PolyLog2

Notations

Traditional name

Dilogarithm

Traditional notation

$\mathrm{Li}_2(z)$

Mathematica StandardForm notation

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PolyLog[2, z]
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Primary definition

10.07.02.0001.01

$$\mathrm{Li}_2(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^2} \;/;\; |z| < 1$$

Specific values

Specialized values

10.07.03.0001.01

$$\mathrm{Li}_2\left(e^{\frac{2\pi i p}{q}}\right) = \frac{1}{q^2} \sum_{k=1}^{q} e^{\frac{2\pi i p k}{q}} \zeta\left(2, \frac{k}{q}\right) \;/;\; p \in \mathbb{N}^+ \land q \in \mathbb{N}^+ \land p \leq q$$

10.07.03.0002.01

$$\mathrm{Li}_2\left(\exp\left(\frac{2\pi i p}{q}\right)\right) = \frac{1}{q^2} \sum_{k=1}^{q-1} \exp\left(\frac{2\pi i k p}{q}\right) \psi^{(1)}\left(\frac{k}{q}\right) + \frac{\pi^2}{6 q^2} \;/;\; p \in \mathbb{N}^+ \land q \in \mathbb{N}^+$$

Values at fixed points

10.07.03.0003.01

$$\mathrm{Li}_2(0) = 0$$

10.07.03.0004.01

$$\mathrm{Li}_2(-1) = -\frac{\pi^2}{12}$$

10.07.03.0005.01

$$\mathrm{Li}_\nu(1) = \frac{\pi^2}{6}$$

10.07.03.0006.01

$$\mathrm{Li}_2\left(\frac{1}{2}\right) = \frac{\pi^2}{12} - \frac{\log^2(2)}{2}$$

(L.Euler)

10.07.03.0007.01

$$\mathrm{Li}_2(2) = \frac{\pi^2}{4} - \pi\, i \log(2)$$

10.07.03.0008.01

$$\mathrm{Li}_2(i) = i\, C - \frac{\pi^2}{48}$$

10.07.03.0009.01

$$\mathrm{Li}_2(-i) = -i\, C - \frac{\pi^2}{48}$$

10.07.03.0010.01

$$\mathrm{Li}_2(1-i) = \frac{\pi^2}{16} - i\, C - \frac{\pi\, i}{4} \log(2)$$

10.07.03.0011.01

$$\mathrm{Li}_2(1+i) = \frac{\pi^2}{16} + i\, C + \frac{\pi\, i}{4} \log(2)$$

10.07.03.0012.01

$$\mathrm{Li}_2\left(\frac{1-\sqrt{5}}{2}\right) = \frac{1}{2} \log^2\left(\frac{\sqrt{5}-1}{2}\right) - \frac{\pi^2}{15}$$

10.07.03.0013.01

$$\mathrm{Li}_2\left(-\frac{1+\sqrt{5}}{2}\right) = -\log^2\left(\frac{1+\sqrt{5}}{2}\right) - \frac{\pi^2}{10}$$

10.07.03.0014.01

$$\mathrm{Li}_2\left(\frac{3-\sqrt{5}}{2}\right) = \frac{\pi^2}{15} - \log^2\left(\frac{\sqrt{5}-1}{2}\right)$$

10.07.03.0015.01

$$\mathrm{Li}_2\left(\frac{\sqrt{5}-1}{2}\right) = \frac{\pi^2}{10} - \log^2\left(\frac{\sqrt{5}-1}{2}\right)$$

10.07.03.0016.01

$$\mathrm{Li}_2\left(\frac{1-i}{2}\right) = -\frac{1}{8} \log^2(2) + \frac{\pi\, i}{8} \log(2) + \frac{5\pi^2}{96} - i\, C$$

General characteristics

Domain and analyticity

$\text{Li}_2(z)$ is an analytical function of z which is defined in $\mathbb{C}$.

10.07.04.0001.01

$$(2 * z) \longrightarrow \text{Li}_2(z) :: (\{2\} \otimes \mathbb{C}) \longrightarrow \mathbb{C}$$

Symmetries and periodicities

Mirror symmetry

10.07.04.0002.01

$$\text{Li}_2(\bar{z}) = \overline{\text{Li}_2(z)} \;/;\; z \notin (-\infty, 0)$$

Periodicity

No periodicity

Poles and essential singularities

The function $\text{Li}_2(z)$ does not have poles and essential singularities.

10.07.04.0003.01

$$\mathcal{Sing}_z(\text{Li}_\nu(z)) = \{\}$$

Branch points

The function $\text{Li}_2(z)$ has two branch points: $z = 1$, $z = \tilde{\infty}$.

10.07.04.0004.01

$$\mathcal{BP}_z(\text{Li}_\nu(z)) = \{1, \tilde{\infty}\}$$

10.07.04.0005.01

$$\mathcal{R}_z(\text{Li}_2(z), 1) = \log$$

Branch cuts

The function $\text{Li}_2(z)$ is a single-valued function on the z-plane cut along the interval $\{1, \infty\}$, where it is continuous from below.

10.07.04.0006.01

$$\mathcal{BC}_z(\text{Li}_2(z)) = \{\{\{1, \infty\}, i\}\}$$

10.07.04.0007.01

$$\lim_{\epsilon \to +0} \text{Li}_2(x - i\,\epsilon) = \text{Li}_2(x) \;/;\; x > 1$$

10.07.04.0008.01

$$\lim_{\epsilon \to +0} \text{Li}_2(x + i\,\epsilon) = \text{Li}_2(x) + 2\,i\,\pi \log(x) \;/;\; x > 1$$

Series representations

Generalized power series

Expansions at $z == 0$

For the function itself

10.07.06.0001.02

$$\text{Li}_2(z) \propto z + \frac{z^2}{4} + \frac{z^3}{9} + \ldots /; (z \to 0)$$

10.07.06.0015.01

$$\text{Li}_2(z) \propto z + \frac{z^2}{4} + \frac{z^3}{9} + O\left(z^4\right)$$

10.07.06.0002.01

$$\text{Li}_2(z) == \sum_{k=1}^{\infty} \frac{z^k}{k^2} /; |z| < 1$$

10.07.06.0016.01

$$\text{Li}_2(z) == z\, {}_3F_2(1, 1, 1; 2, 2; z)$$

10.07.06.0003.02

$$\text{Li}_2(z) \propto z\, (1 + O(z))$$

10.07.06.0017.01

$$\text{Li}_2(z) == F_\infty(z) /; \left(\left(F_m(z) == \sum_{k=1}^{m} \frac{z^k}{k^2} == \text{Li}_2(z) - z^{m+1}\, \Phi(z, 2, m+1)\right) \bigwedge m \in \mathbb{N}^+\right)$$

Summed form of the truncated series expansion.

