StruveL

Notations

Traditional name

Struve function **L**

Traditional notation

$\mathbf{L}_\nu(z)$

Mathematica StandardForm notation

StruveL[ν, z]

Primary definition

03.10.02.0001.01

$$\mathbf{L}_\nu(z) == \left(\frac{z}{2}\right)^{\nu+1} \sum_{k=0}^{\infty} \frac{1}{\Gamma\left(k+\frac{3}{2}\right)\Gamma\left(k+\nu+\frac{3}{2}\right)} \left(\frac{z}{2}\right)^{2k}$$

Specific values

Specialized values

For fixed ν

03.10.03.0001.01

$$\mathbf{L}_\nu(0) == 0 \;/;\; \mathrm{Re}(\nu) > -1$$

03.10.03.0002.01

$$\mathbf{L}_\nu(0) == \tilde{\infty} \;/;\; \mathrm{Re}(\nu) < -1$$

03.10.03.0003.01

$$\mathbf{L}_\nu(0) == ¿ \;/;\; \mathrm{Re}(\nu) == -1$$

For fixed z

Explicit rational ν

03.10.03.0008.01

$$\mathbf{L}_{-\frac{11}{2}}(z) == \frac{\sqrt{\frac{2}{\pi}} \left(z\left(z^4+105 z^2+945\right)\cosh(z) - 15\left(z^4+28 z^2+63\right)\sinh(z)\right)}{z^{11/2}}$$

03.10.03.0009.01

$$\mathbf{L}_{-\frac{9}{2}}(z) = \frac{\sqrt{\frac{2}{\pi}} \left(\left(z^4 + 45 z^2 + 105\right) \sinh(z) - 5 z \left(2 z^2 + 21\right) \cosh(z)\right)}{z^{9/2}}$$

03.10.03.0010.01

$$\mathbf{L}_{-\frac{7}{2}}(z) = \frac{\sqrt{\frac{2}{\pi}} \left(z \left(z^2 + 15\right) \cosh(z) - 3 \left(2 z^2 + 5\right) \sinh(z)\right)}{z^{7/2}}$$

03.10.03.0011.01

$$\mathbf{L}_{-\frac{5}{2}}(z) = \frac{\sqrt{\frac{2}{\pi}} \left(\left(z^2 + 3\right) \sinh(z) - 3 z \cosh(z)\right)}{z^{5/2}}$$

03.10.03.0012.01

$$\mathbf{L}_{-\frac{3}{2}}(z) = \frac{\sqrt{\frac{2}{\pi}} \left(z \cosh(z) - \sinh(z)\right)}{z^{3/2}}$$

03.10.03.0005.01

$$\mathbf{L}_{-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \sinh(z)$$

03.10.03.0004.01

$$\mathbf{L}_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \left(\cosh(z) - 1\right)$$

03.10.03.0013.01

$$\mathbf{L}_{\frac{3}{2}}(z) = -\frac{z^2 - 2 \sinh(z) z + 2 \cosh(z) - 2}{\sqrt{2 \pi}\, z^{3/2}}$$

03.10.03.0014.01

$$\mathbf{L}_{\frac{5}{2}}(z) = \frac{-z^4 + 4 z^2 - 24 \sinh(z) z + 8 \left(z^2 + 3\right) \cosh(z) - 24}{4 \sqrt{2 \pi}\, z^{5/2}}$$

03.10.03.0015.01

$$\mathbf{L}_{\frac{7}{2}}(z) = \frac{-z^6 + 6 z^4 - 72 z^2 + 48 \left(z^2 + 15\right) \sinh(z) z - 144 \left(2 z^2 + 5\right) \cosh(z) + 720}{24 \sqrt{2 \pi}\, z^{7/2}}$$

03.10.03.0016.01

$$\mathbf{L}_{\frac{9}{2}}(z) = \frac{1}{192 \sqrt{2 \pi}\, z^{9/2}} \left(-z^8 + 8 z^6 - 144 z^4 + 2880 z^2 - 1920 \left(2 z^2 + 21\right) \sinh(z) z + 384 \left(z^4 + 45 z^2 + 105\right) \cosh(z) - 40\,320\right)$$

03.10.03.0017.01

$$\mathbf{L}_{\frac{11}{2}}(z) = \frac{1}{1920 \sqrt{2 \pi}\, z^{11/2}} \left(-z^{10} + 10 z^8 - 240 z^6 + 7200 z^4 - 201\,600 z^2 + 3840 \left(z^4 + 105 z^2 + 945\right) \sinh(z) z - 57\,600 \left(z^4 + 28 z^2 + 63\right) \cosh(z) + 3\,628\,800\right)$$

Symbolic rational ν

03.10.03.0006.01

$$\mathbf{L}_\nu(z) = -\frac{1}{\sqrt{z}} e^{\frac{1}{2}\pi i\left(\nu+\frac{1}{2}\right)} \sqrt{\frac{2}{\pi}} \left(\sinh\left(\frac{1}{2} i\pi\left(\nu+\frac{1}{2}\right)-z\right) \sum_{k=0}^{\left\lfloor -\frac{1}{4}(2\nu+1)\right\rfloor} \frac{\left(2k-\nu-\frac{1}{2}\right)!}{(2k)!\left(-2k-\nu-\frac{1}{2}\right)!(2z)^{2k}} + \cosh\left(\frac{1}{2} i\pi\left(\nu+\frac{1}{2}\right)-z\right) \sum_{k=0}^{\left\lfloor -\frac{1}{4}(2\nu+3)\right\rfloor} \frac{\left(2k-\nu+\frac{1}{2}\right)!(2z)^{-2k-1}}{(2k+1)!\left(-2k-\nu-\frac{3}{2}\right)!} \right) /; -\nu-\frac{1}{2} \in \mathbb{N}$$

03.10.03.0007.01

$$\mathbf{L}_\nu(z) = -\frac{2^{1-\nu} z^{\nu-1}}{\sqrt{\pi}\left(\nu-\frac{1}{2}\right)!} \sum_{k=0}^{\nu-\frac{1}{2}} \left(\frac{1}{2}\right)_k \left(\frac{1}{2}-\nu\right)_k \left(\frac{z^2}{4}\right)^{-k} + -\frac{1}{\sqrt{z}} e^{\frac{1}{2}\pi i\left(\nu+\frac{1}{2}\right)} \sqrt{\frac{2}{\pi}} \left(\sinh\left(\frac{1}{2} i\pi\left(\nu+\frac{1}{2}\right)-z\right) \sum_{k=0}^{\left\lfloor \frac{1}{4}(2|\nu|-1)\right\rfloor} \frac{\left(2k+|\nu|-\frac{1}{2}\right)!}{(2k)!\left(|\nu|-2k-\frac{1}{2}\right)!(2z)^{2k}} + \cosh\left(\frac{1}{2} i\pi\left(\nu+\frac{1}{2}\right)-z\right) \sum_{k=0}^{\left\lfloor \frac{1}{4}(2|\nu|-3)\right\rfloor} \frac{\left(2k+|\nu|+\frac{1}{2}\right)!(2z)^{-2k-1}}{(2k+1)!\left(|\nu|-2k-\frac{3}{2}\right)!} \right) /; \nu-\frac{1}{2} \in \mathbb{Z}$$

Values at fixed points

03.10.03.0018.01

$$\mathbf{L}_{-1}(0) = \frac{2}{\pi}$$

General characteristics

Domain and analyticity

$\mathbf{L}_\nu(z)$ is an analytical function of ν and z which is defined over $\mathbb{C}^2$.

03.10.04.0001.01

$$(\nu * z) \longrightarrow \mathbf{L}_\nu(z) :: (\mathbb{C} \otimes \mathbb{C}) \longrightarrow \mathbb{C}$$

Symmetries and periodicities

Parity

03.10.04.0002.01

$$\mathbf{L}_\nu(-z) = -(-z)^\nu z^{-\nu} \mathbf{L}_\nu(z)$$

Mirror symmetry

03.10.04.0003.02

$$\mathbf{L}_{\bar{\nu}}(\bar{z}) = \overline{\mathbf{L}_\nu(z)} /; z \notin (-\infty, 0)$$

Periodicity

No periodicity

Poles and essential singularities

With respect to z

For fixed ν, the function $\boldsymbol{L}_\nu(z)$ has an essential singularity at $z = \tilde{\infty}$. At the same time, the point $z = \tilde{\infty}$ is a branch point for generic ν.

03.10.04.0004.01

$$\mathcal{Sing}_z(\boldsymbol{L}_\nu(z)) = \{\{\tilde{\infty}.\infty\}\}$$

With respect to ν

For fixed z, the function $\boldsymbol{L}_\nu(z)$ has only one singular point at $\nu = \tilde{\infty}$. It is an essential singular point.

03.10.04.0005.01

$$\mathcal{Sing}_\nu(\boldsymbol{L}_\nu(z)) = \{\{\tilde{\infty}, \infty\}\}$$

Branch points

With respect to z

For fixed noninteger ν, the function $\boldsymbol{L}_\nu(z)$ has two branch points: $z = 0$, $z = \tilde{\infty}$. At the same time, the point $z = \tilde{\infty}$ is an essential singularity.

For integer ν, the function $\boldsymbol{L}_\nu(z)$ does not have branch points.

03.10.04.0006.01

$$\mathcal{BP}_z(\boldsymbol{L}_\nu(z)) = \{0, \tilde{\infty}\} /; \nu \notin \mathbb{Z}$$

03.10.04.0007.01

$$\mathcal{BP}_z(\boldsymbol{L}_\nu(z)) = \{\} /; \nu \in \mathbb{Z}$$

03.10.04.0008.01

$$\mathcal{R}_z(\boldsymbol{L}_\nu(z), 0) = \log /; \nu \notin \mathbb{Q}$$

03.10.04.0009.01

$$\mathcal{R}_z\left(\boldsymbol{L}_{\frac{p}{q}}(z), 0\right) = q /; p \in \mathbb{Z} \bigwedge q - 1 \in \mathbb{N}^+ \bigwedge \gcd(p, q) = 1$$

03.10.04.0010.01

$$\mathcal{R}_z(\boldsymbol{L}_\nu(z), \tilde{\infty}) = \log /; \nu \notin \mathbb{Q}$$

03.10.04.0011.01

$$\mathcal{R}_z\left(\boldsymbol{L}_{\frac{p}{q}}(z), \tilde{\infty}\right) = q /; p \in \mathbb{Z} \bigwedge q - 1 \in \mathbb{N}^+ \bigwedge \gcd(p, q) = 1$$

With respect to ν

For fixed z, the function $\boldsymbol{L}_\nu(z)$ does not have branch points.

03.10.04.0012.01

$$\mathcal{BP}_\nu(\boldsymbol{L}_\nu(z)) = \{\}$$

Branch cuts

With respect to z

When ν is an integer, $\boldsymbol{L}_\nu(z)$ is an entire function of z. For fixed noninteger ν, it has one infinitely long branch cut. For fixed noninteger ν, the function $\boldsymbol{L}_\nu(z)$ is a single-valued function on the z-plane cut along the interval $(-\infty, 0)$, where it is continuous from above.

03.10.04.0013.01

$$\mathcal{BC}_z(\boldsymbol{L}_\nu(z)) = \{\{(-\infty, 0), -i\}\} /; \nu \notin \mathbb{Z}$$

03.10.04.0014.01

$$\mathcal{BC}_z(\boldsymbol{L}_\nu(z)) = \{\} /; \nu \in \mathbb{Z}$$

03.10.04.0015.01

$$\lim_{\epsilon \to +0} \boldsymbol{L}_\nu(x + i\,\epsilon) = \boldsymbol{L}_\nu(x) /; x < 0$$

03.10.04.0016.01

$$\lim_{\epsilon \to +0} \boldsymbol{L}_\nu(x - i\,\epsilon) = -e^{-i\,\pi\,\nu}\,\boldsymbol{L}_\nu(-x) /; x < 0$$

With respect to ν

For fixed z, the function $\boldsymbol{L}_\nu(z)$ is an entire function of ν and does not have branch cuts.

03.10.04.0017.01

$$\mathcal{BC}_\nu(\boldsymbol{L}_\nu(z)) = \{\}$$

Series representations

Generalized power series

Expansions at generic point $z = z_0$

For the function itself

03.10.06.0017.01

$$\boldsymbol{L}_\nu(z) \propto \left(\frac{1}{z_0}\right)^{\nu \left\lfloor \frac{\arg(z-z_0)}{2\pi} \right\rfloor} z_0^{\nu \left\lfloor \frac{\arg(z-z_0)}{2\pi} \right\rfloor} \left(\boldsymbol{L}_\nu(z_0) + \left(\boldsymbol{L}_{\nu-1}(z_0) - \frac{\nu\,\boldsymbol{L}_\nu(z_0)}{z_0}\right)(z - z_0) + \frac{1}{2\,z_0^2}\left(z_0\left(\frac{2^{1-\nu}\,z_0^\nu}{\sqrt{\pi}\,\Gamma\left(\nu + \frac{1}{2}\right)} - \boldsymbol{L}_{\nu-1}(z_0)\right) + \boldsymbol{L}_\nu(z_0)\left(\nu^2 + \nu + z_0^2\right)\right)(z - z_0)^2 + \ldots\right) /; (z \to z_0)$$

03.10.06.0018.01

$$\boldsymbol{L}_\nu(z) \propto \left(\frac{1}{z_0}\right)^{\nu \left\lfloor \frac{\arg(z-z_0)}{2\pi} \right\rfloor} z_0^{\nu \left\lfloor \frac{\arg(z-z_0)}{2\pi} \right\rfloor} \left(\boldsymbol{L}_\nu(z_0) + \left(\boldsymbol{L}_{\nu-1}(z_0) - \frac{\nu\,\boldsymbol{L}_\nu(z_0)}{z_0}\right)(z - z_0) + \frac{1}{2\,z_0^2}\left(z_0\left(\frac{2^{1-\nu}\,z_0^\nu}{\sqrt{\pi}\,\Gamma\left(\nu + \frac{1}{2}\right)} - \boldsymbol{L}_{\nu-1}(z_0)\right) + \boldsymbol{L}_\nu(z_0)\left(\nu^2 + \nu + z_0^2\right)\right)(z - z_0)^2 + O\left((z - z_0)^3\right)\right)$$

03.10.06.0019.01

$$\boldsymbol{L}_{\nu}(z) \propto \sqrt{\pi}\, \Gamma(\nu+2) \left(\frac{z_0}{4}\right)^{\nu+1} \left(\frac{1}{z_0}\right)^{\nu \left\lfloor \frac{\arg(z-z_0)}{2\pi} \right\rfloor} z_0^{\nu \left\lfloor \frac{\arg(z-z_0)}{2\pi} \right\rfloor} \sum_{k=0}^{\infty} \frac{2^k z_0^{-k}}{k!} \, {}_3\tilde{F}_4\left(1, \frac{\nu}{2}+1, \frac{\nu+3}{2}; \frac{3}{2}, \frac{\nu-k}{2}+1, \frac{1}{2}(-k+\nu+3), \nu+\frac{3}{2}; \frac{z_0^2}{4}\right) (z-z_0)^k$$

03.10.06.0020.01

$$\boldsymbol{L}_{\nu}(z) \propto \left(\frac{1}{z_0}\right)^{\nu \left\lfloor \frac{\arg(z-z_0)}{2\pi} \right\rfloor} z_0^{\nu \left\lfloor \frac{\arg(z-z_0)}{2\pi} \right\rfloor} \sum_{k=0}^{\infty} \left(\frac{1}{k!} z_0^{-k} \sum_{m=0}^{k} (-1)^{k+m} \binom{k}{m} (-\nu)_{k-m} \sum_{p=0}^{m} \frac{(-1)^{p-1} 2^{2p-m} (-m)_{2(m-p)} (\nu)_p}{(m-p)!} \left(\frac{1}{2} z_0 \sum_{j=0}^{p-1} \frac{(-j+p-1)!}{j!\,(-2j+p-1)!\,(-p-\nu+1)_j\,(\nu)_{j+1}} \left(-\frac{z_0^2}{4}\right)^j \boldsymbol{L}_{\nu-1}(z_0) - \sum_{j=0}^{p} \frac{(p-j)!}{j!\,(p-2j)!\,(-p-\nu+1)_j\,(\nu)_j} \left(-\frac{z_0^2}{4}\right)^j \boldsymbol{L}_{\nu}(z_0) \right) + \frac{2^{-\nu} z_0^{-k+\nu+1}}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right) k!} \sum_{i=1}^{k-1} \sum_{m=0}^{i} (-1)^{i+m} \binom{i}{m} (-\nu)_{i-m} \sum_{p=0}^{m} \frac{(-1)^{p-1} 2^{2p-m} (-m)_{2(m-p)} (\nu)_p}{(m-p)!} \sum_{j=0}^{p-1} \frac{(-1)^j 2^{-2j} (-j+p-1)!\,(2j-k+\nu+2)_{-i+k-1} z_0^{2j}}{j!\,(-2j+p-1)!\,(-p-\nu+1)_j\,(\nu)_{j+1}} \right) (z-z_0)^k$$

