Prime

Notations

Traditional name

The *n*th prime number

Traditional notation

$\text{prime}(n)$

Mathematica StandardForm notation

`Prime[n]`

Primary definition

13.03.02.0002.01

$\text{prime}(1) = 2$

13.03.02.0002.01

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Prime[n] == m /;
 n > 1 ⋀ m ∈ Integers ⋀ m > Prime[n - 1] ⋀ Not[Exists[p, p ∈ Primes, Prime[n - 1] < p < m]] ⋀
  Not[Exists[k, k ∈ Integers ⋀ 1 < k < m, m/k ∈ Integers]]
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13.03.02.0002.01

$$\text{prime}(n) = m \,/;\, n > 1 \bigwedge m \in \mathbb{Z} \bigwedge m > \text{prime}(n-1) \bigwedge \left(\neg \exists_{p, p \in \mathbb{P}}\, \text{prime}(n-1) < p < m\right) \bigwedge \left(\neg \exists_{k, k \in \mathbb{Z} \land 1 < k < m} \frac{m}{k} \in \mathbb{Z}\right)$$

For positive integer n, the function $\text{prime}(n)$ is the smallest integer greater than $\text{prime}(n-1)$ which cannot be divided by any integer greater than 1 and smaller than itself.
The first prime $\text{prime}(1)$ is the integer 2.

13.03.02.0001.01

$\text{prime}(n)$

$\text{prime}(n)$ is the nth prime number $(n > 0)$ (for which $\text{divisors}(\text{prime}(n)) = \{1, \text{prime}(n)\}$ holds).

Examples: The primes under 25 are 2, 3, 5, 7, 11, 13, 17, 19 and 23, so $\text{prime}(1) = 2$, $\text{prime}(2) = 3$, …, $\text{prime}(9) = 23$.