Expansions at $z == 1$

For the function itself

10.07.06.0018.01

$$\text{Li}_2(z) \propto \frac{\pi^2}{6} + z - 1 - \frac{(z-1)^2}{4} + \frac{(z-1)^3}{9} + \ldots - \text{Log}[1-z]\, (z-1) \left(1 - \frac{z-1}{2} + \frac{(z-1)^2}{3} + \ldots\right) /; (z \to 1)$$

10.07.06.0004.02

$$\text{Li}_2(z) \propto \frac{\pi^2}{6} + z - 1 - \frac{(z-1)^2}{4} + \frac{(z-1)^3}{9} + O\left((z-1)^4\right) - \text{Log}[1-z]\, (z-1) \left(1 - \frac{z-1}{2} + \frac{(z-1)^2}{3} + O\left((z-1)^3\right)\right)$$

10.07.06.0005.02

$$\text{Li}_2(z) == \frac{\pi^2}{6} - \sum_{k=1}^{\infty} \frac{(1-z)^k}{k^2} + \log(1-z) \sum_{k=1}^{\infty} \frac{(1-z)^k}{k} /; |z-1| < 1$$

(L.Euler, 1768)

10.07.06.0006.01

$$\text{Li}_2(z) == -\text{Li}_2(1-z) - \log(1-z) \log(z) + \frac{\pi^2}{6}$$

(L.Euler, 1768)

10.07.06.0019.01

$$\mathrm{Li}_2(z) == \log(z)\,(1 - \log(-\log(z))) + \frac{\pi^2}{6} + \sum_{j=2}^{\infty} \frac{\zeta(2-j)}{j!} \log^j(z)$$

10.07.06.0020.01

$$\mathrm{Li}_2(z) == \frac{\pi^2}{6} + (z-1)\,{}_3F_2(1, 1, 1; 2, 2; 1-z) - \log(1-z)\,(z-1)\,{}_2F_1(1, 1; 2; 1-z)$$

10.07.06.0007.02

$$\mathrm{Li}_2(z) \propto \frac{\pi^2}{6} + O(z-1) - \log(1-z)\,(z-1)\,(1 + O(z-1))$$

10.07.06.0021.01

$$\mathrm{Li}_2(z) == F_\infty(z) \;/;\; \left(\left(F_m(z) == \log(1-z) \sum_{k=1}^{m} \frac{(1-z)^k}{k} + \frac{\pi^2}{6} - \sum_{k=1}^{m} \frac{(1-z)^k}{k^2} == \Phi(1-z, 2, m+1)\,(1-z)^{m+1} + \frac{\pi^2}{6} - \log(1-z)\,(\mathrm{B}_{1-z}(m+1, 0) + \log(z)) - \mathrm{Li}_2(1-z)\right) \bigwedge m \in \mathbb{N}^+\right)$$

Summed form of the truncated series expansion.

Expansions at $z == \infty$

For the function itself

10.07.06.0008.02

$$\mathrm{Li}_2(z) \propto -\frac{1}{2} \log^2(-z) - \frac{\pi^2}{6} - \frac{1}{z}\left(1 + \frac{1}{4\,z} + \ldots\right) \;/;\; (|z| \to \infty)$$

10.07.06.0022.01

$$\mathrm{Li}_2(z) \propto -\frac{1}{2} \log^2(-z) - \frac{\pi^2}{6} - \frac{1}{z}\left(1 + \frac{1}{4\,z} + O\left(\frac{1}{z^2}\right)\right)$$

10.07.06.0009.01

$$\mathrm{Li}_2(z) == -\frac{1}{2} \log^2(-z) - \frac{\pi^2}{6} - \sum_{k=1}^{\infty} \frac{1}{k^2\,z^k} \;/;\; |z| > 1$$

10.07.06.0010.01

$$\mathrm{Li}_2(z) == -\frac{1}{2} \log^2(-z) - \frac{\pi^2}{6} - \mathrm{Li}_n\left(\frac{1}{z}\right) \;/;\; z \notin (0, 1)$$

10.07.06.0011.02

$$\mathrm{Li}_2(z) == -\frac{1}{2} \log^2(-z) - \frac{\pi^2}{6} + O\left(\frac{1}{z}\right)$$

10.07.06.0023.01

$$\mathrm{Li}_2(z) == F_\infty(z) \;/;\; \left(\left(F_n(z) == -\frac{1}{2} \log^2(-z) - \frac{\pi^2}{6} - \sum_{k=1}^{n} \frac{1}{k^2\,z^k} == -\frac{1}{2} \log^2(-z) - \frac{\pi^2}{6} + \Phi\left(\frac{1}{z}, 2, n+1\right) z^{-n-1} - \mathrm{Li}_2\left(\frac{1}{z}\right)\right) \bigwedge n \in \mathbb{N}^+\right)$$

Summed form of the truncated series expansion.

For the function itself OTHER TITLE?

10.07.06.0012.01

$$\mathrm{Li}_2(-e^z) \propto -\frac{z^2}{2} - \sum_{k=1}^{\infty} \frac{(-1)^k e^{-kz}}{k^2} - \frac{\pi^2}{6} \text{ /; } (|z| \to \infty)$$

Residue representations

10.07.06.0013.01

$$\mathrm{Li}_2(z) = -\sum_{j=1}^{\infty} \mathrm{res}_s\left(\frac{\Gamma(-s)^3\,(-z)^{-s}}{\Gamma(1-s)^2}\,\Gamma(s+1)\right)(-j) \text{ /; } |z| < 1$$

10.07.06.0014.02

$$\mathrm{Li}_2(z) = \sum_{j=1}^{\infty} \mathrm{res}_s\left(\frac{\Gamma(s+1)\,(-z)^{-s}}{s^2}\,\Gamma(-s)\right)(j) \text{ /; } |z| > 1$$

Integral representations

On the real axis

Of the direct function

10.07.07.0001.01

$$\mathrm{Li}_2(z) = z\int_0^{\infty} \frac{t}{e^t - z}\,dt$$

10.07.07.0002.01

$$\mathrm{Li}_2(z) = z\int_1^{\infty} \frac{\log(t)}{t\,(t-z)}\,dt$$

10.07.07.0003.01

$$\mathrm{Li}_2(z) = -\int_0^1 \frac{\log(1-t\,z)}{t}\,dt \text{ /; } |z| < 1$$

10.07.07.0004.01

$$\mathrm{Li}_2(z) = \int_0^{\infty} \frac{e^{t/z}}{t}\left(\sum_{k=1}^{\infty} \frac{(-t)^k}{k\,k!}\right)dt \text{ /; } \mathrm{Re}(z) < 0$$

10.07.07.0005.01

$$\mathrm{Li}_2(z) = -\int_0^z \frac{\log(1-t)}{t}\,dt$$

10.07.07.0006.01

$$\mathrm{Li}_2(z) = \frac{\pi^2}{6} - \int_1^z \frac{\log(1-t)}{t}\,dt$$

Contour integral representations

10.07.07.0007.01

$$\mathrm{Li}_2(z) = -\frac{1}{2\pi i}\int_{\mathcal{L}} \frac{\Gamma(s+1)\,\Gamma(-s)^3\,(-z)^{-s}}{\Gamma(1-s)^2}\,ds \,/;\, |\arg(-z)| < \pi$$

10.07.07.0008.01

$$\mathrm{Li}_2(z) = -\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(s)\,\Gamma(-s)\,(-z)^{-s}}{s}\,ds \,/;\, -1 < \gamma < 0 \bigwedge |\arg(-z)| < \pi$$

Multiple integral representations

10.07.07.0009.01

$$\mathrm{Li}_2(z) = -\int_0^z \frac{1}{w}\left(\int_1^{1-w} \frac{1}{t}\,dt\right) dw$$

Limit representations

10.07.09.0001.01

$$\mathrm{Li}_2(z) = \lim_{\varepsilon\to 0} \frac{{}_2F_1(\varepsilon, \varepsilon; 1; z) - 1}{\varepsilon^2}$$

