Zeta2

Notations

Traditional name

Hurwitz zeta function

Traditional notation

$\zeta(s, a)$

Mathematica StandardForm notation

`Zeta[s, a]`

Primary definition

10.02.02.0001.01

$$\zeta(s, a) = \sum_{k=0}^{\infty} \frac{1}{\left((a+k)^2\right)^{s/2}} \;/;\; -a \notin \mathbb{N} \land \operatorname{Re}(s) > 1$$

10.02.02.0002.01

$$\zeta(s, -n) = \sum_{k=0}^{n-1} \frac{1}{\left((k-n)^2\right)^{s/2}} + \sum_{k=n+1}^{\infty} \frac{1}{\left((k-n)^2\right)^{s/2}} \;/;\; n \in \mathbb{N} \land \operatorname{Re}(s) > 1$$

10.02.02.0003.01

$$\hat{\zeta}(s, a) = \sum_{k=0}^{\infty} \frac{1}{(a+k)^s} \;/;\; \operatorname{Re}(s) > 1$$

10.02.02.0004.01

$$\tilde{\zeta}(s, a) = \sum_{k=0}^{\infty} \frac{1}{(a+k)^s} \;/;\; -a \notin \mathbb{N}$$

10.02.02.0005.01

$$\tilde{\zeta}(s, -n) = \sum_{k=0}^{n-1} \frac{1}{(k-n)^s} + \sum_{k=n+1}^{\infty} \frac{1}{(k-n)^s} \;/;\; n \in \mathbb{N}$$

10.02.02.0006.01

$$\zeta(s, a) = \tilde{\zeta}(s, a) - \sum_{k=0}^{\lfloor -\operatorname{Re}(a) \rfloor} \left(\frac{1}{(a+k)^s} - \frac{1}{\left((a+k)^2\right)^{s/2}} \right) \;/;\; -a \notin \mathbb{N}$$

10.02.02.0007.01

$$\zeta(s,-n) == \tilde{\zeta}(s,-n) - \sum_{k=0}^{n-1}\left(\frac{1}{(k-n)^s} - \frac{1}{\left((k-n)^2\right)^{s/2}}\right) /; n \in \mathbb{N}$$

In *Mathematica* for the definition of the function $\zeta(s, a)$ the relations $\zeta(s,a) == \sum_{k=0}^{\infty} \frac{1}{\left((a+k)^2\right)^{s/2}} /; -a \notin \mathbb{N} \bigwedge \mathrm{Re}(s) > 1$ and $\zeta(s,-n) == \sum_{k=0}^{n-1} \frac{1}{\left((k-n)^2\right)^{s/2}} + \sum_{k=n+1}^{\infty} \frac{1}{\left((k-n)^2\right)^{s/2}} /; n \in \mathbb{N} \bigwedge \mathrm{Re}\,\{s\} > 1$ are used:

10.02.02.0008.01

$$\zeta(s,a) == \tilde{\zeta}(s,a) /; \mathrm{Re}(a) > 0$$

Specific values

Specialized values

For fixed s

For $\zeta(s, a)$

10.02.03.0001.01

$$\zeta(s,-n) == \zeta(s) + \sum_{k=1}^{n} \frac{1}{k^s} /; n \in \mathbb{N}$$

10.02.03.0002.01

$$\zeta(s,-5) == \zeta(s) + 2^{-s} + 3^{-s} + 4^{-s} + 5^{-s} + 1$$

10.02.03.0003.01

$$\zeta(s,-4) == \zeta(s) + 2^{-s} + 3^{-s} + 4^{-s} + 1$$

10.02.03.0004.01

$$\zeta(s,-3) == \zeta(s) + 2^{-s} + 3^{-s} + 1$$

10.02.03.0005.01

$$\zeta(s,-2) == \zeta(s) + 2^{-s} + 1$$

10.02.03.0006.01

$$\zeta(s,-1) == \zeta(s) + 1$$

10.02.03.0007.01

$$\zeta(s,0) == \zeta(s)$$

10.02.03.0008.01

$$\zeta(s,n) == \zeta(s) - \sum_{k=1}^{n-1} \frac{1}{k^s} /; n \in \mathbb{N}^+$$

10.02.03.0009.01

$$\zeta(s,1) == \zeta(s)$$

10.02.03.0010.01

$$\zeta(s,2) == \zeta(s) - 1$$

10.02.03.0011.01

$$\zeta(s,3) == \zeta(s) - 2^{-s} - 1$$

10.02.03.0012.01

$$\zeta(s, 4) = \zeta(s) - 2^{-s} - 3^{-s} - 1$$

10.02.03.0013.01

$$\zeta(s, 5) = \zeta(s) - 2^{-s} - 3^{-s} - 4^{-s} - 1$$

10.02.03.0088.01

$$\zeta\left(s, \frac{1}{2} - n\right) = 2^s \sum_{k=1}^{n} \frac{1}{(2k-1)^s} + (2^s - 1)\,\zeta(s) \;/;\; n \in \mathbb{N}^+$$

10.02.03.0089.01

$$\zeta\left(s, -\frac{7}{2}\right) = 2^s (1 + 3^{-s} + 5^{-s} + 7^{-s}) + (-1 + 2^s)\,\zeta(s)$$

10.02.03.0090.01

$$\zeta\left(s, -\frac{5}{2}\right) = 2^s (1 + 3^{-s} + 5^{-s}) + (-1 + 2^s)\,\zeta(s)$$

10.02.03.0091.01

$$\zeta\left(s, -\frac{3}{2}\right) = 2^s (1 + 3^{-s}) + (-1 + 2^s)\,\zeta(s)$$

10.02.03.0092.01

$$\zeta\left(s, -\frac{1}{2}\right) = (-1 + 2^s)\,\zeta(s) + 2^s$$

10.02.03.0014.01

$$\zeta\left(s, n + \frac{1}{2}\right) = (2^s - 1)\,\zeta(s) - 2^s \sum_{k=0}^{n-1} \frac{1}{(2k+1)^s} \;/;\; n \in \mathbb{N}$$

10.02.03.0015.01

$$\zeta\left(s, \frac{1}{2}\right) = (2^s - 1)\,\zeta(s)$$

10.02.03.0016.01

$$\zeta\left(s, \frac{3}{2}\right) = (2^s - 1)\,\zeta(s) - 2^s$$

10.02.03.0017.01

$$\zeta\left(s, \frac{5}{2}\right) = (2^s - 1)\,\zeta(s) - 2^s (1 + 3^{-s})$$

10.02.03.0018.01

$$\zeta\left(s, \frac{7}{2}\right) = (2^s - 1)\,\zeta(s) - 2^s (1 + 3^{-s} + 5^{-s})$$

10.02.03.0019.01

$$\zeta\left(s, \frac{9}{2}\right) = (2^s - 1)\,\zeta(s) - 2^s (1 + 3^{-s} + 5^{-s} + 7^{-s})$$

10.02.03.0020.01

$$\zeta\left(s, \frac{m}{n}\right) = \frac{1}{n} \sum_{k=1}^{n} n^s \operatorname{Li}_s\left(e^{\frac{2\pi i k}{n}}\right) e^{-\frac{2\pi i k m}{n}} \;/;\; m \in \mathbb{N}^+ \land n \in \mathbb{N}^+ \land m \leq n$$

For $\hat{\zeta}(s, a)$

10.02.03.0021.01

$$\hat{\zeta}\left(s, \frac{1}{2} - n\right) = (2^s - 1)\,\zeta(s) + 2^s\, e^{-\pi i s} \sum_{k=1}^{n} \frac{1}{(2k-1)^s} \;/;\; n \in \mathbb{N}^+$$