03.10.06.0021.01

$$\boldsymbol{L}_{\nu}(z) \propto \left(\frac{1}{z_0}\right)^{\nu \left\lfloor \frac{\arg(z-z_0)}{2\pi} \right\rfloor} z_0^{\nu \left\lfloor \frac{\arg(z-z_0)}{2\pi} \right\rfloor} \boldsymbol{L}_{\nu}(z_0)\,(1+O(z-z_0))$$

Expansions on branch cuts

For the function itself

03.10.06.0022.01

$$\boldsymbol{L}_{\nu}(z) \propto e^{2\nu\pi i \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} \left(\boldsymbol{L}_{\nu}(x) + \left(\boldsymbol{L}_{\nu-1}(x) - \frac{\nu \boldsymbol{L}_{\nu}(x)}{x} \right)(z-x) + \frac{1}{2x^2} \left(x \left(\frac{2^{1-\nu} x^{\nu}}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)} - \boldsymbol{L}_{\nu-1}(x) \right) + \left(x^2+\nu^2+\nu\right) \boldsymbol{L}_{\nu}(x) \right)(z-x)^2 + \ldots \right) /; (z \to x) \land x \in \mathbb{R} \land x < 0$$

03.10.06.0023.01

$$\boldsymbol{L}_{\nu}(z) \propto e^{2\nu\pi i \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} \left(\boldsymbol{L}_{\nu}(x) + \left(\boldsymbol{L}_{\nu-1}(x) - \frac{\nu \boldsymbol{L}_{\nu}(x)}{x} \right)(z-x) + \frac{1}{2x^2} \left(x \left(\frac{2^{1-\nu} x^{\nu}}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)} - \boldsymbol{L}_{\nu-1}(x) \right) + \left(x^2+\nu^2+\nu\right) \boldsymbol{L}_{\nu}(x) \right)(z-x)^2 + O\left((z-x)^3\right) \right) /; x \in \mathbb{R} \land x < 0$$

03.10.06.0024.01

$$\mathbf{L}_\nu(z) == \sqrt{\pi}\ \Gamma(\nu+2)\left(\frac{x}{4}\right)^{\nu+1} e^{2\nu\pi i\left\lfloor\frac{\arg(z-x)}{2\pi}\right\rfloor}\sum_{k=0}^{\infty}\frac{2^k x^{-k}}{k!}\,{}_3\tilde{F}_4\left(1,\frac{\nu}{2}+1,\frac{\nu+3}{2};\frac{3}{2},\frac{\nu-k}{2}+1,\frac{1}{2}(-k+\nu+3),\nu+\frac{3}{2};\frac{x^2}{4}\right)(z-x)^k\ /;\ x\in\mathbb{R}\land x<0$$

03.10.06.0025.01

$$\mathbf{L}_\nu(z) == e^{2\pi i\nu\left\lfloor\frac{\arg(z-x)}{2\pi}\right\rfloor}\sum_{k=0}^{\infty}\left(\frac{1}{k!}x^{-k}\sum_{m=0}^{k}(-1)^{k+m}\binom{k}{m}(-\nu)_{k-m}\sum_{p=0}^{m}\frac{(-1)^{p-1}2^{2p-m}(-m)_{2(m-p)}(\nu)_p}{(m-p)!}\left(\frac{1}{2}x\sum_{j=0}^{p-1}\frac{(-j+p-1)!}{j!(-2j+p-1)!(-p-\nu+1)_j(\nu)_{j+1}}\left(-\frac{x^2}{4}\right)^j\mathbf{L}_{\nu-1}(x)-\sum_{j=0}^{p}\frac{(p-j)!}{j!(p-2j)!(-p-\nu+1)_j(\nu)_j}\left(-\frac{x^2}{4}\right)^j\mathbf{L}_\nu(x)\right)+\frac{2^{-\nu}x^{-k+\nu+1}}{\sqrt{\pi}\ \Gamma\left(\nu+\frac{1}{2}\right)k!}\sum_{i=1}^{k-1}\sum_{m=0}^{i}(-1)^{i+m}\binom{i}{m}(-\nu)_{i-m}\sum_{p=0}^{m}\frac{(-1)^{p-1}2^{2p-m}(-m)_{2(m-p)}(\nu)_p}{(m-p)!}\sum_{j=0}^{p-1}\frac{(-1)^j2^{-2j}(-j+p-1)!(2j-k+\nu+2)_{-i+k-1}z^{2j}}{j!(-2j+p-1)!(-p-\nu+1)_j(\nu)_{j+1}}\right)(z-x)^k\ /;\ x\in\mathbb{R}\land x<0$$

03.10.06.0026.01

$$\mathbf{L}_\nu(z) \propto e^{2\nu\pi i\left\lfloor\frac{\arg(z-x)}{2\pi}\right\rfloor}\mathbf{L}_\nu(x)\ (1+O(z-x))\ /;\ x\in\mathbb{R}\land x<0$$

Expansions at $z == 0$

For the function itself

General case

03.10.06.0001.02

$$\mathbf{L}_\nu(z) \propto \frac{2^{-\nu}z^{\nu+1}}{\sqrt{\pi}\ \Gamma\left(\nu+\frac{3}{2}\right)}\left(1+\frac{z^2}{3(2\nu+3)}+\frac{z^4}{15(2\nu+3)(2\nu+5)}+\ldots\right)/;\ (z\to 0)$$

03.10.06.0027.01

$$\mathbf{L}_\nu(z) \propto \frac{2^{-\nu}z^{\nu+1}}{\sqrt{\pi}\ \Gamma\left(\nu+\frac{3}{2}\right)}\left(1+\frac{z^2}{3(2\nu+3)}+\frac{z^4}{15(2\nu+3)(2\nu+5)}+O\left(z^6\right)\right)$$

03.10.06.0002.01

$$\mathbf{L}_\nu(z) == \left(\frac{z}{2}\right)^{\nu+1}\sum_{k=0}^{\infty}\frac{1}{\Gamma\left(k+\frac{3}{2}\right)\Gamma\left(k+\nu+\frac{3}{2}\right)}\left(\frac{z}{2}\right)^{2k}$$

03.10.06.0028.01

$$\mathbf{L}_\nu(z) == \frac{2}{\sqrt{\pi}\ \Gamma\left(\nu+\frac{3}{2}\right)}\left(\frac{z}{2}\right)^{\nu+1}\sum_{k=0}^{\infty}\frac{z^{2k}}{4^k\left(\frac{3}{2}\right)_k\left(\nu+\frac{3}{2}\right)_k}$$

03.10.06.0029.01

$$\mathbf{L}_\nu(z) == \frac{2}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{3}{2}\right)}\left(\frac{z}{2}\right)^{\nu+1} {}_1F_2\left(1; \frac{3}{2}, \nu+\frac{3}{2}; \frac{z^2}{4}\right)$$

03.10.06.0003.01

$$\mathbf{L}_\nu(z) == \left(\frac{z}{2}\right)^{\nu+1} {}_1\tilde{F}_2\left(1; \frac{3}{2}, \nu+\frac{3}{2}; \frac{z^2}{4}\right)$$

03.10.06.0004.02

$$\mathbf{L}_\nu(z) \propto \frac{2^{-\nu} z^{\nu+1}}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{3}{2}\right)} + O\left(z^{\nu+3}\right) /; -\nu-\frac{3}{2} \notin \mathbb{N}$$

03.10.06.0030.01

$$\mathbf{L}_\nu(z) == F_\infty(z, \nu) /; \left(\left(F_n(z, \nu) == \left(\frac{z}{2}\right)^{\nu+1} \sum_{k=0}^{n} \frac{\left(\frac{z}{2}\right)^{2k}}{\Gamma\left(k+\frac{3}{2}\right)\Gamma\left(k+\nu+\frac{3}{2}\right)} == \mathbf{L}_\nu(z) - \frac{1}{\Gamma\left(n+\frac{5}{2}\right)\Gamma\left(n+\nu+\frac{5}{2}\right)}\left(\frac{z}{2}\right)^{2n+\nu+3} {}_1F_2\left(1; n+\frac{5}{2}, n+\nu+\frac{5}{2}; \frac{z^2}{4}\right)\right) \bigwedge n \in \mathbb{N}\right)$$

Summed form of the truncated series expansion.

Special cases

03.10.06.0031.01

$$\mathbf{L}_\nu(z) \propto \frac{1}{\Gamma(1-\nu)}\left(\frac{z}{2}\right)^{-\nu}\left(1+\frac{z^2}{4(1-\nu)}+\frac{z^4}{32(1-\nu)(2-\nu)}+\ldots\right) /; (z \to 0) \bigwedge -\nu-\frac{3}{2} \in \mathbb{N}^+$$

03.10.06.0032.01

$$\mathbf{L}_\nu(z) \propto \frac{1}{\Gamma(1-\nu)}\left(\frac{z}{2}\right)^{-\nu}\left(1+\frac{z^2}{4(1-\nu)}+\frac{z^4}{32(1-\nu)(2-\nu)}+O\left(z^6\right)\right) /; -\nu-\frac{3}{2} \in \mathbb{N}^+$$

03.10.06.0033.01

$$\mathbf{L}_\nu(z) == \sum_{k=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2k-\nu}}{\Gamma(k-\nu+1)\,k!} /; -\nu-\frac{3}{2} \in \mathbb{N}$$

03.10.06.0034.01

$$\mathbf{L}_\nu(z) == \frac{1}{\Gamma(1-\nu)}\left(\frac{z}{2}\right)^{-\nu} \sum_{k=0}^{\infty} \frac{z^{2k}}{4^k (1-\nu)_k\, k!} /; -\nu-\frac{3}{2} \in \mathbb{N}$$

03.10.06.0035.01

$$\mathbf{L}_\nu(z) == \frac{1}{\Gamma(1-\nu)}\left(\frac{z}{2}\right)^{-\nu} {}_0F_1\left(; 1-\nu; \frac{z^2}{4}\right) /; -\nu-\frac{3}{2} \in \mathbb{N}$$

03.10.06.0036.01

$$\mathbf{L}_\nu(z) == \left(\frac{z}{2}\right)^{-\nu} {}_0\tilde{F}_1\left(; 1-\nu; \frac{z^2}{4}\right) /; -\nu-\frac{3}{2} \in \mathbb{N}$$

03.10.06.0037.01

$$\mathbf{L}_\nu(z) = I_{-\nu}(z) \;/;\; -\nu - \frac{3}{2} \in \mathbb{N}$$

03.10.06.0005.02

$$\mathbf{L}_\nu(z) \propto \frac{2^\nu z^{-\nu}}{\text{Gamma}[1-\nu]} + O\left(z^{2-\nu}\right) \;/;\; -\nu - \frac{3}{2} \in \mathbb{N}$$

Asymptotic series expansions

Expansions inside Stokes sectors

Expansions containing $z \to \infty$

In exponential form ||| In exponential form

03.10.06.0038.01

$$\mathbf{L}_\nu(z) \propto \frac{1}{\sqrt{2\pi z}}\left(e^z\left(1 + \frac{1-4\nu^2}{8z} + \frac{9-40\nu^2+16\nu^4}{128z^2} + \ldots\right) + e^{-z-i\pi\nu}\, i\left(1 - \frac{1-4\nu^2}{8z} + \frac{9-40\nu^2+16\nu^4}{128z^2} + \ldots\right)\right) - \frac{2^{1-\nu} z^{\nu-1}}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)}\left(1 + \frac{1-2\nu}{z^2} + \frac{3\left(3-8\nu+4\nu^2\right)}{z^4} + \ldots\right) \;/;\; -\pi < \arg(z) < \frac{\pi}{2} \bigwedge (|z| \to \infty)$$

03.10.06.0039.01

$$\mathbf{L}_\nu(z) \propto \frac{1}{\sqrt{2\pi z}}\left(e^z\left(\sum_{k=0}^{n} \frac{\left(\nu+\frac{1}{2}\right)_k \left(\frac{1}{2}-\nu\right)_k}{k!}\left(\frac{1}{2z}\right)^k + O\left(\frac{1}{z^{n+1}}\right)\right) + e^{-z-i\pi\nu}\, i\left(\sum_{k=0}^{n} \frac{\left(\nu+\frac{1}{2}\right)_k \left(\frac{1}{2}-\nu\right)_k}{k!}\left(-\frac{1}{2z}\right)^k + O\left(\frac{1}{z^{n+1}}\right)\right)\right) - \frac{2^{1-\nu} z^{\nu-1}}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)}\left(\sum_{k=0}^{n}\left(\frac{1}{2}\right)_k\left(\frac{1}{2}-\nu\right)_k\left(\frac{4}{z^2}\right)^k + O\left(\frac{1}{z^{2n+2}}\right)\right) \;/;\; -\pi < \arg(z) < \frac{\pi}{2} \bigwedge (|z| \to \infty)$$

03.10.06.0040.01

$$\mathbf{L}_\nu(z) \propto \frac{1}{\sqrt{2\pi z}}\left(e^z\, {}_2F_0\left(\nu+\frac{1}{2}, \frac{1}{2}-\nu;\,;\, \frac{1}{2z}\right) + e^{-z-i\pi\nu}\, i\, {}_2F_0\left(\nu+\frac{1}{2}, \frac{1}{2}-\nu;\,;\, -\frac{1}{2z}\right)\right) - \frac{2^{1-\nu} z^{\nu-1}}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)}\, {}_3F_0\left(1, \frac{1}{2}, \frac{1}{2}-\nu;\,;\, \frac{4}{z^2}\right) \;/;\; -\pi < \arg(z) < \frac{\pi}{2} \bigwedge (|z| \to \infty)$$

03.10.06.0041.01

$$\mathbf{L}_\nu(z) \propto \frac{1}{\sqrt{2\pi z}}\left(e^z\left(1 + O\left(\frac{1}{z}\right)\right) + e^{-z-i\pi\nu}\, i\left(1 + O\left(\frac{1}{z}\right)\right)\right) - \frac{2^{1-\nu} z^{\nu-1}}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)}\left(1 + O\left(\frac{1}{z^2}\right)\right) \;/;\; -\pi < \arg(z) < \frac{\pi}{2} \bigwedge (|z| \to \infty)$$

03.10.06.0006.02

$$\mathbf{L}_\nu(z) \propto \frac{e^z}{\sqrt{2\pi z}}\left(1 + O\left(\frac{1}{z}\right)\right) - \frac{2^{1-\nu} z^{\nu-1}}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)}\left(1 + O\left(\frac{1}{z^2}\right)\right) \;/;\; \text{Re}(z) \geq 0 \land (|z| \to \infty)$$

In hyperbolic form ||| In hyperbolic form

03.10.06.0042.01

$$\mathbf{L}_\nu(z) \propto \sqrt{\frac{2}{\pi}}\, z^{\nu+1} \left(-z^2\right)^{-\frac{2\nu+3}{4}} \left(\sin\!\left(\sqrt{-z^2} - \frac{(2\nu+1)\pi}{4}\right) \left(1 + \frac{16\nu^4 - 40\nu^2 + 9}{128 z^2} + \frac{256\nu^8 - 5376\nu^6 + 31584\nu^4 - 51664\nu^2 + 11025}{98304 z^4} + \ldots\right) + \frac{4\nu^2 - 1}{8\sqrt{-z^2}} \cos\!\left(\sqrt{-z^2} - \frac{(2\nu+1)\pi}{4}\right) \left(1 + \frac{16\nu^4 - 136\nu^2 + 225}{384 z^2} + \frac{256\nu^8 - 10496\nu^6 + 137824\nu^4 - 656784\nu^2 + 893025}{491520 z^4} + \ldots\right)\right) - \frac{2^{1-\nu} z^{\nu-1}}{\sqrt{\pi}\,\Gamma\!\left(\nu + \frac{1}{2}\right)} \left(1 - \frac{2\nu - 1}{z^2} + \frac{3\left(4\nu^2 - 8\nu + 3\right)}{z^4} + \ldots\right) /; -\pi < \arg(z) < \frac{\pi}{2} \bigwedge (|z| \to \infty)$$