Differential equations

Ordinary linear differential equations and Wronskians

For the direct function itself

10.07.13.0002.01

$$(1-z)\,z\,w''(z) + (1-z)\,w'(z) = 1 \,/;\, w(z) = c_1 + c_2 \log(z) + \mathrm{Li}_2(z)$$

10.07.13.0003.01

$$z\,(1-z)\,w^{(3)}(z) + (2-3z)\,w''(z) - w'(z) = 0 \,/;\, w(z) = c_1 + c_2 \log(z) + c_3\,\mathrm{Li}_2(z) = 0$$

10.07.13.0004.01

$$W_z(1, \log(z), \mathrm{Li}_2(z)) = \frac{1}{(1-z)\,z^2}$$

10.07.13.0005.01

$$(1-z)^2\,z\,w^{(4)}(z) + (1-z)(3-7z)\,w^{(3)}(z) - 2(4-5z)\,w''(z) + 2\,w'(z) = 0 \,/;\, w(z) = c_1 + c_2 \log(1-z) + c_3 \log(z) + c_4\,\mathrm{Li}_2(z)$$

10.07.13.0006.01

$$W_z(1, \log(z), \log(1-z), \mathrm{Li}_2(z)) = \frac{1}{(z-1)^4\,z^3}$$

10.07.13.0001.01

$$\left(\prod_{k=1}^{3}\left(z\frac{d}{dz}+1\right) - \frac{d}{dz}\prod_{l=1}^{2}\left(z\frac{d}{dz}+1\right)\right) w(z) = 0 \,/;\, w(z) = \frac{\mathrm{Li}_2(z)}{z}$$

There is no algebraic partial differential equation for $\mathrm{Li}_\nu(z)$ (A. Ostrowski, 1920).

10.07.13.0007.01

$$w^{(3)}(z)+\left(\frac{(3\,g(z)-2)\,g'(z)}{(g(z)-1)\,g(z)}-\frac{3\,g''(z)}{g'(z)}\right)w''(z)+\left(\frac{g'(z)^2}{(g(z)-1)\,g(z)}+\frac{3\,g''(z)^2}{g'(z)^2}+\frac{2\,g''(z)}{(g(z)-1)\,g(z)}-\frac{3\,g''(z)}{g(z)-1}-\frac{g^{(3)}(z)}{g'(z)}\right)w'(z)=0\;/;\; w(z)=c_1+c_2\log(g(z))+c_3\,\mathrm{Li}_2(g(z))$$

10.07.13.0008.01

$$W_z(1,\log(g(z)),\mathrm{Li}_2(g(z)))=\frac{g'(z)^3}{(1-g(z))\,g(z)^2}$$

10.07.13.0009.01

$$w^{(3)}(z)+\left(\frac{(3\,g(z)-2)\,g'(z)}{(g(z)-1)\,g(z)}-\frac{3\,h'(z)}{h(z)}-\frac{3\,g''(z)}{g'(z)}\right)w''(z)+\left(\frac{6\,h'(z)^2}{h(z)^2}+\frac{6\,g''(z)\,h'(z)}{h(z)\,g'(z)}+\frac{3\,g''(z)^2}{g'(z)^2}+\frac{g'(z)\,(h(z)\,g'(z)+2\,(2-3\,g(z))\,h'(z))}{(g(z)-1)\,g(z)\,h(z)}+\frac{2\,g''(z)}{(g(z)-1)\,g(z)}-\frac{3\,g''(z)}{g(z)-1}-\frac{3\,h''(z)}{h(z)}-\frac{g^{(3)}(z)}{g'(z)}\right)w'(z)+\left(-\frac{6\,h'(z)^3}{h(z)^3}+\frac{6\,g'(z)\,h'(z)^2}{(g(z)-1)\,h(z)^2}-\frac{4\,g'(z)\,h'(z)^2}{(g(z)-1)\,g(z)\,h(z)^2}-\frac{6\,g''(z)\,h'(z)^2}{h(z)^2\,g'(z)}+\frac{6\,h''(z)\,h'(z)}{h(z)^2}-\frac{3\,g''(z)^2\,h'(z)}{h(z)\,g'(z)^2}+\frac{2\,g'(z)\,h''(z)-h'(z)\left(g'(z)^2+2\,g''(z)\right)}{(g(z)-1)\,g(z)\,h(z)}+\frac{3\,(h'(z)\,g''(z)-g'(z)\,h''(z))}{(g(z)-1)\,h(z)}+\frac{3\,g''(z)\,h''(z)+h'(z)\,g^{(3)}(z)}{h(z)\,g'(z)}-\frac{h^{(3)}(z)}{h(z)}\right)w(z)=0\;/;\; w(z)=c_1\,h(z)+c_2\,h(z)\log(g(z))+c_3\,h(z)\,\mathrm{Li}_2(g(z))$$

10.07.13.0010.01

$$W_z(h(z),h(z)\log(g(z)),h(z)\,\mathrm{Li}_2(g(z)))=\frac{h(z)^3\,g'(z)^3}{(1-g(z))\,g(z)^2}$$

10.07.13.0011.01

$$z^3\,w^{(3)}(z)+\left(\frac{r}{a\,z^r-1}-3\,s+3\right)z^2\,w''(z)+\left(3\,s^2-3\,s+\frac{r-2\,r\,s}{a\,z^r-1}+1\right)z\,w'(z)+\frac{s^2\,(-a\,s\,z^r+r+s)}{a\,z^r-1}\,w(z)=0\;/;\; w(z)=c_1\,z^s+c_2\,z^s\log(a\,z^r)+c_3\,z^s\,\mathrm{Li}_2(a\,z^r)$$

10.07.13.0012.01

$$W_z(z^s,z^s\log(a\,z^r),z^s\,\mathrm{Li}_2(a\,z^r))=\frac{a\,r^3\,z^{r+3s-3}}{1-a\,z^r}$$

10.07.13.0013.01

$$w^{(3)}(z)+\left(\frac{\log(r)}{a\,r^z-1}-3\log(s)\right)w''(z)+\left(3\log(s)-\frac{2\log(r)}{a\,r^z-1}\right)\log(s)\,w'(z)+\left(\frac{\log(r)}{a\,r^z-1}-\log(s)\right)\log^2(s)\,w(z)=0\;/;\; w(z)=c_1\,s^z+c_2\,s^z\log(a\,r^z)+c_3\,s^z\,\mathrm{Li}_2(a\,r^z)$$

10.07.13.0014.01

$$W_z(s^z,s^z\log(a\,r^z),s^z\,\mathrm{Li}_2(a\,r^z))=\frac{a\,r^z\,s^{3z}\log^3(r)}{1-a\,r^z}$$

Transformations

Multiple arguments

10.07.16.0001.01

$$\mathrm{Li}_2(z\,w) = \mathrm{Li}_2(w) + \mathrm{Li}_2(z) - \mathrm{Li}_2\left(\frac{z(1-w)}{1-w\,z}\right) - \mathrm{Li}_2\left(\frac{w(1-z)}{1-w\,z}\right) - \log\left(\frac{1-w}{1-w\,z}\right)\log\left(\frac{1-z}{1-w\,z}\right) /;$$
$$(|z| < 1 \land |w| < 1) \lor z\,w < 1 \lor (z > 1 \land w > 1)$$

