ThreeJSymbol

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Notations

Traditional name

Wigner 3 - j symbol

Traditional notation

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$$

Mathematica StandardForm notation

ThreeJSymbol[{j_1, m_1}, {j_2, m_2}, {j_3, m_3}]

Primary definition

07.39.02.0001.01

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \delta_{m_1+m_2+m_3,0} \left((-1)^{-j_1+j_2+m_3} \sqrt{(j_1-j_2+j_3)!}\,\sqrt{(-j_1+j_2+j_3)!}\,\sqrt{(j_3-m_3)!}\,\sqrt{(j_3+m_3)!}\,\sqrt{(j_1+m_1)!}\,\sqrt{(j_2-m_2)!}\right) \Big/ \left(\sqrt{(j_1+j_2-j_3)!}\,\sqrt{(j_1+j_2+j_3+1)!}\,\sqrt{(j_1-m_1)!}\,\sqrt{(j_2+m_2)!}\right) {}_3\tilde{F}_2(-j_1-j_2+j_3, m_1-j_1, -j_2-m_2; -j_2+j_3+m_1+1, -j_1+j_3-m_2+1; 1) \,/;\, \mathcal{P}hysical\mathcal{Q}(j_1, m_1, j_2, m_2, j_3, m_3)$$

07.39.02.0002.01

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = 0 \,/;\, \neg \mathcal{P}hysical\mathcal{Q}(\{j_1, m_1\}, \{j_2, m_2\}, \{j_3, m_3\})$$

$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$ is the $3j$-symbol. The $3j$-symbol appears in the quantum mechanical treatment of angular momentum, where $j_3$is the full angular momentum and $-m_3$is its projection onto a given axis. The values of the $3j$-symbol which are physically realizable (in a Minkowski space-time) are obtained under the additional restrictions:

07.39.02.0003.01

$$\mathcal{P}hysical\mathcal{Q}(\{j_1, m_1\}, \{j_2, m_2\}, \{j_3, m_3\}) = \mathcal{T}riangular\mathcal{Q}(j_1, j_2, j_3) \land j_1 - m_1 \in \mathbb{Z} \land j_2 - m_2 \in \mathbb{Z} \land j_3 - m_3 \in \mathbb{Z} \land -j_1 \le m_1 \le j_1 \land -j_2 \le m_2 \le j_2 \land -j_3 \le m_3 \le j_3$$

07.39.02.0004.01

TriangularQ[j_1, j_2, j_3] == 2 j_1 ∈ Integers ∧ j_1 ≥ 0 ∧ 2 j_2 ∈ Integers ∧ j_2 ≥ 0 ∧ 2 j_3 ∈ Integers ∧ j_3 ≥ 0 ∧ j_1 + j_2 + j_3 ∈ Integers ∧ Abs[j_1 − j_2] ≤ j_3 ≤ j_1 + j_2

07.39.02.0004.01

$$\mathcal{T}riangular\mathcal{Q}(j_1, j_2, j_3) = 2\,j_1 \in \mathbb{N} \land 2\,j_2 \in \mathbb{N} \land 2\,j_3 \in \mathbb{N} \land j_1 + j_2 + j_3 \in \mathbb{N} \land |j_1 - j_2| \le j_3 \le j_1 + j_2$$

Specific values

Specialized values

Nonphysical cases

07.39.03.0001.01

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} == 0 \;/;\; m_1 + m_2 + m_3 \neq 0$$

Fixed j_1, j_2, m_1, m_2

07.39.03.0002.01

$$\begin{pmatrix} j_1 & j_2 & 0 \\ m_1 & m_2 & 0 \end{pmatrix} == \frac{(-1)^{j_2-m_1}}{\sqrt{2 j_1 + 1}} \delta_{j_1,j_2} \delta_{m_1,-m_2} \;/;\; \mathcal{P}hysical\mathcal{Q}(\{j_1, m_1\}, \{j_2, m_2\}, \{0, 0\})$$

07.39.03.0003.01

$$\begin{pmatrix} j_1 & j_2 & j_1 + j_2 \\ m_1 & m_2 & -m_1 - m_2 \end{pmatrix} == (-1)^{-j_1+j_2-m_1-m_2} \frac{\sqrt{(2 j_1)!} \sqrt{(2 j_2)!} \sqrt{(j_1 + j_2 + m_1 + m_2)!} \sqrt{(j_1 + j_2 - m_1 - m_2)!}}{\sqrt{2 j_1 + 2 j_2 + 1} \sqrt{(2 j_1 + 2 j_2)!} \sqrt{(j_1 + m_1)!} \sqrt{(j_1 - m_1)!} \sqrt{(j_2 + m_2)!} \sqrt{(j_2 - m_2)!}} \;/;\; \mathcal{P}hysical\mathcal{Q}(\{j_1, m_1\}, \{j_2, m_2\}, \{j_1 + j_2, -m_1 - m_2\})$$

07.39.03.0004.01

$$\begin{pmatrix} j_1 & j_2 & j_1 + j_2 - 1 \\ m_1 & m_2 & -m_1 - m_2 \end{pmatrix} == (-1)^{-j_1+j_2-m_1-m_2} \, 2 (j_2 m_1 - j_1 m_2) \frac{\sqrt{(2 j_1 - 1)!} \sqrt{(2 j_2 - 1)!} \sqrt{(j_1 + j_2 + m_1 + m_2 - 1)!} \sqrt{(j_1 + j_2 - m_1 - m_2 - 1)!}}{\sqrt{(2 j_1 + 2 j_2)!} \sqrt{(j_1 + m_1)!} \sqrt{(j_1 - m_1)!} \sqrt{(j_2 + m_2)!} \sqrt{(j_2 - m_2)!}} \;/;\; \mathcal{P}hysical\mathcal{Q}(\{j_1, m_1\}, \{j_2, m_2\}, \{j_1 + j_2 - 1, -m_1 - m_2\})$$

07.39.03.0005.01

$$\begin{pmatrix} j_1 & j_2 & j_1 + j_2 - 2 \\ m_1 & m_2 & -m_1 - m_2 \end{pmatrix} == (-1)^{-j_1+j_2-m_1-m_2} \left(\sqrt{2} \sqrt{j_1} \sqrt{2 j_1 - 1} \sqrt{j_2} \sqrt{2 j_2 - 1} \sqrt{(j_1 - m_1)!} \sqrt{(j_1 + m_1)!} \sqrt{(j_2 - m_2)!} \sqrt{(j_2 + m_2)!} \sqrt{(j_1 + j_2 - m_1 - m_2 - 2)!} \sqrt{(j_1 + j_2 + m_1 + m_2 - 2)!}\right) \Big/ \left(\sqrt{(2 j_1)!} \sqrt{(2 j_2)!} \sqrt{(2 j_1 + 2 j_2 - 4)!} \sqrt{2 j_1 + 2 j_2 - 3} \sqrt{2 j_1 + 2 j_2 - 2} \sqrt{2 j_1 + 2 j_2 - 1}\right) \left(\frac{(2 j_1 - 2)! (2 j_2 - 2)!}{(j_1 - m_1)! (j_1 + m_1 - 2)! (j_2 + m_2)! (j_2 - m_2 - 2)!} - \frac{2 (2 j_1 - 2)! (2 j_2 - 2)!}{(j_1 - m_1 - 1)! (j_1 + m_1 - 1)! (j_2 + m_2 - 1)! (j_2 - m_2 - 1)!} + \frac{(2 j_1 - 2)! (2 j_2 - 2)!}{(j_1 - m_1 - 2)! (j_1 + m_1)! (j_2 + m_2 - 2)! (j_2 - m_2)!}\right) \;/;\; \mathcal{P}hysical\mathcal{Q}(\{j_1, m_1\}, \{j_2, m_2\}, \{j_1 + j_2 - 2, -m_1 - m_2\})$$

07.39.03.0006.01

$$\begin{pmatrix} j_1 & j_2 & j_1-j_2+2 \\ m_1 & m_2 & -m_1-m_2 \end{pmatrix} == (-1)^{-j_1+2j_2-m_1} \left(\sqrt{j_2}\,\sqrt{2j_2-1}\,\sqrt{2j_1-2j_2+4}\,\sqrt{2j_1-2j_2+3}\,\sqrt{(j_1-m_1)!}\,\sqrt{(j_1+m_1)!}\,\sqrt{(j_2-m_2)!}\,\sqrt{(j_2+m_2)!}\,\sqrt{(j_1-j_2-m_1-m_2+2)!}\,\sqrt{(j_1-j_2+m_1+m_2+2)!}\right) \Big/ \left(\sqrt{2j_1+1}\,\sqrt{2j_1+2}\,\sqrt{2j_1+3}\,\sqrt{(2j_1)!}\,\sqrt{(2j_2)!}\,\sqrt{(2j_1-2j_2+4)!}\right) \left(\frac{(2j_2-2)!\,(2j_1-2j_2+2)!}{(j_2+m_2)!\,(j_2-m_2-2)!\,(j_1-j_2-m_1-m_2)!\,(j_1-j_2+m_1+m_2+2)!} - \frac{2\,(2j_2-2)!\,(2j_1-2j_2+2)!}{(j_2+m_2-1)!\,(j_2-m_2-1)!\,(j_1-j_2-m_1-m_2+1)!\,(j_1-j_2+m_1+m_2+1)!} + \frac{(2j_2-2)!\,(2j_1-2j_2+2)!}{(j_2+m_2-2)!\,(j_2-m_2)!\,(j_1-j_2-m_1-m_2+2)!\,(j_1-j_2+m_1+m_2)!}\right) /;$$

$\mathcal{P}hysical\mathcal{Q}(\{j_1, m_1\}, \{j_2, m_2\}, \{j_1-j_2+2, -m_1-m_2\})$

07.39.03.0007.01

$$\begin{pmatrix} j_1 & j_2 & j_1-j_2+1 \\ m_1 & m_2 & -m_1-m_2 \end{pmatrix} == (-1)^{-j_1+2j_2-m_1+1}\, 2\,(j_1 m_2 + j_2 m_1 + m_2) \frac{\sqrt{(2j_2-1)!}\,\sqrt{(2j_1-2j_2+1)!}\,\sqrt{(j_1+m_1)!}\,\sqrt{(j_1-m_1)!}}{\sqrt{(2j_1+2)!}\,\sqrt{(j_2+m_2)!}\,\sqrt{(j_2-m_2)!}\,\sqrt{(j_1-j_2+m_1+m_2+1)!}\,\sqrt{(j_1-j_2-m_1-m_2+1)!}} /;$$

$\mathcal{P}hysical\mathcal{Q}(\{j_1, m_1\}, \{j_2, m_2\}, \{j_1-j_2+1, -m_1-m_2\})$

07.39.03.0008.01

$$\begin{pmatrix} j_1 & j_2 & j_1-j_2 \\ m_1 & m_2 & -m_1-m_2 \end{pmatrix} == (-1)^{-j_1+2j_2-m_1} \frac{\sqrt{(j_1+m_1)!}\,\sqrt{(j_1-m_1)!}\,\sqrt{(2j_2)!}\,\sqrt{(2j_1-2j_2)!}}{\sqrt{(2j_1+1)!}\,\sqrt{(j_2+m_2)!}\,\sqrt{(j_2-m_2)!}\,\sqrt{(j_1-j_2+m_1+m_2)!}\,\sqrt{(j_1-j_2-m_1-m_2)!}} /;$$

$\mathcal{P}hysical\mathcal{Q}(\{j_1, m_1\}, \{j_2, m_2\}, \{j_1-j_2, -m_1-m_2\})$

07.39.03.0009.01

$$\begin{pmatrix} j_1 & j_2 & m_1+m_2 \\ m_1 & m_2 & -m_1-m_2 \end{pmatrix} == (-1)^{j_2-2m_1-m_2} \left(\sqrt{(2m_1+2m_2)!}\,\sqrt{(j_1+j_2-m_1-m_2)!}\,\sqrt{(j_1+m_1)!}\,\sqrt{(j_2+m_2)!}\right) \Big/ \left(\sqrt{(j_1+j_2+m_1+m_2+1)!}\,\sqrt{(j_1-j_2+m_1+m_2)!}\,\sqrt{(-j_1+j_2+m_1+m_2)!}\,\sqrt{(j_1-m_1)!}\,\sqrt{(j_2-m_2)!}\right) /;$$

$\mathcal{P}hysical\mathcal{Q}(\{j_1, m_1\}, \{j_2, m_2\}, \{m_1+m_2, -m_1-m_2\})$

07.39.03.0010.01

$$\begin{pmatrix} j_1 & j_2 & m_1+m_2+1 \\ m_1 & m_2 & -m_1-m_2 \end{pmatrix} == (-1)^{j_2-2m_1-m_2} \left((j_2-m_2)(j_2+m_2+1)-(j_1-m_1)(j_1+m_1+1)\right) \left(\sqrt{(2m_1+2m_2+1)!}\,\sqrt{(j_1+j_2-m_1-m_2-1)!}\,\sqrt{(j_1+m_1)!}\,\sqrt{(j_2+m_2)!}\right) \Big/ \left(\sqrt{(j_1+j_2+m_1+m_2+2)!}\,\sqrt{(j_1-j_2+m_1+m_2+1)!}\,\sqrt{(-j_1+j_2+m_1+m_2+1)!}\,\sqrt{(j_1-m_1)!}\,\sqrt{(j_2-m_2)!}\right) /;$$