10.02.03.0022.01

$$\hat{\zeta}\left(s, -\frac{1}{2}\right) = (2^s - 1)\,\zeta(s) + 2^s\, e^{-i\pi s}$$

10.02.03.0023.01

$$\hat{\zeta}\left(s, -\frac{3}{2}\right) = (2^s - 1)\,\zeta(s) + 2^s\,(1 + 3^{-s})\, e^{-i\pi s}$$

10.02.03.0024.01

$$\hat{\zeta}\left(s, -\frac{5}{2}\right) = (2^s - 1)\,\zeta(s) + 2^s\,(1 + 3^{-s} + 5^{-s})\, e^{-i\pi s}$$

10.02.03.0025.01

$$\hat{\zeta}\left(s, -\frac{7}{2}\right) = (2^s - 1)\,\zeta(s) + 2^s\,(1 + 3^{-s} + 5^{-s} + 7^{-s})\, e^{-i\pi s}$$

10.02.03.0026.01

$$\hat{\zeta}\left(s, -\frac{9}{2}\right) = (2^s - 1)\,\zeta(s) + 2^s\,(1 + 3^{-s} + 5^{-s} + 7^{-s} + 9^{-s})\, e^{-i\pi s}$$

For fixed *a*

For $\zeta(s, a)$

10.02.03.0027.01

$$\zeta(0, a) = \frac{1}{2} - a$$

10.02.03.0028.01

$$\zeta(1, a) = \tilde{\infty}$$

10.02.03.0093.01

$$\zeta(n, a) = \sum_{k=0}^{\lfloor -\mathrm{Re}(a) \rfloor} \left(\left((a+k)^2\right)^{-\frac{n}{2}} - (a+k)^{-n} \right) - \theta(-n)\, \frac{B_{1-n}(a)}{1-n} + \theta(n-1)\, \frac{(-1)^n\, \psi^{(n-1)}(a)}{(n-1)!} \;/;\; n \in \mathbb{Z} \land n \neq 1$$

10.02.03.0094.01

$$\zeta(n, a) = \frac{(-1)^n\, \psi^{(n-1)}(a)}{(n-1)!} + \sum_{k=0}^{\lfloor -\mathrm{Re}(a) \rfloor} \left(\left((a+k)^2\right)^{-\frac{n}{2}} - (a+k)^{-n} \right) /;\; n - 1 \in \mathbb{N}^+$$

10.02.03.0095.01

$$\zeta(-n, a) = -\frac{1}{n+1}\, B_{n+1}(a) + \sum_{k=0}^{\lfloor -\mathrm{Re}(a) \rfloor} \left(\left((a+k)^2\right)^{n/2} - (a+k)^n \right) /;\; n \in \mathbb{N}$$

For $\hat{\zeta}(s, a)$

10.02.03.0029.01

$$\hat{\zeta}(2, a) = \psi^{(1)}(a)$$

10.02.03.0030.01

$$\hat{\zeta}(n, a) = \frac{(-1)^n}{(n-1)!} \psi^{(n-1)}(a) /; n-1 \in \mathbb{N}^+ \bigwedge \operatorname{Re}(a) > 0$$

10.02.03.0031.01

$$\hat{\zeta}(-n, a) = -\frac{1}{n+1} B_{n+1}(a) /; n \in \mathbb{N}$$

10.02.03.0032.01

$$2^{n+1} \hat{\zeta}(-n, a) - \hat{\zeta}(-n, 2a) = \frac{1}{2} E_n(2a) /; n \in \mathbb{N}$$

10.02.03.0033.01

$$\hat{\zeta}(-n, a) - \hat{\zeta}\left(-n, a + \frac{1}{2}\right) = 2^{-n-1} E_n(2a) /; n \in \mathbb{N}$$

Values at fixed points

Values at fixed points

For $\zeta(s, a)$

10.02.03.0034.01

$$\zeta(-n, 1) = \frac{(-1)^n}{n+1} B_{n+1} /; n \in \mathbb{N}$$

10.02.03.0035.01

$$\zeta\left(-n, \frac{1}{4}\right) = 2^{-2n-2} \left(E_n - 2 \left(2^n - 1\right) \zeta(-n)\right) /; n \in \mathbb{N}^+$$

10.02.03.0036.01

$$\zeta\left(2, \frac{1}{4}\right) = \pi^2 + 8C$$

10.02.03.0037.01

$$\zeta\left(2, \frac{3}{4}\right) = \pi^2 - 8C$$

10.02.03.0038.01

$$\zeta(2, 1) = \frac{\pi^2}{6}$$

10.02.03.0039.01

$$\zeta(2, 2) = \frac{\pi^2}{6} - 1$$

10.02.03.0040.01

$$\zeta(2, 3) = \frac{\pi^2}{6} - \frac{5}{4}$$

10.02.03.0041.01

$$\zeta(2, 4) = \frac{\pi^2}{6} - \frac{49}{36}$$

10.02.03.0042.01

$$\zeta(2,5) == \frac{\pi^2}{6} - \frac{205}{144}$$

10.02.03.0043.01

$$\zeta(2,6) == \frac{\pi^2}{6} - \frac{5269}{3600}$$

10.02.03.0044.01

$$\zeta(2,7) == \frac{\pi^2}{6} - \frac{5369}{3600}$$

10.02.03.0045.01

$$\zeta(2,8) == \frac{\pi^2}{6} - \frac{266681}{176400}$$

10.02.03.0046.01

$$\zeta(2,9) == \frac{\pi^2}{6} - \frac{1077749}{705600}$$

10.02.03.0047.01

$$\zeta(2,10) == \frac{\pi^2}{6} - \frac{9778141}{6350400}$$

10.02.03.0048.01

$$\zeta(4,1) == \frac{\pi^4}{90}$$

10.02.03.0049.01

$$\zeta(4,2) == \frac{\pi^4}{90} - 1$$

10.02.03.0050.01

$$\zeta(4,3) == \frac{\pi^4}{90} - \frac{17}{16}$$

10.02.03.0051.01

$$\zeta(4,4) == \frac{\pi^4}{90} - \frac{1393}{1296}$$

10.02.03.0052.01

$$\zeta(4,5) == \frac{\pi^4}{90} - \frac{22369}{20736}$$

10.02.03.0053.01

$$\zeta(4,6) == \frac{\pi^4}{90} - \frac{14001361}{12960000}$$

10.02.03.0054.01

$$\zeta(4,7) == \frac{\pi^4}{90} - \frac{14011361}{12960000}$$

10.02.03.0055.01

$$\zeta(4, 8) = \frac{\pi^4}{90} - \frac{33\,654\,237\,761}{31\,116\,960\,000}$$

10.02.03.0056.01

$$\zeta(4, 9) = \frac{\pi^4}{90} - \frac{538\,589\,354\,801}{497\,871\,360\,000}$$

10.02.03.0057.01

$$\zeta(4, 10) = \frac{\pi^4}{90} - \frac{43\,631\,884\,298\,881}{40\,327\,580\,160\,000}$$

10.02.03.0058.01

$$\zeta(6, 1) = \frac{\pi^6}{945}$$

10.02.03.0059.01

$$\zeta(6, 2) = \frac{\pi^6}{945} - 1$$

10.02.03.0060.01

$$\zeta(6, 3) = \frac{\pi^6}{945} - \frac{65}{64}$$

10.02.03.0061.01

$$\zeta(6, 4) = \frac{\pi^6}{945} - \frac{47\,449}{46\,656}$$

10.02.03.0062.01

$$\zeta(6, 5) = \frac{\pi^6}{945} - \frac{3\,037\,465}{2\,985\,984}$$

10.02.03.0063.01

$$\zeta(6, 6) = \frac{\pi^6}{945} - \frac{47\,463\,376\,609}{46\,656\,000\,000}$$