03.10.06.0043.01

$$\mathbf{L}_\nu(z) \propto \frac{\sqrt{2}\, e^{-\frac{\pi i}{4}(1+2\nu)}}{\sqrt{\pi}\,\sqrt{z}} \left(\sinh\!\left(z + \frac{\pi i}{4}(2\nu+1)\right) \left(\sum_{k=0}^{n} \frac{\left(\frac{1}{4}(1-2\nu)\right)_k \left(\frac{1}{4}(3-2\nu)\right)_k \left(\frac{1}{4}(2\nu+1)\right)_k \left(\frac{1}{4}(2\nu+3)\right)_k}{\left(\frac{1}{2}\right)_k k!} \left(\frac{1}{z^2}\right)^k + O\!\left(\frac{1}{z^{2n+2}}\right)\right) + \frac{1 - 4\nu^2}{8z} \cosh\!\left(z + \frac{\pi i}{4}(2\nu+1)\right) \left(\sum_{k=0}^{n} \frac{\left(\frac{1}{4}(3-2\nu)\right)_k \left(\frac{1}{4}(5-2\nu)\right)_k \left(\frac{1}{4}(2\nu+3)\right)_k \left(\frac{1}{4}(2\nu+5)\right)_k}{\left(\frac{3}{2}\right)_k k!} \left(\frac{1}{z^2}\right)^k + O\!\left(\frac{1}{z^{2n+2}}\right)\right)\right) - \frac{2^{1-\nu} z^{\nu-1}}{\sqrt{\pi}\,\Gamma\!\left(\nu + \frac{1}{2}\right)} \left(\sum_{k=0}^{n} \left(\frac{1}{2}\right)_k \left(\frac{1}{2} - \nu\right)_k \left(\frac{4}{z^2}\right)^k + O\!\left(\frac{1}{z^{2n+2}}\right)\right) /; -\pi < \arg(z) < \frac{\pi}{2} \bigwedge (|z| \to \infty)$$

03.10.06.0044.01

$$\mathbf{L}_\nu(z) \propto \frac{\sqrt{2}\, e^{-\frac{\pi i}{4}(1+2\nu)}}{\sqrt{\pi}\,\sqrt{z}} \left(\sinh\!\left(z + \frac{\pi i}{4}(2\nu+1)\right) {}_4F_1\!\left(\frac{1-2\nu}{4}, \frac{3-2\nu}{4}, \frac{2\nu+1}{4}, \frac{2\nu+3}{4}; \frac{1}{2}; \frac{1}{z^2}\right) + \frac{1 - 4\nu^2}{8z} \cosh\!\left(z + \frac{\pi i}{4}(2\nu+1)\right) {}_4F_1\!\left(\frac{3-2\nu}{4}, \frac{5-2\nu}{4}, \frac{3+2\nu}{4}, \frac{5+2\nu}{4}; \frac{3}{2}; \frac{1}{z^2}\right)\right) - \frac{2^{1-\nu} z^{\nu-1}}{\sqrt{\pi}\,\Gamma\!\left(\nu + \frac{1}{2}\right)} {}_3F_0\!\left(\frac{1}{2}, 1, \frac{1}{2} - \nu;\, ; \frac{4}{z^2}\right) /; -\pi < \arg(z) < \frac{\pi}{2} \bigwedge (|z| \to \infty)$$

03.10.06.0045.01

$$\mathbf{L}_\nu(z) \propto \frac{\sqrt{2}\, e^{-\frac{\pi i}{4}(1+2\nu)}}{\sqrt{\pi}\,\sqrt{z}} \left(\sinh\!\left(z + \frac{\pi i}{4}(2\nu+1)\right) \left(1 + O\!\left(\frac{1}{z^2}\right)\right) + \frac{1 - 4\nu^2}{8z} \cosh\!\left(z + \frac{\pi i}{4}(2\nu+1)\right) \left(1 + O\!\left(\frac{1}{z^2}\right)\right)\right) - \frac{2^{1-\nu} z^{\nu-1}}{\sqrt{\pi}\,\Gamma\!\left(\nu + \frac{1}{2}\right)} \left(1 + O\!\left(\frac{1}{z^2}\right)\right) /; -\pi < \arg(z) < \frac{\pi}{2} \bigwedge (|z| \to \infty)$$

Containing Bessel functions

03.10.06.0046.01

$$\mathbf{L}_\nu(z) - \frac{z}{\sqrt{z^2}} I_\nu(z) \propto -\frac{2^{1-\nu} z^{\nu-1}}{\sqrt{\pi}\, \Gamma\left(\nu+\frac{1}{2}\right)} \left(1 - \frac{2\nu-1}{z^2} + \frac{3(4\nu^2-8\nu+3)}{z^4} + \ldots\right) /; |\arg(z)| \neq \frac{\pi}{2} \bigwedge (|z| \to \infty)$$

03.10.06.0047.01

$$\mathbf{L}_\nu(z) - \frac{z}{\sqrt{z^2}} I_\nu(z) \propto -\frac{2^{1-\nu} z^{\nu-1}}{\sqrt{\pi}\, \Gamma\left(\nu+\frac{1}{2}\right)} \left(\sum_{k=0}^{n} \left(\frac{1}{2}\right)_k \left(\frac{1}{2}-\nu\right)_k \left(\frac{4}{z^2}\right)^k + O\left(\frac{1}{z^{2n+2}}\right)\right) /; |\arg(z)| \neq \frac{\pi}{2} \bigwedge (|z| \to \infty)$$

03.10.06.0048.01

$$\mathbf{L}_\nu(z) - \frac{z}{\sqrt{z^2}} I_\nu(z) \propto -\frac{2^{1-\nu} z^{\nu-1}}{\sqrt{\pi}\, \Gamma\left(\nu+\frac{1}{2}\right)} \, {}_3F_0\left(\frac{1}{2}, \frac{1}{2}-\nu, 1;; -\frac{4}{z^2}\right) /; |\arg(z)| \neq \frac{\pi}{2} \bigwedge (|z| \to \infty)$$

03.10.06.0049.01

$$\mathbf{L}_\nu(z) - \frac{z}{\sqrt{z^2}} I_\nu(z) \propto -\frac{2^{1-\nu} z^{\nu-1}}{\sqrt{\pi}\, \Gamma\left(\nu+\frac{1}{2}\right)} \left(1 + O\left(\frac{1}{z^2}\right)\right) /; |\arg(z)| \neq \frac{\pi}{2} \bigwedge (|z| \to \infty)$$

Expansions containing $z \to -\infty$

In exponential form ||| In exponential form

03.10.06.0050.01

$$\mathbf{L}_\nu(z) \propto -\frac{i}{\sqrt{-2\pi z}} \left(e^z \left(1 + \frac{1-4\nu^2}{8z} + \frac{16\nu^4-40\nu^2+9}{128z^2} + \ldots\right) - i\, e^{-z+i\pi\nu} \left(1 - \frac{1-4\nu^2}{8z} + \frac{16\nu^4-40\nu^2+9}{128z^2} + \ldots\right)\right) - \frac{2^{1-\nu} z^{\nu-1}}{\sqrt{\pi}\, \Gamma\left(\nu+\frac{1}{2}\right)} \left(1 - \frac{2\nu-1}{z^2} + \frac{3(4\nu^2-8\nu+3)}{z^4} + \ldots\right) /; 0 < \arg(z) \leq \pi \bigwedge (|z| \to \infty)$$

03.10.06.0051.01

$$\mathbf{L}_\nu(z) \propto -\frac{i}{\sqrt{-2\pi z}} \left(e^z \left(\sum_{k=0}^{n} \frac{\left(\nu+\frac{1}{2}\right)_k \left(\frac{1}{2}-\nu\right)_k}{k!} \left(\frac{1}{2z}\right)^k + O\left(\frac{1}{z^{n+1}}\right)\right) - i\, e^{-z+i\pi\nu} \left(\sum_{k=0}^{n} \frac{\left(\nu+\frac{1}{2}\right)_k \left(\frac{1}{2}-\nu\right)_k}{k!} \left(-\frac{1}{2z}\right)^k + O\left(\frac{1}{z^{n+1}}\right)\right)\right) - \frac{2^{1-\nu} z^{\nu-1}}{\sqrt{\pi}\, \Gamma\left(\nu+\frac{1}{2}\right)} \left(\sum_{k=0}^{n} \left(\frac{1}{2}\right)_k \left(\frac{1}{2}-\nu\right)_k \left(\frac{4}{z^2}\right)^k + O\left(\frac{1}{z^{2n+2}}\right)\right) /; 0 < \arg(z) \leq \pi \bigwedge (|z| \to \infty)$$

03.10.06.0052.01

$$\mathbf{L}_\nu(z) \propto -\frac{i}{\sqrt{-2\pi z}} \left(e^z \, {}_2F_0\left(\nu+\frac{1}{2}, \frac{1}{2}-\nu;; \frac{1}{2z}\right) - i\, e^{-z+i\pi\nu} \, {}_2F_0\left(\nu+\frac{1}{2}, \frac{1}{2}-\nu;; -\frac{1}{2z}\right)\right) - \frac{2^{1-\nu} z^{\nu-1}}{\sqrt{\pi}\, \Gamma\left(\nu+\frac{1}{2}\right)} \, {}_3F_0\left(\frac{1}{2}, \frac{1}{2}-\nu, 1;; \frac{4}{z^2}\right) /; 0 < \arg(z) \leq \pi \bigwedge (|z| \to \infty)$$

03.10.06.0053.01

$$\mathbf{L}_\nu(z) \propto -\frac{i}{\sqrt{-2\pi z}} \left(e^z \left(1 + O\left(\frac{1}{z}\right)\right) - i\, e^{i\pi\nu - z} \left(1 + O\left(\frac{1}{z}\right)\right)\right) - \frac{2^{1-\nu} z^{\nu-1}}{\sqrt{\pi}\, \Gamma\left(\nu+\frac{1}{2}\right)} \left(1 + O\left(\frac{1}{z^2}\right)\right) /; 0 < \arg(z) \leq \pi \bigwedge (|z| \to \infty)$$

In hyperbolic form ||| In hyperbolic form

03.10.06.0054.01

$$\mathbf{L}_\nu(z) \propto \sqrt{-\frac{2}{\pi z}}\, e^{\frac{i\pi}{4}(2\nu-1)} \left(\sinh\left(z - \frac{i\pi}{4}(1+2\nu)\right) \left(1 + \frac{16\nu^4 - 40\nu^2 + 9}{128 z^2} + \frac{256\nu^8 - 5376\nu^6 + 31584\nu^4 - 51664\nu^2 + 11025}{98304 z^4} + \ldots\right) + \frac{1-4\nu^2}{8z} \cosh\left(z - \frac{i\pi}{4}(1+2\nu)\right) \left(1 + \frac{16\nu^4 - 136\nu^2 + 225}{384 z^2} + \frac{256\nu^8 - 10496\nu^6 + 137824\nu^4 - 656784\nu^2 + 893025}{491520 z^4} + \ldots\right)\right) - \frac{2^{1-\nu} z^{\nu-1}}{\sqrt{\pi}\, \Gamma\left(\nu+\frac{1}{2}\right)} \left(1 - \frac{2\nu-1}{z^2} + \frac{3\left(4\nu^2 - 8\nu + 3\right)}{z^4} + \ldots\right) /; \operatorname{Im}(z) \geq 0 \land (|z| \to \infty)$$

03.10.06.0055.01

$$\mathbf{L}_\nu(z) \propto \sqrt{-\frac{2}{\pi z}}\, e^{\frac{\pi i}{4}(2\nu-1)} \left(\sinh\left(z - \frac{\pi i}{4}(2\nu+1)\right) \left(\sum_{k=0}^{n} \frac{\left(\frac{1}{4}(1-2\nu)\right)_k \left(\frac{1}{4}(3-2\nu)\right)_k \left(\frac{1}{4}(2\nu+1)\right)_k \left(\frac{1}{4}(2\nu+3)\right)_k}{\left(\frac{1}{2}\right)_k k!} \left(\frac{1}{z^2}\right)^k + O\left(\frac{1}{z^{2n+2}}\right)\right) + \frac{1-4\nu^2}{8z} \cosh\left(z - \frac{\pi i}{4}(2\nu+1)\right) \left(\sum_{k=0}^{n} \frac{\left(\frac{1}{4}(3-2\nu)\right)_k \left(\frac{1}{4}(5-2\nu)\right)_k \left(\frac{1}{4}(2\nu+3)\right)_k \left(\frac{1}{4}(2\nu+5)\right)_k}{\left(\frac{3}{2}\right)_k k!} \left(\frac{1}{z^2}\right)^k + O\left(\frac{1}{z^{2n+2}}\right)\right)\right) - \frac{2^{1-\nu} z^{\nu-1}}{\sqrt{\pi}\, \Gamma\left(\nu+\frac{1}{2}\right)} \left(\sum_{k=0}^{n} \left(\frac{1}{2}\right)_k \left(\frac{1}{2}-\nu\right)_k \left(\frac{4}{z^2}\right)^k + O\left(\frac{1}{z^{2n+2}}\right)\right) /; \operatorname{Im}(z) \geq 0 \land (|z| \to \infty)$$

03.10.06.0007.02

$$\mathbf{L}_\nu(z) \propto \sqrt{-\frac{2}{\pi z}}\, e^{\frac{1}{4} i\pi(2\nu-1)} \left(\sinh\left(z - \frac{2\nu+1}{4} i\pi\right) {}_4F_1\left(\frac{1}{4}(1-2\nu), \frac{1}{4}(3-2\nu), \frac{1}{4}(2\nu+1), \frac{1}{4}(2\nu+3); \frac{1}{2}; \frac{1}{z^2}\right) + \frac{1-4\nu^2}{8z} \cosh\left(z - \frac{2\nu+1}{4} i\pi\right) {}_4F_1\left(\frac{1}{4}(3-2\nu), \frac{1}{4}(5-2\nu), \frac{1}{4}(2\nu+3), \frac{1}{4}(2\nu+5); \frac{3}{2}; \frac{1}{z^2}\right)\right) - \frac{2^{1-\nu} z^{\nu-1}}{\sqrt{\pi}\, \Gamma\left(\nu+\frac{1}{2}\right)} {}_3F_0\left(\frac{1}{2}, 1, \frac{1}{2}-\nu;; \frac{4}{z^2}\right) /; \operatorname{Im}(z) \geq 0 \land (|z| \to \infty)$$

03.10.06.0008.02

$$\mathbf{L}_\nu(z) \propto \sqrt{-\frac{2}{\pi z}}\, e^{\frac{1}{4} i\pi(2\nu-1)} \left(\sinh\left(z - \frac{2\nu+1}{4} i\pi\right)\left(1 + O\left(\frac{1}{z^2}\right)\right) + \frac{1-4\nu^2}{8z} \cosh\left(z - \frac{2\nu+1}{4} i\pi\right)\left(1 + O\left(\frac{1}{z^2}\right)\right)\right) - \frac{2^{1-\nu} z^{\nu-1}}{\sqrt{\pi}\, \Gamma\left(\nu+\frac{1}{2}\right)} \left(1 + O\left(\frac{1}{z^2}\right)\right) /; \operatorname{Im}(z) \geq 0 \land (|z| \to \infty)$$

The general formulas

03.10.06.0009.01

$$\mathbf{L}_\nu(z) \propto \left(\frac{z}{2}\right)^{\nu+1} \mathcal{A}_{\tilde{F}}\left(\begin{matrix} 1; \\ \frac{3}{2}, \nu+\frac{3}{2}; \end{matrix} \left\{\frac{z^2}{4}, \tilde{\infty}, \infty\right\}\right) /; (|z| \to \infty)$$

03.10.06.0010.01

$$\mathbf{L}_\nu(z) \propto \left(\frac{z}{2}\right)^{\nu+1} \left(\mathcal{A}_{\tilde{F}}^{(\text{power})}\left(\begin{matrix}1;\\ \frac{3}{2},\ \nu+\frac{3}{2};\end{matrix}\ \left\{\frac{z^2}{4},\ \tilde{\infty},\ \infty\right\}\right) + \mathcal{A}_{\tilde{F}}^{(\text{trig})}\left(\begin{matrix}1;\\ \frac{3}{2},\ \nu+\frac{3}{2};\end{matrix}\ \left\{\frac{z^2}{4},\ \tilde{\infty},\ \infty\right\}\right)\right) /;\ (|z| \to \infty)$$