$\mathcal{P}hysical\mathcal{Q}(\{j_1, m_1\}, \{j_2, m_2\}, \{m_1+m_2+1, -m_1-m_2\})$

Fixed j_1, j_2, j, m_2

07.39.03.0011.01

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ j_1 & m_2 & -j_1-m_2 \end{pmatrix} = (-1)^{-2j_1+j_2-m_2} \frac{\sqrt{(2j_1)!}\sqrt{(-j_1+j_2+j_3)!}\sqrt{(j_2-m_2)!}\sqrt{(j_3+j_1+m_2)!}}{\sqrt{(j_1+j_2+j_3+1)!}\sqrt{(j_1-j_2+j_3)!}\sqrt{(j_1+j_2-j_3)!}\sqrt{(j_2+m_2)!}\sqrt{(j_3-j_1-m_2)!}} /;$$

$\mathcal{P}hysical\mathcal{Q}(\{j_1, j_1\}, \{j_2, m_2\}, \{j_3, -j_1-m_2\})$

07.39.03.0012.01

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ j_1-1 & m_2 & -j_1-m_2+1 \end{pmatrix} = (-1)^{-2j_1+j_2-m_2+1}((j_3-j_1-m_2+1)(j_3+j_1+m_2)-(j_2+m_2)(j_2-m_2+1)) \frac{\sqrt{(2j_1-1)!}\sqrt{(-j_1+j_2+j_3)!}\sqrt{(j_2-m_2)!}\sqrt{(j_3+j_1+m_2-1)!}}{\sqrt{(j_1+j_2+j_3+1)!}\sqrt{(j_1-j_2+j_3)!}\sqrt{(j_1+j_2-j_3)!}\sqrt{(j_2+m_2)!}\sqrt{(j_3-j_1-m_2+1)!}} /;$$

$\mathcal{P}hysical\mathcal{Q}(\{j_1, j_1-1\}, \{j_2, m_2\}, \{j_3, -j_1-m_2+1\})$

Fixed j_1, j, m_1

07.39.03.0013.01

$$\begin{pmatrix} j_1 & 0 & j_3 \\ m_1 & 0 & -m_1 \end{pmatrix} = \frac{(-1)^{-j_1-m_1}}{\sqrt{2j_3+1}} \delta_{j_1,j_3} /; \mathcal{P}hysical\mathcal{Q}(\{j_1, m_1\}, \{0, 0\}, \{j_3, -m_1\})$$

07.39.03.0014.01

$$\begin{pmatrix} j_1 & j_1 & j_3 \\ m_1 & m_1 & -2m_1 \end{pmatrix} = 0 /; \frac{2j_1+j_3-1}{2} \in \mathbb{Z}$$

07.39.03.0015.01

$$\begin{pmatrix} j_1 & j_1 & j_3 \\ m_1 & m_1 & -2m_1 \end{pmatrix} = (-1)^{\frac{2j_1-j_3-4m_1}{2}} \frac{\left(\frac{2j_1+j_3}{2}\right)!\sqrt{(j_3+2m_1)!}\sqrt{(j_3-2m_1)!}\sqrt{(2j_1-j_3)!}}{\left(\frac{j_3+2m_1}{2}\right)!\left(\frac{j_3-2m_1}{2}\right)!\left(\frac{2j_1-j_3}{2}\right)!\sqrt{(2j_1+j_3+1)!}} /;$$

$$\frac{2j_1+j_3}{2} \in \mathbb{Z} \bigwedge \mathcal{P}hysical\mathcal{Q}(\{j_1, m_1\}, \{j_1, m_1\}, \{j_3, -2m_1\})$$

07.39.03.0016.01

$$\begin{pmatrix} j_1 & j_1+\frac{1}{2} & j_3+\frac{1}{2} \\ m_1 & m_1+\frac{1}{2} & -2m_1-\frac{1}{2} \end{pmatrix} = (-1)^{\frac{j_3-2j_1-4m_1}{2}} \frac{\left(\frac{2j_1+j_3}{2}\right)!\sqrt{j_3+2m_1+1}\sqrt{2j_1+j_3+2}\sqrt{(2j_1-j_3)!}\sqrt{(j_3+2m_1)!}\sqrt{(j_3-2m_1)!}}{2\left(\frac{2j_1-j_3}{2}\right)!\left(\frac{j_3+2m_1}{2}\right)!\left(\frac{j_3-2m_1}{2}\right)!\sqrt{j_3+1}\sqrt{j_1+m_1+1}\sqrt{(2j_1+j_3+1)!}} /;$$

$$\frac{2j_1+j_3}{2} \in \mathbb{Z} \bigwedge \mathcal{P}hysical\mathcal{Q}\left(\{j_1, m_1\}, \left\{j_1+\frac{1}{2}, m_1+\frac{1}{2}\right\}, \left\{j_3+\frac{1}{2}, -2m_1-\frac{1}{2}\right\}\right)$$

07.39.03.0017.01

$$\left\langle j_1\, j_1+\frac{1}{2}\, m_1\, m_1-\frac{1}{2} \,\middle|\, j_1\, j_1+\frac{1}{2}\, j+\frac{1}{2}\, 2\,m_1-\frac{1}{2}\right\rangle = (-1)^{\frac{j_3-2j_1-4m_1+2}{2}} \frac{\left(\frac{2j_1+j_3}{2}\right)!\, \sqrt{j_3-2m_1+1}\, \sqrt{2j_1+j_3+2}\, \sqrt{(2j_1-j_3)!}\, \sqrt{(j_3+2m_1)!}\, \sqrt{(j_3-2m_1)!}}{2\left(\frac{2j_1-j_3}{2}\right)!\left(\frac{j_3+2m_1}{2}\right)!\left(\frac{j_3-2m_1}{2}\right)!\, \sqrt{j_3+1}\, \sqrt{j_1-m_1+1}\, \sqrt{(2j_1+j_3+1)!}} \;/;\; \frac{2j_1+j}{2}\in\mathbb{Z} \bigwedge \mathcal{P}hysical\mathcal{Q}\left(\{j_1,m_1\},\left\{j_1+\frac{1}{2},\, m_1-\frac{1}{2}\right\},\left\{j_3+\frac{1}{2},\, -2m_1+\frac{1}{2}\right\}\right)$$

07.39.03.0018.01

$$\begin{pmatrix} j_1 & j_1+\frac{1}{2} & j_3+\frac{1}{2} \\ m_1 & m_1+\frac{1}{2} & -2m_1-\frac{1}{2} \end{pmatrix} = (-1)^{\frac{j_3-2j_1-4m_1-1}{2}} \frac{\left(\frac{2j_1+j_3+1}{2}\right)!\, \sqrt{j_3-2m_1+1}\, \sqrt{(2j_1-j_3)!}\, \sqrt{(j_3+2m_1+1)!}\, \sqrt{(j_3-2m_1+1)!}}{2\left(\frac{2j_1-j_3-1}{2}\right)!\left(\frac{j_3+2m_1+1}{2}\right)!\left(\frac{j_3-2m_1+1}{2}\right)!\, \sqrt{j_3+1}\, \sqrt{j_1+m_1+1}\, \sqrt{(2j_1+j_3+2)!}} \;/;\; \frac{2j_1+j_3-1}{2}\in\mathbb{Z} \bigwedge \mathcal{P}hysical\mathcal{Q}\left(\{j_1,m_1\},\left\{j_1+\frac{1}{2},\, m_1+\frac{1}{2}\right\},\left\{j_3+\frac{1}{2},\, -2m_1-\frac{1}{2}\right\}\right)$$

07.39.03.0019.01

$$\begin{pmatrix} j_1 & j_1+\frac{1}{2} & j_3+\frac{1}{2} \\ m_1 & m_1-\frac{1}{2} & \frac{1}{2}-2m_1 \end{pmatrix} = (-1)^{\frac{j_3-2j_1-4m_1+3}{2}} \frac{\left(\frac{2j_1+j_3+1}{2}\right)!\, \sqrt{j_3+2m_1+1}\, \sqrt{(2j_1-j_3)!}\, \sqrt{(j_3+2m_1+1)!}\, \sqrt{(j_3-2m_1+1)!}}{2\left(\frac{2j_1-j_3-1}{2}\right)!\left(\frac{j_3+2m_1+1}{2}\right)!\left(\frac{j_3-2m_1+1}{2}\right)!\, \sqrt{j_3+1}\, \sqrt{j_1-m_1+1}\, \sqrt{(2j_1+j_3+2)!}} \;/;\; \frac{2j_1+j_3-1}{2}\in\mathbb{Z} \bigwedge \mathcal{P}hysical\mathcal{Q}\left(\{j_1,m_1\},\left\{j_1+\frac{1}{2},\, m_1-\frac{1}{2}\right\},\left\{j_3+\frac{1}{2},\, -2m_1+\frac{1}{2}\right\}\right)$$

07.39.03.0020.01

$$\begin{pmatrix} j_1 & j_1+1 & j_3 \\ m_1 & m_1+1 & -2m_1-1 \end{pmatrix} = (-1)^{\frac{j_3-2j_1+j_2-4m_1}{2}} \left(\left(\frac{2j_1+j_3}{2}\right)!\, \sqrt{j_3+2m_1+1}\, \sqrt{j_3-2m_1}\, \sqrt{2j_1+j_3+2}\, \sqrt{(2j_1-j_3+1)!}\, \sqrt{(j_3+2m_1)!}\, \sqrt{(j_3-2m_1)!}\right) \Big/ \left(2\left(\frac{2j_1-j_3}{2}\right)!\left(\frac{j_3+2m_1}{2}\right)!\left(\frac{j_3-2m_1}{2}\right)!\, \sqrt{j_1+m_1+2}\, \sqrt{j_1+m_1+1}\, \sqrt{j_3}\, \sqrt{j_3+1}\, \sqrt{(2j_1+j_3+1)!}\right) \;/;\; \frac{2j_1+j_3}{2}\in\mathbb{Z} \bigwedge \mathcal{P}hysical\mathcal{Q}(\{j_1,m_1\},\{j_1+1,\, m_1+1\},\{j_3,\, -2m_1-1\})$$

07.39.03.0021.01

$$\begin{pmatrix} j_1 & j_1+1 & j_3 \\ m_1 & m_1 & -2m_1 \end{pmatrix} = (-1)^{\frac{j_3-2j_1-4m_1+2}{2}} \frac{m_1\left(\frac{2j_1+j_3}{2}\right)!\, \sqrt{2j_1+j_3+2}\, \sqrt{(2j_1-j_3+1)!}\, \sqrt{(j_3+2m_1)!}\, \sqrt{(j_3-2m_1)!}}{\left(\frac{2j_1-j_3}{2}\right)!\left(\frac{j_3+2m_1}{2}\right)!\left(\frac{j_3-2m_1}{2}\right)!\, \sqrt{j_1+m_1+1}\, \sqrt{j_1-m_1+1}\, \sqrt{j_3}\, \sqrt{j_3+1}\, \sqrt{(2j_1+j_3+1)!}} \;/;\; \frac{2j_1+j_3}{2}\in\mathbb{Z} \bigwedge \mathcal{P}hysical\mathcal{Q}(\{j_1,m_1\},\{j_1+1,\, m_1\},\{j_3,\, -2m_1\})$$

07.39.03.0022.01

$$\begin{pmatrix} j_1 & j_1+1 & j_3 \\ m_1 & m_1-1 & 1-2m_1 \end{pmatrix} = (-1)^{\frac{j_3-2j_1-4m_1}{2}} \left(\left(\frac{2j_1+j_3}{2}\right)! \sqrt{j_3-2m_1+1} \sqrt{j_3+2m_1} \sqrt{2j_1+j_3+2} \sqrt{(2j_1-j_3+1)!} \sqrt{(j_3+2m_1)!} \sqrt{(j_3-2m_1)!}\right) \Big/ \left(2\left(\frac{2j_1-j_3}{2}\right)! \left(\frac{j_3+2m_1}{2}\right)! \left(\frac{j_3-2m_1}{2}\right)! \sqrt{j_1-m_1+2} \sqrt{j_1-m_1+1} \sqrt{j_3} \sqrt{j_3+1} \sqrt{(2j_1+j_3+1)!}\right) /; \frac{2j_1+j_3}{2} \in \mathbb{Z} \bigwedge \mathcal{P}hysical\mathcal{Q}(\{j_1, m_1\}, \{j_1+1, m_1-1\}, \{j_3, -2m_1+1\})$$

07.39.03.0023.01

$$\begin{pmatrix} j_1 & j_1+1 & j_3 \\ m_1 & m_1+1 & -2m_1-1 \end{pmatrix} = (-1)^{\frac{j_3-2j_1-4m_1-1}{2}} (j_3(j_3+1)+(2m_1+1)(2j_1+2)) \left(\left(\frac{2j_1+j_3+1}{2}\right)! \sqrt{(j_3+2m_1+1)!} \sqrt{(j_3-2m_1-1)!} \sqrt{(2j_1-j_3+1)!}\right) \Big/ \left(2\left(\frac{j_3+2m_1+1}{2}\right)! \left(\frac{j_3-2m_1-1}{2}\right)! \left(\frac{2j_1-j_3+1}{2}\right)! \sqrt{j_1+m_1+1} \sqrt{j_1+m_1+2} \sqrt{j_3} \sqrt{j_3+1} \sqrt{(2j_1+j_3+2)!}\right) /; \frac{2j_1+j_3-1}{2} \in \mathbb{Z} \bigwedge \mathcal{P}hysical\mathcal{Q}(\{j_1, m_1\}, \{j_1+1, m_1+1\}, \{j_3, -2m_1-1\})$$