10.02.03.0064.01

$$\zeta(6, 7) = \frac{\pi^6}{945} - \frac{47\,464\,376\,609}{46\,656\,000\,000}$$

10.02.03.0065.01

$$\zeta(6, 8) = \frac{\pi^6}{945} - \frac{5\,584\,183\,099\,672\,241}{5\,489\,031\,744\,000\,000}$$

10.02.03.0066.01

$$\zeta(6, 9) = \frac{\pi^6}{945} - \frac{357\,389\,058\,474\,664\,049}{351\,298\,031\,616\,000\,000}$$

10.02.03.0067.01

$$\zeta(6, 10) = \frac{\pi^6}{945} - \frac{260\,537\,105\,518\,334\,091\,721}{256\,096\,265\,048\,064\,000\,000}$$

10.02.03.0068.01

$$\zeta(8,\ 1) = \frac{\pi^8}{9450}$$

10.02.03.0069.01

$$\zeta(8,\ 2) = \frac{\pi^8}{9450} - 1$$

10.02.03.0070.01

$$\zeta(8,\ 3) = \frac{\pi^8}{9450} - \frac{257}{256}$$

10.02.03.0071.01

$$\zeta(8,\ 4) = \frac{\pi^8}{9450} - \frac{1\,686\,433}{1\,679\,616}$$

10.02.03.0072.01

$$\zeta(8,\ 5) = \frac{\pi^8}{9450} - \frac{431\,733\,409}{429\,981\,696}$$

10.02.03.0073.01

$$\zeta(8,\ 6) = \frac{\pi^8}{9450} - \frac{168\,646\,292\,872\,321}{167\,961\,600\,000\,000}$$

10.02.03.0074.01

$$\zeta(8,\ 7) = \frac{\pi^8}{9450} - \frac{168\,646\,392\,872\,321}{167\,961\,600\,000\,000}$$

10.02.03.0075.01

$$\zeta(8,\ 8) = \frac{\pi^8}{9450} - \frac{972\,213\,062\,238\,348\,973\,121}{968\,265\,199\,641\,600\,000\,000}$$

10.02.03.0076.01

$$\zeta(8,\ 9) = \frac{\pi^8}{9450} - \frac{248\,886\,558\,707\,571\,775\,009\,601}{247\,875\,891\,108\,249\,600\,000\,000}$$

10.02.03.0077.01

$$\zeta(8,\ 10) = \frac{\pi^8}{9450} - \frac{1\,632\,944\,749\,460\,578\,249\,437\,992\,161}{1\,626\,313\,721\,561\,225\,625\,600\,000\,000}$$

10.02.03.0078.01

$$\zeta(10,\ 1) = \frac{\pi^{10}}{93\,555}$$

10.02.03.0079.01

$$\zeta(10,\ 2) = \frac{\pi^{10}}{93\,555} - 1$$

10.02.03.0080.01

$$\zeta(10,\ 3) = \frac{\pi^{10}}{93\,555} - \frac{1025}{1024}$$

10.02.03.0081.01

$$\zeta(10,4) == \frac{\pi^{10}}{93555} - \frac{60526249}{60466176}$$

10.02.03.0082.01

$$\zeta(10,5) == \frac{\pi^{10}}{93555} - \frac{61978938025}{61917364224}$$

10.02.03.0083.01

$$\zeta(10,6) == \frac{\pi^{10}}{93555} - \frac{605263128567754849}{604661760000000000}$$

10.02.03.0084.01

$$\zeta(10,7) == \frac{\pi^{10}}{93555} - \frac{605263138567754849}{604661760000000000}$$

10.02.03.0085.01

$$\zeta(10,8) == \frac{\pi^{10}}{93555} - \frac{170971856382109814342232401}{170801981216778240000000000}$$

10.02.03.0086.01

$$\zeta(10,9) == \frac{\pi^{10}}{93555} - \frac{175075181098169912564190119249}{174901228765980917760000000000}$$

10.02.03.0087.01

$$\zeta(10,10) == \frac{\pi^{10}}{93555} - \frac{10338014371627802833957102351534201}{10327742657402407212810240000000000}$$

General characteristics

Domain and analyticity

Domain and analyticity

For $\zeta(s, a)$

$\zeta(s, a)$ is an analytical function of s, a which is defined in $\mathbb{C}^2$. For fixed a, it is an entire function of s.

10.02.04.0001.01

$$(s * a) \longrightarrow \zeta(s, a) :: (\mathbb{C} \otimes \mathbb{C}) \longrightarrow \mathbb{C}$$

Symmetries and periodicities

Mirror symmetry

For $\zeta(s, a)$

10.02.04.0002.01

$$\zeta(\bar{s}, \bar{a}) == \overline{\zeta(s, a)}$$

Periodicity

No periodicity

Poles and essential singularities

With respect to *a*

For $\zeta(s, a)$

For fixed s, the function $\zeta(s, a)$ does not have poles and essential singularities.

10.02.04.0003.01

$$\mathcal{S}ing_a(\zeta(s, a)) == \{\}$$

For $\hat{\zeta}(s, a)$

For fixed s /; $s \notin \mathbb{N}^+$, the function $\hat{\zeta}(s, a)$ does not have poles and essential singularities.

10.02.04.0004.01

$$\mathcal{S}ing_a\left(\hat{\zeta}(s, a)\right) == \{\} \;/;\; s \notin \mathbb{N}^+$$

For positive integer s, the function $\hat{\zeta}(s, a)$ has an infinite set of singular points:

a) $a == -n$ /; $s == 1 \bigwedge n \in \mathbb{N}$ are the simple poles with residues 1;

b) $a == -n$ /; $s > 1 \bigwedge n \in \mathbb{N}$ are the poles of order s with residues 0;

c) $a == \tilde{\infty}$ is the point of convergence of poles, which is an essential singular point.

10.02.04.0005.01

$$\mathcal{S}ing_a\left(\hat{\zeta}(s, a)\right) == \{\{\{-n, s\} \;/;\; n \in \mathbb{N}\}, \{\tilde{\infty}, \infty\}\} \;/;\; s \in \mathbb{N}^+$$

10.02.04.0006.01

$$\operatorname{res}_a\left(\hat{\zeta}(s, a)\right)(-n) == \delta_{s,1} \;/;\; n \in \mathbb{N}$$

With respect to *s*

For $\zeta(s, a)$

For fixed a, the function $\zeta(s, a)$ has only one singular point at $s = \tilde{\infty}$. It is an essential singular point.

10.02.04.0007.01

$$\mathcal{S}ing_s(\zeta(s, a)) == \{\{\tilde{\infty}, \infty\}\}$$

For $\hat{\zeta}(s, a)$

For fixed a, the function $\hat{\zeta}(s, a)$ has only one singular point at $s = \tilde{\infty}$. It is an essential singular point.

10.02.04.0008.01

$$\mathcal{S}ing_s\left(\hat{\zeta}(s, a)\right) == \{\{\tilde{\infty}, \infty\}\}$$

Branch points

With respect to *a*

For $\zeta(s, a)$

For fixed s, the function $\zeta(s, a)$ does not have branch points.

10.02.04.0009.01

$$\mathcal{BP}_a(\zeta(s, a)) == \{\}$$

For $\hat{\zeta}(s, a)$

For noninteger s, the function $\hat{\zeta}(s, a)$ has infinitely many branch points: $a == -n /; n \in \mathbb{N}$ and $a == \tilde{\infty}$. All these are power-type branch points.
For integer s, the function $\hat{\zeta}(s, a)$ does not have branch points.

