathbb{R})\lor(z>w\land z\,w>1)$$

(E.Kummer, 1840)

Involving six dilogarithms

10.08.17.0040.01

$$\mathrm{Li}_2(1-x)+\mathrm{Li}_2(1-y)+\mathrm{Li}_2(1-z)=\frac{1}{2}\left(\mathrm{Li}_2(1-x\,y)+\mathrm{Li}_2(1-x\,z)+\mathrm{Li}_2(1-y\,z)\right)\,/;$$
$$x+y+z=x\,y\,z+2\bigwedge|x|<1\bigwedge|y|<1\bigwedge|z|<1$$

10.08.17.0041.01

$$\mathrm{Li}_2(x)+\mathrm{Li}_2(y)+\mathrm{Li}_2(z)=\frac{1}{2}\left(\mathrm{Li}_2(-y\,x+x+y)+\mathrm{Li}_2(-z\,x+x+z)+\mathrm{Li}_2(-z\,y+y+z)\right)\,/;$$
$$x\,y+z\,y+x\,z=x\,y\,z\bigwedge|x|<1\bigwedge|y|<1\bigwedge|z|<1$$

10.08.17.0042.01

$$\mathrm{Li}_2(x)+\mathrm{Li}_2(y)+\mathrm{Li}_2(z)=\frac{1}{2}\left(\mathrm{Li}_2\left(-\frac{y\,z}{x}\right)+\mathrm{Li}_2\left(-\frac{x\,z}{y}\right)+\mathrm{Li}_2\left(-\frac{x\,y}{z}\right)\right)\,/;\,\frac{1}{y}+\frac{1}{z}+\frac{1}{x}=1\bigwedge|x|<1\bigwedge|y|<1\bigwedge|z|<1$$

Newman's formula

10.08.17.0043.01

$$\mathrm{Li}_2\left(-x^2\right)+\mathrm{Li}_2\left(-y^2\right)+\mathrm{Li}_2\left(-z^2\right)=2\left(\mathrm{Li}_2(x\,y)+\mathrm{Li}_2(x\,z)+\mathrm{Li}_2(y\,z)\right)\,/;\,x+y+z=x\,y\,z\land|x|<1\land|y|<1\land|z|<1$$

Involving nine dilogarithms

10.08.17.0044.01

$$\mathrm{Li}_2\left(\frac{v\,w}{x\,y}\right) = \mathrm{Li}_2\left(\frac{v}{x}\right) + \mathrm{Li}_2\left(\frac{w}{x}\right) + \mathrm{Li}_2\left(\frac{v}{y}\right) + \mathrm{Li}_2\left(\frac{w}{y}\right) + \mathrm{Li}_2(x) + \mathrm{Li}_2(y) - \mathrm{Li}_2(v) - \mathrm{Li}_2(w) + \frac{1}{2}\log^2\left(-\frac{x}{y}\right) /; (1-v)(1-w) = (1-x)(1-y) \land 0 < x < 1 \land 0 < y < 1 \land 0 < v < 1 \land 0 < w < 1$$

(W.Mantel, 1898)

Involving several dilogarithms

10.08.17.0045.01

$$\mathrm{Li}_2(z) = \sum_{k=1}^{n}\sum_{p=1}^{n}\left(\mathrm{Li}_2(z_k\,\lambda_p) - \mathrm{Li}_2\left(\frac{\lambda_k}{\lambda_p}\right)\right) + \frac{\pi^2}{6} /; \prod_{p=1}^{n}(1-\lambda_p\,z_k) = 1 - z \bigwedge 1 \le k \le n$$

Trilogarithmical cases

Involving two trilogarithms

10.08.17.0046.01

$$\mathrm{Li}_3(z) = \mathrm{Li}_3\left(\frac{1}{z}\right) - \frac{1}{6}\log^3(-z) - \frac{\pi^2}{6}\log(-z) /; z \notin (0, 1)$$

10.08.17.0047.01

$$\mathrm{Li}_3(z) = \mathrm{Li}_3\left(\frac{1}{z}\right) - \frac{1}{6}\log^3(z) - \frac{\pi\sqrt{-(z-1)^2}}{2(z-1)}\log^2(z) + \frac{\pi^2}{3}\log(z)$$