07.39.03.0024.01

$$\begin{pmatrix} j_1 & j_1+1 & j_3 \\ m_1 & m_1 & -2m_1 \end{pmatrix} = (-1)^{\frac{j_3-2j_1-4m_1+1}{2}} \left((2j_1+2)\left(\frac{2j_1+j_3+1}{2}\right)! \sqrt{(j_3+2m_1)!} \sqrt{(j_3-2m_1)!} \sqrt{(2j_1-j_3+1)!}\right) \Big/ \left(\left(\frac{j_3+2m_1-1}{2}\right)! \left(\frac{j_3-2m_1-1}{2}\right)! \left(\frac{2j_1-j_3+1}{2}\right)! \sqrt{j_1+m_1+1} \sqrt{j_1-m_1+1} \sqrt{j_3} \sqrt{j_3+1} \sqrt{(2j_1+j_3+2)!}\right) /; \frac{2j_1+j_3-1}{2} \in \mathbb{Z} \bigwedge \mathcal{P}hysical\mathcal{Q}(\{j_1, m_1\}, \{j_1+1, m_1\}, \{j_3, -2m_1\})$$

07.39.03.0025.01

$$\begin{pmatrix} j_1 & j_1+1 & j_3 \\ m_1 & m_1-1 & 1-2m_1 \end{pmatrix} = (-1)^{\frac{j_3-2j_1-4m_1+3}{2}} (j_3(j_3+1)-(2m_1-1)(2j_1+2)) \left(\left(\frac{2j_1+j_3+1}{2}\right)! \sqrt{(j_3+2m_1-1)!} \sqrt{(j_3-2m_1+1)!} \sqrt{(2j_1-j_3+1)!}\right) \Big/ \left(2\left(\frac{2j_1-j_3+1}{2}\right)! \left(\frac{j_3-2m_1+1}{2}\right)! \left(\frac{j_3+2m_1-1}{2}\right)! \sqrt{j_1-m_1+1} \sqrt{j_1-m_1+2} \sqrt{j_3} \sqrt{j_3+1} \sqrt{(2j_1+j_3+2)!}\right) /; \frac{2j_1+j_3-1}{2} \in \mathbb{Z} \bigwedge \mathcal{P}hysical\mathcal{Q}(\{j_1, m_1\}, \{j_1+1, m_1-1\}, \{j_3, -2m_1+1\})$$

Fixed j_1, j_2, j

07.39.03.0026.01

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ 0 & 0 & 0 \end{pmatrix} = 0 /; \frac{j_1+j_2+j_3-1}{2} \in \mathbb{Z}$$

07.39.03.0027.01

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ 0 & 0 & 0 \end{pmatrix} = (-1)^{\frac{-j_1-j_2-j_3}{2}} \frac{\left(\frac{j_1+j_2+j_3}{2}\right)! \sqrt{(-j_1+j_2+j_3)!} \sqrt{(j_1-j_2+j_3)!} \sqrt{(j_1+j_2-j_3)!}}{\left(\frac{-j_1+j_2+j_3}{2}\right)! \left(\frac{j_1-j_2+j_3}{2}\right)! \left(\frac{j_1+j_2-j_3}{2}\right)! \sqrt{(j_1+j_2+j_3+1)!}} /; \frac{j_1+j_2+j_3}{2} \in \mathbb{Z} \bigwedge \mathcal{T}riangular\mathcal{Q}(j_1, j_2, j_3)$$

Fixed j_1, j_2

07.39.03.0028.01

$$\begin{pmatrix} j_1 & j_2 & j_1+j_2 \\ 0 & 0 & 0 \end{pmatrix} = (-1)^{-j_1+j_2} \frac{(j_1+j_2)! \sqrt{(2 j_1)!} \sqrt{(2 j_2)!}}{j_1! \, j_2! \sqrt{2 j_1 + 2 j_2 + 1} \sqrt{(2 j_1 + 2 j_2)!}} /; j_1 \in \mathbb{N} \bigwedge j_2 \in \mathbb{N}$$

07.39.03.0029.01

$$\begin{pmatrix} j_1 & j_2 & j_1-j_2 \\ 0 & 0 & 0 \end{pmatrix} = (-1)^{j_1} \frac{j_1! \sqrt{(2 j_2)!} \sqrt{(2 j_1 - 2 j_2 + 1)!}}{j_2! \, (j_1-j_2)! \sqrt{2 j_1 - 2 j_2 + 1} \sqrt{(2 j_1 + 1)!}} /; j_1 \in \mathbb{N} \bigwedge j_2 \in \mathbb{N} \bigwedge j_1 \geq j_2$$

07.39.03.0030.01

$$\begin{pmatrix} j_1 & j_2 & j_1+j_2 \\ j_1 & j_2 & -j_1-j_2 \end{pmatrix} = \frac{(-1)^{2 j_2}}{\sqrt{2 j_1 + 2 j_2 + 1}} /; 2 j_1 \in \mathbb{N} \bigwedge 2 j_2 \in \mathbb{N}$$

07.39.03.0031.01

$$\begin{pmatrix} j_1 & j_2 & j_1+j_2 \\ j_1 & -j_2 & j_2-j_1 \end{pmatrix} = (-1)^{-2 j_1 + 2 j_2} \frac{\sqrt{(2 j_1)!} \sqrt{(2 j_2)!}}{\sqrt{2 j_1 + 2 j_2 + 1} \sqrt{(2 j_1 + 2 j_2)!}} /; 2 j_1 \in \mathbb{N} \bigwedge 2 j_2 \in \mathbb{N}$$

07.39.03.0032.01

$$\begin{pmatrix} j_1 & j_2 & j_1+j_2-1 \\ m_1 & m_2 & -m_1-m_2 \end{pmatrix} = 0 /; j_1 \, m_2 = j_2 \, m_1$$

07.39.03.0033.01

$$\begin{pmatrix} j_1 & j_2 & j_1+j_2-1 \\ j_1 & -j_2 & j_2-j_1 \end{pmatrix} = (-1)^{-2 j_1 + 2 j_2} \frac{\sqrt{(2 j_1)!} \sqrt{(2 j_2)!}}{\sqrt{(2 j_1 + 2 j_2)!}} /; 2 j_1 \in \mathbb{N}^+ \bigwedge 2 j_2 \in \mathbb{N}^+$$

07.39.03.0034.01

$$\begin{pmatrix} j_1 & j_2 & j_1+j_2-1 \\ j_1-1 & j_2 & -j_1-j_2+1 \end{pmatrix} = (-1)^{2 j_2} \frac{\sqrt{j_2}}{\sqrt{2 j_1 + 2 j_2 - 1} \sqrt{j_1 + j_2}} /; 2 j_1 \in \mathbb{N}^+ \bigwedge 2 j_2 \in \mathbb{N}$$

07.39.03.0035.01

$$\begin{pmatrix} j_1 & j_2 & j_1+j_2-1 \\ j_1 & j_2-1 & -j_1-j_2+1 \end{pmatrix} = (-1)^{-2 j_1 + 1} \frac{\sqrt{j_1}}{\sqrt{2 j_1 + 2 j_2 - 1} \sqrt{j_1 + j_2}} /; 2 j_1 \in \mathbb{N} \bigwedge 2 j_2 \in \mathbb{N}^+$$

07.39.03.0036.01

$$\begin{pmatrix} j_1 & j_2 & j_1-j_2+1 \\ j_1 & -j_2 & j_2-j_1 \end{pmatrix} = (-1)^{-2 j_1 + 2 j_2} \frac{\sqrt{j_2}}{\sqrt{j_1 + 1} \sqrt{2 j_1 + 1}} /; 2 j_1 \in \mathbb{N}^+ \bigwedge 2 j_2 \in \mathbb{N}^+ \bigwedge j_1 \geq j_2$$

07.39.03.0037.01

$$\begin{pmatrix} j_1 & j_2 & j_1-j_2 \\ j_1 & -j_2 & j_2-j_1 \end{pmatrix} = (-1)^{-2 j_1 + 2 j_2} \frac{1}{\sqrt{2 j_1 + 1}} /; 2 j_1 \in \mathbb{N} \bigwedge 2 j_2 \in \mathbb{N} \bigwedge j_1 \geq j_2$$

07.39.03.0038.01

$$\begin{pmatrix} j_1 & j_2 & j_1+j_2-2 \\ j_1 & -j_2 & j_2-j_1 \end{pmatrix} = (-1)^{-2 j_1+2 j_2} \frac{\sqrt{(2 j_1)!} \sqrt{(2 j_2)!}}{\sqrt{2} \sqrt{(2 j_1+2 j_2-1)!}} \; /; \; 2 j_1 - 2 \in \mathbb{N} \bigwedge 2 j_2 - 2 \in \mathbb{N}$$

07.39.03.0039.01

$$\begin{pmatrix} j_1 & j_2 & j_1-j_2+2 \\ j_1 & -j_2 & j_2-j_1 \end{pmatrix} = (-1)^{-2 j_1+2 j_2} \frac{\sqrt{j_2} \sqrt{2 j_2-1}}{\sqrt{j_1+1} \sqrt{2 j_1+1} \sqrt{2 j_1+3}} \bigwedge 2 j_1 \in \mathbb{N}^+ \bigwedge 2 j_2 - 2 \in \mathbb{N} \bigwedge j_1 > j_2 - 1$$

Fixed j_1, m_1

07.39.03.0040.01

$$\begin{pmatrix} j_1 & j_1 & 2 m_1+1 \\ m_1 & m_1 & -2 m_1 \end{pmatrix} = 0$$

General characteristics

Domain and analyticity

$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$ is a six-argument function. The condition $\mathcal{P}hysical\mathcal{Q}$ restricts the arguments to integers or half-integers (interpreted as quantum-mechanical spin quantum numbers) that fulfill certain inequalities. Without the condition $\mathcal{P}hysical\mathcal{Q}$ and the prefactor ensuring $m_1 + m_2 + m_3 = 0$, $\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$ would be an analytic function in all six arguments over $\mathbb{C}^6$.

07.39.04.0001.01

$$(\{j_1 * m_1\} * \{j_2 * m_2\} * \{j_3 * m_3\}) \longrightarrow \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} :: (\{\mathbb{Q} \otimes \mathbb{Q}\} \otimes \{\mathbb{Q} \otimes \mathbb{Q}\} \otimes \{\mathbb{Q} \otimes \mathbb{Q}\}) \longrightarrow \mathbb{C}$$

Symmetries and periodicities

Permutation symmetry

07.39.04.0002.01

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \begin{pmatrix} j_2 & j_3 & j_1 \\ m_2 & m_3 & m_1 \end{pmatrix}$$

07.39.04.0003.01

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \begin{pmatrix} j_3 & j_1 & j_2 \\ m_3 & m_1 & m_2 \end{pmatrix}$$

07.39.04.0004.01

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{j_1+j_2+j_3} \begin{pmatrix} j_1 & j_3 & j_2 \\ m_1 & m_3 & m_2 \end{pmatrix}$$

07.39.04.0005.01

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{j_1+j_2+j_3} \begin{pmatrix} j_2 & j_1 & j_3 \\ m_2 & m_1 & m_3 \end{pmatrix}$$

07.39.04.0006.01

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{j_1+j_2+j_3} \begin{pmatrix} j_3 & j_2 & j_1 \\ m_3 & m_2 & m_1 \end{pmatrix}$$

Reflection symmetry

07.39.04.0007.01

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{j_1+j_2+j_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ -m_1 & -m_2 & -m_3 \end{pmatrix}$$

Series representations

Generalized power series

07.39.06.0001.01

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \frac{\delta_{-m_3,m_1+m_2}}{\sqrt{(j_1+j_2+j_3+1)!}} (-1)^{-j_1+j_2+m_3} \sqrt{(j_1+j_2-j_3)!} \sqrt{(j_1-j_2+j_3)!} \sqrt{(-j_1+j_2+j_3)!} \sqrt{(j_1+m_1)!} \sqrt{(j_1-m_1)!} \sqrt{(j_2+m_2)!} \sqrt{(j_2-m_2)!} \sqrt{(j_3+m_3)!} \sqrt{(j_3-m_3)!} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(j_1+j_2-j_3-k)!\,(j_1-m_1-k)!\,(j_2+m_2-k)!\,(j_3-j_2+m_1+k)!\,(j_3-j_1-m_2+k)!} /; \mathcal{PhysicalQ}(\{j_1, m_1\}, \{j_2, m_2\}, \{j_3, m_3\})$$

07.39.06.0002.01

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \frac{\delta_{-m_3,m_1+m_2}}{\sqrt{(j_1+j_2+j_3+1)!}} (-1)^{-j_1+j_2+m_3} \sqrt{(j_3-m)!} \sqrt{(m+j_3)!} \sqrt{(j_1-j_2+j_3)!} \sqrt{(-j_1+j_2+j_3)!} \sqrt{(j_1+j_2-j_3)!} \sqrt{(j_1-m_1)!} \sqrt{(j_1+m_1)!} \sqrt{(j_2-m_2)!} \sqrt{(j_2+m_2)!} \sum_{k=-\infty}^{\infty} \theta(k, -k+j_1+j_2-j_3, -k+j_1-m_1, -k+j_2+m_2, k-j_2+j_3+m_1, k-j_1+j_3-m_2) / (k!\,(j_1+j_2-j_3-k)!\,(j_1-m_1-k)!\,(j_2+m_2-k)!\,(k-j_2+j_3+m_1)!\,(k-j_1+j_3-m_2)!) /; \mathcal{PhysicalQ}(\{j_1, m_1\}, \{j_2, m_2\}, \{j_3, m_3\})$$