Specific values

Values at fixed points

13.03.03.0001.01

prime(1) == 2

13.03.03.0002.01

prime(2) == 3

13.03.03.0003.01

prime(3) == 5

13.03.03.0004.01

prime(4) == 7

13.03.03.0005.01

prime(5) == 11

13.03.03.0006.01

prime(6) == 13

13.03.03.0007.01

prime(7) == 17

13.03.03.0008.01

prime(8) == 19

13.03.03.0009.01

prime(9) == 23

13.03.03.0010.01

prime(10) == 29

13.03.03.0011.01

prime(11) == 31

13.03.03.0012.01

prime(12) == 37

13.03.03.0013.01

prime(13) == 41

13.03.03.0014.01

prime(14) == 43

13.03.03.0015.01

prime(15) == 47

13.03.03.0016.01

prime(16) == 53

13.03.03.0017.01

prime(17) == 59

13.03.03.0018.01

prime(18) == 61

13.03.03.0019.01

prime(19) == 67

13.03.03.0020.01

prime(20) == 71

13.03.03.0021.01
prime(21) == 73

13.03.03.0022.01
prime(22) == 79

13.03.03.0023.01
prime(23) == 83

13.03.03.0024.01
prime(24) == 89

13.03.03.0025.01
prime(25) == 97

13.03.03.0026.01
prime(26) == 101

13.03.03.0027.01
prime(27) == 103

13.03.03.0028.01
prime(28) == 107

13.03.03.0029.01
prime(29) == 109

13.03.03.0030.01
prime(30) == 113

13.03.03.0031.01
prime(31) == 127

13.03.03.0032.01
prime(32) == 131

13.03.03.0033.01
prime(33) == 137

13.03.03.0034.01
prime(34) == 139

13.03.03.0035.01
prime(35) == 149

13.03.03.0036.01
prime(36) == 151

13.03.03.0037.01
prime(37) == 157

13.03.03.0038.01
prime(38) == 163

13.03.03.0039.01
prime(39) == 167

13.03.03.0040.01
prime(40) == 173

13.03.03.0041.01

$\text{prime}(41) = 179$

13.03.03.0042.01

$\text{prime}(42) = 181$

13.03.03.0043.01

$\text{prime}(43) = 191$

13.03.03.0044.01

$\text{prime}(44) = 193$

13.03.03.0045.01

$\text{prime}(45) = 197$

13.03.03.0046.01

$\text{prime}(46) = 199$

13.03.03.0047.01

$\text{prime}(47) = 211$

13.03.03.0048.01

$\text{prime}(48) = 223$

13.03.03.0049.01

$\text{prime}(49) = 227$

13.03.03.0050.01

$\text{prime}(50) = 229$

13.03.03.0051.01

$\text{prime}(100) = 541$

13.03.03.0052.01

$\text{prime}(1000) = 7919$

13.03.03.0055.01

$\text{prime}(10\,000) = 104\,729$

13.03.03.0053.01

$\text{prime}(1\,000\,000) = 15\,485\,863$

13.03.03.0054.01

$\text{prime}(1\,000\,000\,000) = 22\,801\,763\,489$

General characteristics

Domain and analyticity

prime(n) is a nonanalytical function which is defined only for positive integers $\mathbb{N}^+$.

13.03.04.0001.01

$n \longrightarrow \text{prime}(n) :: \mathbb{Z} \longrightarrow \mathbb{Z}$

Symmetries and periodicities

Symmetry

No symmetry

Periodicity

No periodicity

Series representations

Other series representations

13.03.06.0001.01

$$\text{prime}(n) = \sum_{k=1}^{2^n} \left\lfloor \left(\frac{n}{\pi(k)+1} \right)^{1/n} \right\rfloor + 1$$

13.03.06.0008.01

$$\text{prime}(n) = 2 + \sum_{k=2}^{\lfloor 2n \log(n)+2 \rfloor} \left(1 - \left\lfloor \frac{\pi(k)}{n} \right\rfloor \right)$$

13.03.06.0002.01

$$\text{prime}(n) = \sum_{m=2}^{2^n} m \left\lfloor \frac{1}{\left| n - \left\lfloor \frac{1}{\sum_{i=1}^{m-1} \left\lfloor \frac{\lfloor \frac{m}{i} \rfloor}{\frac{m}{i}} \right\rfloor} \right\rfloor \sum_{k=2}^{m} \left\lfloor \frac{1}{\sum_{i=1}^{k-1} \left\lfloor \frac{\lfloor \frac{k}{i} \rfloor}{\frac{k}{i}} \right\rfloor} \right\rfloor \right| + 1} \right\rfloor$$

13.03.06.0009.01

$$\text{prime}(n) = 2 + \sum_{k=2}^{\lfloor 2n \log(n)+2 \rfloor} \left(1 - \left\lfloor \frac{1}{n} \sum_{j=2}^{k} \left(\left\lfloor \frac{2 - \sum_{i=1}^{j} \left(\left\lfloor \frac{j}{i} \right\rfloor - \left\lfloor \frac{j-1}{i} \right\rfloor \right)}{j} \right\rfloor + 1 \right) \right\rfloor \right)$$

13.03.06.0003.01

$$\text{prime}(n) = \sum_{k=0}^{n^2} g\left(1 - g\left(\sum_{j=0}^{k} r\left(g(j-1)!^2,\ j\right) - n\right)\right) /;\ g(m) = m\,\theta(m) \bigwedge r(a,\ b) = a\,\delta_{b,0} + (1 - \delta_{b,0})\,(a \bmod b)$$

13.03.06.0004.01

$$\text{prime}(n) \propto n \left(\log(n) + \log(\log(n)) - 1 + \frac{\log(\log(n)) - 2}{\log(n)} - \frac{\log^2(\log(n)) - 6\log(\log(n)) + 11}{2\log^2(n)} + \ldots \right) /;\ n > 2$$

13.03.06.0005.01

$$\text{prime}(n) = \left\lfloor 1 - \frac{1}{\log(2)} \log\left(-\frac{1}{2} \sum_{d_j \mid \prod_{k=1}^{n-1} p_k} \frac{\mu(d_j)}{2^{d_j} - 1} \right) \right\rfloor /;\ p_k \in \mathbb{P}$$