10.08.17.0063.01

$$\mathrm{Li}_3(z) = \mathrm{Li}_3\left(\frac{1}{z}\right) - \frac{1}{6}\log^3(-z) - \frac{i\,\pi}{2}\left(\sqrt{\frac{z-1}{z}}\sqrt{\frac{z}{z-1}} - 1\right)\log^2(z) - \frac{1}{6}\pi^2\log(-z)$$

Involving three trilogarithms

10.08.17.0048.01

$$\mathrm{Li}_3(z) = -\mathrm{Li}_3\left(\frac{z}{z-1}\right) - \mathrm{Li}_3(1-z) + \frac{1}{6}\log^3(1-z) - \frac{1}{2}\log(z)\log^2(1-z) + \frac{1}{6}\pi^2\log(1-z) + \zeta(3) /; z \notin (1, \infty)$$

(J.Landen,1780)

10.08.17.0049.01

$$\mathrm{Li}_3(z) = -\mathrm{Li}_3\left(\frac{z}{z-1}\right) - \mathrm{Li}_3\left(\frac{1}{1-z}\right) + \frac{1}{3}\log^3(1-z) - \frac{1}{2}\log(-z)\log^2(1-z) - \frac{1}{6}\pi^2\log(1-z) + \zeta(3) /; z \notin (1, \infty)$$

Involving six trilogarithms

10.08.17.0050.01

$$\mathrm{Li}_3\left(\frac{1-z}{1+z}\right) - \mathrm{Li}_3\left(-\frac{1-z}{1+z}\right) = 2\,\mathrm{Li}_3(1-z) + 2\,\mathrm{Li}_3\left(\frac{1}{z+1}\right) - \frac{1}{2}\mathrm{Li}_3(1-z^2) - \frac{7}{4}\mathrm{Li}_3(1) - \frac{1}{3}\log^3(z+1) + \frac{1}{6}\pi^2\log(z+1) /; \mathrm{Re}(z) > 0$$

10.08.17.0051.01

$$\mathrm{Li}_3\left(\frac{1-z}{1+z}\right)-\mathrm{Li}_3\left(-\frac{1-z}{1+z}\right)=-2\,\mathrm{Li}_3\left(\frac{z}{z-1}\right)-2\,\mathrm{Li}_3\left(\frac{z}{z+1}\right)+\frac{1}{2}\mathrm{Li}_3\left(\frac{z^2}{z^2-1}\right)+\frac{7}{4}\mathrm{Li}_3(1)+\frac{1}{4}\log\left(\frac{1-z^2}{z^2}\right)\log^2\left(\frac{1+z}{1-z}\right)+\frac{1}{4}\pi^2\log\left(\frac{1-z}{1+z}\right)\text{ /; }\mathrm{Re}(z)>0\bigwedge z\notin(1,\infty)$$

10.08.17.0052.01

$$\mathrm{Li}_3\left(\left(\frac{z}{1-z}\right)^3\right)-2\,\mathrm{Li}_3\left(\frac{z^2}{z-1}\right)-\frac{3}{2}\mathrm{Li}_3\left(\left(\frac{z}{1-z}\right)^2\right)-3\,\mathrm{Li}_3\left(\frac{z}{1-z}\right)+\mathrm{Li}_3(z\,(1-z))-2\,\mathrm{Li}_3\left(-\frac{z}{(1-z)^2}\right)=-\log^3(1-z)\text{ /; }|z|<1\lor\mathrm{Re}(z)\le 0$$

Involving six trilogarithms

10.08.17.0053.01

$$\mathrm{Li}_3(z)+\mathrm{Li}_3(1-z)+\frac{1}{4}\mathrm{Li}_3\left(\frac{z^2}{(1-z)^2}\right)-\mathrm{Li}_3\left(\frac{z}{1-z}\right)=\frac{1}{6}\log^3(1-z)-\frac{1}{2}\log(z)\log^2(1-z)+\frac{\pi^2}{6}\log(1-z)+\zeta(3)\text{ /; }z\notin(1,\infty)$$

Involving seven trilogarithms

10.08.17.0054.01

$$\mathrm{Li}_3\left(-\left(\frac{1-z}{1+z}\right)^2\right)-\frac{1}{2}\mathrm{Li}_3\left(\left(\frac{1-z}{1+z}\right)^2\right)+\mathrm{Li}_3\left(-z^2\right)-\frac{1}{2}\mathrm{Li}_3\left(z^2\right)-2\,\mathrm{Li}_3\left(\frac{z\,(1-z)}{1+z}\right)-2\,\mathrm{Li}_3\left(-\frac{1-z}{z\,(1+z)}\right)+\frac{5}{4}\mathrm{Li}_3(1)=-\frac{1}{3}\log^3(z)+\log\left(\frac{1-z}{1+z}\right)\log^2(z)-\frac{1}{3}\pi^2\log(z)\text{ /; }\mathrm{Re}(z)>0$$

