Signature

Notations

Traditional name

Levi–Civita symbol

Traditional notation

$\varepsilon_{n_1, n_2, \ldots, n_d}$

Mathematica StandardForm notation

`Signature[{n1, n2, …, nd}]`

Primary definition

04.21.02.0001.01

$$\varepsilon_{n_1, n_2, \ldots, n_d} = (-1)^t$$

where t is the number of permutations from the sorted version of $\{n_1, n_2, \ldots, n_d\}$ to $\{n_1, n_2, \ldots, n_d\}$.

04.21.02.0002.01

$$\varepsilon_{n_1, n_2, \ldots, n_d} = 0 \;/;\; n_i = n_j \bigwedge 1 \le i \le d \bigwedge 1 \le j \le d$$

Specific values

Values at fixed points

04.21.03.0001.01

$$\varepsilon_0 = 1$$

04.21.03.0002.01

$$\varepsilon_n = 1$$

04.21.03.0003.01

$$\varepsilon_{1,1} = 0$$

04.21.03.0004.01

$$\varepsilon_{1,2} = 1$$

04.21.03.0005.01

$$\varepsilon_{2,1} = -1$$

04.21.03.0006.01

$$\varepsilon_{z,a} = -1$$

04.21.03.0007.01

$$\varepsilon_{a,z} == 1$$

04.21.03.0008.01

$$\varepsilon_{1,1,2} == 0$$

04.21.03.0009.01

$$\varepsilon_{1,2,3} == 1$$

04.21.03.0010.01

$$\varepsilon_{1,3,2} == -1$$

04.21.03.0011.01

$$\varepsilon_{2,3,1} == 1$$

04.21.03.0012.01

$$\varepsilon_{2,1,3} == -1$$

04.21.03.0013.01

$$\varepsilon_{3,1,2} == 1$$

04.21.03.0014.01

$$\varepsilon_{3,2,1} == -1$$

04.21.03.0015.01

$$\varepsilon_{1,2,3,4} == 1$$

04.21.03.0016.01

$$\varepsilon_{1,2,4,3} == -1$$

General characteristics

Domain and analyticity

$\varepsilon_{n_1, n_2, \ldots, n_d}$ is a nonanalytical function, defined on the set of tuples of integers with possible values 0, ± 1.

04.21.04.0001.01

$$(n_1 * n_2 * \ldots * n_d) \longrightarrow \varepsilon_{n_1, n_2, \ldots, n_d} :: \mathbb{Z}^n \longrightarrow \{-1, 0, 1\}$$

Symmetries and periodicities

Quasi-permutation symmetry

04.21.04.0002.01

$$\varepsilon_{n_1, n_2, \ldots, n_d} == -\varepsilon_{n_2, n_1, \ldots, n_d}$$

04.21.04.0003.01

$$\varepsilon_{n_1, n_2, \ldots, n_k, \ldots, n_j, \ldots, n_d} == -\varepsilon_{n_1, n_2, \ldots, n_j, \ldots, n_k, \ldots, n_d}$$

04.21.04.0004.01

$$\varepsilon_{n_1, n_2, \ldots, n_d} == (-1)^{d+1} \varepsilon_{n_2, n_3, \ldots, n_d, n_1}$$

Transformations

Products, sums, and powers of the direct function

Products of the direct function

04.21.16.0001.01

$$\varepsilon_{n_1,n_2,\ldots,n_d}\,\varepsilon_{m_1,m_2,\ldots,m_d} = -\sum_{\text{permutations}(m_1,m_2,\ldots,m_d)} \varepsilon_{m_1,m_2,\ldots,m_d} \prod_{k=1}^{d} \delta_{n_k,m_k}$$

04.21.16.0002.01

$$\varepsilon_{n_1,n_2,\ldots,n_{r-1},n_r,n_{r+1},\ldots,n_d}\,\varepsilon_{n_1,n_2,\ldots,n_{r-1},m_r,m_{r+1},\ldots,m_d} = -\frac{(d-r)!\,r!}{d!} \sum_{\text{permutations}(m_r,m_{r+1},\ldots,m_d)} \varepsilon_{m_r,m_{r+1},\ldots,m_d} \prod_{k=r}^{d} \delta_{n_k,m_k}$$

Complex characteristics

Real part

04.21.19.0001.01

$$\operatorname{Re}(\varepsilon_{n_1,n_2,\ldots,n_d}) = \varepsilon_{n_1,n_2,\ldots,n_d}$$

Imaginary part

04.21.19.0002.01

$$\operatorname{Im}(\varepsilon_{n_1,n_2,\ldots,n_d}) = 0$$

Absolute value

04.21.19.0003.01

$$|\varepsilon_{n_1,n_2,\ldots,n_d}| = \sqrt{\varepsilon_{n_1,n_2,\ldots,n_d}^{\,2}}$$

Argument

04.21.19.0004.01

$$\arg(\varepsilon_{n_1,n_2,\ldots,n_d}) = \tan^{-1}(\varepsilon_{n_1,n_2,\ldots,n_d},\, 0)$$

Conjugate value

04.21.19.0005.01

$$\overline{\varepsilon_{n_1,n_2,\ldots,n_d}} = \varepsilon_{n_1,n_2,\ldots,n_d}$$

Summation

Finite summation

04.21.23.0001.01

$$\sum_{\tau_1=1}^{n}\sum_{\tau_2=1}^{n}\cdots\sum_{\tau_r=1}^{n} \varepsilon_{\tau_1,\tau_2,\ldots,\tau_r,\nu_{r+1},\ldots,\nu_n}\,\varepsilon_{\tau_1,\tau_2,\ldots,\tau_r,\mu_{r+1},\ldots,\mu_n} = r!\sum_{\mu_{r+1}=1}^{n}\sum_{\mu_{r+2}=1}^{n}\cdots\sum_{\mu_n=1}^{n} \varepsilon_{\mu_{r+1},\ldots,\mu_n} \prod_{k=r+1}^{n} \delta_{\nu_k,\mu_k}$$

Theorems

The determinant of a matrix with entries elements

The determinant $\det(A)$ of the $n \times n$ matrix A with entries $a_{i\,j}$ can be expressed as

$$\det(A) = \sum_{k_1=1}^{n} \sum_{k_2=1}^{n} \ldots \sum_{k_n=1}^{n} \varepsilon_{k_1\,k_2,\ldots,k_n}\, a_{1\,k_1}\, a_{2\,k_2} \ldots a_{n\,k_n}.$$

Antisymmetric Levi-Civita tensor

The set of all $\varepsilon_{n_1,n_2,\ldots,n_d}$ /; $1 \leq n_1 \leq d,\ \ldots,\ 1 \leq n_d \leq d$ forms the completly antisymmetric Levi-Civita tensor of dimension d.

History

– T. Levi–Civita (1896)

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