07.39.06.0004.01

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{j_2-j_1+m_3} \frac{\delta_{-m_3,m_1+m_2}}{\sqrt{(j_1+j_2+j_3+1)!}} \sqrt{(j_1+j_2-j_3)!} \sqrt{(j_1-j_2+j_3)!} \sqrt{(-j_1+j_2+j_3)!} \sqrt{(j_1+m_1)!} \sqrt{(j_1-m_1)!} \sqrt{(j_2+m_2)!} \sqrt{(j_2-m_2)!} \sqrt{(j_3-m_3)!} \sqrt{(j_3+m_3)!} \sum_{k=\max(j_2-j_3-m_1,j_1-j_3+m_2,0)}^{\min(j_1-m_1,j_2+m_2)} \frac{(-1)^k}{k!\,(j_1+j_2-j_3-k)!\,(j_1-m_1-k)!\,(j_2+m_2-k)!\,(j_3-j_2+m_1+k)!\,(j_3-j_1-m_2+k)!} /; \mathcal{PhysicalQ}(\{j_1, m_1\}, \{j_2, m_2\}, \{j_3, m_3\})$$

07.39.06.0005.01

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{j_2-j_1+m_3} \delta_{-m_3,m_1+m_2} \frac{\sqrt{(j_1+j_2-j_3)!}}{\sqrt{(1+j_1+j_2+j_3)!} \sqrt{(j_1-j_2+j_3)!} \sqrt{(-j_1+j_2+j_3)!}} \frac{\sqrt{(j_1-m_1)!} \sqrt{(j_2-m_2)!} \sqrt{(j_3+m_3)!} \sqrt{(j_3-m_3)!}}{\sqrt{(j_1+m_1)!} \sqrt{(j_2+m_2)!}} \sum_{k=\max(0,-j_2+j_3-m_1)}^{\min(j_1-m_1,j_3+m_3)} \frac{(-1)^{j_1-m_1+k} (j_1+m_1+k)!\,(j_3+j_2-m_1-k)!}{k!\,(j_1-m_1-k)!\,(j_3+m_3-k)!\,(j_2-j_3+m_1+k)!} /; \mathcal{PhysicalQ}(\{j_1, m_1\}, \{j_2, m_2\}, \{j_3, m_3\})$$

07.39.06.0006.01

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{j_2-j_1+m_3} \delta_{-m_3,m_1+m_2} \frac{\sqrt{(j_1+j_2-j_3)!} \sqrt{(j_1-j_2+j_3)!} \sqrt{(-j_1+j_2+j_3)!}}{\sqrt{(1+j_1+j_2+j_3)!}} \frac{\sqrt{(j_3+m_3)!} \sqrt{(j_3-m_3)!}}{\sqrt{(j_1+m_1)!} \sqrt{(j_1-m_1)!} \sqrt{(j_2+m_2)!} \sqrt{(j_2-m_2)!}} \sum_{k=\max(0,j_2-j_3-m_3)}^{\min(-j_1+j_2+j_3,j_3-m_3)} \frac{(-1)^{j_2+m_2+k} (j_2+j_3+m_1-k)! (j_1-m_1+k)!}{k! (j_3-j_1+j_2-k)! (j_3-m_3-k)! (j_1-j_2+m_3+k)!} /; \mathcal{P}hysical\mathcal{Q}(\{j_1, m_1\}, \{j_2, m_2\}, \{j_3, m_3\})$$

07.39.06.0007.01

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{j_2-j_1+m_3} \delta_{-m_3,m_1+m_2} \frac{\sqrt{(-j_1+j_2+j_3)!}}{\sqrt{(j_1-j_2+j_3)!} \sqrt{(j_1+j_2-j_3)!} \sqrt{(1+j_1+j_2+j_3)!}} \frac{\sqrt{(j_1+m_1)!} \sqrt{(j_1-m_1)!} \sqrt{(j_2-m_2)!} \sqrt{(j_3-m_3)!}}{\sqrt{(j_2+m_2)!} \sqrt{(j_3+m_3)!}} \sum_{k=\max(0,-j_1+j_3+m_2)}^{\min(-j_1+j_2+j_3,j_3-m_3)} \frac{(-1)^{j_2+m_2+k} (2 j_3-k)! (j_1+j_2-j_3+k)!}{k! (j_3-j_1+j_2-k)! (j_3-m_3-k)! (j_1-j_3-m_2+k)!} /; \mathcal{P}hysical\mathcal{Q}(\{j_1, m_1\}, \{j_2, m_2\}, \{j_3, m_3\})$$

07.39.06.0008.01

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{j_2-j_1+m_3} \delta_{-m_3,m_1+m_2} \frac{\sqrt{(-j_1+j_2+j_3)!} \sqrt{(j_1+j_2-j_3)!} \sqrt{(1+j_1+j_2+j_3)!}}{\sqrt{(j_1-j_2+j_3)!}} \frac{\sqrt{(j_1-m_1)!} \sqrt{(j_3-m_3)!}}{\sqrt{(j_1+m_1)!} \sqrt{(j_2+m_2)!} \sqrt{(j_2-m_2)!} \sqrt{(j_3+m_3)!}} \sum_{k=0}^{\min(-j_1+j_2+j_3,j_3-m_3)} \frac{(-1)^{j_2+m_2+k} (2 j_3-k)! (j_2+j_3+m_1-k)!}{k! (j_3-j_1+j_2-k)! (j_3-m_3-k)! (j_1+j_2+j_3+1-k)!} /; \mathcal{P}hysical\mathcal{Q}(\{j_1, m_1\}, \{j_2, m_2\}, \{j_3, m_3\})$$

07.39.06.0009.01

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{j_2-j_1+m_3} \delta_{-m_3,m_1+m_2} \frac{\sqrt{(1+j_1+j_2+j_3)!}}{\sqrt{(j_1+j_2-j_3)!} \sqrt{(j_1-j_2+j_3)!} \sqrt{(-j_1+j_2+j_3)!}} \frac{\sqrt{(j_1+m_1)!} \sqrt{(j_1-m_1)!} \sqrt{(j_3+m_3)!} \sqrt{(j_3-m_3)!}}{\sqrt{(j_2+m_2)!} \sqrt{(j_2-m_2)!}} \sum_{k=0}^{\min(j_3+m_3,j_1-m_1)} \frac{(-1)^{j_1-m_1+k} (j_1+j_2+m_3-k)! (j_2+j_3-m_1-k)!}{k! (j_1-m_1-k)! (j_3+m_3-k)! (j_1+j_2+j_3+1-k)!} /; \mathcal{P}hysical\mathcal{Q}(\{j_1, m_1\}, \{j_2, m_2\}, \{j_3, m_3\})$$

Other series representations

Series of binomial coefficients

07.39.06.0003.01

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \frac{(-1)^{-j_1+j_2+m_3}}{\sqrt{2 j_3+1}} \delta_{m_1+m_2,-m_3} \frac{\sqrt{\binom{2 j_1}{j_1+j_2-j_3}} \sqrt{\binom{2 j_2}{j_1+j_2-j_3}}}{\sqrt{\binom{j_1+j_2+j_3+1}{j_1+j_2-j_3}} \sqrt{\binom{2 j_1}{j_1-m_1}} \sqrt{\binom{2 j_2}{j_2-m_2}} \sqrt{\binom{2 j_3}{j_3+m_3}}} \sum_{k=\max(0,j_2-j_3-m_1,j_1-j_3+m_2)}^{\min(j_1+j_2-j_3,j_1-m_1,j_2+m_2)} (-1)^k \binom{j_1+j_2-j_3}{k} \binom{j_1-j_2+j_3}{-k+j_1-m_1} \binom{-j_1+j_2+j_3}{-k+j_2+m_2} /; \mathcal{PhysicalQ}(\{j_1, m_1\}, \{j_2, m_2\}, \{j_3, m_3\})$$

Integral representations

On the real axis

Of the direct function

07.39.07.0001.01

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \delta_{m_1+m_2,-m_3} \frac{(-1)^{j_2-j_3+m_2+m_3}}{2^{j_1+j_2+j_3+1}} \left(\sqrt{(j_3-m_3)!} \sqrt{(j_1+j_2-j_3)!} \sqrt{(j_1+j_2+j_3+1)!}\right) \Big/ \left(\sqrt{(j_1-m_1)!} \sqrt{(j_1+m_1)!} \sqrt{(j_2-m_2)!} \sqrt{(j_2+m_2)!} \sqrt{(j_3+m_3)!} \sqrt{(-j_1+j_2+j_3)!} \sqrt{(j_1-j_2+j_3)!}\right) \int_{-1}^{1} (1-t)^{j_1-m_1} (1+t)^{j_2-m_2} \frac{\partial^{j_3+m_3} \left((1-t)^{-j_1+j_2+j_3} (1+t)^{j_1-j_2+j_3}\right)}{\partial t^{j_3+m_3}} \, dt /; \mathcal{PhysicalQ}(\{j_1, m_1\}, \{j_2, m_2\}, \{j_3, m_3\})$$

07.39.07.0002.01

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \delta_{m_1+m_2,-m_3} \frac{(-1)^{j_2-j_3+m_2+m_3}}{2^{j_1+j_2+j_3+1}} \left(\sqrt{(j_1-j_2+j_3)!} \sqrt{(j_1+j_2-j_3)!} \sqrt{(j_1+j_2+j_3+1)!}\right) \Big/ \left(\sqrt{(j_1-m_1)!} \sqrt{(j_1+m_1)!} \sqrt{(j_2-m_2)!} \sqrt{(j_2+m_2)!} \sqrt{(j_3+m_3)!} \sqrt{(j_3-m_3)!} \sqrt{(-j_1+j_2+j_3)!}\right) \int_{-1}^{1} (1-t)^{j_2+m_2} (1+t)^{j_2-m_2} \frac{\partial^{-j_1+j_2+j_3} \left((1-t)^{j_3+m_3} (1+t)^{j_3-m_3}\right)}{\partial t^{-j_1+j_2+j_3}} \, dt /; \mathcal{PhysicalQ}(\{j_1, m_1\}, \{j_2, m_2\}, \{j_3, m_3\})$$

07.39.07.0003.01

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ 0 & 0 & 0 \end{pmatrix} = \frac{(-1)^{j_2-j_3}}{2^{j_1+j_2+j_3+1}} \frac{\sqrt{(j_1-j_2+j_3)!} \sqrt{(j_1+j_2-j_3)!} \sqrt{(j_1+j_2+j_3+1)!}}{j_1!\, j_2!\, j_3! \sqrt{(-j_1+j_2+j_3)!}} \int_{-1}^{1} \left(1-t^2\right)^{j_2} \frac{\partial^{-j_1+j_2+j_3} \left(1-t^2\right)^{j_3}}{\partial t^{-j_1+j_2+j_3}} \, dt /; j_1 \in \mathbb{N} \land j_2 \in \mathbb{N} \land j_3 \in \mathbb{N} \land |j_1-j_2| \le j_3 \le j_1+j_2$$

07.39.07.0004.01

$$\langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m \rangle = \delta_{m_1+m_2,-m_3} \frac{(-1)^{j_2+j_3-m_1+2 m_3}}{2^{j_1+j_2+1}} \frac{\sqrt{(j_1+j_2-j_3)!} \sqrt{(j_1+j_2+j_3+1)!}}{\sqrt{(j_1+m_1)!} \sqrt{(j_1-m_1)!} \sqrt{(j_2+m_2)!} \sqrt{(j_2-m_2)!}} \int_{-1}^{1} \left(1-t^2\right)^{\frac{j_1+j_2}{2}} \left(\frac{1-t}{1+t}\right)^{\frac{m_1-m_2}{2}} d_{j_2-j_1,-m_3}^{j_3}\left(\cos^{-1}(t)\right) dt /; \mathcal{PhysicalQ}(\{j_1, m_1\}, \{j_2, m_2\}, \{j_3, m_3\})$$

Multiple integral representations

For the direct function itself

07.39.07.0005.01

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \frac{(-1)^{j_2-j_1+j_3} (2\,i)^{j_1+j_2+j_3}}{\pi^2} \frac{\sqrt{(j_3+m_3)!}\,\sqrt{(j_3-m_3)!}\,\sqrt{(j_1-m_1)!}\,\sqrt{(j_1+m_1)!}\,\sqrt{(j_2-m_2)!}\,\sqrt{(j_2+m_2)!}}{\sqrt{(j_3+j_1-j_2)!}\,\sqrt{(j_3-j_1+j_2)!}\,\sqrt{(j_1+j_2-j_3)!}\,\sqrt{(j_1+j_2+j_3+1)!}}$$
$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{2\,i\,\varphi\,m_1+2\,i\,\vartheta\,m_2} \sin^{-j_1+j_2+j_3}(\vartheta) \sin^{j_1+j_2-j_3}(\vartheta-\varphi) \sin^{j_1-j_2+j_3}(\varphi)\,d\varphi\,d\vartheta \;/;\; \mathcal{PhysicalQ}(\{j_1, m_1\}, \{j_2, m_2\}, \{j_3, m_3\})$$