Involving nine trilogarithms

10.08.17.0055.01

$$\mathrm{Li}_3\left(-x^2\right)+\mathrm{Li}_3\left(-y^2\right)+\mathrm{Li}_3\left(-z^2\right)-2\,\mathrm{Li}_3(x\,y)-2\,\mathrm{Li}_3(x\,z)-2\,\mathrm{Li}_3(y\,z)-2\,\mathrm{Li}_3\left(-\frac{x}{y}\right)-2\,\mathrm{Li}_3\left(-\frac{x}{z}\right)-2\,\mathrm{Li}_3\left(-\frac{y}{z}\right)=\frac{1}{3}\log^3\left(-\frac{z}{y}\right)+\log\left(-\frac{x}{z}\right)\log^2\left(-\frac{z}{y}\right)+\frac{1}{3}\pi^2\log\left(-\frac{z}{y}\right)-2\,\mathrm{Li}_3(1)\text{ /; }x+y+z=x\,y\,z$$

Relations of special kind

10.08.17.0056.01

$$f(z\,w)=f(w)-f\left(\frac{z\,(1-w)}{1-z\,w}\right)-f\left(\frac{w\,(1-z)}{1-z\,w}\right)+f(z)\bigwedge f(z)+f(1-z)=\frac{\pi^2}{6}\text{ /; }f(z)=\mathrm{Li}_2(z)+\frac{1}{2}\log(1-z)\log(z)\bigwedge 0<z<1$$

$\mathrm{Li}_2(z)+\frac{1}{2}\log(1-z)\log(z)$ is the unique solution of class $C^3((0,1))$ of the functional equations $f(z\,w)=f(w)-f\left(\frac{z\,(1-w)}{1-z\,w}\right)-f\left(\frac{w\,(1-z)}{1-z\,w}\right)+f(z)\bigwedge f(z)+f(1-z)=\frac{\pi^2}{6}$ for all real $0<z<1\bigwedge 0<w<1$.

Complex characteristics

Real part

10.08.19.0001.01

$$\operatorname{Re}(\operatorname{Li}_4(2)) = -\operatorname{Li}_4\left(\frac{1}{2}\right) - \frac{1}{24}\log^4(2) + \frac{\pi^2}{6}\log^2(2) + \frac{\pi^4}{45}$$

10.08.19.0002.01

$$\operatorname{Re}(\operatorname{Li}_2(x + i\,y)) = -\frac{1}{4}\left(\log\left(x^2 + y^2\right)\left(\log\left(x^2 - 2\,x + y^2 + 1\right) - \log\left(\frac{x^2 + y^2}{\sqrt{-y^2} - x} + 1\right) - \log\left(1 - \frac{x^2 + y^2}{x + \sqrt{-y^2}}\right)\right) - 2\operatorname{Li}_2\left(\frac{x^2 + y^2}{x - \sqrt{-y^2}}\right) - 2\operatorname{Li}_2\left(\frac{x^2 + y^2}{x + \sqrt{-y^2}}\right)\right)$$

Imaginary part

10.08.19.0003.01

$$\operatorname{Im}(\operatorname{Li}_4(2)) = -\frac{\pi}{6}\log^3(2)$$

Differentiation

Low-order differentiation

With respect to ν

10.08.20.0001.01

$$\frac{\partial \operatorname{Li}_\nu(z)}{\partial \nu} = -\sum_{k=2}^{\infty} \frac{\log(k)\,z^k}{k^\nu} \;/;\; |z| < 1$$

10.08.20.0002.01

$$\frac{\partial^2 \operatorname{Li}_\nu(z)}{\partial \nu^2} = \sum_{k=2}^{\infty} \frac{\log^2(k)\,z^k}{k^\nu} \;/;\; |z| < 1$$

With respect to z

10.08.20.0003.01

$$\frac{\partial \operatorname{Li}_\nu(z)}{\partial z} = \frac{1}{z}\operatorname{Li}_{\nu-1}(z)$$