10.02.04.0010.01

$$\mathcal{BP}_a\left(\hat{\zeta}(s, a)\right) == \{\{-n /; n \in \mathbb{N}\}, \tilde{\infty}\}$$

10.02.04.0011.01

$$\mathcal{BP}_a\left(\hat{\zeta}(s, a)\right) == \{\} /; s \in \mathbb{Z}$$

10.02.04.0012.01

$$\mathcal{R}_a\left(\hat{\zeta}(s, a), -n\right) == \log /; n \in \mathbb{N} \land s \notin \mathbb{Z} \land s \notin \mathbb{Q}$$

10.02.04.0013.01

$$\mathcal{R}_a\left(\hat{\zeta}(s, a), -n\right) == q /; n \in \mathbb{N} \bigwedge s == \frac{p}{q} \bigwedge p \in \mathbb{Z} \bigwedge q - 1 \in \mathbb{N}^+ \bigwedge \gcd(p, q) == 1$$

10.02.04.0014.01

$$\mathcal{R}_a\left(\hat{\zeta}(s, a), \tilde{\infty}\right) == \log /; s \notin \mathbb{Z} \land s \notin \mathbb{Q}$$

10.02.04.0015.01

$$\mathcal{R}_a\left(\hat{\zeta}(s, a), \tilde{\infty}\right) == q /; s == \frac{p}{q} \bigwedge p \in \mathbb{Z} \bigwedge q - 1 \in \mathbb{N}^+ \bigwedge \gcd(p, q) == 1$$

With respect to *s*

For $\zeta(s, a)$

For fixed a, the function $\zeta(s, a)$ does not have branch points.

10.02.04.0016.01

$$\mathcal{BP}_s(\zeta(s, a)) == \{\}$$

For $\hat{\zeta}(s, a)$

For fixed a, the function $\hat{\zeta}(s, a)$ does not have branch points.

10.02.04.0017.01

$$\mathcal{BP}_s\left(\hat{\zeta}(s, a)\right) == \{\}$$

Branch cuts

With respect to *a*

For $\zeta(s, a)$

For fixed s /; $\frac{s}{2} \notin \mathbb{Z}$, the function $\zeta(s, a)$ has infinitely many branch cuts parallel to the imaginary axis at $\operatorname{Re}(a) = -n$ /; $n \in \mathbb{N}$. In these cases the function $\zeta(s, a)$ is continuous from the left on the interval $(-i\infty - n, -n)$ /; $n \in \mathbb{N}$ and from the right on the interval $(-n, -n + i\infty)$ /; $n \in \mathbb{N}$.

For integer $\frac{s}{2}$, the function $\zeta(s, a)$ does not have branch cuts.

10.02.04.0018.01

$$\mathcal{BC}_a(\zeta(s, a)) = \{\{\{(-i\infty - n, -n), 1\} /; n \in \mathbb{N}\}, \{\{(-n, -n + i\infty), -1\} /; n \in \mathbb{N}\}\}$$

10.02.04.0019.01

$$\mathcal{BC}_a(\zeta(s, a)) = \{\}$$

10.02.04.0020.01

$$\lim_{\epsilon \to +0} \zeta(s, -n - i x - \epsilon) = \zeta(s, -n - i x) /; n \in \mathbb{N} \land x > 0$$

10.02.04.0021.01

$$\lim_{\epsilon \to +0} \zeta(s, -n - i x + \epsilon) = \zeta(s, -n - i x) + 2 e^{\frac{i\pi s}{2}} i \sin\left(\frac{\pi s}{2}\right) \left(-x^2\right)^{-\frac{s}{2}} /; n \in \mathbb{N} \land x > 0$$

10.02.04.0022.01

$$\lim_{\epsilon \to +0} \zeta(s, -n + i x + \epsilon) = \zeta(s, -n + i x) /; n \in \mathbb{N} \land x > 0$$

10.02.04.0023.01

$$\lim_{\epsilon \to +0} \zeta(s, -n + i x - \epsilon) = \zeta(s, -n + i x) + 2 e^{\frac{i\pi s}{2}} i \sin\left(\frac{\pi s}{2}\right) \left(-x^2\right)^{-\frac{s}{2}} /; n \in \mathbb{N} \land x > 0$$

For $\hat{\zeta}(s, a)$

For fixed noninteger s, the function $\hat{\zeta}(s, a)$ is a single-valued function on the a-plane cut along the interval $(-\infty, 0)$, where $\hat{\zeta}(s, a)$ is continuous from above. This interval includes an infinite set of branch cut lines of power type along $(-\infty, -n)$ /; $n \in \mathbb{N}$.

For integer s, the function $\hat{\zeta}(s, a)$ does not have branch cuts.

10.02.04.0024.01

$$\mathcal{BC}_a\left(\hat{\zeta}(s, a)\right) = \{\{(-\infty, 0), -i\}\}$$

10.02.04.0025.01

$$\mathcal{BC}_a\left(\hat{\zeta}(s, a)\right) = \{\}$$

10.02.04.0026.01

$$\lim_{\epsilon \to +0} \hat{\zeta}(s, x + i\epsilon) = \hat{\zeta}(s, x) /; x < 0$$

10.02.04.0027.01

$$\lim_{\epsilon\to+0}\hat{\zeta}(s, x - i\,\epsilon) == \hat{\zeta}(s, x) + \left(e^{2 i \pi s} - 1\right)\sum_{k=0}^{\lfloor -x \rfloor}\frac{1}{(k+x)^s} \;/;\; x < 0$$

For $\tilde{\zeta}(s, a)$

For fixed non integer s, the function $\tilde{\zeta}(s, a)$ is a single-valued function on the a-plane cut along the interval $(-\infty, 0)$, where $\tilde{\zeta}(s, a)$ is continuous from above. This interval includes an infinite set of branch cut lines of power type along $(-\infty, -n) \;/;\; n \in \mathbb{N}$.

For fixed a, the function $\tilde{\zeta}(s, a)$ does not have branch cuts.

10.02.04.0028.01

$$\mathcal{BC}_a\left(\tilde{\zeta}(s, a)\right) == \{\{(-\infty, 0), -i\}\}$$

10.02.04.0029.01

$$\mathcal{BC}_a\left(\hat{\zeta}(s, a)\right) == \{\}$$

10.02.04.0030.01

$$\lim_{\epsilon\to+0}\tilde{\zeta}(s, x + i\,\epsilon) == \tilde{\zeta}(s, x) \;/;\; x < 0$$

10.02.04.0031.01

$$\lim_{\epsilon\to+0}\tilde{\zeta}(s, x - i\,\epsilon) == \tilde{\zeta}(s, x) + \left(e^{2 i \pi s} - 1\right)\sum_{k=0}^{\lfloor -x \rfloor}\frac{1}{(k+x)^s} \;/;\; x < 0$$

With respect to s

For $\zeta(s, a)$

For fixed a, the function $\zeta(s, a)$ does not have branch cuts.

10.02.04.0032.01

$$\mathcal{BC}_s(\zeta(s, a)) == \{\}$$

For $\hat{\zeta}(s, a)$

For fixed a, the function $\hat{\zeta}(s, a)$ does not have branch cuts.

10.02.04.0033.01

$$\mathcal{BC}_s\left(\hat{\zeta}(s, a)\right) == \{\}$$

For $\tilde{\zeta}(s, a)$

For fixed a, the function $\tilde{\zeta}(s, a)$ does not have branch cuts.