07.39.07.0006.01

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = 2\,(-1)^{m_3-j_1+j_2}\, i^{j_1+j_2-j_3} \sqrt{\pi} \left(\left(\frac{j_1-j_2+j_3}{2}\right)! \left(\frac{-j_1+j_2+j_3}{2}\right)! \left(\frac{j_1+j_2-j_3}{2}\right)! \sqrt{(1+j_1+j_2+j_3)!}\right) \Big/$$
$$\left(\left(\frac{j_1+j_2+j_3}{2}\right)! \sqrt{(j_1-j_2+j_3)!}\,\sqrt{(-j_1+j_2+j_3)!}\,\sqrt{(j_1+j_2-j_3)!}\,\sqrt{1+2\,j_1}\,\sqrt{1+2\,j_2}\,\sqrt{1+2\,j_3}\right)$$
$$\int_0^{\pi}\int_0^{2\pi} \sin(\vartheta)\, Y_{j_1}^{m_1}(\vartheta, \varphi)\, Y_{j_2}^{m_2}(\vartheta, \varphi)\, \overline{Y_{j_3}^{-m_3}(\vartheta, \varphi)}\,d\varphi\,d\vartheta \;/;$$
$$\frac{j_1+j_2+j_3}{2} \in \mathbb{N} \bigwedge m_1 \in \mathbb{Z} \bigwedge m_2 \in \mathbb{Z} \bigwedge m_3 \in \mathbb{Z} \bigwedge \mathcal{PhysicalQ}(\{j_1, m_1\}, \{j_2, m_2\}, \{j_3, m_3\})$$

07.39.07.0007.01

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \frac{(-1)^{j_2-j_1+m_3}}{8\,\pi^2} \frac{\sqrt{(j_1+j_2+j_3+1)!}\,\sqrt{(j_1+j_2-j_3)!}}{\sqrt{(2\,j_1)!}\,\sqrt{(2\,j_2)!}}$$
$$\int_0^{2\pi}\int_0^{\pi}\int_0^{2\pi} \sin(\beta)\, D_{m_1,j_1}^{j_1}(\alpha, \beta, \gamma)\, D_{m_2,-j_2}^{j_2}(\alpha, \beta, \gamma)\, \overline{D_{-m_3,j_1-j_2}^{j_3}(\alpha, \beta, \gamma)}\,d\gamma\,d\beta\,d\alpha \;/;\; \mathcal{PhysicalQ}(\{j_1, m_1\}, \{j_2, m_2\}, \{j_3, m_3\})$$

Involving the direct function

07.39.07.0008.01

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}\begin{pmatrix} j_1 & j_2 & j_3 \\ n_1 & n_2 & n_3 \end{pmatrix} = \frac{(-1)^{-m_3+n_3}}{8\,\pi^2} \int_0^{2\pi}\int_0^{\pi}\int_0^{2\pi} \sin(\beta)\, D_{m_1,n_1}^{j_1}(\alpha, \beta, \gamma)\, D_{m_2,n_2}^{j_2}(\alpha, \beta, \gamma)\, \overline{D_{-m_3,-n_3}^{j_3}(\alpha, \beta, \gamma)}\,d\gamma\,d\beta\,d\alpha \;/;$$
$$\mathcal{PhysicalQ}(\{j_1, m_1\}, \{j_2, m_2\}, \{j, m\}) \land \mathcal{PhysicalQ}(\{j_1, n_1\}, \{j_2, n_2\}, \{j, n\})$$

07.39.07.0009.01

$$\left|\begin{pmatrix} j_1 & j_2 & j_3 \\ 0 & 0 & 0 \end{pmatrix}\right|^2 = \frac{1}{2}\int_0^{\pi} \sin(\vartheta)\, P_{j_1}(\cos(\vartheta))\, P_{j_2}(\cos(\vartheta))\, P_{j_3}(\cos(\vartheta))\,d\vartheta \;/;\; j_1 \in \mathbb{N} \land j_2 \in \mathbb{N} \land j_3 \in \mathbb{N} \land |j_1-j_2| \le j_3 \le j_1+j_2$$

Integral representations of negative integer order

07.39.07.0010.01

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \delta_{-m_3,m_1+m_2}\,(-1)^{2\,j_2-j+m_3}$$
$$\frac{\sqrt{(j_1-j_2+j_3)!}}{\sqrt{(j_1+j_2-j_3)!}\,\sqrt{(-j_1+j_2+j_3)!}\,\sqrt{(j_1+j_2+j_3+1)!}} \frac{\sqrt{(j_1+m_1)!}\,\sqrt{(j_1-m_1)!}\,\sqrt{(j_2+m_2)!}\,\sqrt{(j_3-m_3)!}}{(j_1-j_2-m_3)!\,\sqrt{(j_2-m_2)!}\,\sqrt{(j_3+m_3)!}}$$
$$\frac{\partial^{j_2-m_2}\left((1-t)^{j_1+j_2-j_3}\,{}_2F_1(j_1-j_2-j_3, j_3-m_3; j_1-j_2-m_3+1; t)\right)}{\partial t^{j_2-m_2}}\Big|_{t=0} \;/;\; \mathcal{PhysicalQ}(\{j_1, m_1\}, \{j_2, m_2\}, \{j_3, m_3\})$$

Generating functions

07.39.11.0001.01

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \left(\sqrt{(j_1+j_2+j_3+1)!}\,\sqrt{(j_1+m_1)!}\,\sqrt{(j_1-m_1)!}\,\sqrt{(j_2+m_2)!}\,\sqrt{(j_2-m_2)!}\,\sqrt{(j_3+m_3)!}\,\sqrt{(j_3-m_3)!}\right) \Big/ \left(\sqrt{(j_1+j_2-j_3)!}\,\sqrt{(j_2+j_3-j_1)!}\,\sqrt{(j_3+j_1-j_2)!}\right) \left(\left[x^{j_1+m_1},\ x_1^{j_1-m_1},\ y^{j_2+m_2},\ y_1^{j_2-m_2},\ z^{j_3+m_3},\ z_1^{j_3-m_3}\right] e^{z x_1 - y_1 z - y x_1 - x z_1 + x y_1 + y z_1}\right) /; \mathcal{PhysicalQ}(\{j_1, m_1\}, \{j_2, m_2\}, \{j_3, m_3\})$$

Identities

Recurrence identities

Consecutive neighbors

07.39.17.0003.01

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = -\frac{\sqrt{j_1-m_1}\,\sqrt{j_1+m_1+1}}{\sqrt{j_3+m_3}\,\sqrt{j_3-m_3+1}} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1+1 & m_2 & m_3-1 \end{pmatrix} - \frac{\sqrt{j_2-m_2}\,\sqrt{j_2+m_2+1}}{\sqrt{j_3+m_3}\,\sqrt{j_3-m_3+1}} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2+1 & m_3-1 \end{pmatrix}$$

07.39.17.0004.01

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = -\frac{\sqrt{j_1+m_1}\,\sqrt{j_1-m_1+1}}{\sqrt{j_3-m_3}\,\sqrt{j_3+m_3+1}} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1-1 & m_2 & m_3+1 \end{pmatrix} - \frac{\sqrt{j_2+m_2}\,\sqrt{j_2-m_2+1}}{\sqrt{j_3-m_3}\,\sqrt{j_3+m_3+1}} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2-1 & m_3+1 \end{pmatrix}$$

07.39.17.0009.01

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \frac{(2 j_1 - 1)(2 m_2 j_1 (j_1-1) + m_1 (j_1 (j_1-1) + j_2 (j_2+1) - j_3 (j_3+1)))}{(j_1-1)\sqrt{j_1+m_1}\,\sqrt{j_1-m_1}\,\sqrt{-j_1+j_2+j_3+1}\,\sqrt{j_1-j_2+j_3}\,\sqrt{j_1+j_2-j_3}\,\sqrt{j_1+j_2+j_3+1}} \begin{pmatrix} j_1-1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} - \left(j_1 \sqrt{j_1+m_1-1}\,\sqrt{j_1-m_1-1}\,\sqrt{-j_1+j_2+j_3+2}\,\sqrt{j_1-j_2+j_3-1}\,\sqrt{j_1+j_2-j_3-1}\,\sqrt{j_1+j_2+j_3}\right) \Big/ \left((j_1-1)\sqrt{j_1+m_1}\,\sqrt{j_1-m_1}\,\sqrt{-j_1+j_2+j_3+1}\,\sqrt{j_1-j_2+j_3}\,\sqrt{j_1+j_2-j_3}\,\sqrt{j_1+j_2+j_3+1}\right) \begin{pmatrix} j_1-2 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$$

07.39.17.0010.01

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = ((2 j_1 + 3)(2 m_2 (j_1+1)(j_1+2) + m_1 ((j_1+1)(j_1+2) + j_2 (j_2+1) - j_3 (j_3+1)))) \Big/ \left((j_1+2)\sqrt{j_1-m_1+1}\,\sqrt{j_1+m_1+1}\,\sqrt{-j_1+j_2+j_3}\,\sqrt{j_1-j_2+j_3+1}\,\sqrt{j_1+j_2-j_3+1}\,\sqrt{j_1+j_2+j_3+2}\right) \begin{pmatrix} j_1+1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} - \left((j_1+1)\sqrt{j_1-m_1+2}\,\sqrt{j_1+m_1+2}\,\sqrt{-j_1+j_2+j_3-1}\,\sqrt{j_1-j_2+j_3+2}\,\sqrt{j_1+j_2-j_3+2}\,\sqrt{j_1+j_2+j_3+3}\right) \Big/ \left((j_1+2)\sqrt{j_1-m_1+1}\,\sqrt{j_1+m_1+1}\,\sqrt{-j_1+j_2+j_3}\,\sqrt{j_1-j_2+j_3+1}\,\sqrt{j_1+j_2-j_3+1}\,\sqrt{j_1+j_2+j_3+2}\right) \begin{pmatrix} j_1+2 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$$

07.39.17.0011.01

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = -((2 j_2 - 1)(2 m_1 j_2 (j_2 - 1) + m_2 (j_2 (j_2 - 1) + j_1 (j_1 + 1) - j_3 (j_3 + 1)))) \Big/ \Big((j_2 - 1) \sqrt{j_2 - m_2} \sqrt{j_2 + m_2} \sqrt{j_1 - j_2 + j_3 + 1} \sqrt{-j_1 + j_2 + j_3} \sqrt{j_1 + j_2 - j_3} \sqrt{j_1 + j_2 + j_3 + 1}\Big) \begin{pmatrix} j_1 & j_2 - 1 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} - \Big(j_2 \sqrt{j_2 - m_2 - 1} \sqrt{j_2 + m_2 - 1} \sqrt{j_1 - j_2 + j_3 + 2} \sqrt{-j_1 + j_2 + j_3 - 1} \sqrt{j_1 + j_2 - j_3 - 1} \sqrt{j_1 + j_2 + j_3}\Big) \Big/ \Big((j_2 - 1) \sqrt{j_2 - m_2} \sqrt{j_2 + m_2} \sqrt{j_1 - j_2 + j_3 + 1} \sqrt{-j_1 + j_2 + j_3} \sqrt{j_1 + j_2 - j_3} \sqrt{j_1 + j_2 + j_3 + 1}\Big) \begin{pmatrix} j_1 & j_2 - 2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$$

07.39.17.0012.01

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = -((2 j_2 + 3)(2 m_1 (j_2 + 1)(j_2 + 2) + m_2 ((j_2 + 1)(j_2 + 2) + j_1 (j_1 + 1) - j_3 (j_3 + 1)))) \Big/ \Big((j_2 + 2) \sqrt{j_2 - m_2 + 1} \sqrt{j_2 + m_2 + 1} \sqrt{j_1 - j_2 + j_3} \sqrt{-j_1 + j_2 + j_3 + 1} \sqrt{j_1 + j_2 - j_3 + 1} \sqrt{j_1 + j_2 + j_3 + 2}\Big) \begin{pmatrix} j_1 & j_2 + 1 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} - \Big((j_2 + 1) \sqrt{j_2 - m_2 + 2} \sqrt{j_2 + m_2 + 2} \sqrt{j_1 - j_2 + j_3 - 1} \sqrt{-j_1 + j_2 + j_3 + 2} \sqrt{j_1 + j_2 - j_3 + 2} \sqrt{j_1 + j_2 + j_3 + 3}\Big) \Big/ \Big((j_2 + 2) \sqrt{j_2 - m_2 + 1} \sqrt{j_2 + m_2 + 1} \sqrt{j_1 - j_2 + j_3} \sqrt{-j_1 + j_2 + j_3 + 1} \sqrt{j_1 + j_2 - j_3 + 1} \sqrt{j_1 + j_2 + j_3 + 2}\Big) \begin{pmatrix} j_1 & j_2 + 2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$$