Expansions for any *z* in exponential form

Using exponential function with branch cut-containing arguments

03.10.06.0056.01

$$\mathbf{L}_\nu(z) \propto \frac{z^{\nu+1}}{\sqrt{2\pi}} \left(-z^2\right)^{-\frac{1}{4}(2\nu+3)} \left(e^{-i\sqrt{-z^2}+\frac{1}{4}(2\nu+3)\pi i}\left(1 - \frac{i\left(4\nu^2-1\right)}{8\sqrt{-z^2}} + \frac{16\nu^4-40\nu^2+9}{128 z^2} + \ldots\right) + e^{i\sqrt{-z^2}-\frac{1}{4}(2\nu+3)\pi i}\left(1 + \frac{i\left(4\nu^2-1\right)}{8\sqrt{-z^2}} + \frac{16\nu^4-40\nu^2+9}{128 z^2} + \ldots\right)\right) - \frac{2^{1-\nu} z^{\nu-1}}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)}\left(1 - \frac{2\nu-1}{z^2} + \frac{3\left(4\nu^2-8\nu+3\right)}{z^4} + \ldots\right) /;\ (|z| \to \infty)$$

03.10.06.0057.01

$$\mathbf{L}_\nu(z) \propto \frac{1}{\sqrt{2\pi}} z^{\nu+1} \left(-z^2\right)^{-\frac{2\nu+3}{4}} \left(e^{-i\sqrt{-z^2}+\frac{2\nu+3}{4}\pi i}\left(\sum_{k=0}^{n} \frac{\left(\nu+\frac{1}{2}\right)_k \left(\frac{1}{2}-\nu\right)_k}{k!}\left(\frac{i}{2\sqrt{-z^2}}\right)^k + O\left(\frac{1}{z^{n+1}}\right)\right) + e^{i\sqrt{-z^2}-\frac{2\nu+3}{4}\pi i}\left(\sum_{k=0}^{n} \frac{\left(\nu+\frac{1}{2}\right)_k \left(\frac{1}{2}-\nu\right)_k}{k!}\left(-\frac{i}{2\sqrt{-z^2}}\right)^k + O\left(\frac{1}{z^{n+1}}\right)\right)\right) - \frac{2^{1-\nu} z^{\nu-1}}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)}\left(\sum_{k=0}^{n}\left(\frac{1}{2}\right)_k\left(\frac{1}{2}-\nu\right)_k\left(\frac{4}{z^2}\right)^k + O\left(\frac{1}{z^{2n+2}}\right)\right) /;\ (|z| \to \infty)$$

03.10.06.0058.01

$$\mathbf{L}_\nu(z) \propto \frac{z^{\nu+1}}{\sqrt{2\pi}} \left(-z^2\right)^{-\frac{2\nu+3}{4}} \left(e^{-i\sqrt{-z^2}+\frac{2\nu+3}{4}\pi i}\,{}_2F_0\left(\nu+\frac{1}{2},\ \frac{1}{2}-\nu;\ ;\ \frac{i}{2\sqrt{-z^2}}\right) + e^{i\sqrt{-z^2}-\frac{2\nu+3}{4}\pi i}\,{}_2F_0\left(\nu+\frac{1}{2},\ \frac{1}{2}-\nu;\ ;\ -\frac{i}{2\sqrt{-z^2}}\right)\right) - \frac{2^{1-\nu} z^{\nu-1}}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)}\,{}_3F_0\left(\frac{1}{2},\ \frac{1}{2}-\nu,\ 1;\ ;\ \frac{4}{z^2}\right) /;\ (|z| \to \infty)$$

03.10.06.0059.01

$$\mathbf{L}_\nu(z) \propto \frac{1}{\sqrt{2\pi}} z^{\nu+1} \left(-z^2\right)^{-\frac{2\nu+3}{4}} \left(e^{-i\sqrt{-z^2}+\frac{2\nu+3}{4}\pi i}\left(1+O\left(\frac{1}{z}\right)\right) + e^{i\sqrt{-z^2}-\frac{2\nu+3}{4}\pi i}\left(1+O\left(\frac{1}{z}\right)\right)\right) - \frac{2^{1-\nu} z^{\nu-1}}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)}\left(1+O\left(\frac{1}{z^2}\right)\right) /;\ (|z| \to \infty)$$

Using exponential function with branch cut-free arguments

03.10.06.0060.01

$$\mathbf{L}_\nu(z) \propto \frac{1}{2\sqrt{2\pi}} z^{\nu+1} \left(-z^2\right)^{-\frac{2\nu+3}{4}} \left(e^{-\frac{2\nu+3}{4} i\pi} \left(e^z \left(1 + \frac{i\sqrt{-z^2}}{z}\right) + e^{-z}\left(1 - \frac{i\sqrt{-z^2}}{z}\right)\right)\left(1 + \frac{i\left(4\nu^2-1\right)}{8\sqrt{-z^2}} + \frac{16\nu^4 - 40\nu^2 + 9}{128 z^2} + \ldots\right) + e^{\frac{2\nu+3}{4} i\pi}\left(e^z\left(1 - \frac{i\sqrt{-z^2}}{z}\right) + e^{-z}\left(1 + \frac{i\sqrt{-z^2}}{z}\right)\right)\left(1 - \frac{i\left(4\nu^2-1\right)}{8\sqrt{-z^2}} + \frac{16\nu^4-40\nu^2+9}{128z^2} + \ldots\right)\right) - \frac{2^{1-\nu} z^{\nu-1}}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)}\left(1 - \frac{2\nu-1}{z^2} + \frac{3\left(4\nu^2-8\nu+3\right)}{z^4} + \ldots\right) /; (|z| \to \infty)$$

03.10.06.0061.01

$$\mathbf{L}_\nu(z) \propto \frac{1}{2\sqrt{2\pi}} z^{\nu+1}\left(-z^2\right)^{-\frac{3+2\nu}{4}}\left(e^{\frac{3+2\nu}{4} i\pi}\left(e^z\left(1 - \frac{i\sqrt{-z^2}}{z}\right) + e^{-z}\left(1 + \frac{i\sqrt{-z^2}}{z}\right)\right)\left(\sum_{k=0}^{n} \frac{\left(\nu+\frac{1}{2}\right)_k \left(\frac{1}{2}-\nu\right)_k}{k!}\left(\frac{i}{2\sqrt{-z^2}}\right)^k + O\left(\frac{1}{z^{n+1}}\right)\right) + e^{-\frac{3+2\nu}{4} i\pi}\left(e^z\left(1 + \frac{i\sqrt{-z^2}}{z}\right) + e^{-z}\left(1 - \frac{i\sqrt{-z^2}}{z}\right)\right)\left(\sum_{k=0}^{n}\frac{\left(\nu+\frac{1}{2}\right)_k\left(\frac{1}{2}-\nu\right)_k}{k!}\left(-\frac{i}{2\sqrt{-z^2}}\right)^k + O\left(\frac{1}{z^{n+1}}\right)\right)\right) - \frac{2^{1-\nu}z^{\nu-1}}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)}\left(\sum_{k=0}^{n}\left(\frac{1}{2}\right)_k\left(\frac{1}{2}-\nu\right)_k\left(\frac{4}{z^2}\right)^k + O\left(\frac{1}{z^{2n+2}}\right)\right) /; (|z| \to \infty)$$

03.10.06.0062.01

$$\mathbf{L}_\nu(z) \propto \frac{1}{2\sqrt{2\pi}} z^{\nu+1}\left(-z^2\right)^{-\frac{3+2\nu}{4}}\left(e^{\frac{3+2\nu}{4} i\pi}\left(e^z\left(1 - \frac{i\sqrt{-z^2}}{z}\right) + e^{-z}\left(1 + \frac{i\sqrt{-z^2}}{z}\right)\right) {}_2F_0\left(\nu+\frac{1}{2}, \frac{1}{2}-\nu;; \frac{i}{2\sqrt{-z^2}}\right) + e^{-\frac{3+2\nu}{4} i\pi}\left(e^z\left(1 + \frac{i\sqrt{-z^2}}{z}\right) + e^{-z}\left(1 - \frac{i\sqrt{-z^2}}{z}\right)\right) {}_2F_0\left(\nu+\frac{1}{2}, \frac{1}{2}-\nu;; -\frac{i}{2\sqrt{-z^2}}\right)\right) - \frac{2^{1-\nu}z^{\nu-1}}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)} {}_3F_0\left(\frac{1}{2}, \frac{1}{2}-\nu, 1;; \frac{4}{z^2}\right) /; (|z| \to \infty)$$

03.10.06.0063.01

$$\mathbf{L}_\nu(z) \propto \frac{1}{2\sqrt{2\pi}} z^{\nu+1}\left(-z^2\right)^{-\frac{3+2\nu}{4}}\left(e^{\frac{3+2\nu}{4} i\pi}\left(e^z\left(1 - \frac{i\sqrt{-z^2}}{z}\right) + e^{-z}\left(1 + \frac{i\sqrt{-z^2}}{z}\right)\right)\left(1 + O\left(\frac{1}{z}\right)\right) + e^{-\frac{3+2\nu}{4} i\pi}\left(e^{-z}\left(1 - \frac{i\sqrt{-z^2}}{z}\right) + e^z\left(1 + \frac{i\sqrt{-z^2}}{z}\right)\right)\left(1 + O\left(\frac{1}{z}\right)\right)\right) - \frac{2^{1-\nu}z^{\nu-1}}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)}\left(1 + O\left(\frac{1}{z^2}\right)\right) /; (|z| \to \infty)$$

Expansions for any z in trigonometric and hyperbolic forms

Using trigonometric functions with branch cut-containing arguments

03.10.06.0064.01

$$\mathbf{L}_\nu(z) \propto \sqrt{\frac{2}{\pi}}\, z^{\nu+1} \left(-z^2\right)^{-\frac{2\nu+3}{4}} \left(\sin\left(\sqrt{-z^2} - \frac{(2\nu+1)\pi}{4}\right)\left(1 + \frac{16\nu^4 - 40\nu^2 + 9}{128 z^2} + \frac{256\nu^8 - 5376\nu^6 + 31584\nu^4 - 51664\nu^2 + 11025}{98304 z^4} + \ldots\right) + \frac{4\nu^2 - 1}{8\sqrt{-z^2}} \cos\left(\sqrt{-z^2} - \frac{(2\nu+1)\pi}{4}\right) \left(1 + \frac{16\nu^4 - 136\nu^2 + 225}{384 z^2} + \frac{256\nu^8 - 10496\nu^6 + 137824\nu^4 - 656784\nu^2 + 893025}{491520 z^4} + \ldots\right)\right) - \frac{2^{1-\nu} z^{\nu-1}}{\sqrt{\pi}\, \Gamma\left(\nu + \frac{1}{2}\right)} \left(1 - \frac{2\nu - 1}{z^2} + \frac{3\left(4\nu^2 - 8\nu + 3\right)}{z^4} + \ldots\right) /; (|z| \to \infty)$$

03.10.06.0065.01

$$\mathbf{L}_\nu(z) \propto \sqrt{\frac{2}{\pi}} \left(-z^2\right)^{-\frac{2\nu+3}{4}} z^{\nu+1} \left(\sin\left(\sqrt{-z^2} - \frac{(2\nu+1)\pi}{4}\right)\left(\sum_{k=0}^{n} \frac{\left(\frac{1}{4}(1-2\nu)\right)_k \left(\frac{1}{4}(3-2\nu)\right)_k \left(\frac{1}{4}(2\nu+1)\right)_k \left(\frac{1}{4}(2\nu+3)\right)_k}{\left(\frac{1}{2}\right)_k k!} \left(\frac{1}{z^2}\right)^k + O\left(\frac{1}{z^{2n+2}}\right)\right) + \frac{4\nu^2 - 1}{8\sqrt{-z^2}} \cos\left(\sqrt{-z^2} - \frac{(2\nu+1)\pi}{4}\right) \left(\sum_{k=0}^{n} \frac{\left(\frac{1}{4}(3-2\nu)\right)_k \left(\frac{1}{4}(5-2\nu)\right)_k \left(\frac{1}{4}(2\nu+3)\right)_k \left(\frac{1}{4}(2\nu+5)\right)_k}{\left(\frac{3}{2}\right)_k k!} \left(\frac{1}{z^2}\right)^k + O\left(\frac{1}{z^{2n+2}}\right)\right)\right) - \frac{2^{1-\nu} z^{\nu-1}}{\sqrt{\pi}\, \Gamma\left(\nu + \frac{1}{2}\right)} \left(\sum_{k=0}^{n} \left(\frac{1}{2}\right)_k \left(\frac{1}{2} - \nu\right)_k \left(\frac{4}{z^2}\right)^k + O\left(\frac{1}{z^{2n+2}}\right)\right) /; (|z| \to \infty)$$

03.10.06.0011.01

$$\mathbf{L}_\nu(z) \propto \sqrt{\frac{2}{\pi}}\, z^{\nu+1} \left(-z^2\right)^{-\frac{2\nu+3}{4}} \left(\sin\left(\sqrt{-z^2} - \frac{2\nu+1}{4}\pi\right) {}_4F_1\left(\frac{1-2\nu}{4}, \frac{3-2\nu}{4}, \frac{2\nu+1}{4}, \frac{2\nu+3}{4}; \frac{1}{2}; \frac{1}{z^2}\right) + \frac{4\nu^2 - 1}{8\sqrt{-z^2}} \cos\left(\sqrt{-z^2} - \frac{2\nu+1}{4}\pi\right) {}_4F_1\left(\frac{3-2\nu}{4}, \frac{5-2\nu}{4}, \frac{3+2\nu}{4}, \frac{5+2\nu}{4}; \frac{3}{2}; \frac{1}{z^2}\right)\right) - \frac{2^{1-\nu} z^{\nu-1}}{\sqrt{\pi}\, \Gamma\left(\nu + \frac{1}{2}\right)} {}_3F_0\left(\frac{1}{2}, 1, \frac{1}{2} - \nu;; \frac{4}{z^2}\right) /; (|z| \to \infty)$$

03.10.06.0012.01

$$\mathbf{L}_\nu(z) \propto \sqrt{\frac{2}{\pi}}\, z^{\nu+1} \left(-z^2\right)^{-\frac{2\nu+3}{4}} \left(\sin\left(\sqrt{-z^2} - \frac{2\nu+1}{4}\pi\right)\left(1 + O\left(\frac{1}{z^2}\right)\right) + \frac{4\nu^2-1}{8\sqrt{-z^2}} \cos\left(\sqrt{-z^2} - \frac{2\nu+1}{4}\pi\right)\left(1 + O\left(\frac{1}{z^2}\right)\right)\right) - \frac{2^{1-\nu} z^{\nu-1}}{\sqrt{\pi}\, \Gamma\left(\nu + \frac{1}{2}\right)} \left(1 + O\left(\frac{1}{z^2}\right)\right) /; (|z| \to \infty)$$

Using hyperbolic functions with branch cut-free arguments

03.10.06.0066.01

$$\mathbf{L}_\nu(z) \propto \sqrt{\frac{2}{\pi}}\, z^{\nu+1} \left(-z^2\right)^{-\frac{2\nu+3}{4}} \left(-\left(\frac{z}{\sqrt{-z^2}} \cos\left(\frac{2\nu+1}{4}\pi\right) \sinh(z) + \sin\left(\frac{2\nu+1}{4}\pi\right) \cosh(z)\right) \left(1 + \frac{16\nu^4 - 40\nu^2 + 9}{128 z^2} + \frac{256\nu^8 - 5376\nu^6 + 31584\nu^4 - 51664\nu^2 + 11025}{98304 z^4} + \ldots\right) + \frac{4\nu^2-1}{8} \left(\frac{1}{\sqrt{-z^2}} \cos\left(\frac{2\nu+1}{4}\pi\right) \cosh(z) + \frac{1}{z} \sin\left(\frac{2\nu+1}{4}\pi\right) \sinh(z)\right) \left(1 + \frac{16\nu^4 - 136\nu^2 + 225}{384 z^2} + \frac{256\nu^8 - 10496\nu^6 + 137824\nu^4 - 656784\nu^2 + 893025}{491520 z^4} + \ldots\right)\right) - \frac{2^{1-\nu} z^{\nu-1}}{\sqrt{\pi}\, \Gamma\left(\nu + \frac{1}{2}\right)} \left(1 - \frac{2\nu-1}{z^2} + \frac{3\left(4\nu^2 - 8\nu + 3\right)}{z^4} + \ldots\right) /; (|z| \to \infty)$$