10.02.04.0034.01

$$\mathcal{BC}_s\left(\tilde{\zeta}(s, a)\right) == \{\}$$

Series representations

Generalized power series

Expansions at $s == s_0$ /; $s_0 \neq 1$

For $\zeta(s, a)$

10.02.06.0021.01

$$\zeta(s, a) \propto \zeta(s_0, a) + \zeta^{(1,0)}(s_0, a)(s - s_0) + \frac{1}{2}\zeta^{(2,0)}(s_0, a)(s - s_0)^2 + \ldots \;/;\; (s \to s_0) \bigwedge s_0 \neq 1$$

10.02.06.0022.01

$$\zeta(s, a) \propto \zeta(s_0, a) + \zeta^{(1,0)}(s_0, a)(s - s_0) + \frac{1}{2}\zeta^{(2,0)}(s_0, a)(s - s_0)^2 + O\left((s - s_0)^3\right) \;/;\; s_0 \neq 1$$

10.02.06.0023.01

$$\zeta(s, a) == \sum_{k=0}^{\infty} \frac{1}{k!}\zeta^{(k,0)}(s_0, a)(s - s_0)^k \;/;\; s_0 \neq 1$$

10.02.06.0024.01

$$\zeta(s, a) \propto \zeta(s_0, a)(1 + O(s - s_0)) \;/;\; s_0 \neq 1$$

For $\hat{\zeta}(s, a)$

10.02.06.0025.01

$$\hat{\zeta}(s, a) \propto \hat{\zeta}(s_0, a) + \hat{\zeta}^{(1,0)}(s_0, a)(s - s_0) + \frac{1}{2}\hat{\zeta}^{(2,0)}(s_0, a)(s - s_0)^2 + \ldots \;/;\; (s \to s_0) \bigwedge s_0 \neq 1$$

10.02.06.0026.01

$$\hat{\zeta}(s, a) \propto \hat{\zeta}(s_0, a) + \hat{\zeta}^{(1,0)}(s_0, a)(s - s_0) + \frac{1}{2}\hat{\zeta}^{(2,0)}(s_0, a)(s - s_0)^2 + O\left((s - s_0)^3\right) \;/;\; s_0 \neq 1$$

10.02.06.0027.01

$$\hat{\zeta}(s, a) == \sum_{k=0}^{\infty} \frac{1}{k!}\hat{\zeta}^{(k,0)}(s_0, a)(s - s_0)^k \;/;\; s_0 \neq 1$$

10.02.06.0028.01

$$\hat{\zeta}(s, a) \propto \hat{\zeta}(s_0, a)(1 + O(s - s_0)) \;/;\; s_0 \neq 1$$

Expansions at $s == 1$

For $\hat{\zeta}(s, a)$

10.02.06.0029.01

$$\hat{\zeta}(s, a) \propto \frac{1}{s-1} - \psi(a) + \left(-\gamma_1 - \sum_{k=1}^{\infty}\left(\frac{\log^j(a+k-1)}{a+k-1} - \frac{\log^j(k)}{k}\right)\right)(s-1) + \frac{1}{2}\left(\gamma_2 - \sum_{k=1}^{\infty}\left(\frac{\log^j(a+k-1)}{a+k-1} - \frac{\log^j(k)}{k}\right)\right)(s-1)^2 + \ldots \;/;$$
$$(z \to 1)$$

10.02.06.0030.01

$$\hat{\zeta}(s,a)\propto\frac{1}{s-1}-\psi(a)+\left(-\gamma_1-\sum_{k=1}^{\infty}\left(\frac{\log^j(a+k-1)}{a+k-1}-\frac{\log^j(k)}{k}\right)\right)(s-1)+\frac{1}{2}\left(\gamma_2-\sum_{k=1}^{\infty}\left(\frac{\log^j(a+k-1)}{a+k-1}-\frac{\log^j(k)}{k}\right)\right)(s-1)^2+O\left((s-1)^3\right)$$

10.02.06.0001.01

$$\hat{\zeta}(s,a)=\frac{1}{s-1}-\psi(a)+\sum_{j=1}^{\infty}\frac{1}{j!}\left((-1)^j\gamma_j-\sum_{k=1}^{\infty}\left(\frac{\log^j(a+k-1)}{a+k-1}-\frac{\log^j(k)}{k}\right)\right)(s-1)^j$$

10.02.06.0002.02

$$\hat{\zeta}(s,a)\propto\frac{1}{s-1}-\psi(a)\,(1+O(s-1))$$

Expansions at $a = a_0$ /; $a_0 \neq -n$

For $\zeta(s, a)$

10.02.06.0031.01

$$\zeta(s,a)\propto\zeta(s,a_0)-\zeta(s,\lfloor-\mathrm{Re}(a_0)\rfloor+a_0+1)+\zeta(s,\max(0,\lfloor-\mathrm{Re}(a_0)\rfloor+1)+a_0)-s\,(\zeta(s+1,a_0)-\zeta(s+1,\lfloor-\mathrm{Re}(a_0)\rfloor+a_0+1)+\zeta(s+1,\max(0,\lfloor-\mathrm{Re}(a_0)\rfloor+1)+a_0))\,(a-a_0)+\frac{1}{2}(s+1)\,s\left(\zeta\left(-2\left(-\frac{s}{2}-1\right),a_0\right)-\zeta\left(-2\left(-\frac{s}{2}-1\right),\lfloor-\mathrm{Re}(a_0)\rfloor+a_0+1\right)+\zeta(s+2,\max(0,\lfloor-\mathrm{Re}(a_0)\rfloor+1)+a_0)\right)(a-a_0)^2+\ldots\;/;\;(a\to a_0)$$

10.02.06.0032.01

$$\zeta(s,a)=\sum_{k=0}^{\infty}\frac{(1-k-s)_k}{k!}\left(\sum_{j=0}^{\lfloor-\mathrm{Re}(a_0)\rfloor}\frac{1}{(j+a_0)^k\left((j+a_0)^2\right)^{s/2}}+\zeta(k+s,\max(\lfloor-\mathrm{Re}(a_0)\rfloor+1,0)+a_0)\right)(a-a_0)^k$$

10.02.06.0033.01

$$\zeta(s,a)\propto(\zeta(s,a_0)-\zeta(s,\lfloor-\mathrm{Re}(a_0)\rfloor+a_0+1)+\zeta(s,\max(0,\lfloor-\mathrm{Re}(a_0)\rfloor+1)+a_0))\,(1+O(a-a_0))$$

For $\hat{\zeta}(s, a)$

10.02.06.0034.01

$$\hat{\zeta}(s,a)\propto\hat{\zeta}(s,a_0)-s\,\hat{\zeta}(s+1,a_0)\,(a-a_0)+\frac{1}{2}s\,(s+1)\,\hat{\zeta}(s+2,a_0)\,(a-a_0)^2+\ldots\;/;\;(a\to a_0)$$

10.02.06.0035.01

$$\hat{\zeta}(s,a)\propto\hat{\zeta}(s,a_0)-s\,\hat{\zeta}(s+1,a_0)\,(a-a_0)+\frac{1}{2}s\,(s+1)\,\hat{\zeta}(s+2,a_0)\,(a-a_0)^2+O\left((a-a_0)^3\right)$$

10.02.06.0036.01

$$\hat{\zeta}(s,a)=\sum_{k=0}^{\infty}\frac{(1-k-s)_k}{k!}\,\hat{\zeta}(k+s,a_0)\,(a-a_0)^k$$

10.02.06.0037.01

$$\hat{\zeta}(s,a)\propto\hat{\zeta}(s,a_0)\,(1+O(a-a_0))$$

Expansions at $a = -n$

For $\zeta(s, a)$

10.02.06.0038.01

$$\zeta(s, a) \propto \frac{1}{\left((a+n)^2\right)^{s/2}} + \sum_{k=0}^{n-1} \frac{1}{(-k+n-(a+n))^s} + \zeta(s) - s\,\zeta(s+1)\,(a+n) + \frac{1}{2}\,(s+1)\,s\,\zeta(s+2)\,(a+n)^2 + \ldots\;/;\;(a \to -n) \land n \in \mathbb{N}$$