10.08.20.0004.01

$$\frac{\partial^2 \operatorname{Li}_\nu(z)}{\partial z^2} = \frac{1}{z^2}\left(\operatorname{Li}_{\nu-2}(z) - \operatorname{Li}_{\nu-1}(z)\right)$$

Symbolic differentiation

With respect to ν

10.08.20.0005.02

$$\frac{\partial^m \operatorname{Li}_\nu(z)}{\partial \nu^m} = (-1)^m \sum_{k=2}^{\infty} \frac{\log^m(k)\,z^k}{k^\nu} \;/;\; |z| < 1 \land m \in \mathbb{N}$$

With respect to z

10.08.20.0007.02

$$\frac{\partial^m \operatorname{Li}_\nu(z)}{\partial z^m} == z^{-m} \sum_{j=0}^{m} S_m^{(j)} \operatorname{Li}_{\nu-j}(z) /; n \in \mathbb{N}$$

10.08.20.0006.02

$$\frac{\partial^m \operatorname{Li}_\nu(z)}{\partial z^m} == \sum_{k=0}^{\infty} \frac{(k+1)_m z^k}{(k+m)^\nu} /; |z| < 1 \land n \in \mathbb{N}$$

10.08.20.0008.02

$$\frac{\partial^m \operatorname{Li}_\nu(z)}{\partial z^m} == \sum_{j=0}^{m} S_m^{(j)} \Phi(z, \nu - j, m) /; n \in \mathbb{N}$$

10.08.20.0009.01

$$\frac{\partial^m \operatorname{Li}_n(z)}{\partial z^m} == \Gamma(m)^n \, {}_n\tilde{F}_{n-1}(a_1, a_2, \ldots, a_n; a_1 + 1, a_2 + 1, \ldots, a_{n-1} + 1; z) /; a_1 == a_2 == \ldots == a_n == m \bigwedge m \in \mathbb{N}^+ \bigwedge n \in \mathbb{N}^+$$

Fractional integro-differentiation

With respect to ν

10.08.20.0010.01

$$\frac{\partial^\alpha \operatorname{Li}_\nu(z)}{\partial \nu^\alpha} == \nu^{-\alpha} \sum_{k=1}^{\infty} \frac{(-\nu \log(k))^\alpha Q(-\alpha, 0, -\nu \log(k)) z^k}{k^\nu} /; |z| < 1$$

With respect to z

10.08.20.0012.01

$$\frac{\partial^\alpha \operatorname{Li}_n(z)}{\partial z^\alpha} == z^{1-\alpha} \, {}_{n+1}\tilde{F}_n(a_1, a_2, \ldots, a_{n+1}; 2 - \alpha, \; a_1 + 1, a_2 + 1, \ldots, a_{n-1} + 1; z) /; a_1 == a_2 == \ldots == a_{n+1} == 1 \bigwedge n \in \mathbb{N}^+$$

10.08.20.0011.01

$$\frac{\partial^\alpha \operatorname{Li}_\nu(z)}{\partial z^\alpha} == \sum_{k=1}^{\infty} \frac{(k-1)! \, z^{k-\alpha}}{\Gamma(k - \alpha + 1) \, k^{\nu-1}} /; |z| < 1$$

Integration

Indefinite integration

Involving only one direct function

10.08.21.0001.01

$$\int \operatorname{Li}_\nu(z) \, dz == \sum_{k=1}^{\infty} \frac{z^{k+1}}{(k+1) k^\nu} /; |z| < 1$$

10.08.21.0002.01

$$\int \operatorname{Li}_n(z) \, dz == (-1)^n \left((z-1) \log(1-z) - z + z \sum_{k=2}^{n} (-1)^k \operatorname{Li}_k(z) \right) /; n - 1 \in \mathbb{N}^+$$

10.08.21.0003.01

$$\int \operatorname{Li}_2(z) \, dz == \operatorname{Li}_2(z) \, z - z + (z-1) \log(1-z)$$

Involving one direct function and elementary functions

Involving power function

10.08.21.0004.01

$$\int z^{\alpha-1}\,\mathrm{Li}_\nu(z)\,dz = \sum_{k=1}^{\infty}\frac{z^{k+\alpha}}{(k+\alpha)\,k^\nu}\;/;\;|z|<1$$