13.03.06.0006.01

$$\operatorname{prime}(n+1) = \left\lfloor 1 - \log_2\left(\sum_{r=1}^{n} \sum_{j_r=j_{r-1}}^{n} \ldots \sum_{j_1=1}^{j_2} \frac{(-1)^r}{2^{\prod_{k=1}^{r} \operatorname{prime}(j_k)} - 1} + \frac{1}{2}\right)\right\rfloor$$

13.03.06.0007.01

$$\operatorname{prime}(n) = \left\lfloor 10^{2^n} \alpha \right\rfloor - 10^{2^{n-1}} \left\lfloor 10^{2^{n-1}} \alpha \right\rfloor /; \alpha = \sum_{k=1}^{\infty} \operatorname{prime}(k)\, 10^{-2^k}$$

Identities

Functional identities

13.03.17.0001.01

$$\sin^2\left(\frac{\pi\,(\Gamma(\operatorname{prime}(n)) + 1)}{\operatorname{prime}(n)}\right) + \sin^2(\pi \operatorname{prime}(n)) = 0$$

13.03.17.0002.01

$$\frac{e^{\frac{2\pi i \Gamma(p)}{p}} - 1}{e^{-\frac{2\pi i}{p}} - 1} = 1 /; p \in \mathbb{P}$$

13.03.17.0003.01

$$\frac{e^{\frac{2\pi i \Gamma(p)}{p}} - 1}{e^{-\frac{2\pi i}{p}} - 1} = 0 /; p \notin \mathbb{P} \land p > 4$$

Products

Finite products

13.03.24.0001.01

$$\prod_{k=1}^{n} p_k > p_{n+1}\, p_{n+2} /; n \geq 4 \land p_k \in \mathbb{P}$$

Infinite products

13.03.24.0004.01

$$\prod_{k=1}^{\infty} \left(1 - p_k^{-s}\right) = \frac{1}{\zeta(s)} /; \operatorname{Re}(s) > 1 \land p_k = \operatorname{prime}(k)$$

13.03.24.0005.01

$$\prod_{k=1}^{\infty} \left(p_k^{-s} + 1\right) = \frac{\zeta(s)}{\zeta(2\,s)} /; \operatorname{Re}(s) > 1 \land p_k = \operatorname{prime}(k)$$

13.03.24.0002.01

$$\prod_{k=1}^{\infty} \frac{p_k^2 + 1}{p_k^2 - 1} = \frac{5}{2} /; p_k = \operatorname{prime}(k)$$

Regularized products

13.03.24.0003.01

$$\prod_{k=1}^{\infty} p_k == 4\pi^2 \;/;\; p_k == \text{prime}(k)$$

Operations

Limit operation

13.03.25.0001.01

$$\lim_{n\to\infty} \frac{\text{prime}(n)}{n\log(n)} == 1$$

Representations through equivalent functions

With related functions

13.03.27.0001.01

$$\text{prime}(n) == \sum_{k=1}^{2^n} \left\lfloor \left(\frac{n}{\pi(k)+1}\right)^{1/n} \right\rfloor + 1$$

Inequalities

13.03.29.0001.01

$$\text{prime}(n+1) < 2\,\text{prime}(n)$$

13.03.29.0002.01

$$n\,(\log(n) + \log(\log(n)) - 1) < \text{prime}(n) < n\,(\log(n) + \log(\log(n))) \;/;\; n > 5$$

13.03.29.0003.01

$$\text{prime}(n+1) < \text{prime}(n)^{23/42} + \text{prime}(n) \;/;\; n > 9$$

13.03.29.0004.01

$$\text{prime}(n) > \frac{n\,m}{\phi(m)} - \frac{m^2}{\phi(m)} \;/;\; n > 2 \land m > 2$$