07.39.17.0013.01

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \frac{(2 j_3 - 1)(j_3 (j_3 - 1)(m_1 - m_2) + m_3 j_1 (j_1 + 1) - m_3 j_2 (j_2 + 1))}{(j_3 - 1) \sqrt{j_3 - m_3} \sqrt{j_3 + m_3} \sqrt{-j_1 + j_2 + j_3} \sqrt{j_1 - j_2 + j_3} \sqrt{j_1 + j_2 - j_3 + 1} \sqrt{j_1 + j_2 + j_3 + 1}} \begin{pmatrix} j_1 & j_2 & j_3 - 1 \\ m_1 & m_2 & m_3 \end{pmatrix} - \Big(j_3 \sqrt{j_3 + m_3 - 1} \sqrt{j_3 - m_3 - 1} \sqrt{-j_1 + j_2 + j_3 - 1} \sqrt{j_1 - j_2 + j_3 - 1} \sqrt{j_1 + j_2 - j_3 + 2} \sqrt{j_1 + j_2 + j_3}\Big) \Big/ \Big((j_3 - 1) \sqrt{j_3 - m_3} \sqrt{j_3 + m_3} \sqrt{-j_1 + j_2 + j_3} \sqrt{j_1 - j_2 + j_3} \sqrt{j_1 + j_2 - j_3 + 1} \sqrt{j_1 + j_2 + j_3 + 1}\Big) \begin{pmatrix} j_1 & j_2 & j_3 - 2 \\ m_1 & m_2 & m_3 \end{pmatrix}$$

07.39.17.0014.01

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = ((2 j_3 + 3)((j_3 + 1)(j_3 + 2)(m_1 - m_2) + m_3 j_1 (j_1 + 1) - m_3 j_2 (j_2 + 1))) \Big/ \Big((j_3 + 2) \sqrt{j_3 - m_3 + 1} \sqrt{j_3 + m_3 + 1} \sqrt{-j_1 + j_2 + j_3 + 1} \sqrt{j_1 - j_2 + j_3 + 1} \sqrt{j_1 + j_2 - j_3} \sqrt{j_1 + j_2 + j_3 + 2}\Big) \begin{pmatrix} j_1 & j_2 & j_3 + 1 \\ m_1 & m_2 & m_3 \end{pmatrix} - \Big((j_3 + 1) \sqrt{j_3 - m_3 + 2} \sqrt{j_3 + m_3 + 2} \sqrt{-j_1 + j_2 + j_3 + 2} \sqrt{j_1 - j_2 + j_3 + 2} \sqrt{j_1 + j_2 - j_3 - 1} \sqrt{j_1 + j_2 + j_3 + 3}\Big) \Big/ \Big((j_3 + 2) \sqrt{j_3 - m_3 + 1} \sqrt{j_3 + m_3 + 1} \sqrt{-j_1 + j_2 + j_3 + 1} \sqrt{j_1 - j_2 + j_3 + 1} \sqrt{j_1 + j_2 - j_3} \sqrt{j_1 + j_2 + j_3 + 2}\Big) \begin{pmatrix} j_1 & j_2 & j_3 + 2 \\ m_1 & m_2 & m_3 \end{pmatrix}$$

Functional identities

General relations

07.39.17.0015.01

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \left(\sqrt{(j_1+j_2-j_3)!}\,\sqrt{(-j_1+j_2+j_3)!}\,\sqrt{(j_1+j_2+j_3+1)!}\,\sqrt{(-2n+j_2+m_2)!}\,\sqrt{(j_3-m_3)!}\right)\Big/ \left(\sqrt{(j_1-j_2+j_3)!}\,\sqrt{(j_2+m_2)!}\,\sqrt{(j_3+m_3)!}\right) \sum_{k=j_3-n}^{j_3+n} (-1)^{-j_3+k+n}\left((2k+1)(j_3+k-n)!\,(2n)!\,\sqrt{(k--m_3+n)!}\,\sqrt{(j_1-j_2+k+n)!}\right)\Big/\left((j_3-k+n)!\,(j_3+k+n+1)!\,(-j_3+k+n)!\,\sqrt{(k-m_3-n)!}\,\sqrt{(-j_1+j_2+k-n)!}\,\sqrt{(j_1+j_2-k-n)!}\,\sqrt{(j_1+j_2+k-n+1)!}\right) \begin{pmatrix} j_1 & j_2-n & k \\ m_1 & m_2-n & n+m_3 \end{pmatrix} /;\ 2n\in\mathbb{N} \bigwedge n \le \frac{j_2+m_2}{2} \bigwedge \mathcal{P}hysical\mathcal{Q}(\{j_1, m_1\}, \{j_2, m_2\}, \{j_3, m_3\})$$

07.39.17.0016.01

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \left(\sqrt{(j_1+j_2-j_3)!}\,\sqrt{(j_1-j_2+j_3)!}\,\sqrt{(1+j_1+j_2+j_3)!}\,\sqrt{(j_2-m_2)!}\,\sqrt{(j_1-j_2+m_1-m_2)!}\,\sqrt{(j_2+m_2)!}\,\sqrt{(j_1-j_2-m_1+m_2)!}\right)\Big/\left(\sqrt{(-j_1+j_2+j_3)!}\,\sqrt{(j_1-m_1)!}\,\sqrt{(j_1+m_1)!}\right) \sum_{k=|m_2|}^{j_2} (-1)^{j_2+k}\left((4k+1)(j_2+k)!\,\sqrt{(-j_1+j_2+j_3+2k)!}\,\sqrt{(2k+2m_2)!}\,\sqrt{(2k-2m_2)!}\right)\Big/\left((2j_2+2k+1)!\,(j_2-k)!\,(k+m_2)!\,(k-m_2)!\,\sqrt{(j_1-j_2+j_3+2k+1)!}\,\sqrt{(j_1-j_2+j_3-2k)!}\,\sqrt{(j_1-j_2-j_3+2k)!}\right) \begin{pmatrix} j_1-j_2 & 2k & j_3 \\ m_1-m_2 & 2m_2 & m_3 \end{pmatrix} /;\ |m_1-m_2| \le j_1-j_2 \bigwedge \mathcal{P}hysical\mathcal{Q}(\{j_1, m_1\}, \{j_2, m_2\}, \{j_3, m_3\})$$

Involving two 3 *j* symbols

07.39.17.0017.01

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{j_1+j_2+j_3} \begin{pmatrix} \frac{1}{2}(j_2+j_3+m_1) & \frac{1}{2}(j_1+j_3+m_2) & \frac{1}{2}(j_1+j_2+m_3) \\ j_1-\frac{1}{2}(j_2+j_3-m_1) & j_2-\frac{1}{2}(j_1+j_3-m_2) & j_3-\frac{1}{2}(j_1+j_2-m_3) \end{pmatrix}$$

07.39.17.0018.01

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \begin{pmatrix} j_1 & \frac{1}{2}(j_2+j_3-m_1) & \frac{1}{2}(j_2+j_3+m_1) \\ j_3-j_2 & \frac{1}{2}(j_2-j_3-m_1)-m_3 & \frac{1}{2}(j_2-j_3+m_1)+m_3 \end{pmatrix}$$

Involving three 3 *j* symbols

07.39.17.0006.02

$$\sqrt{j_1-m_1+1}\,\sqrt{j_1+m_1}\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1-1 & m_2 & m_3 \end{pmatrix} + \sqrt{j_2-m_2+1}\,\sqrt{j_2+m_2}\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2-1 & m_3 \end{pmatrix} + \sqrt{j_3-m_3+1}\,\sqrt{j_3+m_3}\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3-1 \end{pmatrix} = 0$$

07.39.17.0007.02

$$\sqrt{j_1+m_1+1}\,\sqrt{j_1-m_1}\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1+1 & m_2 & m_3 \end{pmatrix} + \sqrt{j_2+m_2+1}\,\sqrt{j_2-m_2}\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2+1 & m_3 \end{pmatrix} + \sqrt{j_3+m_3+1}\,\sqrt{j_3-m_3}\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3+1 \end{pmatrix} = 0$$

07.39.17.0008.01

$$\sqrt{j_2-m_2}\sqrt{j_3+m_3}\begin{pmatrix} j_1 & j_2-\frac{1}{2} & j_3-\frac{1}{2} \\ m_1 & m_2+\frac{1}{2} & m_3-\frac{1}{2} \end{pmatrix} - \sqrt{j_2+m_2}\sqrt{j_3-m_3}\begin{pmatrix} j_1 & j_2-\frac{1}{2} & j_3-\frac{1}{2} \\ m_1 & m_2-\frac{1}{2} & m_3+\frac{1}{2} \end{pmatrix} + \sqrt{j_1+j_2+j_3+1}\sqrt{j_2+j_3-j_1}\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = 0$$

Summation

Finite summation

Involving one 3 *j* symbol

07.39.23.0003.01

$$\sum_{m_1=-j_1}^{j_1} (-1)^{j_1-m_1}\begin{pmatrix} j_1 & j_1 & j_3 \\ m_1 & -m_1 & 0 \end{pmatrix} = \sqrt{2 j_1+1}\,\delta_{j_3,0} \;/;\; 2 j_1 \in \mathbb{N}$$

07.39.23.0004.01

$$\sum_{m_1=-j_1}^{j_1}\sum_{m_2=-j_2}^{j_2}\sum_{m_3=-j_3}^{j_3} \frac{\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}}{\sqrt{(j_1+m_1)!}\sqrt{(j_1-m_1)!}\sqrt{(j_2+m_2)!}\sqrt{(j_2-m_2)!}\sqrt{(j_3+m_3)!}\sqrt{(j_3-m_3)!}} = 0$$

07.39.23.0005.01

$$\sum_{j_1=|j_2-j_3|}^{j_2+j_3} \frac{2 j_1+1}{j_1(j_1+1)-n(n+1)}\begin{pmatrix} j_1 & j_2 & j_3 \\ 0 & 0 & 0 \end{pmatrix}^2 = 0 \;/;\; n \in \mathbb{Z} \bigwedge |j-j_2| \le n \le j+j_2 \bigwedge \frac{j+n+j_2+1}{2} \in \mathbb{Z}$$

07.39.23.0006.01

$$\sum_{j_2=0}^{j_1} \frac{1}{2 j_2-1}\begin{pmatrix} j_1 & j_2+j_3 & j_1-j_2+j_3 \\ 0 & 0 & 0 \end{pmatrix}^2 = -\frac{\delta_{j_1,0}}{2 j_3+1} \;/;\; j \in \mathbb{N}$$

07.39.23.0007.01

$$\sum_{j_2=0}^{j_1} \left(\frac{1}{2 j_2+3} - \frac{j_1+1}{(2 j_1+3)(2 j_2+1)}\right)\begin{pmatrix} j_1 & j_2+j_3 & j_1-j_2+j_3 \\ 0 & 0 & 0 \end{pmatrix}^2 = 0$$

07.39.23.0024.01

$$\sum_{j_2=0}^{j_1} \frac{1}{2 j_2-1}\begin{pmatrix} j_1 & j_2 & j_1-j_2 \\ 0 & 0 & 0 \end{pmatrix}^2 = -\delta_{j_1,0}$$

07.39.23.0025.01

$$\sum_{j_2=0}^{j_1} \frac{1}{2 j_1-2 j_2-1}\begin{pmatrix} j_1 & j_2 & j_1-j_2 \\ 0 & 0 & 0 \end{pmatrix}^2 = -\delta_{j_1,0}$$

Involving two 3 *j* symbols

07.39.23.0008.01

$$\sum_{m_2=-j_2}^{j_2}\sum_{m_3=-j_3}^{j_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}\begin{pmatrix} j_4 & j_2 & j_3 \\ m_4 & m_2 & m_3 \end{pmatrix} = \frac{\delta_{j_1,j_4}\,\delta_{m_1,m_4}}{2 j_1+1} \;/;\; \mathcal{TriangularQ}(j_1,\, j_2,\, j_3) \bigwedge j_1-m_1 \in \mathbb{Z} \bigwedge -j_1 \le m_1 \le j_1$$

07.39.23.0009.01

$$\sum_{m_1=-j_1}^{j_1} \sum_{m_3=-j_3}^{j_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_4 & j_3 \\ m_1 & m_4 & m_3 \end{pmatrix} == \frac{\delta_{j_2,j_4}\, \delta_{m_2,m_4}}{2\, j_2 + 1} \;/;\; \mathcal{T}riangular\mathcal{Q}(j_1,\, j_2,\, j_3) \bigwedge j_2 - m_2 \in \mathbb{Z} \bigwedge -j_2 \le m_2 \le j_2$$

07.39.23.0001.01

$$\sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_4 \\ m_1 & m_2 & m_4 \end{pmatrix} == \frac{\delta_{j_3,j_4}\, \delta_{m_3,m_4}}{2\, j_3 + 1} \;/;\; \mathcal{T}riangular\mathcal{Q}(j_1,\, j_2,\, j_3) \bigwedge j_3 - m_3 \in \mathbb{Z} \bigwedge -j_3 \le m_3 \le j_3$$

07.39.23.0010.01

$$\sum_{j_1=|j_2-j_3|}^{j_2+j_3} \sum_{m_1=-j_1}^{j_1} (2\, j_1 + 1) \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_4 & m_5 \end{pmatrix} == \delta_{m_2,m_4}\, \delta_{m_3,m_5} \;/;\; \mathcal{P}hysical\mathcal{Q}(\{j_2 + j_3,\, -m_2 - m_3\},\, \{j_2,\, m_2\},\, \{j_3,\, m_3\})$$

07.39.23.0011.01

$$\sum_{j_2=|j_1-j_2|}^{j_1+j_3} \sum_{m_2=-j_2}^{j_2} (2\, j_2 + 1) \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_4 & m_2 & m_5 \end{pmatrix} == \delta_{m_1,m_4}\, \delta_{m_3,m_5} \;/;\; \mathcal{P}hysical\mathcal{Q}(\{j_1,\, m_1\},\, \{j_1 + j_3,\, -m_1 - m_3\},\, \{j_3,\, m_3\})$$