(L.Rogers, 1906)

10.07.16.0002.01

$$\mathrm{Li}_2(z\,w) = \mathrm{Li}_2(z) + \mathrm{Li}_2(w) + \mathrm{Li}_2\left(\frac{w(z-1)}{1-w}\right) + \mathrm{Li}_2\left(\frac{z(w-1)}{1-z}\right) + \frac{1}{2}\log^2\left(\frac{1-w}{1-z}\right) /;$$
$$\neg\left(z \in \mathbb{R} \bigwedge w \in \mathbb{R} \bigwedge z > 1 \bigwedge w > 1 \bigwedge \frac{z(w-1)}{1-z} > 1 \bigwedge \frac{w(z-1)}{1-w} > 1\right) \bigvee$$
$$(z \in \mathbb{R} \land w \in \mathbb{R} \land (z > 1 \land w > 1) \lor (z < 1 \land w < 1))$$

(C.J.Hill,1830)

10.07.16.0003.01

$$\mathrm{Li}_2(z\,w) = \mathrm{Li}_2(z) + \mathrm{Li}_2(w) - \mathrm{Li}_2\left(\frac{z(1-w)}{1-w\,z}\right) - \mathrm{Li}_2\left(\frac{w(1-z)}{1-w\,z}\right) + \frac{1}{2}\left(\log(1-w)\log(w) + \right.$$
$$\left.\log(1-z)\log(z) - \log\left(\frac{1-z}{1-w\,z}\right)\log\left(\frac{z(1-w)}{1-w\,z}\right) - \log\left(\frac{1-w}{1-w\,z}\right)\log\left(\frac{w(1-z)}{1-w\,z}\right) - \log(w\,z)\log(1-w\,z)\right) /;$$
$$(|z| < 1 \land 0 < w < 1) \lor (|w| < 1 \land 0 < z < 1) \lor (z \in \mathbb{R} \land w \in \mathbb{R} \land (0 < z < 1 \land w < 1) \lor (0 < w < 1 \land z < 1) \lor (z > 1 \land w > 1))$$

Power of arguments

10.07.16.0004.01

$$\mathrm{Li}_2(z^2) = 2(\mathrm{Li}_2(z) + \mathrm{Li}_2(-z))$$

10.07.16.0005.01

$$\mathrm{Li}_2(z^m) = m \sum_{k=0}^{m-1} \mathrm{Li}_2\left(e^{\frac{2\pi i k}{m}} z\right) /; m \in \mathbb{N}^+$$

Identities

Functional identities

Involving two dilogarithms

10.07.17.0001.01

$$\mathrm{Li}_2(-1) = -\frac{1}{2}\mathrm{Li}_2(1)$$

10.07.17.0002.01

$$\mathrm{Li}_2(z) = -\mathrm{Li}_2(1-z) - \log(1-z)\log(z) + \frac{\pi^2}{6}$$

(L.Euler, 1768)

10.07.17.0003.01

$$\mathrm{Li}_2(z) = -\mathrm{Li}_2\left(\frac{1}{z}\right) - \frac{1}{2}\log^2(-z) - \frac{\pi^2}{6} /; z \notin (0, 1)$$

10.07.17.0035.01

$$\mathrm{Li}_2(z) = -\mathrm{Li}_2\left(\frac{1}{z}\right) - \frac{1}{2}\log^2(-z) - i\,\pi\left(\sqrt{\frac{z-1}{z}}\,\sqrt{\frac{z}{z-1}} - 1\right)\log(z) - \frac{\pi^2}{6}$$

10.07.17.0036.01

$$\mathrm{Li}_2(z) = -\frac{1}{2}\log^2(z) - \frac{\pi\sqrt{-(z-1)^2}}{z-1}\log(z) + \frac{\pi^2}{3} - \mathrm{Li}_2\left(\frac{1}{z}\right)$$

10.07.17.0004.01

$$\mathrm{Li}_2(z) = -\mathrm{Li}_2\left(\frac{1}{z}\right) - \frac{1}{2}\log^2(z) - \frac{\pi}{z-1}\sqrt{-(z-1)^2}\,\log(z) + \frac{\pi^2}{3}$$

10.07.17.0005.01

$$\mathrm{Li}_2(z) = \mathrm{Li}_2\left(\frac{1}{1-z}\right) + \frac{1}{2}\log(1-z)\log\left(\frac{1-z}{z^2}\right) - \frac{\pi^2}{6} \;/;\; \mathrm{Re}(z) \le 0 \bigvee z \notin (1, \infty)$$

10.07.17.0006.01

$$\mathrm{Li}_2(z) = \mathrm{Li}_2\left(\frac{1}{1-z}\right) + \frac{1}{2}\log^2(1-z) - \log(-z)\log(1-z) - \frac{\pi^2}{6}$$

10.07.17.0007.01

$$\mathrm{Li}_2(z) = -\mathrm{Li}_2\left(\frac{z}{z-1}\right) - \frac{1}{2}\log^2(1-z) \;/;\; z \notin (1, \infty)$$

(J.Landen,1780)

10.07.17.0008.01

$$\mathrm{Li}_2(z) = -\mathrm{Li}_2\left(\frac{z}{z-1}\right) - \frac{1}{2}\log^2(z-1) + \frac{\pi^2}{2} + i\,\pi\log\left(\frac{z-1}{z^2}\right) \;/;\; z > 1$$

10.07.17.0009.01

$$\mathrm{Li}_2(z) = -\mathrm{Li}_2\left(\frac{z}{z-1}\right) - \frac{1}{2}\log^2(z-1) + i\,\pi\log(z-1) + \frac{\pi^2}{2} - 2\,\pi\,i\log(z) \;/;\; z > 1$$

Involving three dilogarithms

10.07.17.0010.01

$$\mathrm{Li}_2(z) = \mathrm{Li}_2(z+1) - \frac{1}{2}\mathrm{Li}_2\left(\frac{1}{z^2}\right) + \log\left(\frac{z+1}{z}\right)\log(-z) \;/;\; z \notin (0, 1)$$

10.07.17.0011.01

$$\mathrm{Li}_2(z) = \mathrm{Li}_2\left(\frac{z}{z+1}\right) + \frac{1}{2}\left(+\mathrm{Li}_2\left(z^2\right) - \log(1-z)\log(z) + \log\left(\frac{1}{z+1}\right)\log\left(\frac{z}{z+1}\right) + \frac{1}{2}\log\left(z^2\right)\log\left(1-z^2\right)\right) \;/;\; \mathrm{Im}(z) > 0 \lor z > 0$$

Abel's duplication formula

10.07.17.0012.01

$$\mathrm{Li}_2\left(2\,z - z^2\right) = -\log^2(2-z) + \frac{\pi^2}{6} - 2\,\mathrm{Li}_2\left(\frac{1}{2-z}\right) + 2\,\mathrm{Li}_2(z) \;/;\; |z| < 1$$

10.07.17.0013.01

$$\mathrm{Li}_2\left(\frac{z}{z-1}\right) + \mathrm{Li}_2\left(\frac{z}{z+1}\right) - \frac{1}{2}\mathrm{Li}_2\left(\frac{z^2}{z^2-1}\right) = -\frac{1}{4}\log^2\left(\frac{1+z}{1-z}\right) \;/;\; |z| < 1 \lor \mathrm{Re}(z) \le 0 \lor \mathrm{Im}(z) \ge 0$$