10.02.06.0039.01

$$\zeta(s, a) \propto \frac{1}{\left((a+n)^2\right)^{s/2}} + \sum_{k=0}^{n-1} \frac{1}{(-k+n-(a+n))^s} + \zeta(s) - s\,\zeta(s+1)\,(a+n) + \frac{1}{2}\,(s+1)\,s\,\zeta(s+2)\,(a+n)^2 + O\left((a+n)^3\right)\;/;$$
$$(a \to -n) \land n \in \mathbb{N}$$

10.02.06.0040.01

$$\zeta(s, a) = \frac{1}{\left((a+n)^2\right)^{s/2}} + \sum_{k=0}^{n-1} \frac{1}{(n-k-(a+n))^s} + \zeta(s) + \sum_{k=1}^{\infty} \frac{(1-k-s)_k}{k!}\,\zeta(k+s)\,(a+n)^k\;/;\;n \in \mathbb{N}$$

10.02.06.0041.01

$$\zeta(s, a) \propto \frac{1}{\left((a+n)^2\right)^{s/2}} + \left(\sum_{k=0}^{n-1} \frac{1}{(-k+n-(a+n))^s} + \zeta(s)\right)(1 + O(a+n))\;\;/;\;n \in \mathbb{N}$$

For $\hat{\zeta}(s, a)$

10.02.06.0042.01

$$\hat{\zeta}(s, a) \propto \frac{1}{(a+n)^s} + \sum_{k=0}^{n-1} \frac{1}{(k-n+(a+n))^s} + \zeta(s) - s\,\zeta(s+1)\,(a+n) + \frac{s\,(s+1)}{2}\,\zeta(s+2)\,(a+n)^2 + \ldots\;/;\;(a \to -n) \land n \in \mathbb{N}$$

10.02.06.0043.01

$$\hat{\zeta}(s, a) \propto \frac{1}{(a+n)^s} + \sum_{k=0}^{n-1} \frac{1}{(k-n+(a+n))^s} + \zeta(s) - s\,\zeta(s+1)\,(a+n) + \frac{s\,(s+1)}{2}\,\zeta(s+2)\,(a+n)^2 + O\left((a+n)^3\right)\;/;\;(a \to -n) \land n \in \mathbb{N}$$

10.02.06.0003.01

$$\hat{\zeta}(s, a) = \frac{1}{(a+n)^s} + \left(\sum_{k=0}^{n-1} \frac{1}{(k-n+(a+n))^s} + \zeta(s)\right) + \sum_{j=1}^{\infty} \frac{(1-j-s)_j\,\zeta(j+s)}{j!}\,(a+n)^j\;/;\;n \in \mathbb{N}$$

10.02.06.0004.01

$$\hat{\zeta}(s, a) \propto \frac{1}{(a+n)^s} + \left(\sum_{k=0}^{n-1} \frac{1}{(k-n+(a+n))^s} + \zeta(s)\right)(1 + O(a+n))\;/;\;(a \to -n) \land n \in \mathbb{N}$$

Expansions on branch cuts

For $\zeta(s, a)$

10.02.06.0044.01

$$\zeta(s,a)=i^{-s}(-i(n+a_0))^{-s}e^{-i\pi s\left\lfloor-\frac{1}{\pi}\left(\arg\left(\frac{a+n}{n+a_0}\right)+\arg(-i(n+a_0))\right)\right\rfloor}\sum_{k=0}^{\infty}\frac{(-n-a_0)^{-k}(s)_k}{k!}(a-a_0)^k+\sum_{k=0}^{n-1}\frac{1}{(-k-a_0)^s}\sum_{j=0}^{\infty}\frac{(k+a_0)^{-j}(s)_j}{j!}(a-a_0)^j+\zeta(s,a+n+1)\,/;\,(a\to a_0)\bigwedge\operatorname{Re}(a_0)=-n\bigwedge n\in\mathbb{N}$$

10.02.06.0045.01

$$\zeta(s,a)=i^{-s}(-i(n+a_0))^{-s}e^{-i\pi s\left\lfloor-\frac{1}{\pi}\left(\arg\left(\frac{a+n}{n+a_0}\right)+\arg(-i(n+a_0))\right)\right\rfloor}\sum_{k=0}^{\infty}\frac{(-n-a_0)^{-k}(s)_k}{k!}(a-a_0)^k+\sum_{k=0}^{n-1}\frac{1}{(-a-k)^s}+\zeta(s,a+n+1)\,/;\,(a\to a_0)\bigwedge\operatorname{Re}(a_0)=-n\bigwedge n\in\mathbb{N}$$

10.02.06.0046.01

$$\zeta(s,a)\propto i^{-s}(-i(n+a_0))^{-s}e^{-i\pi s\left\lfloor-\frac{1}{\pi}\left(\arg\left(\frac{a+n}{n+a_0}\right)+\arg(-i(n+a_0))\right)\right\rfloor}(1+O(a-a_0))+\left(\sum_{k=0}^{n-1}\frac{1}{(-k-a_0)^s}+\zeta(s,n+a_0+1)\right)(1+O(a-a_0))\,/;\,\operatorname{Re}(a_0)=-n\bigwedge n\in\mathbb{N}$$

For $\hat{\zeta}(s,a)$

10.02.06.0047.01

$$\hat{\zeta}(s,a)=\hat{\zeta}(s,a+n+1)+(n+a_0)^{-s}\sum_{k=0}^{\infty}\frac{(n+a_0)^{-k}(s)_k}{k!}(a-a_0)^k+\sum_{k=0}^{n-1}e^{-2i\pi s\left\lfloor-\frac{1}{2\pi}\left(\arg(-k-a_0)+\arg\left(\frac{a+k}{k+a_0}\right)\right)\right\rfloor}(k+a_0)^{-s}\sum_{j=0}^{\infty}\frac{(k+a_0)^{-j}(s)_j}{j!}(a-a_0)^j\,/;\,(a\to a_0)\bigwedge -n<a_0<1-n\bigwedge n\in\mathbb{N}^+$$

10.02.06.0048.01

$$\hat{\zeta}(s,a)\propto\left((n+a_0)^{-s}+\hat{\zeta}(s,a_0+n+1)\right)(1+O(a-a_0))+\sum_{k=0}^{n-1}e^{-2i\pi s\left\lfloor-\frac{1}{2\pi}\left(\arg(-k-a_0)+\arg\left(\frac{a+k}{k+a_0}\right)\right)\right\rfloor}(k+a_0)^{-s}(1+O(a-a_0))\,/;\,-n<a_0<1-n\bigwedge n\in\mathbb{N}^+$$

Asymptotic series expansions

For $\zeta(\pm n,a)$ by a

10.02.06.0049.01

$$\zeta(n,a)\propto\sum_{k=0}^{\lfloor-\operatorname{Re}(a)\rfloor}\left(\left((a+k)^2\right)^{-\frac{n}{2}}-(a+k)^{-n}\right)-\theta(-n)\frac{B_{1-n}(a)}{1-n}+\theta(n-1)\frac{(-1)^n\psi^{(n-1)}(a)}{(n-1)!}\,/;\,(|a|\to\infty)\land n\in\mathbb{Z}\land n\neq 1$$

10.02.06.0050.01

$$\zeta(n,a)\propto-\frac{(i\pi)^n 2^{n-1}}{(n-1)!}\left\lfloor\frac{|\arg(a)|}{\pi}\right\rfloor(i\cot(\pi a)-1)\sum_{k=0}^{n-1}\frac{(-1)^k k!\,\mathcal{S}_{n-1}^{(k)}(i\cot(\pi a)+1)^k}{2^k}+\frac{1}{(n-1)!}\sum_{k=1}^{\infty}\frac{(2k+n-2)!\,B_{2k}}{(2k)!\,a^{2k+n-1}}+\frac{2a+n-1}{2a^n(n-1)}+\sum_{k=0}^{\lfloor-\operatorname{Re}(a)\rfloor}\left(\left((a+k)^2\right)^{-\frac{n}{2}}-(a+k)^{-n}\right)\,/;\,(|a|\to\infty)\land n\in\mathbb{Z}\land n>1$$