07.39.23.0002.01

$$\sum_{j_3=|j_1-j_2|}^{j_1+j_2} \sum_{m_3=-j_3}^{j_3} (2\, j_3 + 1) \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_4 & m_5 & m_3 \end{pmatrix} == \delta_{m_1,m_4}\, \delta_{m_2,m_5} \;/;\; \mathcal{P}hysical\mathcal{Q}(\{j_1,\, m_1\},\, \{j_2,\, m_2\},\, \{j_1 + j_2,\, -m_1 - m_2\})$$

Involving three 3 *j* symbols

07.39.23.0012.01

$$\sum_{m_6=-j_6}^{j_6} \sum_{m_5=-j_5}^{j_5} \sum_{m_4=-j_4}^{j_4} (-1)^{j_4+j_5+j_6-m_4-m_5-m_6} \begin{pmatrix} j_5 & j_1 & j_6 \\ m_5 & -m_1 & -m_6 \end{pmatrix} \begin{pmatrix} j_6 & j_2 & j_4 \\ m_6 & -m_2 & -m_4 \end{pmatrix} \begin{pmatrix} j_4 & j_3 & j_5 \\ m_4 & -m_3 & -m_5 \end{pmatrix} ==$$
$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix}$$

07.39.23.0013.01

$$\sum_{m_3=-j_3}^{j_3} \sum_{m_5=-j_5}^{j_5} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_3 & j_4 & j_5 \\ -m_3 & -m_4 & -m_5 \end{pmatrix} \begin{pmatrix} j_5 & j_6 & j_7 \\ m_5 & m_6 & m_7 \end{pmatrix} ==$$
$$\sum_{k=|j_1-j_6|}^{j_1+j_6} \sum_{l=\max(|j_4-k|,|j_2-j_7|)}^{\min(j_4+k,j_2+j_7)} \sum_{\kappa=-k}^{k} \sum_{\lambda=-l}^{l} (2\, k + 1)\, (2\, l + 1) \begin{pmatrix} j_1 & j_6 & k \\ m_1 & m_6 & \kappa \end{pmatrix} \begin{pmatrix} k & j_4 & l \\ -\kappa & -m_4 & -\lambda \end{pmatrix} \begin{pmatrix} l & j_2 & j_7 \\ \lambda & m_2 & m_7 \end{pmatrix} \begin{Bmatrix} j_1 & j_2 & j_3 \\ j_6 & j_7 & j_5 \\ k & l & j_4 \end{Bmatrix}$$

Involving four 3 *j* symbols

07.39.23.0014.01

$$\sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} \sum_{m_4=-j_4}^{j_4} \sum_{m_5=-j_5}^{j_5} \sum_{m_6=-j_6}^{j_6} (-1)^{j_1+j_2+j_4+j_5+j_6-m_1-m_2-m_4-m_5-m_6} \begin{pmatrix} j_2 & j_3 & j_1 \\ m_2 & -m_3 & m_1 \end{pmatrix}$$
$$\begin{pmatrix} j_1 & j_5 & j_6 \\ -m_1 & m_5 & m_6 \end{pmatrix} \begin{pmatrix} j_5 & j_3' & j_4 \\ -m_5 & m_3' & m_4 \end{pmatrix} \begin{pmatrix} j_4 & j_2 & j_6 \\ -m_4 & -m_2 & -m_6 \end{pmatrix} == \frac{(-1)^{j_3-m_3}}{2\, j_3 + 1}\, \delta_{j_3,j_3'}\, \delta_{m_3,m_3'} \begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix}$$

07.39.23.0015.01

$$\sum_{m_3=-j_3}^{j_3}\sum_{m_4=-j_4}^{j_4}\sum_{m_5=-j_5}^{j_5}\sum_{m_6=-j_6}^{j_6}(-1)^{j_3+j_4+j_5+j_6-m_3-m_4-m_5-m_6}\begin{pmatrix} j_5 & j_1 & j_6 \\ m_5 & m_1 & -m_6 \end{pmatrix}\begin{pmatrix} j_6 & j_2 & j_4 \\ m_6 & m_2 & -m_4 \end{pmatrix}\begin{pmatrix} j_4 & j_3 & j_5 \\ m_4 & m_3 & -m_5 \end{pmatrix}\begin{pmatrix} j_3 & j_7 & j_8 \\ -m_3 & m_7 & m_8 \end{pmatrix} = \begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix}\sum_{m_3=-j_3}^{j_3}(-1)^{j_3-m_3}\begin{pmatrix} j_1 & j_3 & j_2 \\ m_1 & m_3 & m_2 \end{pmatrix}\begin{pmatrix} j_8 & j_3 & j_7 \\ m_8 & -m_3 & m_7 \end{pmatrix}$$

Involving five 3 *j* symbols

07.39.23.0016.01

$$\sum_{m_1=-j_1}^{j_1}\sum_{m_2=-j_2}^{j_2}\sum_{m_3=-j_3}^{j_3}\sum_{m_4=-j_4}^{j_4}\sum_{m_5=-j_5}^{j_5}\sum_{m_6=-j_6}^{j_6}\sum_{m_7=-j_7}^{j_7}(-1)^{j_1+j_2+j_3+j_4+j_5+j_6+j_7-m_1-m_2-m_3-m_4-m_5-m_6-m_7}\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & -m_3 \end{pmatrix}\begin{pmatrix} j_3 & j_4 & j_5 \\ m_3 & m_4 & -m_5 \end{pmatrix}\begin{pmatrix} j_5 & j_8 & j_7 \\ m_5 & m_8 & -m_7 \end{pmatrix}\begin{pmatrix} j_7 & j_6 & j_1 \\ m_7 & m_6 & -m_1 \end{pmatrix}\begin{pmatrix} j_6 & j_4 & j_2 \\ -m_6 & -m_4 & -m_2 \end{pmatrix} = \frac{(-1)^{2 j_5}}{\sqrt{2 j_5+1}}\delta_{j_8,0}\,\delta_{m_8,0}\,\delta_{j_5,j_7}\begin{Bmatrix} j_3 & j_4 & j_5 \\ j_6 & j_1 & j_2 \end{Bmatrix}$$

07.39.23.0017.01

$$\sum_{m_4=-j_4}^{j_4}\sum_{m_5=-j_5}^{j_5}\sum_{m_6=-j_6}^{j_6}\sum_{m_7=-j_7}^{j_7}\sum_{m_8=-j_8}^{j_8}\sum_{m_9=-j_9}^{j_9}(-1)^{j_4+j_5+j_6+j_7+j_8+j_9-m_4-m_5-m_6-m_7-m_8-m_9}\begin{pmatrix} j_4 & j_2 & j_6 \\ m_4 & m_2 & -m_6 \end{pmatrix}\begin{pmatrix} j_5 & j_3 & j_4 \\ m_5 & m_3 & -m_4 \end{pmatrix}\begin{pmatrix} j_7 & j_1 & j_8 \\ m_7 & m_1 & -m_8 \end{pmatrix}\begin{pmatrix} j_8 & j_6 & j_9 \\ m_8 & m_6 & -m_9 \end{pmatrix}\begin{pmatrix} j_9 & j_5 & j_7 \\ m_9 & -m_5 & -m_7 \end{pmatrix} = \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix}\begin{Bmatrix} j_1 & j_7 & j_8 \\ j_9 & j_6 & j_5 \end{Bmatrix}$$

07.39.23.0018.01

$$\sum_{m_4=-j_4}^{j_4}\sum_{m_5=-j_5}^{j_5}\sum_{m_6=-j_6}^{j_6}\sum_{m_7=-j_7}^{j_7}\sum_{m_8=-j_8}^{j_8}\sum_{m_9=-j_9}^{j_9}(-1)^{j_4+j_5+j_6+j_7+j_8+j_9-m_4-m_5-m_6-m_7-m_8-m_9}\begin{pmatrix} j_4 & j_1 & j_7 \\ m_4 & -m_1 & m_7 \end{pmatrix}\begin{pmatrix} j_5 & j_4 & j_6 \\ -m_5 & -m_4 & -m_6 \end{pmatrix}\begin{pmatrix} j_6 & j_3 & j_9 \\ m_6 & -m_3 & m_9 \end{pmatrix}\begin{pmatrix} j_7 & j_9 & j_8 \\ -m_7 & -m_9 & -m_8 \end{pmatrix}\begin{pmatrix} j_8 & j_2 & j_5 \\ m_8 & -m_2 & m_5 \end{pmatrix} = (-1)^{j_2+j_5+j_8}\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \\ j_7 & j_8 & j_9 \end{Bmatrix}$$

Involving six 3 *j* symbols

07.39.23.0019.01

$$\sum_{m_2=-j_2}^{j_2}\sum_{m_3=-j_3}^{j_3}\sum_{m_4=-j_4}^{j_4}\sum_{m_5=-j_5}^{j_5}\sum_{m_6=-j_6}^{j_6}\sum_{m_7=-j_7}^{j_7}\sum_{m_8=-j_8}^{j_8}\sum_{m_9=-j_9}^{j_9}(-1)^{j_2+j_3+j_4+j_5+j_6+j_7+j_8+j_9-m_2-m_3-m_4-m_5-m_6-m_7-m_8-m_9}\begin{pmatrix} j_2 & j_1 & j_3 \\ m_2 & -m_1 & m_3 \end{pmatrix}\begin{pmatrix} j_3 & j_5 & j_7 \\ -m_3 & m_5 & m_7 \end{pmatrix}\begin{pmatrix} j_5 & j_1' & j_6 \\ -m_5 & m_1' & m_6 \end{pmatrix}\begin{pmatrix} j_6 & j_2 & j_4 \\ -m_6 & -m_2 & -m_4 \end{pmatrix}\begin{pmatrix} j_7 & j_8 & j_9 \\ -m_7 & m_8 & m_9 \end{pmatrix}\begin{pmatrix} j_8 & j_9 & j_4 \\ -m_8 & -m_9 & m_4 \end{pmatrix} = \frac{(-1)^{j_1+2 j_4-m_1}}{(2 j_1+1)(2 j_4+1)}\delta_{j_1,j_1'}\,\delta_{j_4,j_7}\,\delta_{m_1,m_1'}\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} /;\ \mathcal{TriangularQ}(j_4, j_8, j_9)$$

07.39.23.0020.01

$$\sum_{m_1=-j_1}^{j_1}\sum_{m_2=-j_2}^{j_2}\sum_{m_3=-j_3}^{j_3}\sum_{m_4=-j_4}^{j_4}\sum_{m_5=-j_5}^{j_5}\sum_{m_6=-j_6}^{j_6}\sum_{m_7=-j_7}^{j_7}\sum_{m_8=-j_8}^{j_8}(-1)^{j_1+j_2+j_3+j_4+j_5+j_6+j_7+j_8-m_1-m_2-m_3-m_4-m_5-m_6-m_7-m_8}\begin{pmatrix} j_1 & j_2 & j_7 \\ -m_1 & -m_2 & -m_7 \end{pmatrix}$$
$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}\begin{pmatrix} j_3 & j_6 & j_9 \\ m_3 & m_6 & m_9 \end{pmatrix}\begin{pmatrix} j_4 & j_5 & j_8 \\ -m_4 & -m_5 & -m_8 \end{pmatrix}\begin{pmatrix} j_4 & j_5 & j_6 \\ m_4 & m_5 & m_6 \end{pmatrix}\begin{pmatrix} j_7 & j_8 & j_9' \\ -m_7 & -m_8 & -m_9' \end{pmatrix}=$$
$$(-1)^{j_9-m_9}\frac{\delta_{j_3,j_7}\,\delta_{j_6,j_8}\,\delta_{j_9,j_9'}\,\delta_{m_9,m_9'}}{(2\,j_3+1)\,(2\,j_6+1)\,(2\,j_9+1)}\;/;\;\mathcal{T}riangular\mathcal{Q}(j_1,\,j_2,\,j_3)\land\mathcal{T}riangular\mathcal{Q}(j_4,\,j_5,\,j_6)\land$$
$$\mathcal{T}riangular\mathcal{Q}(j_3,\,j_6,\,j_9)$$

07.39.23.0021.01

$$\sum_{m_2=-j_2}^{j_2}\sum_{m_3=-j_3}^{j_3}\sum_{m_4=-j_4}^{j_4}\sum_{m_5=-j_5}^{j_5}\sum_{m_6=-j_6}^{j_6}\sum_{m_7=-j_7}^{j_7}\sum_{m_8=-j_8}^{j_8}\sum_{m_9=-j_9}^{j_9}(-1)^{j_2+j_3+j_4+j_5+j_6+j_7+j_8+j_9-m_2-m_3-m_4-m_5-m_6-m_7-m_8-m_9}\begin{pmatrix} j_1 & j_3 & j_2 \\ -m_1 & m_3 & m_2 \end{pmatrix}$$
$$\begin{pmatrix} j_2 & j_6 & j_4 \\ -m_2 & m_6 & -m_4 \end{pmatrix}\begin{pmatrix} j_4 & j_7 & j_9 \\ m_4 & -m_7 & m_9 \end{pmatrix}\begin{pmatrix} j_5 & j_6 & j_1' \\ m_5 & -m_6 & m_1' \end{pmatrix}\begin{pmatrix} j_8 & j_7 & j_5 \\ -m_8 & m_7 & -m_5 \end{pmatrix}\begin{pmatrix} j_9 & j_8 & j_3 \\ -m_9 & m_8 & -m_3 \end{pmatrix}=$$
$$\frac{(-1)^{-j_2-j_4+j_5-m_1}}{2\,j_1+1}\,\delta_{j_1,j_1'}\,\delta_{m_1,m_1'}\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix}\begin{Bmatrix} j_3 & j_4 & j_5 \\ j_7 & j_8 & j_9 \end{Bmatrix}$$

