SpheroidalQS

Notations

Traditional name

Angular spheroidal function of the second kind

Traditional notation

$QS_{\nu,\mu}(\gamma, z)$

Mathematica StandardForm notation

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SpheroidalQS[ν, μ, γ, z]
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Primary definition

11.09.02.0001.01

$QS_{\nu,\mu}(\gamma, z)$

$QS_{\nu,\mu}(\gamma, z)$ is the angular spheroidal function of the first kind with variable z and parameters ν, μ, γ. It is defined as the normalizable solution $w(z) == QS_{\nu,\mu}(\gamma, z)$ of the wave differential equation $\left(1-z^2\right) w''(z) - 2\, z\, w'(z) + \left(\lambda + \gamma^2 \left(1-z^2\right) - \mu^2 \big/ \left(1-z^2\right)\right) w(z) == 0$ with parameter λ equal to spheroidal eigenvalue $\lambda == \lambda_{\nu,\mu}(\gamma)$. The parameter ν enumerates the spheroidal eigenvalues in such a manner that in the limit $(\gamma \to 0)$, the eigenvalues are $\lambda_{\nu,\mu}(0) == \nu(\nu+1)$ and $QS_{\nu,\mu}(0, z) = Q_\nu^\mu(z)$, where $Q_\nu^\mu(z)$ is the Legendre function of the second kind. The angular spheroidal functions are normalized according to the Meixner-Schäfke normalization scheme. $QS_{\nu,\mu}(\gamma, z)$ is an analytical function in the variables ν, μ, γ and z.

Specific values

Specialized values

For fixed ν, μ, z

11.09.03.0001.01

$QS_{\nu,\mu}(0, z) = Q_\nu^\mu(z)$

For fixed ν, γ, z

11.09.03.0002.01

$$QS_{\nu,\frac{1}{2}}(\gamma, z) = -\frac{\sqrt{\pi}}{\sqrt{2}\,\sqrt[4]{1-z^2}}\,\mathrm{Se}\left(b_{\nu+\frac{1}{2}}\left(\frac{\gamma^2}{4}\right), \frac{\gamma^2}{4}, \cos^{-1}(z)\right)$$

For fixed ν, z

11.09.03.0003.01

$$QS_{\nu,0}(0, z) = Q_\nu(z)$$

11.09.03.0004.01

$$QS_{\nu,\frac{1}{2}}(0, z) = -\frac{\sqrt{\pi}}{\sqrt{2}\,\sqrt[4]{1-z^2}}\sin\left(\left(\nu+\frac{1}{2}\right)\cos^{-1}(z)\right)$$

11.09.03.0005.01

$$QS_{\nu,\frac{1}{2}}(0, z) = -\frac{\sqrt{\pi}\,\sqrt[4]{1-z^2}}{\sqrt{2}}\,U_{\nu-\frac{1}{2}}(z)$$

General characteristics

Domain and analyticity

$QS_{\nu,\mu}(\gamma, z)$ is an analytical function of ν, μ, γ, z which is defined in $\mathbb{C}^4$.

11.09.04.0001.01

$$(\nu * \mu * \gamma * z) \longrightarrow QS_{\nu,\mu}(\gamma, z) :: (\mathbb{C} \otimes \mathbb{C} \otimes \mathbb{C} \otimes \mathbb{C}) \longrightarrow \mathbb{C}$$

Symmetries and periodicities

Parity

$QS_{\nu,\mu}(\gamma, z)$ is an even function with respect to γ.

11.09.04.0002.01

$$QS_{\nu,\mu}(-\gamma, z) = QS_{\nu,\mu}(\gamma, z)$$

Mirror symmetry

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$$QS_{\overline{\nu},\overline{\mu}}(\overline{\gamma}, \overline{z}) == \overline{QS_{\nu,\mu}(\gamma, z)}$$