10.07.17.0014.01

$$\mathrm{Li}_2\left(\frac{z}{z-1}\right)+\mathrm{Li}_2\left(\frac{z}{z+1}\right)-\frac{1}{2}\,\mathrm{Li}_2\left(\frac{z^2}{z^2-1}\right)=-\frac{1}{4}\log^2\left(\frac{1+z}{1-z}\right)\,/;\,|z|<1\lor\mathrm{Re}(z)\le 0\lor\mathrm{Im}(z)\ge 0$$

10.07.17.0015.01

$$\mathrm{Li}_2\left(z^2\right)-2\,(\mathrm{Li}_2(z)+\mathrm{Li}_2(-z))=0$$

10.07.17.0016.01

$$\mathrm{Li}_2\left(z^2\right)-2\,\mathrm{Li}_2(z)+2\,\mathrm{Li}_2\left(\frac{z}{z+1}\right)=\log(1-z)\log(z)-\log\left(\frac{1}{z+1}\right)\log\left(\frac{z}{z+1}\right)-\frac{1}{2}\log\left(z^2\right)\log\left(1-z^2\right)\,/;$$
$$\mathrm{Re}(z)>0\lor(\mathrm{Re}(z)=0\land\mathrm{Im}(z)>0)$$

Involving four dilogarithms

10.07.17.0017.01

$$\mathrm{Li}_2(z)-\mathrm{Li}_2(-z)+\mathrm{Li}_2\left(\frac{1-z}{1+z}\right)-\mathrm{Li}_2\left(-\frac{1-z}{1+z}\right)=\log\left(\frac{1+z}{1-z}\right)\log(z)+\frac{\pi^2}{4}\,/;\,z\notin(-\infty,-1)\bigwedge z\notin(1,\infty)$$

Involving five dilogarithms

10.07.17.0018.01

$$L(z)+L(w)-L(z\,w)-L\left(\frac{z\,(1-w)}{1-z\,w}\right)-L\left(\frac{w\,(1-z)}{1-z\,w}\right)=0\,/;$$
$$\left(L(z)=\mathrm{Li}_2(z)+\frac{1}{2}\log(1-z)\log(z)\,/;\,(|z|<1\land 0<w<1)\lor(|w|<1\land 0<z<1)\lor\right.$$
$$\left.(z<1\land 0<w<1)\lor(w<1\land 0<z<1)\lor(z\,w<1\land z>0\land w>0)\lor(z>1\land w>1)\right)$$

10.07.17.0019.01

$$\mathrm{Li}_2\left(\frac{z\,w}{(1-z)\,(1-w)}\right)=$$
$$+\mathrm{Li}_2\left(\frac{z}{1-w}\right)+\mathrm{Li}_2\left(\frac{w}{1-z}\right)-\mathrm{Li}_2(z)-\mathrm{Li}_2(w)+\frac{1}{2}\left(\log\left(\frac{z}{1-w}\right)\log\left(\frac{w+z-1}{w-1}\right)+\log\left(\frac{w}{1-z}\right)\log\left(\frac{w+z-1}{z-1}\right)-\right.$$
$$\left.\log\left(\frac{w\,z}{(w-1)\,(z-1)}\right)\log\left(-\frac{w+z-1}{(w-1)\,(z-1)}\right)-\log(1-z)\log(z)-\log(1-w)\log(w)\right)\,/;$$
$$(|z|<1\land 0<w<1)\lor|w|<1\land 0<z<1\lor z<1\land 0<w<1\lor w<1\land 0<z<1\lor w+z>1\land w<0\lor w+z>1\land z<0$$

10.07.17.0020.01

$$\mathrm{Li}_2\left(\frac{z\,w}{(1-z)\,(1-w)}\right)=\mathrm{Li}_2\left(\frac{w}{w-1}\right)+\mathrm{Li}_2\left(\frac{z}{z-1}\right)+\mathrm{Li}_2\left(\frac{z}{1-w}\right)+\mathrm{Li}_2\left(\frac{w}{1-z}\right)+\frac{1}{2}\log^2\left(\frac{1-z}{1-w}\right)\,/;$$
$$(|z|<1\land|w|<1)\lor(w<1\land z<1)\lor(w<1\land z<1)\lor 0<w<-z$$

(W.Spence, 1809)

10.07.17.0021.01

$$\mathrm{Li}_2\left(\frac{z\,w}{(1-z)\,(1-w)}\right)=\mathrm{Li}_2\left(\frac{z}{1-w}\right)+\mathrm{Li}_2\left(\frac{w}{1-z}\right)-\mathrm{Li}_2(z)-\mathrm{Li}_2(w)-\log(1-z)\log(1-w)\,/;$$
$$|z|<1\land|w|<1\lor(w\in\mathbb{R}\land w<1\land z\in\mathbb{R}\land z<1)$$

(N.Abel, 1830)

10.07.17.0022.01

$$\mathrm{Li}_2(z\,w)-\mathrm{Li}_2(w)-\mathrm{Li}_2(z)+\mathrm{Li}_2\left(\frac{z\,(1-w)}{1-w\,z}\right)+\mathrm{Li}_2\left(\frac{w\,(1-z)}{1-w\,z}\right)=-\log\left(\frac{1-w}{1-w\,z}\right)\log\left(\frac{1-z}{1-w\,z}\right)\,/;\,(|z|<1\land|w|<1)\lor z\,w<1\lor(z>1\land w>1)$$

(L.Rogers, 1906)

10.07.17.0023.01

$$\mathrm{Li}_2(z\,w)-\mathrm{Li}_2(z)-\mathrm{Li}_2(w)-\mathrm{Li}_2\left(\frac{w\,(z-1)}{1-w}\right)-\mathrm{Li}_2\left(\frac{z\,(w-1)}{1-z}\right)=\frac{1}{2}\log^2\left(\frac{1-w}{1-z}\right)\,/;\,\neg\left(z\in\mathbb{R}\bigwedge w\in\mathbb{R}\bigwedge z>1\bigwedge w>1\bigwedge\frac{z\,(w-1)}{1-z}>1\bigwedge\frac{w\,(z-1)}{1-w}>1\right)\bigvee(z\in\mathbb{R}\land w\in\mathbb{R}\land(z>1\land w>1)\lor(z<1\land w<1))$$

(C.J.Hill,1830)

10.07.17.0024.01

$$\mathrm{Li}_2(z\,w)-\mathrm{Li}_2(z)-\mathrm{Li}_2(w)+\mathrm{Li}_2\left(\frac{z\,(1-w)}{1-w\,z}\right)+\mathrm{Li}_2\left(\frac{w\,(1-z)}{1-w\,z}\right)=\frac{1}{2}\left(\log(1-w)\log(w)+\log(1-z)\log(z)-\log\left(\frac{1-z}{1-w\,z}\right)\log\left(\frac{z\,(1-w)}{1-w\,z}\right)-\log\left(\frac{1-w}{1-w\,z}\right)\log\left(\frac{w\,(1-z)}{1-w\,z}\right)-\log(w\,z)\log(1-w\,z)\right)\,/;\,(|z|<1\land 0<w<1)\lor(|w|<1\land 0<z<1)\lor(z\in\mathbb{R}\land w\in\mathbb{R}\land(0<z<1\land w<1)\lor(0<w<1\land z<1)\lor(z>1\land w>1))$$