10.08.21.0005.01

$$\int z^{\alpha-1}\,\mathrm{Li}_n(z)\,dz = \frac{z^{\alpha+1}}{\alpha+1}\,{}_{n+2}\tilde{F}_{n+1}(\alpha+1,\,1,\,a_1,\,a_2,\,\ldots,\,a_n;\,\alpha+2,\,a_1+1,\,a_2+1,\,\ldots,\,a_n+1;\,z)\;/;$$
$$a_1 = a_2 = \ldots = a_n = 1 \bigwedge n-1\in\mathbb{N}^+$$

Involving rational functions

10.08.21.0006.01

$$\int \frac{\mathrm{Li}_2(z)}{1-z}\,dz = -\log(z)\log^2(1-z) - 2\,\mathrm{Li}_2(1-z)\log(1-z) - \mathrm{Li}_2(z)\log(1-z) + 2\,\mathrm{Li}_3(1-z)$$

10.08.21.0007.01

$$\int \frac{\mathrm{Li}_2(z)}{(z-1)^2}\,dz = \frac{z\,\mathrm{Li}_2(z)}{1-z} - \frac{1}{2}\log^2(1-z)$$

Involving exponential function

10.08.21.0008.01

$$\int \mathrm{Li}_2(e^{-z})\,dz = -\mathrm{Li}_3(e^{-z})$$

Involving rational functions and logarithm

10.08.21.0009.01

$$\int \frac{\log(z)\,\mathrm{Li}_2(z)}{z}\,dz = \log(z)\,\mathrm{Li}_3(z) - \mathrm{Li}_4(z)$$

10.08.21.0010.01

$$\int \frac{\log(1-z)\,\mathrm{Li}_2(z)}{1-z}\,dz =$$
$$\frac{1}{2}\left(-\log(z)\log^3(1-z) - 3\,\mathrm{Li}_2(1-z)\log^2(1-z) - \mathrm{Li}_2(z)\log^2(1-z) + 6\,\mathrm{Li}_3(1-z)\log(1-z) - 6\,\mathrm{Li}_4(1-z)\right)$$

Involving only one direct function with respect to ν

10.08.21.0011.01

$$\int \mathrm{Li}_\nu(z)\,d\nu = z\,\nu - \sum_{k=2}^{\infty}\frac{z^k}{\log(k)\,k^\nu}\;/;\;|z|<1$$

Involving one direct function and elementary functions with respect to ν

Involving power function

10.08.21.0012.01

$$\int \nu^{\alpha-1}\,\mathrm{Li}_\nu(z)\,d\nu = \frac{z\,\nu^\alpha}{\alpha} - \nu^\alpha \sum_{k=2}^{\infty} \frac{\Gamma(\alpha,\,\nu\log(k))\,z^k}{(\nu\log(k))^\alpha} \,/;\, |z|<1$$

Definite integration

For the direct function itself

10.08.21.0013.01

$$\int_0^z \frac{1}{t}\,\mathrm{Li}_\nu(t)\,dt = \mathrm{Li}_{\nu+1}(z)$$

10.08.21.0014.01

$$\int_0^1 t^\alpha\,\mathrm{Li}_2(t)\,dt = \frac{\alpha\left(\pi^2(\alpha+1)^2-12\right)-6\,\alpha\,(\alpha+1)\,(\psi(\alpha)+\gamma)-6}{6\,\alpha\,(\alpha+1)^3} \,/;\, \mathrm{Re}(\alpha) > -2$$

10.08.21.0015.01

$$\int_0^1 t^n\,\mathrm{Li}_2(t)\,dt = \frac{\pi^2}{6\,(n+1)} - \frac{1}{(n+1)^2}\sum_{k=1}^{n+1}\frac{1}{k} \,/;\, n \in \mathbb{N}$$

10.08.21.0016.01

$$\int_0^\infty t^{-3/2}\,\mathrm{Li}_2(-t)^2\,dt = \frac{8\,\pi}{3}\left(24\log(2)+\pi^2\right)$$

10.08.21.0017.01

$$\int_0^\infty t^{-5/2}\,\mathrm{Li}_2(-t)^2\,dt = \frac{8\,\pi}{27}\left(-8\log(2)-\pi^2+20\right)$$