03.10.06.0067.01

$$\mathbf{L}_\nu(z) \propto \sqrt{\frac{2}{\pi}}\, z^{\nu+1} \left(-z^2\right)^{-\frac{2\nu+3}{4}} \left(-\left(\frac{z}{\sqrt{-z^2}} \cos\left(\frac{2\nu+1}{4}\pi\right) \sinh(z) + \sin\left(\frac{2\nu+1}{4}\pi\right) \cosh(z)\right) \left(\sum_{k=0}^{n} \frac{\left(\frac{1}{4}(1-2\nu)\right)_k \left(\frac{1}{4}(3-2\nu)\right)_k \left(\frac{1}{4}(2\nu+1)\right)_k \left(\frac{1}{4}(2\nu+3)\right)_k}{\left(\frac{1}{2}\right)_k k!} \left(\frac{1}{z^2}\right)^k + O\left(\frac{1}{z^{2n+2}}\right)\right) + \frac{4\nu^2-1}{8} \left(\frac{1}{\sqrt{-z^2}} \cos\left(\frac{2\nu+1}{4}\pi\right) \cosh(z) + \frac{1}{z} \sin\left(\frac{2\nu+1}{4}\pi\right) \sinh(z)\right) \left(\sum_{k=0}^{n} \frac{\left(\frac{1}{4}(3-2\nu)\right)_k \left(\frac{1}{4}(5-2\nu)\right)_k \left(\frac{1}{4}(2\nu+3)\right)_k \left(\frac{1}{4}(2\nu+5)\right)_k}{\left(\frac{3}{2}\right)_k k!} \left(\frac{1}{z^2}\right)^k + O\left(\frac{1}{z^{2n+2}}\right)\right)\right) - \frac{2^{1-\nu} z^{\nu-1}}{\sqrt{\pi}\, \Gamma\left(\nu + \frac{1}{2}\right)} \left(\sum_{k=0}^{n} \left(\frac{1}{2}\right)_k \left(\frac{1}{2} - \nu\right)_k \left(\frac{4}{z^2}\right)^k + O\left(\frac{1}{z^{2n+2}}\right)\right) /; (|z| \to \infty)$$

03.10.06.0068.01

$$\mathbf{L}_\nu(z) \propto \sqrt{\frac{2}{\pi}}\, z^{\nu+1} \left(-z^2\right)^{-\frac{2\nu+3}{4}} \left(-\left(\frac{z}{\sqrt{-z^2}} \cos\left(\frac{2\nu+1}{4}\pi\right) \sinh(z) + \sin\left(\frac{2\nu+1}{4}\pi\right) \cosh(z)\right) {}_4F_1\left(\frac{1}{4}(1-2\nu), \frac{1}{4}(3-2\nu), \frac{1}{4}(2\nu+1), \frac{1}{4}(2\nu+3); \frac{1}{2}; \frac{1}{z^2}\right) + \frac{4\nu^2-1}{8} \left(\frac{1}{\sqrt{-z^2}} \cos\left(\frac{2\nu+1}{4}\pi\right) \cosh(z) + \frac{1}{z} \sin\left(\frac{2\nu+1}{4}\pi\right) \sinh(z)\right) {}_4F_1\left(\frac{1}{4}(3-2\nu), \frac{1}{4}(5-2\nu), \frac{1}{4}(2\nu+3), \frac{1}{4}(2\nu+5); \frac{3}{2}; \frac{1}{z^2}\right)\right) - \frac{2^{1-\nu} z^{\nu-1}}{\sqrt{\pi}\, \Gamma\left(\nu+\frac{1}{2}\right)} {}_3F_0\left(1, \frac{1}{2}, \frac{1}{2}-\nu; ; \frac{4}{z^2}\right) /; (|z| \to \infty)$$

03.10.06.0069.01

$$\mathbf{L}_\nu(z) \propto \sqrt{\frac{2}{\pi}}\, z^{\nu+1} \left(-z^2\right)^{-\frac{2\nu+3}{4}} \left(-\left(\frac{z}{\sqrt{-z^2}} \cos\left(\frac{2\nu+1}{4}\pi\right) \sinh(z) + \sin\left(\frac{2\nu+1}{4}\pi\right) \cosh(z)\right) \left(1 + O\left(\frac{1}{z^2}\right)\right) + \frac{4\nu^2-1}{8} \left(\frac{1}{\sqrt{-z^2}} \cos\left(\frac{2\nu+1}{4}\pi\right) \cosh(z) + \frac{1}{z} \sin\left(\frac{2\nu+1}{4}\pi\right) \sinh(z)\right) \left(1 + O\left(\frac{1}{z^2}\right)\right)\right) - \frac{2^{1-\nu} z^{\nu-1}}{\sqrt{\pi}\, \Gamma\left(\nu+\frac{1}{2}\right)} \left(1 + O\left(\frac{1}{z^2}\right)\right) /; (|z| \to \infty)$$

Residue representations

03.10.06.0013.01

$$\mathbf{L}_\nu(z) = -\pi \csc\left(\frac{\pi\nu}{2}\right) z^{\nu-1} \left(z^2\right)^{\frac{1-\nu}{2}} \sum_{j=0}^{\infty} \operatorname{res}_s\left(\frac{\Gamma\left(\frac{1-\nu}{2}-s\right)\left(\frac{z^2}{4}\right)^{-s}}{\Gamma\left(s+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}-s\right)\Gamma\left(1+\frac{\nu}{2}-s\right)\Gamma\left(1-\frac{\nu}{2}-s\right)} \Gamma\left(\frac{\nu+1}{2}+s\right)\right)\left(-\frac{\nu+1}{2}-j\right)$$

03.10.06.0014.01

$$\mathbf{L}_\nu(z) = -\pi \csc\left(\frac{\pi\nu}{2}\right) \sum_{j=0}^{\infty} \operatorname{res}_s\left(\frac{\Gamma\left(\frac{1-\nu}{2}-s\right)\left(\frac{z}{2}\right)^{-2s}}{\Gamma\left(s+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}-s\right)\Gamma\left(1+\frac{\nu}{2}-s\right)\Gamma\left(1-\frac{\nu}{2}-s\right)} \Gamma\left(\frac{\nu+1}{2}+s\right)\right)\left(-\frac{\nu+1}{2}-j\right)$$

Other series representations

03.10.06.0015.01

$$\mathbf{L}_0(z) = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{I_{2k+1}(z)}{2k+1}$$

03.10.06.0016.01

$$\mathbf{L}_1(z) = \frac{2}{\pi}(I_0(z)-1) + \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{I_{2k}(z)}{4k^2-1}$$

Integral representations

On the real axis

Of the direct function

03.10.07.0001.01

$$\boldsymbol{L}_\nu(z) = \frac{2^{1-\nu} z^\nu}{\sqrt{\pi}\, \Gamma\left(\nu + \frac{1}{2}\right)} \int_0^1 \left(1 - t^2\right)^{\nu - \frac{1}{2}} \sinh(t z)\, dt \,/;\, \mathrm{Re}(\nu) > -\frac{1}{2}$$

03.10.07.0002.01

$$\boldsymbol{L}_\nu(z) = \frac{2^{1-\nu} z^\nu}{\sqrt{\pi}\, \Gamma\left(\nu + \frac{1}{2}\right)} \int_0^{\frac{\pi}{2}} \sin^{2\nu}(t) \sinh(z \cos(t))\, dt \,/;\, \mathrm{Re}(\nu) > -\frac{1}{2}$$

03.10.07.0003.01

$$\boldsymbol{L}_\nu(z) = I_{-\nu}(z) - \frac{2^{1-\nu} z^\nu}{\sqrt{\pi}\, \Gamma\left(\nu + \frac{1}{2}\right)} \int_0^\infty \sin(t z) \left(t^2 + 1\right)^{\nu - \frac{1}{2}} dt \,/;\, z > 0 \bigwedge \mathrm{Re}(\nu) < \frac{1}{2}$$

Contour integral representations

03.10.07.0004.01

$$\boldsymbol{L}_\nu(z) = -\pi \csc\left(\frac{\pi \nu}{2}\right) z^{\nu-1} z^2 \frac{1}{2 \pi i} \int_{\mathcal{L}} \frac{\Gamma\left(\frac{\nu+1}{2} + s\right) \Gamma\left(\frac{1-\nu}{2} - s\right)}{\Gamma\left(s + \frac{1}{2}\right) \Gamma\left(\frac{1}{2} - s\right) \Gamma\left(1 + \frac{\nu}{2} - s\right) \Gamma\left(1 - \frac{\nu}{2} - s\right)} \left(\frac{z^2}{4}\right)^{-s} ds$$

03.10.07.0005.01

$$\boldsymbol{L}_\nu(z) = -\pi \csc\left(\frac{\pi \nu}{2}\right) \frac{1}{2 \pi i} \int_{\mathcal{L}} \frac{\Gamma\left(\frac{\nu+1}{2} + s\right) \Gamma\left(\frac{1-\nu}{2} - s\right)}{\Gamma\left(s + \frac{1}{2}\right) \Gamma\left(\frac{1}{2} - s\right) \Gamma\left(1 + \frac{\nu}{2} - s\right) \Gamma\left(1 - \frac{\nu}{2} - s\right)} \left(\frac{z}{2}\right)^{-2s} ds$$

Differential equations

Ordinary linear differential equations and wronskians

For the direct function itself

03.10.13.0001.01

$$w''(z) z^2 + w'(z) z - \left(z^2 + \nu^2\right) w(z) = \frac{4}{\sqrt{\pi}\, \Gamma\left(\nu + \frac{1}{2}\right)} \left(\frac{z}{2}\right)^{\nu+1} /;\, w(z) = c_1 I_\nu(z) + c_2 K_\nu(z) + \boldsymbol{L}_\nu(z)$$

03.10.13.0002.01

$$W_z(I_\nu(z), K_\nu(z)) = -\frac{1}{z}$$

03.10.13.0003.01

$$w''(z) z^2 + w'(z) z - \left(z^2 + \nu^2\right) w(z) = \frac{4}{\sqrt{\pi}\, \Gamma\left(\nu + \frac{1}{2}\right)} \left(\frac{z}{2}\right)^{\nu+1} /;\, w(z) = c_1 I_\nu(z) + c_2 I_{-\nu}(z) + \boldsymbol{L}_\nu(z) \bigwedge \nu \notin \mathbb{Z}$$

03.10.13.0004.01

$$W_z(I_\nu(z), I_{-\nu}(z)) = -\frac{2 \sin(\pi \nu)}{\pi z}$$

03.10.13.0005.01

$$z^3 w^{(3)}(z) - (\nu - 2) z^2 w''(z) - \left(z^2 + \nu^2 + \nu\right) z\, w'(z) + \left((\nu - 1) z^2 + \nu^2 (\nu + 1)\right) w(z) = 0 \,/;\, w(z) = \boldsymbol{L}_\nu(z) c_1 + c_2 I_\nu(z) + c_3 K_\nu(z)$$

03.10.13.0006.01

$$W_z(\boldsymbol{L}_\nu(z),\, I_\nu(z),\, K_\nu(z)) == -\frac{2^{1-\nu}\, z^{\nu-2}}{\sqrt{\pi}\; \Gamma\left(\nu+\frac{1}{2}\right)}$$

03.10.13.0007.01

$$w^{(3)}(z) - \frac{((\nu-2)\, g'(z))\,(3\, g''(z))}{g(z)\, g'(z)}\, w''(z) + \left(-\frac{\nu\,(\nu+1)\, g'(z)^2}{g(z)^2} - g'(z)^2 + \frac{3\, g''(z)^2}{g'(z)^2} + \frac{(\nu-2)\, g''(z)}{g(z)} - \frac{g^{(3)}(z)}{g'(z)}\right) w'(z) + \left(\frac{(\nu-1)\, g'(z)^3}{g(z)} + \frac{\nu^2\,(\nu+1)\, g'(z)^3}{g(z)^3}\right) w(z) = 0 \;/;\; w(z) = c_1\, \boldsymbol{L}_\nu(g(z)) \;+ c_2\, I_\nu(g(z)) \;+ c_3\, K_\nu(g(z))$$

03.10.13.0008.01

$$W_z(\boldsymbol{L}_\nu(g(z)),\, I_\nu(g(z)),\, K_\nu(g\,(z))) == -\frac{2^{1-\nu}\, g(z)^{\nu-2}\, g'(z)^3}{\sqrt{\pi}\; \Gamma\left(\nu+\frac{1}{2}\right)}$$

03.10.13.0009.01

$$w^{(3)}(z) - \left(\frac{(\nu-2)\, g'(z)}{g(z)} + \frac{3\, h'(z)}{h(z)} + \frac{3\, g''(z)}{g'(z)}\right) w''(z) + \left(-\frac{\nu\,(\nu+1)\, g'(z)^2}{g(z)^2} - g'(z)^2 + \frac{2\,(\nu-2)\, h'(z)\, g'(z)}{g(z)\, h(z)} + \frac{6\, h'(z)^2}{h(z)^2} + \frac{3\, g''(z)^2}{g'(z)^2} + \frac{6\, h'(z)\, g''(z)}{h(z)\, g'(z)} + \frac{(\nu-2)\, g''(z)}{g(z)} - \frac{3\, h''(z)}{h(z)} - \frac{g^{(3)}(z)}{g'(z)}\right) w'(z) + \left(\frac{(\nu-1)\, g'(z)^3}{g(z)} + \frac{\nu^2\,(\nu+1)\, g'(z)^3}{g(z)^3} + \frac{\nu\,(\nu+1)\, h'(z)\, g'(z)^2}{g(z)^2\, h(z)} - \frac{2\,(\nu-2)\, h'(z)^2\, g'(z)}{g(z)\, h(z)^2} + \frac{6\, h'(z)\, h''(z)}{h(z)^2} + \frac{3\, g''(z)\, h''(z) + h'(z)\, g^{(3)}(z)}{h(z)\, g'(z)} + \frac{g'(z)^2\, h'(z) - h^{(3)}(z)}{h(z)} - \frac{(\nu-2)\,(h'(z)\, g''(z) - g'(z)\, h''(z))}{g(z)\, h(z)} - \frac{6\, h'(z)^3}{h(z)^3} - \frac{6\, h'(z)^2\, g''(z)}{h(z)^2\, g'(z)} - \frac{3\, h'(z)\, g''(z)^2}{h(z)\, g'(z)^2}\right) w(z) \;= 0 \;/;\; w(z) = c_1\, h(z)\, \boldsymbol{L}_\nu(g(z)) \;+ c_2\, h(z)\, I_\nu(g(z)) + c_3\, h(z)\, K_\nu(g(z))$$

03.10.13.0010.01

$$W_z(h(z)\, \boldsymbol{L}_\nu(g(z)),\, h(z)\, I_\nu(g(z)),\, h(z)\, K_\nu(g\,(z))) == -\frac{2^{1-\nu}\, g(z)^{\nu-2}\, h(z)^3\, g'(z)^3}{\sqrt{\pi}\; \Gamma\left(\nu+\frac{1}{2}\right)}$$

03.10.13.0011.01

$$z^3\, w^{(3)}(z) \;- (\nu\, r + r + 3\, s - 3)\, z^2\, w''(z) \;+ \left(-\left(a^2\, z^{2\,r} + \nu^2\right) r^2 + (2\, s - 1)\,(\nu+1)\, r + 3\,(s-1)\, s + 1\right) z\, w'(z) + \left(\left(a^2\,(\nu-1)\, z^{2\,r} + \nu^2\,(\nu+1)\right) r^3 + s\left(a^2\, z^{2\,r} + \nu^2\right) r^2 - s^2\,(\nu+1)\, r - s^3\right) w(z) = 0 \;/;\; w(z) = c_1\, z^s\, \boldsymbol{L}_\nu(a\, z^r) \;+ c_2\, z^s\, I_\nu(a\, z^r) \;+ c_3\, z^s\, K_\nu(a\, z^r)$$

03.10.13.0012.01

$$W_z(z^s\, \boldsymbol{L}_\nu(a\, z^r),\, z^s\, I_\nu(a\, z^r),\, z^s\, K_\nu(a\, z^r)) == -\frac{2^{1-\nu}\, a\, r^3\, z^{r+3\,s-3}\, (a\, z^r)^\nu}{\sqrt{\pi}\; \Gamma\left(\nu+\frac{1}{2}\right)}$$