10.07.17.0025.01

$$\mathrm{Li}_2(z)+\mathrm{Li}_2(w-z)+\mathrm{Li}_2\left(\frac{w-z}{w-1}\right)-\mathrm{Li}_2\left(\frac{z\,(w-z)}{w-1}\right)+\mathrm{Li}_2\left(\frac{z}{w-1}\right)=-\frac{1}{2}\log^2(1-w)\,/;\,|z|<1\land|w|<1$$

10.07.17.0026.01

$$\mathrm{Li}_2\left(\frac{z\,(1-w)}{w\,(1-z)}\right)=\mathrm{Li}_2(w)-\mathrm{Li}_2(z)+\mathrm{Li}_2\left(\frac{z}{w}\right)+\mathrm{Li}_2\left(\frac{1-w}{1-z}\right)+\log(w)\log\left(\frac{1-w}{1-z}\right)-\frac{\pi^2}{6}\,/;\,(0<z<1\land|w|<1)\lor(0<w<1\land|z|<1)\lor(w\in\mathbb{R}\land w>0\land z\in\mathbb{R}\land z<1)$$

(W.Schaeffer, 1846)

10.07.17.0027.01

$$\mathrm{Li}_2\left(\frac{z\,(1-w)^2}{w\,(1-z)^2}\right)=\mathrm{Li}_2\left(\frac{1-w}{1-z}\right)+\mathrm{Li}_2\left(-\frac{z\,(1-w)}{1-z}\right)+\mathrm{Li}_2\left(\frac{z\,(1-w)}{w\,(1-z)}\right)+\mathrm{Li}_2\left(-\frac{1-w}{w\,(1-z)}\right)+\frac{1}{2}\log^2(w)\,/;\,(w>0\land z\in\mathbb{R})\lor(z>w\land z\,w>1)$$

(E.Kummer, 1840)

Involving six dilogarithms

10.07.17.0028.01

$$\mathrm{Li}_2(1-x)+\mathrm{Li}_2(1-y)+\mathrm{Li}_2(1-z)=\frac{1}{2}(\mathrm{Li}_2(1-x\,y)+\mathrm{Li}_2(1-x\,z)+\mathrm{Li}_2(1-y\,z))\,/;\,x+y+z=x\,y\,z+2\bigwedge|x|<1\bigwedge|y|<1\bigwedge|z|<1$$

10.07.17.0029.01

$$\mathrm{Li}_2(x)+\mathrm{Li}_2(y)+\mathrm{Li}_2(z)=\frac{1}{2}(\mathrm{Li}_2(-y\,x+x+y)+\mathrm{Li}_2(-z\,x+x+z)+\mathrm{Li}_2(-z\,y+y+z))\,/;\,x\,y+z\,y+x\,z=x\,y\,z\bigwedge|x|<1\bigwedge|y|<1\bigwedge|z|<1$$

10.07.17.0030.01

$$\mathrm{Li}_2(x)+\mathrm{Li}_2(y)+\mathrm{Li}_2(z)=\frac{1}{2}\left(\mathrm{Li}_2\left(-\frac{yz}{x}\right)+\mathrm{Li}_2\left(-\frac{xz}{y}\right)+\mathrm{Li}_2\left(-\frac{xy}{z}\right)\right)/;\ \frac{1}{y}+\frac{1}{z}+\frac{1}{x}=1\bigwedge|x|<1\bigwedge|y|<1\bigwedge|z|<1$$

Newman's formula

10.07.17.0031.01

$$\mathrm{Li}_2(-x^2)+\mathrm{Li}_2(-y^2)+\mathrm{Li}_2(-z^2)=2\left(\mathrm{Li}_2(xy)+\mathrm{Li}_2(xz)+\mathrm{Li}_2(yz)\right)/;\ x+y+z=xyz\land|x|<1\land|y|<1\land|z|<1$$

Involving nine dilogarithms

10.07.17.0032.01

$$\mathrm{Li}_2\left(\frac{vw}{xy}\right)=\mathrm{Li}_2\left(\frac{v}{x}\right)+\mathrm{Li}_2\left(\frac{w}{x}\right)+\mathrm{Li}_2\left(\frac{v}{y}\right)+\mathrm{Li}_2\left(\frac{w}{y}\right)+\mathrm{Li}_2(x)+\mathrm{Li}_2(y)-\mathrm{Li}_2(v)-\mathrm{Li}_2(w)+\frac{1}{2}\log^2\left(-\frac{x}{y}\right)/;$$
$$(1-v)(1-w)=(1-x)(1-y)\land 0<x<1\land 0<y<1\land 0<v<1\land 0<w<1$$

(W.Mantel, 1898)

Involving several dilogarithms

10.07.17.0033.01

$$\mathrm{Li}_2(z)=\sum_{k=1}^{n}\sum_{p=1}^{n}\left(\mathrm{Li}_2(z_k\lambda_p)-\mathrm{Li}_2\left(\frac{\lambda_k}{\lambda_p}\right)\right)+\frac{\pi^2}{6}/;\ \prod_{p=1}^{n}(1-\lambda_p z_k)=1-z\bigwedge 1\le k\le n$$

Relations of special kind

10.07.17.0034.01

$$f(zw)=f(w)-f\left(\frac{z(1-w)}{1-zw}\right)-f\left(\frac{w(1-z)}{1-zw}\right)+f(z)\bigwedge f(z)+f(1-z)=\frac{\pi^2}{6}/;\ f(z)=\mathrm{Li}_2(z)+\frac{1}{2}\log(1-z)\log(z)\bigwedge 0<z<1$$

$\mathrm{Li}_2(z)+\frac{1}{2}\log(1-z)\log(z)$ is the unique solution of class $C^3((0,1))$ of the functional equations $f(zw)=f(w)-f\left(\frac{z(1-w)}{1-zw}\right)-f\left(\frac{w(1-z)}{1-zw}\right)+f(z)\bigwedge f(z)+f(1-z)=\frac{\pi^2}{6}$ for all real $0<z<1\land 0<w<1$.

Complex characteristics

Real part

10.07.19.0001.01

$$\mathrm{Re}(\mathrm{Li}_2(x+iy))=-\frac{1}{4}\left(\log(x^2+y^2)\left(\log(x^2-2x+y^2+1)-\log\left(\frac{x^2+y^2}{\sqrt{-y^2}-x}+1\right)-\log\left(1-\frac{x^2+y^2}{x+\sqrt{-y^2}}\right)\right)-\right.$$
$$\left.2\,\mathrm{Li}_2\left(\frac{x^2+y^2}{x-\sqrt{-y^2}}\right)-2\,\mathrm{Li}_2\left(\frac{x^2+y^2}{x+\sqrt{-y^2}}\right)\right)$$

Differentiation

Low-order differentiation

10.07.20.0001.01

$$\frac{\partial \operatorname{Li}_2(z)}{\partial z} = -\frac{\log(1-z)}{z}$$

10.07.20.0002.01

$$\frac{\partial^2 \operatorname{Li}_2(z)}{\partial z^2} = \frac{\log(1-z)}{z^2} + \frac{1}{(1-z)\,z}$$

Symbolic differentiation

10.07.20.0004.02

$$\frac{\partial^m \operatorname{Li}_2(z)}{\partial z^m} = z^{-m} \sum_{j=0}^{m} S_m^{(j)} \left(\operatorname{Li}_{2-j}(z) - \sum_{k=1}^{m-1} \frac{z^k}{k^{2-j}} \right) /;\ n \in \mathbb{N}$$