10.08.21.0018.01

$$\int_0^1 \frac{\mathrm{Li}_2(t)}{(1-t\,z)^2}\,dt = -\frac{\log^2(1-z)}{2\,z} - \frac{\mathrm{Li}_2(z)}{z} + \frac{\pi^2}{6\,(1-z)} \,/;\, z \notin (1,\,\infty)$$

10.08.21.0019.01

$$\int_1^\infty \frac{\mathrm{Li}_2(-t\,z)}{\sqrt{t}\,(t+1)}\,dt = 4\,\pi\,\mathrm{Li}_2\left(-\sqrt{z}\right)$$

10.08.21.0020.01

$$\int_0^\infty \mathrm{Li}_2\left(-z\tan^2(t)\right)dt = 2\,\pi\,\mathrm{Li}_2\left(-\sqrt{z}\right)$$

Involving the direct function

10.08.21.0021.01

$$\int_0^\infty \frac{1}{t}\log(1+a\,t)\,\mathrm{Li}_2\left(-\frac{z}{t^2}\right)dt = \frac{1}{480\,a}$$
$$\left(120\,\pi\sqrt{\frac{1}{z}}\;\Phi\left(-\frac{1}{a^2\,z},\,3,\,\frac{1}{2}\right) - a\left(5\log^4\left(\frac{1}{a^2\,z}\right) + 50\,\pi^2\log^2\left(\frac{1}{a^2\,z}\right) + 120\,\mathrm{Li}_3\left(-\frac{1}{a^2\,z}\right)\log\left(\frac{1}{a^2\,z}\right) + 53\,\pi^4 - 360\,\mathrm{Li}_4\left(-\frac{1}{a^2\,z}\right)\right)\right)$$

10.08.21.0022.01

$$\int_0^\infty t^{-3/4}\log(1+t)\,\mathrm{Li}_2\left(-\frac{1}{t}\right)dt = -2\,\pi\sqrt{2}\left(16\,(3\log(2)+C-4)+\frac{5\,\pi^2}{3}\right)$$

For the products of direct functions

10.08.21.0023.01

$$\int_0^\infty \frac{\text{Li}_3(-t)\,\text{Li}_4(-t)}{t^2}\,dt = 20\,\zeta(3) + \frac{2\,\pi^4}{15} + \frac{10\,\pi^2}{3} + 4\,\zeta(5)$$

10.08.21.0024.01

$$\int_0^\infty \frac{1}{\sqrt{t}}\,\text{Li}_2(-t)\,\text{Li}_2\left(-\frac{1}{t}\right)dt = 16\,\pi\,(3 - 4\log(2))$$

10.08.21.0025.01

$$\int_0^\infty t^{-3/4}\,\text{Li}_2(-t)\,\text{Li}_2\left(-\frac{1}{t}\right)dt = 256\,\pi\,\sqrt{2}\,(-3\log(2) - C + 3)$$

10.08.21.0026.01

$$\int_0^\infty \frac{1}{t}\,\text{Li}_2(-a\,t)\,\text{Li}_2\left(-\frac{z}{t^2}\right)dt = \frac{1}{2880\,a}\left(360\,\pi\,\sqrt{\frac{1}{z}}\,\Phi\left(-\frac{1}{a^2\,z}, 4, \frac{1}{2}\right) - a\left(3\log^5\left(\frac{1}{a^2\,z}\right) + 50\,\pi^2\log^3\left(\frac{1}{a^2\,z}\right) + 159\,\pi^4\log\left(\frac{1}{a^2\,z}\right) + 360\,\text{Li}_4\left(-\frac{1}{a^2\,z}\right)\log\left(\frac{1}{a^2\,z}\right) - 1440\,\text{Li}_5\left(-\frac{1}{a^2\,z}\right)\right)\right)$$

10.08.21.0027.01

$$\int_0^\infty \frac{1}{t}\,\text{Li}_n(-t)\,\text{Li}_m\left(-\frac{1}{t}\right)dt = (m+n)\,\zeta(m+n+1)\;/;\; m \in \mathbb{N}^+ \land n \in \mathbb{N}^+$$

Summation

Infinite summation

10.08.23.0001.01

$$\sum_{k=1}^{\infty}(\text{Li}_{2k}(1) - 1) = \frac{3}{4}$$

10.08.23.0002.01

$$\sum_{k=1}^{\infty}(\text{Li}_{2k+1}(1) - 1) = \frac{1}{4}$$

10.08.23.0003.01

$$\sum_{k=1}^{\infty}\frac{\text{Li}_{2k}(1)}{2^{2k}} = \frac{1}{2}$$

10.08.23.0004.01

$$\sum_{k=1}^{\infty}\frac{\text{Li}_{2k+1}(1)}{2^{2k+1}} = \log(2) - \frac{1}{2}$$