03.10.13.0013.01

$$w^{(3)}(z) + (-(\nu+1)\log(r) - 3\log(s))\, w''(z) + \left(-\left(a^2\, r^{2\,z} + \nu^2\right)\log^2(r) + 2\,(\nu+1)\log(s)\log(r) + 3\log^2(s)\right) w'(z) + \left(\left(a^2\,(\nu-1)\, r^{2\,z} + \nu^2\,(\nu+1)\right)\log^3(r) + \left(a^2\, r^{2\,z} + \nu^2\right)\log(s)\log^2(r) - (\nu+1)\log^2(s)\log(r) - \log^3(s)\right) w(z) = 0 \;/;\; w(z) = c_1\, s^z\, \boldsymbol{L}_\nu(a\, r^z) \;+ c_2\, s^z\, I_\nu(a\, r^z) \;+ c_3\, s^z\, K_\nu(a\, r^z)$$

03.10.13.0014.01

$$W_z(s^z \boldsymbol{L}_\nu(a\, r^z),\, s^z\, I_\nu(a\, r^z),\, s^z\, K_\nu(a\, r^z)) = -\frac{2^{1-\nu}\, (a\, r^z)^{\nu+1}\, s^{3z} \log^3(r)}{\sqrt{\pi}\, \Gamma\!\left(\nu + \frac{1}{2}\right)}$$

Transformations

Transformations and argument simplifications

Argument involving basic arithmetic operations

03.10.16.0001.01

$$\boldsymbol{L}_\nu(-z) = -(-z)^\nu\, z^{-\nu}\, \boldsymbol{L}_\nu(z)$$

03.10.16.0002.01

$$\boldsymbol{L}_\nu(i\, z) = i\, (i\, z)^\nu\, z^{-\nu}\, \boldsymbol{H}_\nu(z)$$

03.10.16.0003.01

$$\boldsymbol{L}_\nu(-i\, z) = -i\, (-i\, z)^\nu\, z^{-\nu}\, \boldsymbol{H}_\nu(z)$$

03.10.16.0004.01

$$\boldsymbol{L}_\nu\!\left(\sqrt{z^2}\right) = z^{-\nu-1}\, \left(z^2\right)^{\frac{\nu+1}{2}}\, \boldsymbol{L}_\nu(z)$$

03.10.16.0005.01

$$\boldsymbol{L}_\nu(c\, (d\, z^n)^m) = \frac{(c\, (d\, z^n)^m)^{\nu+1}}{(c\, d^m\, z^{m\, n})^{\nu+1}}\, \boldsymbol{L}_\nu(c\, d^m\, z^{m\, n}) \;/;\; 2\, m \in \mathbb{Z}$$

Identities

Recurrence identities

Consecutive neighbors

03.10.17.0001.01

$$\boldsymbol{L}_\nu(z) = \frac{2\, (\nu+1)}{z}\, \boldsymbol{L}_{\nu+1}(z) + \boldsymbol{L}_{\nu+2}(z) + \frac{2^{-\nu-1}\, z^{\nu+1}}{\sqrt{\pi}\, \Gamma\!\left(\nu + \frac{5}{2}\right)}$$

03.10.17.0002.01

$$\boldsymbol{L}_\nu(z) = -\frac{2\, (\nu-1)}{z}\, \boldsymbol{L}_{\nu-1}(z) + \boldsymbol{L}_{\nu-2}(z) - \frac{2^{1-\nu}\, z^{\nu-1}}{\sqrt{\pi}\, \Gamma\!\left(\nu + \frac{1}{2}\right)}$$

Distant neighbors

Increasing

03.10.17.0012.01

$$\mathbf{L}_\nu(z) = 2^{n-1} z^{-n} (\nu+1)_{n-1} \left(2(n+\nu) \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(-1)^k (n-k)!}{k!\,(n-2k)!\,(-n-\nu)_k (\nu+1)_k} \left(\frac{z^2}{4}\right)^k \mathbf{L}_{n+\nu}(z) + z \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \frac{(-1)^k (n-k-1)!}{k!\,(n-2k-1)!\,(1-n-\nu)_k (\nu+1)_k} \left(\frac{z^2}{4}\right)^k \mathbf{L}_{n+\nu+1}(z)\right) + \frac{1}{\sqrt{\pi}} \left(\frac{z}{2}\right)^{\nu+1} \sum_{j=0}^{n-1} \frac{(\nu+1)_j}{\Gamma\left(j+\nu+\frac{5}{2}\right)} \sum_{k=0}^{\left\lfloor \frac{j}{2} \right\rfloor} \frac{(-1)^k (j-k)!}{k!\,(j-2k)!\,(-j-\nu)_k (\nu+1)_k} \left(\frac{z^2}{4}\right)^k \; /;\; n \in \mathbb{N}$$

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03.10.17.0013.01

$$\mathbf{L}_\nu(z) = 2^{n-1} (\nu+1)_{n-1} \left(2(n+\nu)\, {}_3F_4\left(1, \frac{1-n}{2}, -\frac{n}{2}; 1, -n, -n-\nu, \nu+1; z^2\right) \mathbf{L}_{n+\nu}(z) + z\, {}_3F_4\left(1, \frac{1-n}{2}, 1-\frac{n}{2}; 1, 1-n, 1-\nu-n, \nu+1; z^2\right) \mathbf{L}_{n+\nu+1}(z)\right) z^{-n} + \frac{2^{-\nu-1} z^{\nu+1}}{\sqrt{\pi}} \sum_{j=0}^{n-1} \frac{(\nu+1)_j}{\Gamma\left(j+\nu+\frac{5}{2}\right)}\, {}_3F_4\left(1, \frac{1-j}{2}, -\frac{j}{2}; 1, -j, -j-\nu, \nu+1; z^2\right) /;\; n \in \mathbb{N}$$

Brychkov Yu.A. (2005)

03.10.17.0004.01

$$\mathbf{L}_\nu(z) = \frac{2^{-\nu-2} (4\nu+7) z^{\nu+1}}{\sqrt{\pi}\, \Gamma\left(\nu+\frac{7}{2}\right)} + \left(\frac{4(\nu+1)(\nu+2)}{z^2} + 1\right) \mathbf{L}_{\nu+2}(z) + \frac{2(\nu+1) \mathbf{L}_{\nu+3}(z)}{z}$$

03.10.17.0005.01

$$\mathbf{L}_\nu(z) = \frac{2^{-\nu-3} \left(z^2 + 12\nu^2 + 54\nu + 57\right) z^{\nu+1}}{\sqrt{\pi}\, \Gamma\left(\nu+\frac{9}{2}\right)} + \frac{4(\nu+2)\left(z^2 + 2\nu^2 + 8\nu + 6\right) \mathbf{L}_{\nu+3}(z)}{z^3} + \left(\frac{4(\nu+1)(\nu+2)}{z^2} + 1\right) \mathbf{L}_{\nu+4}(z)$$

03.10.17.0006.01

$$\mathbf{L}_\nu(z) = \frac{2^{-\nu-4} \left(32\nu^3 + 264\nu^2 + 688\nu + z^2 (6\nu+17) + 561\right) z^{\nu+1}}{\sqrt{\pi}\, \Gamma\left(\nu+\frac{11}{2}\right)} + \left(\frac{4(\nu+1)(\nu+2)}{z^2} + \frac{8(\nu+4)\left(z^2 + 2\nu^2 + 8\nu + 6\right)(\nu+2)}{z^4} + 1\right) \mathbf{L}_{\nu+4}(z) + \frac{4(\nu+2)\left(z^2 + 2\nu^2 + 8\nu + 6\right) \mathbf{L}_{\nu+5}(z)}{z^3}$$

03.10.17.0007.01

$$\mathbf{L}_\nu(z) = \frac{2(\nu+3)\left(3z^4 + 16\left(\nu^2 + 6\nu + 8\right) z^2 + 16\left(\nu^4 + 12\nu^3 + 49\nu^2 + 78\nu + 40\right)\right) \mathbf{L}_{\nu+5}(z)}{z^5} + \left(1 + \frac{4(\nu+1)(\nu+2)}{z^2} + \frac{8(\nu+4)\left(z^2 + 2\nu^2 + 8\nu + 6\right)(\nu+2)}{z^4}\right) \mathbf{L}_{\nu+6}(z) + \frac{2^{-\nu-5} z^{\nu+1} \left(z^4 + \left(24\nu^2 + 160\nu + 259\right) z^2 + 5\left(16\nu^4 + 208\nu^3 + 968\nu^2 + 1898\nu + 1311\right)\right)}{\sqrt{\pi}\, \Gamma\left(\nu+\frac{13}{2}\right)}$$

03.10.17.0014.01

$$\mathbf{L}_\nu(z) \equiv \mathcal{C}_n(\nu, z)\,\mathbf{L}_{\nu+n}(z) + \mathcal{C}_{n-1}(\nu, z)\,\mathbf{L}_{\nu+n+1}(z) + \frac{1}{\sqrt{\pi}} \sum_{j=0}^{n-1} \frac{1}{\Gamma\left(j+\nu+\frac{5}{2}\right)} \left(\frac{z}{2}\right)^{j+\nu+1} \mathcal{C}_j(\nu, z) \;/;$$

$$\mathcal{C}_0(\nu, z) \equiv 1 \bigwedge \mathcal{C}_1(\nu, z) \equiv \frac{2(\nu+1)}{z} \bigwedge \mathcal{C}_n(\nu, z) \equiv \frac{2(n+\nu)}{z}\,\mathcal{C}_{n-1}(\nu, z) + \mathcal{C}_{n-2}(\nu, z) \bigwedge n \in \mathbb{N}^+$$

03.10.17.0015.01

$$\mathbf{L}_\nu(z) = \mathcal{C}_n(\nu, z)\,\mathbf{L}_{n+\nu}(z) + \mathcal{C}_{n-1}(\nu, z)\,\mathbf{L}_{n+\nu+1}(z) + \frac{1}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{5}{2}\right)} \left(\frac{z}{2}\right)^{\nu+1} \sum_{j=0}^{n-1} \frac{(\nu+1)_j}{\left(\nu+\frac{5}{2}\right)_j}\, {}_2F_3\left(\frac{1-j}{2}, -\frac{j}{2}; \nu+1, -j, -j-\nu; z^2\right) /;$$

$$\mathcal{C}_n(\nu, z) = 2^n z^{-n} (\nu+1)_n\, {}_2F_3\left(\frac{1-n}{2}, -\frac{n}{2}; \nu+1, -n, -n-\nu; z^2\right) \bigwedge n \in \mathbb{N}^+$$

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Decreasing

03.10.17.0016.01

$$\mathbf{L}_\nu(z) = 2^{n-1} z^{-n} (1-\nu)_{n-1} \left(2(n-\nu) \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(-1)^k (n-k)!}{k!\,(n-2k)!\,(1-\nu)_k (\nu-n)_k} \left(\frac{z^2}{4}\right)^k \mathbf{L}_{\nu-n}(z) + z \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \frac{(-1)^k (n-k-1)!}{k!\,(n-2k-1)!\,(1-\nu)_k (\nu-n+1)_k} \left(\frac{z^2}{4}\right)^k \mathbf{L}_{\nu-n-1}(z)\right) -$$

$$\frac{1}{\sqrt{\pi}} \left(\frac{z}{2}\right)^{\nu-1} \sum_{j=0}^{n-1} \frac{(1-\nu)_j}{\Gamma\left(\nu-j+\frac{1}{2}\right)\left(\frac{z^2}{4}\right)^j} \sum_{k=0}^{\left\lfloor \frac{j}{2} \right\rfloor} \frac{(-1)^k (j-k)! \left(\frac{z^2}{4}\right)^k}{k!\,(j-2k)!\,(1-\nu)_k (\nu-j)_k} \;/;\, n \in \mathbb{N}$$

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03.10.17.0017.01

$$\mathbf{L}_\nu(z) = 2^{n-1} z^{-n} (1-\nu)_{n-1} \left(z\, {}_3F_4\left(1, \frac{1-n}{2}, 1-\frac{n}{2}; 1, 1-n, 1-\nu, -n+\nu+1; z^2\right) \mathbf{L}_{-n+\nu-1}(z) + 2(n-\nu)\, {}_3F_4\left(1, \frac{1-n}{2}, -\frac{n}{2}; 1, -n, 1-\nu, \nu-n; z^2\right) \mathbf{L}_{\nu-n}(z)\right) -$$

$$\frac{2^{1-\nu} z^{\nu-1}}{\sqrt{\pi}} \sum_{j=0}^{n-1} \frac{4^j z^{-2j} (1-\nu)_j}{\Gamma\left(\nu-j+\frac{1}{2}\right)}\, {}_3F_4\left(1, \frac{1-j}{2}, -\frac{j}{2}; 1, -j, 1-\nu, \nu-j; z^2\right) /;\, n \in \mathbb{N}$$

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03.10.17.0008.01

$$\mathbf{L}_\nu(z) \equiv \frac{2^{1-\nu}\left(-z^2+4\nu^2-6\nu+2\right) z^{\nu-3}}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)} + \left(\frac{4(\nu-2)(\nu-1)}{z^2}+1\right) \mathbf{L}_{\nu-2}(z) - \frac{2(\nu-1)\,\mathbf{L}_{\nu-3}(z)}{z}$$

03.10.17.0009.01

$$\mathbf{L}_\nu(z) = \left(\frac{4(\nu-2)(\nu-1)}{z^2}+1\right)\mathbf{L}_{\nu-4}(z) - \frac{\left(4(\nu-2)\left(z^2+2\nu^2-8\nu+6\right)\right)\mathbf{L}_{\nu-3}(z)}{z^3} - \frac{2^{1-\nu} z^{\nu-5}\left(z^4+(1-2\nu)z^2+4(\nu-2)(\nu-1)(2\nu-3)(2\nu-1)\right)}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)}$$

03.10.17.0010.01

$$\mathbf{L}_\nu(z) = \left(\frac{4(\nu-2)(\nu-1)}{z^2}+\frac{8(\nu-4)(\nu-2)\left(z^2+2\nu^2-8\nu+6\right)}{z^4}+1\right)\mathbf{L}_{\nu-4}(z) - \frac{4(\nu-2)\left(z^2+2\nu^2-8\nu+6\right)\mathbf{L}_{\nu-5}(z)}{z^3} - \frac{1}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)}\left(2^{1-\nu} z^{\nu-7}\left(z^6+(1-2\nu)z^4-4\left(4\nu^4-32\nu^3+83\nu^2-82\nu+24\right)z^2-8\left(8\nu^6-84\nu^5+350\nu^4-735\nu^3+812\nu^2-441\nu+90\right)\right)\right)$$

03.10.17.0011.01

$$\mathbf{L}_\nu(z) = \left(\frac{4(\nu-2)(\nu-1)}{z^2}+\frac{8(\nu-4)(\nu-2)\left(z^2+2\nu^2-8\nu+6\right)}{z^4}+1\right)\mathbf{L}_{\nu-6}(z) - \frac{\left(2(\nu-3)\left(3z^4+16\left(\nu^2-6\nu+8\right)z^2+16\left(\nu^4-12\nu^3+49\nu^2-78\nu+40\right)\right)\right)\mathbf{L}_{\nu-5}(z)}{z^5} - \frac{1}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)}\left(2^{1-\nu} z^{\nu-9}\left(z^8+(1-2\nu)z^6+3\left(4\nu^2-8\nu+3\right)z^4+4\left(32\nu^6-456\nu^5+2540\nu^4-7050\nu^3+10163\nu^2-7029\nu+1710\right)z^2+16\left(16\nu^8-288\nu^7+2184\nu^6-9072\nu^5+22449\nu^4-33642\nu^3+29531\nu^2-13698\nu+2520\right)\right)\right)$$

03.10.17.0018.01

$$\mathbf{L}_\nu(z) = C_n(\nu,z)\,\mathbf{L}_{\nu-n}(z) + C_{n-1}(\nu,z)\,\mathbf{L}_{\nu-n-1}(z) - \frac{1}{\sqrt{\pi}}\sum_{j=0}^{n-1}\frac{1}{\Gamma\left(\nu+\frac{1}{2}-j\right)}\left(\frac{z}{2}\right)^{\nu-j-1} C_j(\nu,z)\ /;$$
$$C_0(\nu,z)=1 \bigwedge C_1(\nu,z) = -\frac{2(\nu-1)}{z} \bigwedge C_n(\nu,z) = -\frac{2(\nu-n)}{z} C_{n-1}(\nu,z) + C_{n-2}(\nu,z) \bigwedge n \in \mathbb{N}^+$$