10.07.20.0003.01

$$\frac{\partial^m \operatorname{Li}_2(z)}{\partial z^m} = \sum_{k=0}^{\infty} \frac{(k+1)_m\, z^k}{(k+m)^2} /;\ |z| < 1 \bigwedge n \in \mathbb{N}^+$$

10.07.20.0005.01

$$\frac{\partial^m \operatorname{Li}_2(z)}{\partial z^m} = \sum_{j=0}^{m} S_m^{(j)}\, \Phi(z,\, 2-j,\, m) /;\ n \in \mathbb{N}^+$$

10.07.20.0006.01

$$\frac{\partial^m \operatorname{Li}_2(z)}{\partial z^m} = z^{-m}\, \Gamma(m)\, \mathrm{B}_z(m,\, 1-m) /;\ m \in \mathbb{N}^+$$

10.07.20.0007.01

$$\frac{\partial^m \operatorname{Li}_2(z)}{\partial z^m} = \Gamma(m)^2\, {}_2\tilde{F}_1(m,\, m;\, m+1;\, z) /;\ m \in \mathbb{N}^+$$

Fractional integro-differentiation

10.07.20.0009.01

$$\frac{\partial^\alpha \operatorname{Li}_2(z)}{\partial z^\alpha} = \frac{z^{1-\alpha}}{\Gamma(2-\alpha)}\, {}_3F_2(1,\, 1,\, 1;\, 2,\, 2-\alpha;\, z)$$

10.07.20.0008.01

$$\frac{\partial^\alpha \operatorname{Li}_2(z)}{\partial z^\alpha} = \sum_{k=1}^{\infty} \frac{(k-1)!\, z^{k-\alpha}}{\Gamma(k-\alpha+1)\, k} /;\ |z| < 1$$

Integration

Indefinite integration

Involving only one direct function

10.07.21.0001.01

$$\int \operatorname{Li}_2(z)\, dz = \operatorname{Li}_2(z)\, z - z + (z-1)\log(1-z)$$

10.07.21.0002.01

$$\int \mathrm{Li}_2(z)\, dz = \sum_{k=1}^{\infty} \frac{z^{k+1}}{(k+1)\, k^2} \;/;\; |z| < 1$$

Involving one direct function and elementary functions

Involving power function

10.07.21.0003.01

$$\int z^{\alpha-1}\, \mathrm{Li}_2(z)\, dz = \sum_{k=1}^{\infty} \frac{z^{k+\alpha}}{(k+\alpha)\, k^2} \;/;\; |z| < 1$$

10.07.21.0004.01

$$\int z^{\alpha-1}\, \mathrm{Li}_2(z)\, dz = \frac{z^{\alpha+1}}{\alpha+1}\, {}_4F_3(\alpha+1, 1, 1, 1; \alpha+2, 2, 2; z)$$

10.07.21.0005.01

$$\int \frac{\mathrm{Li}_\nu(-a\, z)}{z}\, dz = \mathrm{Li}_{\nu+1}(-a\, z)$$

Involving rational functions

10.07.21.0006.01

$$\int \frac{\mathrm{Li}_2(z)}{1-z}\, dz = -\log(z) \log^2(1-z) - 2\, \mathrm{Li}_2(1-z) \log(1-z) - \mathrm{Li}_2(z) \log(1-z) + 2\, \mathrm{Li}_3(1-z)$$

10.07.21.0007.01

$$\int \frac{\mathrm{Li}_2(z)}{(z-1)^2}\, dz = \frac{z\, \mathrm{Li}_2(z)}{1-z} - \frac{1}{2} \log^2(1-z)$$

Involving exponential function

10.07.21.0008.01

$$\int \mathrm{Li}_2(e^{-z})\, dz = -\mathrm{Li}_3(e^{-z})$$

Involving rational functions and logarithm

10.07.21.0009.01

$$\int \frac{\log(z)\, \mathrm{Li}_2(z)}{z}\, dz = \log(z)\, \mathrm{Li}_3(z) - \mathrm{Li}_4(z)$$

10.07.21.0010.01

$$\int \frac{\log(1-z)\, \mathrm{Li}_2(z)}{1-z}\, dz = \frac{1}{2} \left(-\log(z) \log^3(1-z) - 3\, \mathrm{Li}_2(1-z) \log^2(1-z) - \mathrm{Li}_2(z) \log^2(1-z) + 6\, \mathrm{Li}_3(1-z) \log(1-z) - 6\, \mathrm{Li}_4(1-z)\right)$$

Definite integration

For the direct function itself

10.07.21.0011.01

$$\int_0^z \frac{1}{t}\,\mathrm{Li}_2(t)\,dt = \mathrm{Li}_3(z)$$

10.07.21.0012.01

$$\int_0^1 t^\alpha\,\mathrm{Li}_2(t)\,dt = \frac{\alpha\left(\pi^2(\alpha+1)^2-12\right)-6\,\alpha\,(\alpha+1)\,(\psi(\alpha)+\gamma)-6}{6\,\alpha\,(\alpha+1)^3} \;/;\; \mathrm{Re}(\alpha) > -2$$

10.07.21.0013.01

$$\int_0^1 t^n\,\mathrm{Li}_2(t)\,dt = \frac{\pi^2}{6\,(n+1)} - \frac{1}{(n+1)^2}\sum_{k=1}^{n+1}\frac{1}{k} \;/;\; n \in \mathbb{N}$$

10.07.21.0014.01

$$\int_0^\infty t^{-3/2}\,\mathrm{Li}_2(-t)^2\,dt = \frac{8\,\pi}{3}\left(24\log(2)+\pi^2\right)$$

10.07.21.0015.01

$$\int_0^\infty t^{-5/2}\,\mathrm{Li}_2(-t)^2\,dt = \frac{8\,\pi}{27}\left(-8\log(2)-\pi^2+20\right)$$

10.07.21.0016.01

$$\int_0^1 \frac{\mathrm{Li}_2(t)}{(1-t\,z)^2}\,dt = -\frac{\log^2(1-z)}{2\,z} - \frac{\mathrm{Li}_2(z)}{z} + \frac{\pi^2}{6\,(1-z)} \;/;\; z \notin (1,\infty)$$

10.07.21.0017.01

$$\int_1^\infty \frac{\mathrm{Li}_2(-t\,z)}{\sqrt{t}\,(t+1)}\,dt = 4\,\pi\,\mathrm{Li}_2\left(-\sqrt{z}\right)$$

10.07.21.0018.01

$$\int_0^\infty \mathrm{Li}_2\left(-z\tan^2(t)\right)dt = 2\,\pi\,\mathrm{Li}_2\left(-\sqrt{z}\right)$$

Involving the direct function

10.07.21.0019.01

$$\int_0^\infty \frac{1}{t}\log(1+a\,t)\,\mathrm{Li}_2\left(-\frac{z}{t^2}\right)dt = \frac{1}{480\,a}$$

$$\left(120\,\pi\sqrt{\frac{1}{z}}\;\Phi\left(-\frac{1}{a^2\,z},3,\frac{1}{2}\right) - a\left(5\log^4\left(\frac{1}{a^2\,z}\right)+50\,\pi^2\log^2\left(\frac{1}{a^2\,z}\right)+120\,\mathrm{Li}_3\left(-\frac{1}{a^2\,z}\right)\log\left(\frac{1}{a^2\,z}\right)+53\,\pi^4-360\,\mathrm{Li}_4\left(-\frac{1}{a^2\,z}\right)\right)\right)$$