Representations through more general functions

Through hypergeometric functions of two variables

10.08.26.0001.01

$$\text{Li}_n(z) = \frac{z}{2}\,F_{1\,n-2\times 0}^{0\,n\,2}\left(\begin{matrix} ;\, a_1, \ldots, a_n;\, 1, 1; \\ 3;\, a_1+1, \ldots, a_{n-2}+1;; \end{matrix}\; z, 1\right)\;/;\; n \in \mathbb{Z} \land a_k = 1 \land 1 \le k \le n$$

10.08.26.0002.01

$$\mathrm{Li}_2(z) == \frac{z}{2} F_{1\times 0\times 0}^{0\times 2\times 2}\left(\begin{matrix} ;1,1;1,1; \\ 3;;; \end{matrix} z, 1\right)$$

10.08.26.0003.01

$$\mathrm{Li}_2(z) + \mathrm{Li}_2(w) - \mathrm{Li}_2(z + w - z\,w) == \frac{z\,w}{2} F_{1\times 0\times 0}^{0\times 2\times 2}\left(\begin{matrix} ;1,1;1,1; \\ 3;;; \end{matrix} z, w\right)$$

10.08.26.0004.01

$$\mathrm{Li}_2(z^2) == z^2\, F_{1\times 0\times 0}^{0\times 2\times 2}\left(\begin{matrix} ;1,1;1,1; \\ 3;;; \end{matrix} z, -z\right)$$

10.08.26.0005.01

$$\mathrm{Li}_2(z) == \frac{1}{4}\left(\frac{1}{z} - 2\right)^2 F_{1\times 0\times 0}^{0\times 2\times 2}\left(\begin{matrix} ;1,1;1,1; \\ 3;;; \end{matrix} 2 - \frac{1}{z}, 2 - \frac{1}{z}\right) + \frac{\pi^2}{12} - \frac{1}{2}\log^2\left(\frac{1}{z}\right)$$

Through hypergeometric functions

nvolving ${}_pF_q$

10.08.26.0006.01

$$\mathrm{Li}_2(z) == z\, {}_3F_2(1, 1, 1; 2, 2; z)$$

10.08.26.0007.01

$$\mathrm{Li}_n(z) == z\, {}_{n+1}F_n(1, a_1, a_2, \ldots, a_n; a_1 + 1, a_2 + 1, \ldots, a_n + 1; z) \,/;\, a_1 == a_2 == \ldots == a_n == 1 \bigwedge n \in \mathbb{N}^+$$

Through Meijer G

Classical cases for the direct function itself

10.08.26.0008.01

$$\mathrm{Li}_n(z) == -G_{n+1,n+1}^{1,n+1}\left(-z \,\middle|\, \begin{matrix} 1, 1, \ldots, 1 \\ 1, 0, \ldots, 0 \end{matrix}\right) /;\, n \in \mathbb{N}^+$$

Through other functions

10.08.26.0009.01

$$\mathrm{Li}_\nu(z) = S_{\nu-1,1}(z)$$

10.08.26.0010.01

$$\mathrm{Li}_\nu(z) == z\,\Phi(z, \nu, 1)$$

Representations through equivalent functions

With related functions

10.08.27.0001.01

$$\mathrm{Li}_\nu\left(e^{\frac{2\pi i p}{q}}\right) == \frac{1}{q^\nu} \sum_{k=1}^{q} e^{\frac{2\pi i p k}{q}} \zeta\left(\nu, \frac{k}{q}\right) /;\, p \in \mathbb{N}^+ \bigwedge q \in \mathbb{N}^+ \bigwedge p \le q$$

10.08.27.0002.01

$$\mathrm{Li}_\nu(z) = (2\pi)^{\nu-1}\, i\, \Gamma(1-\nu)\left(e^{-\frac{1}{2}(\pi i \nu)} \zeta\left(1-\nu, \left\lfloor -\frac{\arg(z)}{2\pi} \right\rfloor + \frac{\log(z)}{2\pi i} + 1\right) - e^{\frac{\pi i \nu}{2}} \zeta\left(1-\nu, -\left\lfloor -\frac{\arg(z)}{2\pi} \right\rfloor - \frac{\log(z)}{2\pi i}\right)\right) /;\, z \notin (0, 1)$$