10.02.06.0051.01

$$\zeta(-n, a) \propto -\frac{1}{n+1} \sum_{k=0}^{n+1} \binom{n+1}{k} B_{1-k+n}\, a^k + \sum_{k=0}^{\lfloor -\mathrm{Re}(a) \rfloor} \left(\left((a+k)^2 \right)^{n/2} - (a+k)^n \right) /;\, n \in \mathbb{N}$$

10.02.06.0052.01

$$\hat{\zeta}(n, a) \propto \frac{2\,a+n-1}{2\,a^n\,(n-1)} - \left\lfloor \frac{|\arg(a)|}{\pi} \right\rfloor \frac{(i\,\pi)^n\, 2^{n-1}\,(i \cot(\pi\,a) - 1)}{(n-1)!} \sum_{k=0}^{n-1} \frac{(-1)^k\, k!\, \mathcal{S}_{n-1}^{(k)}\,(i \cot(\pi\,a)+1)^k}{2^k} + \frac{1}{(n-1)!} \sum_{k=1}^{\infty} \frac{(2\,k+n-2)!\, B_{2\,k}}{(2\,k)!\, a^{2\,k+n-1}} /;\, (|a| \to \infty) \land n \in \mathbb{Z} \land n > 1$$

10.02.06.0053.01

$$\hat{\zeta}(-n, a) \propto -\frac{1}{n+1} \sum_{k=0}^{n+1} \binom{n+1}{k} B_{1-k+n}\, a^k /;\, n \in \mathbb{N}$$

For $\zeta^{(1,0)}(s, a)$ by a

10.02.06.0013.01

$$\zeta^{(1,0)}(s, a) \propto -\left(\log(a) + \frac{1}{s-1}\right)\zeta(s, a) + \frac{a^{-s}}{2\,(s-1)} + \frac{1}{s-1} \sum_{k=2}^{\infty} B_k \sum_{j=0}^{k-1} \frac{(s-1)_k}{j!\,(k-j)!}\, a^{-k-s+1} /;\, -\frac{\pi}{2} < \arg(a) \le \frac{\pi}{2} \bigwedge (\mathrm{Re}(a) \to \infty)$$

10.02.06.0014.01

$$\zeta^{(1,0)}(-n, a) \propto \frac{B_{n+1}(a)\log(a)}{n+1} + (-1)^n\, n! \sum_{k=0}^{\infty} \frac{a^{-k-1}\, B_{k+n+2}}{(k+1)_{n+2}} - \frac{1}{n+1} \sum_{k=2}^{n+1} B_k \sum_{j=0}^{k-1} \binom{n+1}{k} \frac{(-1)^j\, a^{-k+n+1}}{k-j} - \frac{\frac{1}{2}(n+1)\,a^n + B_{n+1}(a)}{(n+1)^2} /;\, n \in \mathbb{N}^+ \bigwedge -\frac{\pi}{2} < \arg(a) \le \frac{\pi}{2} \bigwedge (\mathrm{Re}(a) \to \infty)$$

10.02.06.0015.01

$$\zeta^{(1,0)}(p, a) \propto a \log(a) - a - \frac{\log(a)}{2} + \sum_{k=2}^{\infty} \frac{B_k}{k\,(k-1)}\, a^{1-k} /;\, -\frac{\pi}{2} < \arg(a) \le \frac{\pi}{2} \bigwedge (\mathrm{Re}(a) \to \infty)$$

Exponential Fourier series

Exponential Fourier series

For $\zeta(s, a)$

10.02.06.0005.01

$$\zeta(s, a) = 2\,(2\,\pi)^{s-1}\,\Gamma(1-s) \left(\sin\left(\frac{\pi\,s}{2}\right) \sum_{k=1}^{\infty} \frac{\cos(2\,\pi\,a\,k)}{k^{1-s}} + \cos\left(\frac{\pi\,s}{2}\right) \sum_{k=1}^{\infty} \frac{\sin(2\,\pi\,a\,k)}{k^{1-s}} \right) /;\, \mathrm{Re}(s) < 1 \land 0 < a \le 1$$

Residue representations

Residue representations

For $\zeta(s, a)$

10.02.06.0016.01

$$\zeta(s,a) = \sum_{j=0}^{\infty} \operatorname{res}_t\left(\pi\left((a-t)^2\right)^{-\frac{s}{2}}(-1)^{-t}\csc(\pi t)\right)(-j) \;/;\; \operatorname{Re}(s)>1 \bigwedge -a \notin \mathbb{N}$$

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10.02.06.0017.01

$$\zeta(s,a) = \frac{1}{s-1}\sum_{j=0}^{\infty} \operatorname{res}_t\left((a-t)\left((a-t)^2\right)^{-\frac{s}{2}}(\pi\csc(\pi t))^2\right)(-j) \;/;\; \operatorname{Re}(s)>1 \bigwedge -a \notin \mathbb{N}$$

Allan Cortzen

For $\hat{\zeta}(s, a)$

10.02.06.0006.02

$$\hat{\zeta}(s,a) = \sum_{j=0}^{\infty} \operatorname{res}_t\left(\Gamma(s)\,\Gamma(1-s)\left(\frac{\Gamma(a-t)}{\Gamma(a-t+1)}\right)^s(-1)^{-s}\right)(-j) \;/;\; \operatorname{Re}(s)>1 \bigwedge -a \notin \mathbb{N}$$

10.02.06.0018.01

$$\hat{\zeta}(s,a) = \frac{1}{s-1}\sum_{j=0}^{\infty} \operatorname{res}_t\left((a-t)^{1-s}(\pi\csc(\pi t))^2\right)(-j) \;/;\; \operatorname{Re}(s)>1 \bigwedge -a \notin \mathbb{N}$$

Allan Cortzen

Other series representations

Other series representations

For $\zeta(s, a)$

10.02.06.0054.01

$$\zeta(s,a) = \frac{1}{s-1}\sum_{k=0}^{\infty}\frac{1}{k+1}\sum_{j=0}^{k}(-1)^j\binom{k}{j}(a+j)^{1-s}$$

10.02.06.0055.01

$$\zeta(s,a) = \sum_{k=0}^{\infty}\frac{(-1)^k (s)_k}{k!}\zeta(k+s)(a-1)^k \;/;\; |a-1|<1$$

10.02.06.0007.01

$$\zeta\left(s, a-\frac{1}{2}\right) - \zeta(s,a) = \sum_{k=1}^{\infty}\frac{(s)_k\,\zeta(k+s)}{k!\,2^k}$$

10.02.06.0008.01

$$\zeta(s,a)^2 = \frac{1}{\Gamma(s)}\sum_{k=0}^{\infty}\int_0^{\infty}\frac{t^{s-1}e^{-a(a+k)t}}{1-e^{-(a+k)t}}\,dt \;/;\; \operatorname{Re}(s)>1 \land a>0$$

10.02.06.0019.01

$$\zeta(s, a) = \sum_{k=0}^{n} \left((a+k)^2\right)^{-\frac{s}{2}} + \frac{(a+n)^{1-s}}{s-1} + \sum_{k=0}^{r-1} \frac{(k+s-1)_k \, B_{k+1}}{(k+1)! \, (a+n)^{k+s}} - \frac{(s)_{r+1}}{(r+1)!} \int_0^\infty \frac{B_{r+1}(t - \lfloor t \rfloor) - B_{r+1}}{(a+n+t)^{r+s+1}} \, dt \,/;$$
$$n \in \mathbb{N} \land r \in \mathbb{N} \land \operatorname{Re}(a) > -n \land -a \notin \mathbb{N} \land \operatorname{Re}(s) > -r \land s \neq 1$$