03.10.17.0019.01

$$\mathbf{L}_\nu(z) = C_n(\nu,z)\,\mathbf{L}_{\nu-n}(z) + C_{n-1}(\nu,z)\,\mathbf{L}_{\nu-n-1}(z) - \frac{1}{\sqrt{\pi}}\left(\frac{z}{2}\right)^{\nu-1}\sum_{j=0}^{n-1}\frac{(1-\nu)_j}{\Gamma\left(\nu-j+\frac{1}{2}\right)\left(\frac{z^2}{4}\right)^j}\,{}_2F_3\left(\frac{1-j}{2},-\frac{j}{2};1-\nu,-j,\nu-j;z^2\right)/;$$
$$C_n(\nu,z) = 2^n z^{-n}(1-\nu)_n\,{}_2F_3\left(\frac{1-n}{2},-\frac{n}{2};1-\nu,-n,\nu-n;z^2\right)/;\ n\in\mathbb{N}^+$$

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Functional identities

Relations between contiguous functions

03.10.17.0003.01

$$\mathbf{L}_\nu(z) = \frac{z}{2\nu}\left(\mathbf{L}_{\nu-1}(z)-\mathbf{L}_{\nu+1}(z)\right) - \frac{2^{-\nu-1}z^{\nu+1}}{\sqrt{\pi}\,\nu\,\Gamma\left(\nu+\frac{3}{2}\right)}$$

Differentiation

Low-order differentiation

With respect to ν

03.10.20.0001.01

$$\boldsymbol{L}_\nu^{(1,0)}(z) = \log\left(\frac{z}{2}\right) \boldsymbol{L}_\nu(z) - \left(\frac{z}{2}\right)^{\nu+1} \sum_{k=0}^{\infty} \frac{1}{\Gamma\left(k+\frac{3}{2}\right)\Gamma\left(k+\nu+\frac{3}{2}\right)} \psi\left(k+\nu+\frac{3}{2}\right)\left(\frac{z}{2}\right)^{2k}$$

03.10.20.0014.01

$$\boldsymbol{L}_n^{(1,0)}(z) = (-1)^n K_n(z) + \frac{(-1)^{n+1} 2^{n-1}}{z^n \pi^2} G_{2,4}^{4,2}\left(\frac{z}{2}, \frac{1}{2} \middle| \begin{array}{c} \frac{1}{2}, \frac{1}{2} \\ n, 0, \frac{1}{2}, \frac{1}{2} \end{array}\right) - \frac{1}{\pi} \sum_{k=0}^{n-1} \frac{(-1)^k}{\left(k+\frac{1}{2}\right)_{n-2k}} \left(\frac{z}{2}\right)^{n-2k-1} \left(\log\left(\frac{z}{2}\right) - \psi\left(n-k+\frac{1}{2}\right)\right) + \frac{1}{2} n! \sum_{k=0}^{n-1} \frac{1}{k!\,(n-k)} \left(-\frac{z}{2}\right)^{k-n} \boldsymbol{L}_{-k}(z) /; n \in \mathbb{N}$$

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03.10.20.0015.01

$$\boldsymbol{L}_{-n}^{(1,0)}(z) = (-1)^n K_n(z) + \frac{(-1)^{n+1} 2^{n-1}}{z^n \pi^2} G_{2,4}^{4,2}\left(\frac{z}{2}, \frac{1}{2} \middle| \begin{array}{c} \frac{1}{2}, \frac{1}{2} \\ n, 0, \frac{1}{2}, \frac{1}{2} \end{array}\right) - \frac{n!}{2} \sum_{k=0}^{n-1} \frac{1}{k!\,(n-k)} \left(-\frac{z}{2}\right)^{k-n} \boldsymbol{L}_{-k}(z) /; n \in \mathbb{N}$$

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03.10.20.0016.01

$$\boldsymbol{L}_{n+\frac{1}{2}}^{(1,0)}(z) = \frac{(-1)^n n!}{2\sqrt{\pi}} \left(\frac{z}{2}\right)^{-n-\frac{1}{2}} \sum_{k=0}^{n-1} \frac{\left(\frac{1}{2}\right)_k}{k!\,(n-k)} - \frac{1}{n!\sqrt{\pi}} \log\left(\frac{z}{2}\right)\left(\frac{z}{2}\right)^{n-\frac{1}{2}} {}_3F_0\left(-n, \frac{1}{2}, 1;; \frac{4}{z^2}\right) + \frac{1}{\sqrt{\pi}} \left(\frac{z}{2}\right)^{n-\frac{1}{2}} \sum_{k=0}^{n-1} \frac{\left(\frac{1}{2}\right)_k \psi(-k+n+1)}{(n-k)!} \left(-\frac{4}{z^2}\right)^k - \frac{(-1)^n \sqrt{\pi}\; n!}{2} \left(\frac{z}{2}\right)^{\frac{1}{2}-n} \sum_{k=0}^{n-1} \frac{1}{k!\,(n-k)} \left(-\frac{z}{2}\right)^k \sum_{p=0}^{n-k-1} \frac{1}{p!} \left(-\frac{z}{2}\right)^p \left(\left(2 I_{p-\frac{1}{2}}(z) - 2^{p+\frac{1}{2}} I_{p-\frac{1}{2}}(2z)\right) I_{-k-\frac{1}{2}}(z) - \left(2 I_{\frac{1}{2}-p}(z) - 2^{p+\frac{1}{2}} I_{\frac{1}{2}-p}(2z)\right) I_{k+\frac{1}{2}}(z)\right) + I_{-n-\frac{1}{2}}(z)\,(2\,\mathrm{Chi}(z) - \mathrm{Chi}(2z)) + \frac{(-1)^{n+1}}{2\pi} \left(\frac{z}{2}\right)^{-n-\frac{1}{2}} \Gamma\left(n+\frac{1}{2}\right)\left(\log(4) + \psi\left(\frac{1}{2}-n\right) + 3\gamma\right) + I_{n+\frac{1}{2}}(z)\,(\mathrm{Shi}(2z) - 2\,\mathrm{Shi}(z)) + \frac{(-1)^n n!}{2} \left(\frac{2}{z}\right)^n \sum_{k=0}^{n-1} \frac{1}{k!\,(n-k)} \left(-\frac{z}{2}\right)^k I_{-k-\frac{1}{2}}(z) /; n \in \mathbb{N}$$

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03.10.20.0017.01

$$\boldsymbol{L}_{-n-\frac{1}{2}}^{(1,0)}(z) = I_{n+\frac{1}{2}}(z)\,(2\,\mathrm{Chi}(z) - \mathrm{Chi}(2\,z)) + I_{-n-\frac{1}{2}}(z)\,(\mathrm{Shi}(2\,z) - 2\,\mathrm{Shi}(z)) - \frac{1}{2}\,n!\sum_{k=0}^{n-1}\frac{1}{k!\,(n-k)}\left(-\frac{z}{2}\right)^{k-n} I_{k+\frac{1}{2}}(z) - \frac{n!\,\sqrt{\pi\,z}}{2}\sum_{k=1}^{n}\frac{1}{(n-k)!\,k}\left(-\frac{2}{z}\right)^{k}\sum_{p=0}^{k-1}\frac{(-z)^p}{p!}\left(\left(2^{\frac{1}{2}-p} I_{p-\frac{1}{2}}(z) - I_{p-\frac{1}{2}}(2\,z)\right) I_{n-k+\frac{1}{2}}(z) - \left(2^{\frac{1}{2}-p} I_{\frac{1}{2}-p}(z) - I_{\frac{1}{2}-p}(2\,z)\right) I_{k-n-\frac{1}{2}}(z)\right) \;/;\; n \in \mathbb{N}$$

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03.10.20.0002.01

$$\boldsymbol{L}_{\nu}^{(1,0)}(z) = -\frac{2^{-\nu}\,z^{\nu+3}}{3\sqrt{\pi}\,(2\,\nu+3)\,\Gamma\left(\nu+\frac{5}{2}\right)}\,F_{3\times0\times1}^{1\times1\times2}\left(\begin{matrix}2;\,1;\,1,\,\nu+\frac{3}{2};\\ 2,\,\frac{5}{2},\,\nu+\frac{5}{2};;\,\nu+\frac{5}{2};\end{matrix}\;\frac{z^2}{4},\,\frac{z^2}{4}\right) + \left(\log(z) - \log(2) - \psi\left(\nu+\frac{3}{2}\right)\right)\boldsymbol{L}_{\nu}(z)$$

With respect to z

03.10.20.0003.01

$$\frac{\partial \boldsymbol{L}_{\nu}(z)}{\partial z} = \boldsymbol{L}_{\nu-1}(z) - \frac{\nu}{z}\,\boldsymbol{L}_{\nu}(z)$$

03.10.20.0004.01

$$\frac{\partial \boldsymbol{L}_{\nu}(z)}{\partial z} = \frac{2^{-\nu}\,z^{\nu}}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{3}{2}\right)} + \boldsymbol{L}_{\nu+1}(z) + \frac{\nu}{z}\,\boldsymbol{L}_{\nu}(z)$$

03.10.20.0005.01

$$\frac{\partial \boldsymbol{L}_{\nu}(z)}{\partial z} = \frac{1}{2}\left(\frac{2^{-\nu}\,z^{\nu}}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{3}{2}\right)} + \boldsymbol{L}_{\nu-1}(z) + \boldsymbol{L}_{\nu+1}(z)\right)$$

03.10.20.0006.01

$$\frac{\partial^2 \boldsymbol{L}_{\nu}(z)}{\partial z^2} = \frac{1}{z^2}\left(z^2\,\boldsymbol{L}_{\nu-2}(z) + (z - 2\,z\,\nu)\,\boldsymbol{L}_{\nu-1}(z) + \nu\,(\nu+1)\,\boldsymbol{L}_{\nu}(z)\right)$$

03.10.20.0007.01

$$\frac{\partial^2 \boldsymbol{L}_{\nu}(z)}{\partial z^2} = \frac{1}{4}\,(\boldsymbol{L}_{\nu-2}(z) + \boldsymbol{L}_{\nu+2}(z) + 2\,\boldsymbol{L}_{\nu}(z)) + \frac{2^{-\nu-1}\left(z^2 + 8\,\nu^2 + 14\,\nu + 3\right) z^{\nu-1}}{\sqrt{\pi}\,(4\,\nu\,(\nu+2)+3)\,\Gamma\left(\nu+\frac{1}{2}\right)}$$

03.10.20.0008.01

$$\frac{\partial (z^{\nu}\,\boldsymbol{L}_{\nu}(z))}{\partial z} = z^{\nu}\,\boldsymbol{L}_{\nu-1}(z)$$

03.10.20.0009.01

$$\frac{\partial (z^{-\nu}\,\boldsymbol{L}_{\nu}(z))}{\partial z} = z^{-\nu}\,\boldsymbol{L}_{\nu+1}(z) + \frac{2^{-\nu}}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{3}{2}\right)}$$

Symbolic differentiation

With respect to z

03.10.20.0018.01

$$\frac{\partial^n \boldsymbol{L}_\nu(z)}{\partial z^n} = \frac{n!}{\left(-\frac{z}{2}\right)^n} \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(-1)^k}{2^{2k}\, k!\,(n-2k)!} \sum_{p=0}^{n-k} \binom{n-k}{p} \left(\frac{\nu}{2}\right)_{-k+n-p} \left(-\frac{z}{2}\right)^p \boldsymbol{L}_{\nu-p}(z) \;/;\; n \in \mathbb{N}$$

Brychkov Yu.A. (2005)

03.10.20.0019.01

$$\frac{\partial^n \boldsymbol{L}_\nu(z)}{\partial z^n} = \frac{n!}{\left(-\frac{z}{2}\right)^n} \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(-1)^k}{2^{2k}\,(k!\,(n-2k)!)} \sum_{p=0}^{n-k} (-1)^p \binom{n-k}{p} \left(-\frac{\nu}{2}\right)_{-k+n-p} \left(\left(\frac{z}{2}\right)^p \boldsymbol{L}_{p+\nu}(z) + \frac{1}{\pi}\left(\frac{z}{2}\right)^{2p+\nu-1} \sum_{r=0}^{p-1} \frac{(-1)^r\, \Gamma\!\left(r+\frac{1}{2}\right)}{\Gamma\!\left(p-r+\nu+\frac{1}{2}\right)} \left(\frac{z}{2}\right)^{-2r}\right) /;\; n \in \mathbb{N}$$

Brychkov Yu.A. (2005)

03.10.20.0020.01

$$\frac{\partial^n \boldsymbol{L}_\nu(z)}{\partial z^n} = z^{-n} \sum_{m=0}^{n} (-1)^{m+n} \binom{n}{m} (-\nu)_{n-m} \sum_{k=0}^{m} \frac{(-1)^{k-1}\, 2^{2k-m}\, (-m)_{2(m-k)}\, (\nu)_k}{(m-k)!} \left(\frac{z}{2} \sum_{j=0}^{k-1} \frac{(k-j-1)!\left(-\frac{z^2}{4}\right)^j}{j!\,(k-2j-1)!\,(1-k-\nu)_j\,(\nu)_{j+1}} \boldsymbol{L}_{\nu-1}(z) - \sum_{j=0}^{k} \frac{(k-j)!\left(-\frac{z^2}{4}\right)^j}{j!\,(k-2j)!\,(1-k-\nu)_j\,(\nu)_j} \boldsymbol{L}_\nu(z)\right) + \frac{2^{-\nu} z^{\nu-n+1}}{\sqrt{\pi}\,\Gamma\!\left(\nu+\frac{1}{2}\right)} \sum_{i=1}^{n-1} \sum_{m=0}^{i} (-1)^{i+m} \binom{i}{m} (-\nu)_{i-m} \sum_{k=0}^{m} \frac{(-1)^{k-1}\, 2^{2k-m}\, (-m)_{2(m-k)}\, (\nu)_k}{(m-k)!} \sum_{j=0}^{k-1} \frac{(-1)^j\, 2^{-2j}\, (k-j-1)!\,(2j-n+\nu+2)_{n-i-1}\, z^{2j}}{j!\,(k-2j-1)!\,(1-k-\nu)_j\,(\nu)_{j+1}} \;/;\; n \in \mathbb{N}$$

03.10.20.0010.02

$$\frac{\partial^n \boldsymbol{L}_\nu(z)}{\partial z^n} = 2^{n-2\nu-2} \sqrt{\pi}\, z^{\nu-n+1}\, \Gamma(\nu+2)\, {}_3\tilde{F}_4\!\left(1, \frac{\nu}{2}+1, \frac{\nu+3}{2}; \frac{3}{2}, \frac{\nu-n}{2}+1, \frac{\nu-n+3}{2}, \nu+\frac{3}{2}; \frac{z^2}{4}\right) /;\; n \in \mathbb{N}$$

Fractional integro-differentiation

With respect to z

03.10.20.0011.01

$$\frac{\partial^\alpha \boldsymbol{L}_\nu(z)}{\partial z^\alpha} = 2^{\alpha-2\nu-2} \sqrt{\pi}\, z^{1-\alpha+\nu}\, \Gamma(\nu+2)\, {}_3\tilde{F}_4\!\left(1, \frac{\nu}{2}+1, \frac{\nu+3}{2}; \frac{3}{2}, \frac{\nu-\alpha}{2}+1, \frac{3+\nu-\alpha}{2}, \nu+\frac{3}{2}; \frac{z^2}{4}\right) /;\; -\nu \notin \mathbb{N}^+$$