10.07.21.0020.01

$$\int_0^\infty t^{-3/4}\log(1+t)\,\mathrm{Li}_2\left(-\frac{1}{t}\right)dt = -2\,\pi\sqrt{2}\left(16\,(3\log(2)+C-4)+\frac{5\,\pi^2}{3}\right)$$

For the products of direct functions

10.07.21.0021.01

$$\int_0^\infty \frac{1}{\sqrt{t}}\,\mathrm{Li}_2(-t)\,\mathrm{Li}_2\left(-\frac{1}{t}\right)dt = 16\,\pi\,(3-4\log(2))$$

10.07.21.0022.01

$$\int_0^\infty t^{-3/4}\,\mathrm{Li}_2(-t)\,\mathrm{Li}_2\left(-\frac{1}{t}\right)dt = 256\,\pi\sqrt{2}\,(-3\log(2)-C+3)$$

10.07.21.0023.01

$$\int_0^{\infty} \frac{1}{t} \operatorname{Li}_2(-a\,t) \operatorname{Li}_2\left(-\frac{z}{t^2}\right) dt = \frac{1}{2880\,a} \left(360\,\pi \sqrt{\frac{1}{z}}\; \Phi\left(-\frac{1}{a^2\,z}, 4, \frac{1}{2}\right) - a\left(3 \log^5\left(\frac{1}{a^2\,z}\right) + 50\,\pi^2 \log^3\left(\frac{1}{a^2\,z}\right) + 159\,\pi^4 \log\left(\frac{1}{a^2\,z}\right) + 360 \operatorname{Li}_4\left(-\frac{1}{a^2\,z}\right) \log\left(\frac{1}{a^2\,z}\right) - 1440 \operatorname{Li}_5\left(-\frac{1}{a^2\,z}\right)\right)\right)$$

10.07.21.0024.01

$$\int_0^{\infty} \frac{1}{t} \operatorname{Li}_n(-t) \operatorname{Li}_2\left(-\frac{1}{t}\right) dt = (n+2)\,\zeta(n+3) \;/;\; n \in \mathbb{N}^+$$

Representations through more general functions

Through hypergeometric functions of two variables

10.07.26.0001.01

$$\operatorname{Li}_2(z) = \frac{z}{2} F_{1\times 0\times 0}^{0\times 2\times 2}\left(\begin{matrix} ;\,1,\,1;\,1,\,1; \\ 3;;; \end{matrix}\; z, 1\right)$$

10.07.26.0002.01

$$\operatorname{Li}_2(z) + \operatorname{Li}_2(w) - \operatorname{Li}_2(z + w - z\,w) = \frac{z\,w}{2} F_{1\times 0\times 0}^{0\times 2\times 2}\left(\begin{matrix} ;\,1,\,1;\,1,\,1; \\ 3;;; \end{matrix}\; z, w\right)$$

10.07.26.0003.01

$$\operatorname{Li}_2\left(z^2\right) = z^2\; F_{1\times 0\times 0}^{0\times 2\times 2}\left(\begin{matrix} ;\,1,\,1;\,1,\,1; \\ 3;;; \end{matrix}\; z, -z\right)$$

10.07.26.0004.01

$$\operatorname{Li}_2(z) = \frac{1}{4}\left(\frac{1}{z} - 2\right)^2 F_{1\times 0\times 0}^{0\times 2\times 2}\left(\begin{matrix} ;\,1,\,1;\,1,\,1; \\ 3;;; \end{matrix}\; 2 - \frac{1}{z}, 2 - \frac{1}{z}\right) + \frac{\pi^2}{12} - \frac{1}{2} \log^2\left(\frac{1}{z}\right)$$

Through hypergeometric functions

Involving ${}_pF_q$

10.07.26.0005.01

$$\operatorname{Li}_2(z) = z\,{}_3F_2(1, 1, 1; 2, 2; z)$$

Through Meijer *G*

Classical cases for the direct function itself

10.07.26.0006.01

$$\operatorname{Li}_2(z) = -G_{3,3}^{1,3}\left(-z \,\middle|\, \begin{matrix} 1, 1, 1 \\ 1, 0, 0 \end{matrix}\right)$$

Through other functions

10.07.26.0007.01

$$\operatorname{Li}_2(z) = S_{1,1}(z)$$

10.07.26.0008.01

$$\operatorname{Li}_2(z) = z\,\Phi(z, 2, 1)$$

Representations through equivalent functions

With related functions

10.07.27.0001.01

$$\mathrm{Li}_2\left(e^{\frac{2\pi i p}{q}}\right) == \frac{1}{q^2}\sum_{k=1}^{q} e^{\frac{2\pi i p k}{q}}\,\zeta\left(2, \frac{k}{q}\right) /;\ p \in \mathbb{N}^+ \bigwedge q \in \mathbb{N}^+ \bigwedge p \leq q$$

Theorems

The Fermi–Dirac integral

The Fermi-Dirac integral $F_\alpha(\mu) == \int_0^\infty \frac{\varepsilon^\alpha}{e^{\varepsilon-\mu}+1}\,d\varepsilon$, describing, for instance, the number of electrons (holes) in the conduction band (valence band) in a semiconductor with density of states $\propto \varepsilon^\alpha$, can be expressed as $F_\alpha(\mu) == -\alpha\,\Gamma(\alpha)\,\mathrm{Li}_{\alpha+1}(-e^\mu)$.

The volume of a Lambert cube

The volume V of a Lambert cube with essential angles α_1, α_2, α_3 and apices of length l_1, l_2, l_3 in three-dimensional hyperbolic space is given by

$$V == \sum_{k=1}^{3} (\mathcal{L}(\alpha_k + \theta) - \mathcal{L}(\alpha_k - \theta)) - 1/4\,\mathcal{L}(2\,\theta) + 1/2\,\mathcal{L}(\pi/2 - \theta) /;\ \mathcal{L}(z) == 1/2\,\mathrm{Im}(\mathrm{Li}_2(\exp(2\,i\,z))) \bigwedge \tan^{-1}\left(\left(\cosh(l_2) - \sin^2(\alpha_1)\sin^2(\alpha_2)\right)/\left(\cos^2(\alpha_1)\cos^2(\alpha_2)\right)\right).$$

Rationality of dilogarithm

The value $\mathrm{Li}_2(z)$ is irrational when zis rational (G.V.Chudnovsky,1979).

History

– G. W. Leibniz defined dilogarithm for the case $\nu = 2$
– L. Euler (1768)
– J. Landen (1760,1780) inverstigated $\mathrm{Li}_2(z)$ and $\mathrm{Li}_3(z)$
– W. Spence (1809)
– N. H. Abel
– E.E. Kummer (1840)
– J. Kummer; L. L. Lindelöf
– N. I. Lobachevski
– C. J. Hill (1828) introduced the name "dilogarithm"

Applications include electrical network problems, number theory, group theory, K-theory, geometry, quantum electrodynamics, group cohomology, mixed Hodge structures, mixed motives, evaluation of volumes of hyperbolic polytopes, celestial mechanics.

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