10.08.27.0003.01

$$\mathrm{Li}_\nu(z) = (2\pi)^{\nu-1}\, i\, \Gamma(1-\nu)\left(e^{-\frac{1}{2}(\pi i \nu)}\, \zeta\left(1-\nu, \frac{\log(-z)+\pi i}{2\pi i}\right) - e^{\frac{\pi i \nu}{2}}\, \zeta\left(1-\nu, 1-\frac{\log(-z)+\pi i}{2\pi i}\right)\right) /; z \notin (0, 1)$$

10.08.27.0004.01

$$\mathrm{Li}_\nu(z) = (2\pi)^{\nu-1}\, i\, \Gamma(1-\nu)\left(e^{-\frac{1}{2}(\pi i \nu)}\, \zeta\left(1-\nu, \frac{\log(z)}{2\pi i}\right) - e^{\frac{\pi i \nu}{2}}\, \zeta\left(1-\nu, 1-\frac{\log(z)}{2\pi i}\right)\right) /;$$

$$(\mathrm{Im}(z) \geq 0 \land |z| < 1) \bigvee (0 < \arg(z) \leq \pi \bigwedge |z| \geq 1)$$

Inequalities

10.08.29.0001.01

$$\mathrm{Li}_\nu(x) \leq \mathrm{Li}_\mu(x) /; \nu \in \mathbb{R} \bigwedge \mu \in \mathbb{R} \bigwedge \nu > \mu \bigwedge x \in \mathbb{R} \bigwedge -\infty < x \leq 1$$

10.08.29.0002.01

$$\mathrm{Li}_\nu(x) \geq 0 /; \nu \in \mathbb{R} \land x \in \mathbb{R} \land 0 \leq x \leq 1$$

10.08.29.0003.01

$$\mathrm{Li}_\nu(x) \geq x /; \nu \in \mathbb{R} \land x \in \mathbb{R} \land -\infty \leq x \leq 1$$

Theorems

The Fermi–Dirac integral

The Fermi-Dirac integral $F_\alpha(\mu) = \int_0^\infty \frac{\varepsilon^\alpha}{e^{\varepsilon-\mu}+1}\, d\varepsilon$, describing, for instance, the number of electrons (holes) in the conduction band (valence band) in a semiconductor with density of states $\propto \varepsilon^\alpha$, can be expressed as $F_\alpha(\mu) = -\alpha\, \Gamma(\alpha)\, \mathrm{Li}_{\alpha+1}(-e^\mu)$.

The volume of a Lambert cube

The volume V of a Lambert cube with essential angles α_1, α_2, α_3 and apices of length l_1, l_2, l_3 in three-dimensional hyperbolic space is given by

$$V = \sum_{k=1}^{3} (\mathcal{L}(\alpha_k + \theta) - \mathcal{L}(\alpha_k - \theta)) - 1/4\, \mathcal{L}(2\theta) + 1/2\, \mathcal{L}(\pi/2 - \theta) /; \mathcal{L}(z) = 1/2\, \mathrm{Im}(\mathrm{Li}_2(\exp(2 i z))) \bigwedge$$
$$\tan^{-1}\left(\left(\cosh(l_2) - \sin^2(\alpha_1)\sin^2(\alpha_2)\right) / \left(\cos^2(\alpha_1)\cos^2(\alpha_2)\right)\right).$$

Rationality of dilogarithm

The value $\mathrm{Li}_2(z)$ is irrational when zis rational (G.V.Chudnovsky,1979).

History

– G. W. Leibniz defined dilogarithm for the case $\nu = 2$
– L. Euler (1768)
– J. Landen (1760,1780) inverstigated $\mathrm{Li}_2(z)$ and $\mathrm{Li}_3(z)$
– W. Spence (1809)
– N. H. Abel
– E.E. Kummer (1840)
– J. Kummer; L. L. Lindelöf
– N. I. Lobachevski
– C. J. Hill (1828) introduced the name "dilogarithm"

Applications include electrical network problems, number theory, group theory, K-theory, geometry, quantum electrodynamics, group cohomology, mixed Hodge structures, mixed motives, evaluation of volumes of hyperbolic polytopes, celestial mechanics.

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