Periodicity

No periodicity

Series representations

Generalized power series

Expansions at $\gamma == 0$

11.09.06.0001.01

$$QS_{\nu,\mu}(\gamma, z) \propto Q_\nu^\mu(z) + \frac{1}{2(2\nu+1)}\left(\frac{(-\mu+\nu+1)(-\mu+\nu+2)}{(2\nu+3)^2} Q_{\nu+2}^\mu(z) - \frac{(\mu+\nu-1)(\mu+\nu)}{(1-2\nu)^2} Q_{\nu-2}^\mu(z)\right)\gamma^2 + \frac{1}{8}\left(\frac{(\mu+\nu-3)(\mu+\nu-2)(\mu+\nu-1)(\mu+\nu)}{(4\nu^2-8\nu-5)(4\nu^2-8\nu+3)^2} Q_{\nu-4}^\mu(z) + \frac{8(4\mu^2-1)(\mu^2+(2\nu-1)\mu+(\nu-1)\nu)}{(1-2\nu)^4(8\nu^3-4\nu^2-34\nu-15)} Q_{\nu-2}^\mu(z) - \frac{1}{2\nu+1}\left(\frac{(\mu-\nu)(\mu-\nu+1)(\mu+\nu-1)(\mu+\nu)}{(1-2\nu)^4(2\nu-3)} + \frac{(-\mu+\nu+1)(-\mu+\nu+2)(\mu+\nu+1)(\mu+\nu+2)}{(2\nu+3)^4(2\nu+5)}\right) Q_\nu^\mu(z) - \frac{8(4\mu^2-1)(\mu^2-(2\nu+3)\mu+\nu^2+3\nu+2)}{(2\nu+3)^4(8\nu^3+28\nu^2-2\nu-7)} Q_{\nu+2}^\mu(z) + \frac{(-\mu+\nu+1)(-\mu+\nu+2)(-\mu+\nu+3)(-\mu+\nu+4)}{(2\nu+1)(2\nu+3)^2(2\nu+5)^2(2\nu+7)} Q_{\nu+4}^\mu(z)\right)\gamma^4 + \ldots /; (\gamma \to 0)$$

11.09.06.0002.01

$$QS_{\nu,\mu}(\gamma, z) \propto Q_\nu^\mu(z)\left(1 + O(\gamma^2)\right)$$

Expansions at generic point $z == z_0$

11.09.06.0003.01

$$QS_{\nu,\mu}(\gamma, z) \propto QS_{\nu,\mu}(\gamma, z_0) + QS_{\nu,\mu}'(\gamma, z_0)(z-z_0) - \frac{1}{2(z_0^2-1)^2}\left(2\,QS_{\nu,\mu}'(\gamma, z_0)\,z_0(z_0^2-1) + QS_{\nu,\mu}(\gamma, z_0)\left(-\mu^2 + \gamma^2(z_0^2-1)^2 - \lambda_{\nu,\mu}(\gamma)(z_0^2-1)\right)\right)(z-z_0)^2 + \frac{1}{6(z_0^2-1)^3}\left(2\,QS_{\nu,\mu}(\gamma, z_0)\,z_0\left(-3\mu^2 + \gamma^2(z_0^2-1)^2 - 2\lambda_{\nu,\mu}(\gamma)(z_0^2-1)\right) - QS_{\nu,\mu}'(\gamma, z_0)(z_0^2-1)\left(\gamma^2 z_0^4 - 2(\gamma^2+3)z_0^2 + \gamma^2 - \mu^2 - \lambda_{\nu,\mu}(\gamma)(z_0^2-1) - 2\right)\right)(z-z_0)^3 + \ldots /; (z \to z_0)$$

11.09.06.0004.01

$$QS_{\nu,\mu}(\gamma, z) \propto QS_{\nu,\mu}(\gamma, z_0)(1 + O(z-z_0))$$

Expansions at $z == 0$

11.09.06.0005.01

$$QS_{\nu,\mu}(\gamma, z) \propto QS_{\nu,\mu}(\gamma, 0) + QS_{\nu,\mu}'(\gamma, 0)\,z - \frac{1}{2}\left(\gamma^2 - \mu^2 + \lambda_{\nu,\mu}(\gamma)\right)QS_{\nu,\mu}(\gamma, 0)\,z^2 + \frac{1}{6}\left(2\,QS_{\nu,\mu}'(\gamma, 0) - \left(\gamma^2 - \mu^2 + \lambda_{\nu,\mu}(\gamma)\right)QS_{\nu,\mu}'(\gamma, 0)\right)z^3 + \ldots /; (z \to 0)$$