03.10.20.0012.01

$$\frac{\partial^\alpha \boldsymbol{L}_\nu(z)}{\partial z^\alpha} = (-1)^{-\left\lfloor \frac{\nu+1}{2} \right\rfloor} 2^{\alpha-2(\nu+1)+4\left\lfloor \frac{\nu+1}{2} \right\rfloor} \sqrt{\pi}\, \Gamma\!\left(\nu-2\left\lfloor \frac{\nu+1}{2} \right\rfloor+2\right) {}_3\tilde{F}_4\!\left(1, \frac{1}{2}\left(\nu-2\left\lfloor \frac{\nu+1}{2} \right\rfloor+2\right), \frac{1}{2}\left(\nu-2\left\lfloor \frac{\nu+1}{2} \right\rfloor+3\right); \frac{3}{2}-\left\lfloor \frac{\nu+1}{2} \right\rfloor, \frac{1}{2}\left(\nu-\alpha-2\left\lfloor \frac{\nu+1}{2} \right\rfloor+2\right), \frac{1}{2}\left(\nu-\alpha-2\left\lfloor \frac{\nu+1}{2} \right\rfloor+3\right), \nu-\left\lfloor \frac{\nu+1}{2} \right\rfloor+\frac{3}{2}; \frac{z^2}{4}\right) z^{\nu-\alpha-2\left\lfloor \frac{\nu+1}{2} \right\rfloor+1} + \sum_{k=0}^{-\left\lfloor \frac{\nu+3}{2} \right\rfloor} \frac{(-1)^\nu\, 2^{-2k-\nu-1}\, z^{2k-\alpha+\nu+1}\,(\log(z)+\psi(-2k-\nu-1)-\psi(2k-\alpha+\nu+2))}{(-2k-\nu-2)!\,\Gamma\!\left(k+\frac{3}{2}\right)\Gamma\!\left(k+\nu+\frac{3}{2}\right)\Gamma(2k-\alpha+\nu+2)} \;/;\; -\nu \in \mathbb{N}^+$$

03.10.20.0013.01

$$\frac{\partial^\alpha \boldsymbol{L}_\nu(z)}{\partial z^\alpha} = \sum_{k=0}^{\infty} \frac{2^{-2k-\nu-1}\, \mathcal{FC}_{\exp}^{(\alpha)}(z,\, 2k+\nu+1)\, z^{2k-\alpha+\nu+1}}{\Gamma\left(k+\frac{3}{2}\right)\Gamma\left(k+\nu+\frac{3}{2}\right)}$$

Integration

Indefinite integration

Involving only one direct function

03.10.21.0001.01

$$\int \boldsymbol{L}_\nu(z)\, dz = \frac{2^{-\nu} z^{\nu+2}}{\sqrt{\pi}\,(\nu+2)\,\Gamma\left(\nu+\frac{3}{2}\right)}\, {}_2F_3\left(1, \frac{\nu}{2}+1; \frac{3}{2}, \frac{\nu}{2}+2, \nu+\frac{3}{2}; \frac{z^2}{4}\right)$$

Involving one direct function and elementary functions

Involving power function

03.10.21.0002.01

$$\int z^{\alpha-1}\, \boldsymbol{L}_\nu(z)\, dz = \frac{2^{-\nu} z^{\alpha+\nu+1}}{\sqrt{\pi}\,(\alpha+\nu+1)\,\Gamma\left(\nu+\frac{3}{2}\right)}\, {}_2F_3\left(1, \frac{\alpha+\nu+1}{2}; \frac{3}{2}, \frac{\alpha+\nu+3}{2}, \nu+\frac{3}{2}; \frac{z^2}{4}\right)$$

03.10.21.0003.01

$$\int z^{1-\nu}\, \boldsymbol{L}_\nu(z)\, dz = z^{1-\nu}\, \boldsymbol{L}_{\nu-1}(z) - \frac{2^{1-\nu} z}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)}$$

03.10.21.0004.01

$$\int z^n\, \boldsymbol{L}_\nu(a z)\, dz = 2^{-\nu-2}\, a\, z^{n+2}\, (a z)^\nu\, \Gamma\left(\frac{1}{2}(n+\nu+2)\right) {}_2\tilde{F}_3\left(1, \frac{1}{2}(n+\nu+2); \nu+\frac{3}{2}, \frac{3}{2}, \frac{1}{2}(n+\nu+4); \frac{a^2 z^2}{4}\right)$$

03.10.21.0005.01

$$\int z^{1-\nu}\, \boldsymbol{L}_\nu(a z)\, dz = \frac{z^{1-\nu}\left(\boldsymbol{L}_{\nu-1}(a z) - \frac{2^{1-\nu}(a z)^\nu}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)}\right)}{a}$$

03.10.21.0006.01

$$\int z^{\nu+1}\, \boldsymbol{L}_\nu(a z)\, dz = \frac{z^{\nu+1}\, \boldsymbol{L}_{\nu+1}(a z)}{a}$$

Involving exponential function and a power function

03.10.21.0007.01

$$\int z^\nu\, e^{-z}\, \boldsymbol{L}_\nu(z)\, dz = \frac{1}{2\nu+1}\left(e^{-z}\left(\boldsymbol{L}_\nu(z) + \boldsymbol{L}_{\nu+1}(z)\right) z^{\nu+1} + \frac{2^{-\nu}\,\Gamma(2\nu+2, z)}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{3}{2}\right)}\right)$$

03.10.21.0008.01

$$\int z^{-\nu} e^{-z} \mathbf{L}_\nu(z)\, dz = \frac{e^{-z}}{2\nu-1}\left(-(\mathbf{L}_{\nu-1}(z)+\mathbf{L}_\nu(z))\, z^{1-\nu} - \frac{2^{1-\nu}}{\sqrt{\pi}\, \Gamma\left(\nu+\frac{1}{2}\right)}\right)$$

03.10.21.0009.01

$$\int z^{\nu} e^{z} \mathbf{L}_\nu(z)\, dz = \frac{z^\nu}{2\nu+1}\left(\frac{2^{-\nu} z^\nu\, \Gamma(2\nu+2, -z)\, (-z)^{-2\nu}}{\sqrt{\pi}\, \Gamma\left(\nu+\frac{3}{2}\right)} + e^z z\, (\mathbf{L}_\nu(z)-\mathbf{L}_{\nu+1}(z))\right)$$

03.10.21.0010.01

$$\int z^{-\nu} e^{z} \mathbf{L}_\nu(z)\, dz = \frac{e^{z}}{2\nu-1}\left(z^{1-\nu}\, (\mathbf{L}_{\nu-1}(z)-\mathbf{L}_\nu(z)) - \frac{2^{1-\nu}}{\sqrt{\pi}\, \Gamma\left(\nu+\frac{1}{2}\right)}\right)$$

Involving direct function and Bessel-, Airy-, Struve-type functions

Involving Bessel functions

Involving Bessel I and power

03.10.21.0011.01

$$\int z^n\, (I_{-\nu}(a z) - \mathbf{L}_\nu(a z))\, dz = 2^{-\nu-2} z^{n+1} (a z)^{-\nu} \left(2^{2\nu+1}\, \Gamma\left(\frac{1}{2}(n-\nu+1)\right) {}_1\tilde{F}_2\left(\frac{1}{2}(n-\nu+1); 1-\nu, \frac{1}{2}(n-\nu+3); \frac{a^2 z^2}{4}\right) - a z\, (a z)^{2\nu}\, \Gamma\left(\frac{1}{2}(n+\nu+2)\right) {}_2\tilde{F}_3\left(1, \frac{1}{2}(n+\nu+2); \nu+\frac{3}{2}, \frac{3}{2}, \frac{1}{2}(n+\nu+4); \frac{a^2 z^2}{4}\right)\right)$$

03.10.21.0012.01

$$\int z^n\, (I_{\nu}(a z) - \mathbf{L}_\nu(a z))\, dz = 2^{-\nu-2} z^{n+1} (a z)^{\nu} \left(2\, \Gamma\left(\frac{1}{2}(n+\nu+1)\right) {}_1\tilde{F}_2\left(\frac{1}{2}(n+\nu+1); \nu+1, \frac{1}{2}(n+\nu+3); \frac{a^2 z^2}{4}\right) - a z\, \Gamma\left(\frac{1}{2}(n+\nu+2)\right) {}_2\tilde{F}_3\left(1, \frac{1}{2}(n+\nu+2); \nu+\frac{3}{2}, \frac{3}{2}, \frac{1}{2}(n+\nu+4); \frac{a^2 z^2}{4}\right)\right)$$

Definite integration

Involving the direct function

03.10.21.0013.01

$$\int_0^\infty t^{\alpha-1} e^{-a t} \mathbf{L}_\nu(b t)\, dt = \frac{2^{-\nu} a^{-\alpha-\nu-1} b^{\nu+1}\, \Gamma(\alpha+\nu+1)}{\sqrt{\pi}\, \Gamma\left(\nu+\frac{3}{2}\right)}\, {}_3F_2\left(1, \frac{1}{2}(\alpha+\nu+2), \frac{1}{2}(\alpha+\nu+1); \nu+\frac{3}{2}, \frac{3}{2}; \frac{b^2}{a^2}\right) /;$$

$\mathrm{Re}(\alpha+\nu) > -1 \land \mathrm{Re}(a) > |\mathrm{Re}(b)|$

Integral transforms

Laplace transforms

03.10.22.0001.01

$$\mathcal{L}_t[\boldsymbol{L}_\nu(t)](z) = \frac{2^{-\nu} z^{-\nu-2} \Gamma(\nu+2)}{\sqrt{\pi}\, \Gamma\left(\nu+\frac{3}{2}\right)} {}_3F_2\left(1, \frac{\nu+3}{2}, \frac{\nu+2}{2}; \nu+\frac{3}{2}, \frac{3}{2}; \frac{1}{z^2}\right) /; \operatorname{Re}(\nu) > -2$$

Representations through more general functions

Through hypergeometric functions

Involving ${}_p\tilde{F}_q$

03.10.26.0001.01

$$\boldsymbol{L}_\nu(z) = \left(\frac{z}{2}\right)^{\nu+1} {}_1\tilde{F}_2\left(1; \frac{3}{2}, \nu+\frac{3}{2}; \frac{z^2}{4}\right)$$

03.10.26.0010.01

$$\boldsymbol{L}_\nu(z) = \left(\frac{z}{2}\right)^{-\nu} {}_0\tilde{F}_1\left(; 1-\nu; \frac{z^2}{4}\right) /; -\nu-\frac{3}{2} \in \mathbb{N}$$

Involving ${}_pF_q$

03.10.26.0002.01

$$\boldsymbol{L}_\nu(z) = \frac{z^{\nu+1}}{2^\nu \sqrt{\pi}\, \Gamma\left(\nu+\frac{3}{2}\right)} {}_1F_2\left(1; \frac{3}{2}, \nu+\frac{3}{2}; \frac{z^2}{4}\right) /; -\nu-\frac{3}{2} \notin \mathbb{N}$$

03.10.26.0011.01

$$\boldsymbol{L}_\nu(z) = \frac{1}{\Gamma(1-\nu)} \left(\frac{z}{2}\right)^{-\nu} {}_0F_1\left(; 1-\nu; \frac{z^2}{4}\right) /; -\nu-\frac{3}{2} \in \mathbb{N}$$

Through Meijer *G*

Classical cases for the direct function itself

03.10.26.0003.01

$$\boldsymbol{L}_\nu(z) = -\pi \csc\left(\frac{\pi \nu}{2}\right) z^{\nu-1} \left(z^2\right)^{\frac{1-\nu}{2}} G_{2,4}^{1,1}\left(\frac{z^2}{4} \,\middle|\, \begin{matrix} \frac{\nu+1}{2}, \frac{1}{2} \\ \frac{\nu+1}{2}, \frac{1}{2}, -\frac{\nu}{2}, \frac{\nu}{2} \end{matrix}\right)$$

03.10.26.0004.01

$$\boldsymbol{L}_\nu(z) = -\pi \csc\left(\frac{\pi \nu}{2}\right) G_{2,4}^{1,1}\left(\frac{z^2}{4} \,\middle|\, \begin{matrix} \frac{\nu+1}{2}, \frac{1}{2} \\ \frac{\nu+1}{2}, \frac{1}{2}, -\frac{\nu}{2}, \frac{\nu}{2} \end{matrix}\right) /; \operatorname{Re}(z) > 0$$

03.10.26.0005.01

$$\boldsymbol{L}_\nu\left(\sqrt{z}\right) = -\pi \csc\left(\frac{\pi \nu}{2}\right) G_{2,4}^{1,1}\left(\frac{z}{4} \,\middle|\, \begin{matrix} \frac{\nu+1}{2}, \frac{1}{2} \\ \frac{\nu+1}{2}, \frac{1}{2}, -\frac{\nu}{2}, \frac{\nu}{2} \end{matrix}\right)$$

Classical cases involving Bessel *I*

03.10.26.0006.01

$$I_\nu\left(\sqrt{z}\right) - \boldsymbol{L}_\nu\left(\sqrt{z}\right) = \frac{1}{\pi} G_{1,3}^{2,1}\left(\frac{z}{4} \middle| \begin{array}{c} \frac{\nu+1}{2} \\ \frac{\nu+1}{2}, \frac{\nu}{2}, -\frac{\nu}{2} \end{array}\right)$$

Generalized cases for the direct function itself

03.10.26.0007.01

$$\boldsymbol{L}_\nu(z) = -\pi \csc\left(\frac{\pi\nu}{2}\right) G_{2,4}^{1,1}\left(\frac{z}{2}, \frac{1}{2} \middle| \begin{array}{c} \frac{\nu+1}{2}, \frac{1}{2} \\ \frac{\nu+1}{2}, \frac{1}{2}, -\frac{\nu}{2}, \frac{\nu}{2} \end{array}\right)$$

Generalized cases involving Bessel *I*

03.10.26.0008.01

$$I_\nu(z) - \boldsymbol{L}_\nu(z) = \frac{1}{\pi} G_{1,3}^{2,1}\left(\frac{z}{2}, \frac{1}{2} \middle| \begin{array}{c} \frac{\nu+1}{2} \\ \frac{\nu+1}{2}, \frac{\nu}{2}, -\frac{\nu}{2} \end{array}\right)$$

Through other functions

03.10.26.0009.01

$$\boldsymbol{L}_\nu(z) = z \csc(\pi\nu) \left(\frac{\sqrt{\pi}}{\Gamma(1-\nu)\,\Gamma\left(\nu+\frac{1}{2}\right)} I_\nu(z)\, {}_1F_2\left(\frac{1}{2}; \frac{3}{2}, 1-\nu; \frac{z^2}{4}\right) - \frac{z^{2\nu}}{\Gamma(2(\nu+1))} I_{-\nu}(z)\, {}_1F_2\left(\nu+\frac{1}{2}; \nu+1, \nu+\frac{3}{2}; \frac{z^2}{4}\right)\right)$$

Representations through equivalent functions

With related functions

03.10.27.0001.01

$$\boldsymbol{L}_\nu(i z) = i\,(i z)^\nu z^{-\nu} \boldsymbol{H}_\nu(z)$$

03.10.27.0002.01

$$\boldsymbol{L}_\nu(-i z) = -i\,(-i z)^\nu z^{-\nu} \boldsymbol{H}_\nu(z)$$

03.10.27.0003.01

$$\boldsymbol{L}_\nu(z) = I_{-\nu}(z) - \frac{2^{1-\nu} z^{\nu-1}}{\sqrt{\pi}\left(\nu-\frac{1}{2}\right)!} \sum_{k=0}^{\nu-\frac{1}{2}} \left(\frac{1}{2}\right)_k \left(\frac{1}{2}-\nu\right)_k \left(\frac{z^2}{4}\right)^{-k} /; \nu - \frac{1}{2} \in \mathbb{Z}$$

03.10.27.0004.01

$$\boldsymbol{L}_\nu(z) = I_{-\nu}(z) /; -\nu - \frac{1}{2} \in \mathbb{N}$$

Inequalities

03.10.29.0001.01

$$\boldsymbol{L}_\nu(x) \geq 0 /; x \geq 0 \land \nu \in \mathbb{R}$$

Theorems

The Coulomb potential

The Coulomb potential, including the order quantum electrodynamical correction, is given by

$$V(r) \propto \frac{1}{r}\left(1+\alpha \frac{1}{72\,\pi^2}\left(\pi\left(3+\frac{r^2}{\lambda_C^2}\right)\frac{r}{\lambda_C}-2\left(4+\frac{r^2}{\lambda_C^2}\right)\frac{r}{\lambda_C}K_1\left(\frac{r}{\lambda_C}\right)-2\left(\frac{r^4}{\lambda_C^4}+\frac{2\,r^2}{\lambda_C^2}-6\right)K_0\left(\frac{r}{\lambda_C}\right)+\pi\frac{r^2}{\lambda_C^2}\left(3+\frac{r^2}{\lambda_C^2}\right)\left(K_1\left(\frac{r}{\lambda_C}\right)\mathbf{L}_0\left(\frac{r}{\lambda_C}\right)-K_0\left(\frac{r}{\lambda_C}\right)\mathbf{L}_1\left(\frac{r}{\lambda_C}\right)\right)\right)\right).$$

Here, α is the fine structure constant and is λ_C is the Compton wavelength.

History

– J. W. Nicholson (1911)

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