Other identities

Congruence properties

13.03.32.0001.01

$$\frac{p}{\text{prime}(n)^2} \in \mathbb{Z} \bigwedge \frac{p}{\text{prime}(n)^2} \geq 0 \;/;\; \frac{p}{q} == \sum_{k=1}^{\text{prime}(n)-1} \frac{1}{k} \bigwedge p \in \mathbb{N}^+ \bigwedge q \in \mathbb{N}^+$$

13.03.32.0002.01

$$\frac{p}{\text{prime}(n)} \in \mathbb{Z} \bigwedge \frac{p}{\text{prime}(n)} \geq 0 \;/;\; \frac{p}{q} == \sum_{k=1}^{\text{prime}(n)-1} \frac{1}{k^2} \bigwedge p \in \mathbb{N}^+ \bigwedge q \in \mathbb{N}^+$$

13.03.32.0003.01

$$\binom{2\,\text{prime}(k)-1}{\text{prime}(k)-1} \bmod \text{prime}(k)^3 == 1 \,/;\, k > 2$$

Various Conjectures

13.03.32.0004.01

$$\overline{\lim_{k\to\infty}}\left(\frac{p_{k+1}-p_k}{\log(k)^2}\right) == 1$$

H. Cramer, 1936

Theorems

Fermats' little theorem

$$a^{p-1} \equiv 1 \pmod{p} \,/;\; p \in \mathbb{P} \bigwedge \frac{a}{p} \notin \mathbb{N}.$$

Representation of positive integers through primes (fundamental theorem of arithmetic)

Every positive integer n can be uniquely represented in the form $n == \prod_{k=1}^{r} p_k^{n_k},\ p_k \in \mathbb{P} \bigwedge n_k \in \mathbb{N}^+,\ p_k < p_{k+1}$.

Bertrand's hypothesis

For all $n \in \mathbb{N}$, $n > 1$ there is at least one prime between n and $2\,n$.

One linearly independent set

The set $\sqrt[k]{p_1^{\alpha_1}\, p_2^{\alpha_2} \ldots p_n^{\alpha_n}}$, $p_i \in \mathbb{P},\ k \in \mathbb{N},\ 0 \le \alpha_i \le k$ is linearly independent over $\mathbb{Z}$.

Giuga's conjecture

If an integer n satisfies $\sum_{k=1}^{n-1} k^{n-1} == -1 \bmod n$, then n is prime.

One linearly independent set

The set of primes $\mathbb{P}$ coincides with the set of positive values of the following polynomial:

$$z(k+2)\Big(-(a\,i-i+k-l+1)^2-\left(\left(a^2-1\right)l^2-m^2+1\right)^2\left(16\,(k+2)\,(n+1)^2\,(k+1)^3-f^2+1\right)^2-\left((a+1)^2\,(e+2)\,e^3-o^2+1\right)^2-\left(-m+l\,(a-n-1)+b\left(-n^2+2\,a\,n-2\,n+2\,a-2\right)+p\right)^2-(l+n+v-y)^2-\left(q+\left(-p^2-2\,p+2\,a+2\,a\,p-2\right)s-x+(a-p-1)\,y\right)^2-\left(-x^2+\left(a^2-1\right)y^2+1\right)^2-\left(16\left(a^2-1\right)r^2\,y^4-u^2+1\right)^2-\left(-(c\,u+x)^2+\left(\left(\left(u^2-a\right)u^2+a\right)^2-1\right)(n+4\,d\,y)^2+1\right)^2-(h+(h+j)\,(k\,g+2\,g+k+1)-z)^2-(-e+2\,n+p+q+z)^2\left(-m\,p+l\,(a-p)\,p+\left(-p^2+2\,a\,p-1\right)t+z\right)^2-(h+j-q+w\,z)^2+1\Big),$$

when $a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, x, y, w, z \in \mathbb{N}$.

History

– Euclid
– P. J. Mollerup (1922)

The prime numbers are encountered often in mathematics and the natural sciences.

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