11.09.06.0006.01

$$QS_{\nu,\mu}(\gamma, z) \propto QS_{\nu,\mu}(\gamma, 0)(1 + O(z))$$

Differential equations

Ordinary linear differential equations and wronskians

For the direct function itself

11.09.13.0001.01

$$\left(1-z^2\right) w''(z)-2\,z\,w'(z)+\left(\left(1-z^2\right)\gamma^2+\lambda_{\nu,\mu}(\gamma)-\frac{\mu^2}{1-z^2}\right)w(z)=0\;/;\;w(z)=c_1\,QS_{\nu,\mu}(\gamma,z)\,+c_2\,PS_{\nu,\mu}(\gamma,z)$$

11.09.13.0002.01

$$W_z\left(QS_{\nu,\mu}(\gamma,z),\,PS_{\nu,\mu}(\gamma,z)\right)=\frac{1}{1-z^2}\left(PS_{\nu,\mu}{}'(\gamma,0)\,QS_{\nu,\mu}(\gamma,0)-PS_{\nu,\mu}(\gamma,0)\,QS_{\nu,\mu}{}'(\gamma,0)\right)$$

11.09.13.0003.01

$$\left(1-g(z)^2\right) w''(z)+\left(\left(g(z)^2-1\right)\frac{g''(z)}{g'(z)}-2\,g(z)\,g'(z)\right)w'(z)+\left(\frac{\left(\gamma\,g(z)^2+\mu-\gamma\right)\left(-\gamma\,g(z)^2+\mu+\gamma\right)g'(z)^2}{g(z)^2-1}+\lambda_{\nu,\mu}(\gamma)\,g'(z)^2\right)w(z)=0\;/;$$
$$w(z)=c_1\,QS_{\nu,\mu}(\gamma,g(z))\,+c_2\,PS_{\nu,\mu}(\gamma,g(z))$$

11.09.13.0004.01

$$W_z\left(QS_{\nu,\mu}(\gamma,g(z)),\,PS_{\nu,\mu}(\gamma,g(z))\right)=\frac{g'(z)}{1-g(z)^2}\left(PS_{\nu,\mu}{}'(\gamma,0)\,QS_{\nu,\mu}(\gamma,0)-PS_{\nu,\mu}(\gamma,0)\,QS_{\nu,\mu}{}'(\gamma,0)\right)$$

11.09.13.0005.01

$$\left(1-g(z)^2\right) w''(z)+\left(\left(g(z)^2-1\right)\left(\frac{2\,h'(z)}{h(z)}+\frac{g''(z)}{g'(z)}\right)-2\,g(z)\,g'(z)\right)w'(z)+$$
$$\left(\frac{\left(\gamma\,g(z)^2+\mu-\gamma\right)\left(-\gamma\,g(z)^2+\mu+\gamma\right)g'(z)^2}{g(z)^2-1}+\lambda_{\nu,\mu}(\gamma)\,g'(z)^2+\frac{2\,g(z)\,h'(z)\,g'(z)}{h(z)}+\frac{\left(g(z)^2-1\right)\left(h(z)\,h''(z)-2\,h'(z)^2\right)}{h(z)^2}-\right.$$
$$\left.\frac{\left(g(z)^2-1\right)h'(z)\,g''(z)}{h(z)\,g'(z)}\right)w(z)=0\;/;\;w(z)=c_1\,h(z)\,QS_{\nu,\mu}(\gamma,g(z))\,+c_2\,h(z)\,PS_{\nu,\mu}(\gamma,g(z))$$

11.09.13.0006.01

$$W_z\left(h(z)\,QS_{\nu,\mu}(\gamma,g(z)),\,h(z)\,PS_{\nu,\mu}(\gamma,g(z))\right)=\frac{h(z)^2\,g'(z)}{1-g(z)^2}\left(PS_{\nu,\mu}{}'(\gamma,0)\,QS_{\nu,\mu}(\gamma,0)-PS_{\nu,\mu}(\gamma,0)\,QS_{\nu,\mu}{}'(\gamma,0)\right)$$