07.39.23.0022.01

$$\sum_{m_2=-j_2}^{j_2}\sum_{m_3=-j_3}^{j_3}\sum_{m_4=-j_4}^{j_4}\sum_{m_5=-j_5}^{j_5}\sum_{m_6=-j_6}^{j_6}\sum_{m_7=-j_7}^{j_7}\sum_{m_8=-j_8}^{j_8}\sum_{m_9=-j_9}^{j_9}(-1)^{j_2+j_3+j_4+j_5+j_6+j_7+j_8+j_9-m_2-m_3-m_4-m_5-m_6-m_7-m_8-m_9}\begin{pmatrix} j_4 & j_3 & j_5 \\ m_4 & m_3 & m_5 \end{pmatrix}$$
$$\begin{pmatrix} j_5 & j_1' & j_6 \\ -m_5 & -m_1' & -m_6 \end{pmatrix}\begin{pmatrix} j_6 & j_2 & j_4 \\ m_6 & -m_2 & -m_4 \end{pmatrix}\begin{pmatrix} j_7 & j_3 & j_8 \\ m_7 & -m_3 & -m_8 \end{pmatrix}\begin{pmatrix} j_8 & j_1 & j_9 \\ m_8 & m_1 & -m_9 \end{pmatrix}\begin{pmatrix} j_9 & j_2 & j_7 \\ m_9 & m_2 & -m_7 \end{pmatrix}=$$
$$\frac{(-1)^{j_1+2\,j_4-m_1}}{2\,j_1+1}\,\delta_{j_1,j_1'}\,\delta_{m_1,m_1'}\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix}\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_7 & j_8 & j_9 \end{Bmatrix}$$

07.39.23.0023.01

$$\sum_{m_2=-j_2}^{j_2}\sum_{m_3=-j_3}^{j_3}\sum_{m_4=-j_4}^{j_4}\sum_{m_5=-j_5}^{j_5}\sum_{m_6=-j_6}^{j_6}\sum_{m_7=-j_7}^{j_7}\sum_{m_8=-j_8}^{j_8}\sum_{m_9=-j_9}^{j_9}(-1)^{j_2+j_3+j_4+j_5+j_6+j_7+j_8+j_9-m_2-m_3-m_4-m_5-m_6-m_7-m_8-m_9}$$
$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}\begin{pmatrix} j_3 & j_6 & j_9 \\ -m_3 & -m_6 & -m_9 \end{pmatrix}\begin{pmatrix} j_4 & j_7 & j_1' \\ -m_4 & -m_7 & -m_1' \end{pmatrix}\begin{pmatrix} j_5 & j_6 & j_4 \\ m_5 & m_6 & m_4 \end{pmatrix}$$
$$\begin{pmatrix} j_8 & j_2 & j_5 \\ -m_8 & -m_2 & -m_5 \end{pmatrix}\begin{pmatrix} j_9 & j_7 & j_8 \\ m_9 & m_7 & m_8 \end{pmatrix}=\frac{(-1)^{j_1-m_1}}{2\,j_1+1}\,\delta_{j_1,j_1'}\,\delta_{m_1,m_1'}\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \\ j_7 & j_8 & j_9 \end{Bmatrix}$$

Representations through more general functions

Through hypergeometric functions

Involving ${}_p\tilde{F}_q$

07.39.26.0001.01

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \delta_{-m_3,m_1+m_2} (-1)^{-j_1+j_2+m_3} \left(\sqrt{(j_1-j_2+j_3)!}\, \sqrt{(-j_1+j_2+j_3)!}\, \sqrt{(j_3-m_3)!}\, \sqrt{(j_3+m_3)!}\, \sqrt{(j_1+m_1)!}\, \sqrt{(j_2-m_2)!}\right) \Big/ \left(\sqrt{(j_1+j_2-j_3)!}\, \sqrt{(j_1+j_2+j_3+1)!}\, \sqrt{(j_1-m_1)!}\, \sqrt{(j_2+m_2)!}\right) {}_3\tilde{F}_2(-j_1-j_2+j_3, m_1-j_1, -j_2-m_2; -j_2+j_3+m_1+1, -j_1+j_3-m_2+1; 1) /; \mathcal{P}hysical\mathcal{Q}(\{j_1, m_1\}, \{j_2, m_2\}, \{j_3, m_3\})$$

Involving ${}_pF_q$

07.39.26.0002.01

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{-j_1+j_2+m_3} \delta_{-m_3,m_1+m_2} \left(\sqrt{(j_1-j_2+j_3)!}\, \sqrt{(-j_1+j_2+j_3)!}\, \sqrt{(j_1+m_1)!}\, \sqrt{(j_2-m_2)!}\, \sqrt{(j_3+m_3)!}\, \sqrt{(j_3-m_3)!}\right) \Big/ \left(\sqrt{(j_1+j_2-j_3)!}\, \sqrt{(j_1+j_2+j_3+1)!}\, (-j_2+j_3+m_1)!\, (-j_1+j_3-m_2)!\, \sqrt{(j_1-m_1)!}\, \sqrt{(j_2+m_2)!}\right) {}_3F_2(-j_1-j_2+j_3, m_1-j_1, -j_2-m_2; -j_1+j_3-m_2+1, -j_2+j_3+m_1+1; 1) /; \mathcal{P}hysical\mathcal{Q}(\{j_1, m_1\}, \{j_2, m_2\}, \{j_3, m_3\})$$

07.39.26.0004.01

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{j_2+m_3-m_1} \delta_{-m_3,m_1+m_2} \left(\sqrt{(j_1+j_2-j_3)!}\, \sqrt{(j_1+m_1)!}\, \sqrt{(j_2-m_2)!}\, \sqrt{(j_3-m_3)!}\, (j_2+j_3-m_1)!\right) \Big/ \left(\sqrt{(j_1-j_2+j_3)!}\, \sqrt{(-j_1+j_2+j_3)!}\, \sqrt{(j_1+j_2+j_3+1)!}\, \sqrt{(j_1-m_1)!}\, \sqrt{(j_2+m_2)!}\, \sqrt{(j_3+m_3)!}\, (j_2-j_3+m_1)!\right) {}_3F_2(j_1+m_1+1, m_1-j_1, -j_3-m_3; -j_2-j_3+m_1, j_2-j_3+m_1+1; 1) /; \mathcal{P}hysical\mathcal{Q}(\{j_1, m_1\}, \{j_2, m_2\}, \{j_3, m_3\})$$

07.39.26.0005.01

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{-j_1-m_2+m_3} \delta_{-m_3,m_1+m_2} \left(\sqrt{(j_1+j_2-j_3)!}\, \sqrt{(j_1-j_2+j_3)!}\, \sqrt{(j_1-m_1)!}\, \sqrt{(j_3+m_3)!}\, (j_2+j_3+m_1)!\right) \Big/ \left(\sqrt{(-j_1+j_2+j_3)!}\, \sqrt{(j_1+j_2+j_3+1)!}\, \sqrt{(j_1+m_1)!}\, \sqrt{(j_2+m_2)!}\, \sqrt{(j_2-m_2)!}\, \sqrt{(j_3-m_3)!}\, (j_1-j_2+m_3)!\right) {}_3F_2(j_1-j_2-j_3, j_1-m_1+1, m_3-j_3; j_1-j_2+m_3+1, -j_2-j_3-m_1; 1) /; \mathcal{P}hysical\mathcal{Q}(\{j_1, m_1\}, \{j_2, m_2\}, \{j_3, m_3\})$$

07.39.26.0006.01

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{-j_1-m_2+m_3} \delta_{-m_3,m_1+m_2} \left(\sqrt{(j_1+j_2-j_3)!}\, \sqrt{(j_1+m_1)!}\, \sqrt{(j_1-m_1)!}\, \sqrt{(j_2-m_2)!}\, (2 j_3)!\right) \Big/ \left(\sqrt{(j_1-j_2+j_3)!}\, \sqrt{(-j_1+j_2+j_3)!}\, \sqrt{(j_1+j_2+j_3+1)!}\, \sqrt{(j_2+m_2)!}\, \sqrt{(j_3+m_3)!}\, \sqrt{(j_3-m_3)!}\, (j_1-j_3-m_2)!\right) {}_3F_2(j_1-j_2-j_3, j_1+j_2-j_3+1, m_3-j_3; -2 j_3, j_1-j_3-m_2+1; 1) /; \mathcal{P}hysical\mathcal{Q}(\{j_1, m_1\}, \{j_2, m_2\}, \{j_3, m_3\})$$

07.39.26.0007.01

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{-j_1-m_2+m_3} \delta_{-m_3,m_1+m_2} \left(\sqrt{(j_1+j_2-j_3)!}\, \sqrt{(j_1-m_1)!}\, (2 j_3)!\, (j_2+j_3+m_1)!\right) \Big/ \left(\sqrt{(j_1-j_2+j_3)!}\, \sqrt{(-j_1+j_2+j_3)!}\, \sqrt{(j_1+j_2+j_3+1)!}\, \sqrt{(j_1+m_1)!}\, \sqrt{(j_2+m_2)!}\, \sqrt{(j_2-m_2)!}\, \sqrt{(j_3+m_3)!}\, \sqrt{(j_3-m_3)!}\right) {}_3F_2(j_1-j_2-j_3, -j_1-j_2-j_3-1, m_3-j_3; -2 j_3, -j_2-j_3-m_1; 1) /; \mathcal{P}hysical\mathcal{Q}(\{j_1, m_1\}, \{j_2, m_2\}, \{j_3, m_3\})$$

07.39.26.0008.01

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{j_2-m_1+m_3} \delta_{-m_3,m_1+m_2} \left((j_1+j_2+m_3)!\,(j_2+j_3-m_1)!\,\sqrt{(j_1+m_1)!}\,\sqrt{(j_3-m_3)!}\right) \Big/ \left(\sqrt{(j_1+j_2-j_3)!}\,\sqrt{(j_1-j_2+j_3)!}\,\sqrt{(-j_1+j_2+j_3)!}\,\sqrt{(j_1+j_2+j_3+1)!}\,\sqrt{(j_1-m_1)!}\,\sqrt{(j_2+m_2)!}\,\sqrt{(j_2-m_2)!}\,\sqrt{(j_3+m_3)!}\right) {}_3F_2(-j_1-j_2-j_3-1, m_1-j_1, -j_3-m_3; -j_1-j_2-m_3, -j_2-j_3+m_1; 1) \;/;\; \mathcal{P}hysical\mathcal{Q}(\{j_1, m_1\}, \{j_2, m_2\}, \{j_3, m_3\})$$

Through Meijer *G*

Classical cases for the direct function itself

07.39.26.0003.01

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{-j_1+j_2+m_3} \delta_{-m_3,m_1+m_2} \left(\sqrt{(j_3-m_3)!}\,\sqrt{(j_3+m_3)!}\,\sqrt{(j_1-j_2+j_3)!}\,\sqrt{(-j_1+j_2+j_3)!}\,\sqrt{(j_1+m_1)!}\,\sqrt{(j_2-m_2)!}\right) \left(\sqrt{(j_1+j_2-j_3)!}\,\sqrt{(j_1+j_2+j_3+1)!}\,\sqrt{(j_1-m_1)!}\,\sqrt{(j_2+m_2)!}\,(-j_1+m_1-1)!\,(-j_2-m_2-1)!\,(j-j_1-j_2-1)!\right) G_{3,3}^{1,3}\!\left(-1 \,\middle|\, \begin{matrix} j_1+j_2-j_3+1,\ j_1-m_1+1,\ j_2+m_2+1 \\ 0,\ j_2-j_3-m_1,\ j_1-j_3+m_2 \end{matrix}\right) /;\; \mathcal{P}hysical\mathcal{Q}(\{j_1, m_1\}, \{j_2, m_2\}, \{j_3, m_3\})$$

Representations through equivalent functions

With related functions

07.39.27.0001.01

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \frac{(-1)^{-j_1+j_2+m_3}}{\sqrt{2\,j_3+1}} \langle j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j_3\, -m_3\rangle$$

Theorems

Representation as a magic square

For a physically realizable $3j$ -symbol $\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$ the following 3×3 matrix is a magic square, all rows and columns summing up to $j_1 + j_2 + j_3$:

$$\begin{pmatrix} -j_1+j_2+j_3 & j_1-j_2+j_3 & j_1+j_2-j_3 \\ j_1-m_1 & j_2-m_2 & j_3-m_3 \\ j_1+m_1 & j_2+m_2 & j_3+m_3 \end{pmatrix}$$

The quantum chemistry Gaunt coefficients

The quantum chemistry Gaunt coefficients

$$a(m, n, \mu, \nu, p) = \frac{(2\,p+1)\,(p-m-\mu)}{2\,(p+m+\mu)} \int_{-1}^{1} P_n^m(x)\,P_\nu^\mu(x)\,P_p^{m+\mu}(x)\,dx$$

can be expressed in $3j$ symbols.

History

– E. P. Wigner (1931, 1940)
– G. Racah (1942)

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