11.09.13.0007.01

$$\left(1-a^2\,z^{2\,r}\right) w''(z)-\frac{a^2\,(r-2\,s+1)\,z^{2\,r}+r+2\,s-1}{z}\,w'(z)+$$
$$\left(a^2\,r^2\,\lambda_{\nu,\mu}(\gamma)\,z^{2\,r-2}+\frac{-a^2\,r^2\left(\left(a^2\,z^{2\,r}-1\right)^2\gamma^2-\mu^2\right)z^{2\,r}-s^2\left(a^2\,z^{2\,r}-1\right)^2+r\,s\left(a^4\,z^{4\,r}-1\right)}{z^2\left(a^2\,z^{2\,r}-1\right)}\right)w(z)=$$
$$0\;/;\;w(z)=c_1\,z^s\,QS_{\nu,\mu}(\gamma,a\,z^r)\,+\,c_2\,z^s\,PS_{\nu,\mu}(\gamma,a\,z^r)$$

11.09.13.0008.01

$$W_z\left(z^s\,QS_{\nu,\mu}(\gamma,a\,z^r),\,z^s\,PS_{\nu,\mu}(\gamma,a\,z^r)\right)=\frac{a\,r\,z^{r+2\,s-1}}{1-a^2\,z^{2\,r}}\left(PS_{\nu,\mu}{}'(\gamma,0)\,QS_{\nu,\mu}(\gamma,0)-PS_{\nu,\mu}(\gamma,0)\,QS_{\nu,\mu}{}'(\gamma,0)\right)$$

11.09.13.0009.01

$$\left(1-a^2\,r^{2\,z}\right) w''(z)+\left(-a^2\log(r)\,r^{2\,z}+2\,a^2\log(s)\,r^{2\,z}-\log(r)-2\log(s)\right)w'(z)+$$
$$\left(a^2\log^2(r)\,\lambda_{\nu,\mu}(\gamma)\,r^{2\,z}+\frac{1}{a^2\,r^{2\,z}-1}\left(-a^2\left(\left(a^2\,r^{2\,z}-1\right)^2\gamma^2-\mu^2\right)\log^2(r)\,r^{2\,z}-\left(a^2\,r^{2\,z}-1\right)^2\log^2(s)+\left(a^4\,r^{4\,z}-1\right)\log(r)\log(s)\right)\right)$$
$$w(z)=0\;/;\;w(z)=c_1\,s^z\,QS_{\nu,\mu}(\gamma,a\,r^z)\,+c_2\,s^z\,PS_{\nu,\mu}(\gamma,a\,r^z)$$

11.09.13.0010.01

$$W_z\left(s^z\, QS_{\nu,\mu}(\gamma,\, a\, r^z),\, s^z\, PS_{\nu,\mu}(\gamma,\, a\, r^z)\right) = \frac{a\, r^z\, s^{2\, z} \log(r)}{1 - a^2\, r^{2\, z}} \left(PS_{\nu,\mu}{}'(\gamma,\, 0)\, QS_{\nu,\mu}(\gamma,\, 0) - PS_{\nu,\mu}(\gamma,\, 0)\, QS_{\nu,\mu}{}'(\gamma,\, 0)\right)$$

Differentiation

Low-order differentiation

With respect to z

11.09.20.0001.01

$$\frac{\partial QS_{\nu,\mu}(\gamma,\, z)}{\partial z} = QS_{\nu,\mu}{}'(\gamma,\, z)$$

11.09.20.0002.01

$$\frac{\partial^2 QS_{\nu,\mu}(\gamma,\, z)}{\partial z^2} = \frac{1}{1 - z^2} \left(2\, z\, QS_{\nu,\mu}{}'(\gamma,\, z) - \left(\left(1 - z^2\right) \gamma^2 + \lambda_{\nu,\mu}(\gamma) - \frac{\mu^2}{1 - z^2}\right) QS_{\nu,\mu}(\gamma,\, z)\right)$$

Integration

Representations through equivalent functions

Theorems

History

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