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For $\hat{\zeta}(s, a)$

10.02.06.0009.01

$$\hat{\zeta}(s, a) = a^{-s} + (a+1)^{-s} + (a+2)^{-s} + \ldots \,/; \operatorname{Re}(s) > 1$$

10.02.06.0010.01

$$\hat{\zeta}(s, a) = \sum_{k=0}^{\infty} \frac{1}{(a+k)^s} \,/; \operatorname{Re}(s) > 1$$

10.02.06.0011.01

$$\hat{\zeta}(s, a) = -\sum_{k=m}^{\infty} \left(\frac{(a+k+1)^{1-s} - (a+k)^{1-s}}{1-s} - (a+k+1)^{-s} \right) + \sum_{k=0}^{m} \frac{1}{(a+k)^s} - \frac{1}{(1-s)\,(a+m)^{s-1}} \,/; \operatorname{Re}(s) > 1 \land m \in \mathbb{N}$$

10.02.06.0012.01

$$\hat{\zeta}(s, a) = \sum_{k=0}^{\infty} \frac{(-1)^k \, (s)_k \, \zeta(k+s) \, a^k}{k!} + \frac{1}{a^s} \,/; |a| < 1$$

10.02.06.0020.01

$$\hat{\zeta}(s, a) = \sum_{k=0}^{n} (a+k)^{-s} + \frac{(a+n)^{1-s}}{s-1} + \sum_{k=0}^{r-1} \frac{(k+s-1)_k \, B_{k+1}}{(k+1)! \, (a+n)^{k+s}} - \frac{(s)_{r+1}}{(r+1)!} \int_0^\infty \frac{B_{r+1}(t - \lfloor t \rfloor) - B_{r+1}}{(a+n+t)^{r+s+1}} \, dt \,/;$$
$$n \in \mathbb{N} \land r \in \mathbb{N} \land \operatorname{Re}(a) > -n \land -a \notin \mathbb{N}^+ \land \operatorname{Re}(s) > -r \land s \neq 1$$

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Integral representations

On the real axis

Of the direct function

For $\zeta(s, a)$

10.02.07.0001.01

$$\zeta(s, a) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1} \, e^{-a t}}{1 - e^{-t}} \, dt \,/; \operatorname{Re}(s) > 1 \land \operatorname{Re}(a) > 0$$

10.02.07.0002.01

$$\zeta(s, a) = \frac{a^{1-s}}{s-1} + 2 \int_0^\infty \frac{\sin\left(s \tan^{-1}\left(\frac{t}{a}\right)\right)}{\left(a^2 + t^2\right)^{s/2} \left(e^{2 \pi t} - 1\right)} \, dt + \frac{1}{2 a^s} \,/; \operatorname{Re}(a) > 0$$

Contour integral representations

Contour integral representations

For $\hat{\zeta}(s, a)$

10.02.07.0003.01

$$\hat{\zeta}(s, a) == \frac{1}{2\pi i}\int_{\mathcal{L}} \frac{\Gamma(t)\,\Gamma(1-t)\,\Gamma(a-t)^s\,(-1)^{-t}}{\Gamma(1+a-t)^s}\,dt \;/;\; s \in \mathbb{N}^+$$

10.02.07.0004.02

$$\hat{\zeta}(s, a) == \frac{1}{2\pi i}\int_{\gamma-i\infty}^{i\infty+\gamma} \Gamma(t)\,\Gamma(1-t)\left(\frac{\Gamma(a-t)}{\Gamma(a-t+1)}\right)^s (-1)^{-t}\,dt \;/;\; 0<\gamma<1 \bigwedge \mathrm{Re}(a)>\gamma \bigwedge \mathrm{Re}(s)>1$$

10.02.07.0005.01

$$\hat{\zeta}(s, a-n) == \frac{\pi}{2 i (s-1)}\int_{\gamma-i\infty}^{i\infty+\gamma} (a-t)^{1-s}\csc^2(\pi t)\,dt \;/;\; n \in \mathbb{Z} \bigwedge n<\gamma<n+1 \bigwedge \mathrm{Re}(a)>\gamma \bigwedge s \neq 1$$

Allan Cortzen

Transformations

Transformations and argument simplifications

Argument involving basic arithmetic operations

For $\zeta(s, a)$

10.02.16.0001.01

$$\zeta(s, a+1) == \zeta(s, a) - \frac{1}{\left(a^2\right)^{s/2}}$$

10.02.16.0002.01

$$\zeta(s, a-1) == \zeta(s, a) + \frac{1}{\left((a-1)^2\right)^{s/2}}$$

10.02.16.0003.01

$$\zeta(s, a+n) == \zeta(s, a) - \sum_{k=0}^{n-1} \frac{1}{\left((a+k)^2\right)^{s/2}} \;/;\; n \in \mathbb{N}$$

10.02.16.0004.01

$$\zeta(s, a-n) == \zeta(s, a) + \sum_{k=0}^{n-1} \frac{1}{\left((a+k-n)^2\right)^{s/2}} \;/;\; n \in \mathbb{N}$$

10.02.16.0005.01

$$\zeta\left(1-s, \frac{n}{m}\right) == \frac{2\,\Gamma(s)}{(2\pi m)^s}\sum_{j=1}^{m} \cos\left(\frac{\pi s}{2} - \frac{2\pi j n}{m}\right) \zeta\left(s, \frac{j}{m}\right) \;/;\; n \in \mathbb{N}^+ \bigwedge m \in \mathbb{N}^+ \bigwedge n \leq m$$

For $\hat{\zeta}(s, a)$

10.02.16.0006.01

$$\hat{\zeta}(s, a+1) = \hat{\zeta}(s, a) - \frac{1}{a^s}$$

10.02.16.0007.01

$$\hat{\zeta}(s, a-1) = \hat{\zeta}(s, a) + \frac{1}{a^s}$$

10.02.16.0008.01

$$\hat{\zeta}(s, a+n) = \hat{\zeta}(s, a) - \sum_{k=0}^{n-1} \frac{1}{(a+k)^s} \;/;\; n \in \mathbb{N}$$

10.02.16.0009.01

$$\hat{\zeta}(s, a-n) = \hat{\zeta}(s, a) + \sum_{k=1}^{n} \frac{1}{(a-k)^s} \;/;\; n \in \mathbb{N}$$

Multiple arguments

Argument involving numeric multiples of variable

For $\hat{\zeta}(s, a)$

10.02.16.0010.01

$$\hat{\zeta}(s, 2a) = 2^{-s}\left(\hat{\zeta}(s, a) + \hat{\zeta}\left(s, a+\frac{1}{2}\right)\right)$$

10.02.16.0011.01

$$\hat{\zeta}(s, 3a) = 3^{-s}\left(\hat{\zeta}(s, a) + \hat{\zeta}\left(s, a+\frac{1}{3}\right) + \hat{\zeta}\left(s, a+\frac{2}{3}\right)\right)$$

10.02.16.0012.01

$$\hat{\zeta}^{(1,0)}(s, 2a) = 2^{-s}\left(\hat{\zeta}^{(1,0)}\left(s, a+\frac{1}{2}\right) + \hat{\zeta}^{(1,0)}(s, a)\right) - \log(2)\, 2^{-s}\left(\hat{\zeta}\left(s, a+\frac{1}{2}\right) + \hat{\zeta}(s, a)\right)$$

10.02.16.0013.01

$$\hat{\zeta}^{(1,0)}(s, 3a) = 3^{-s}\left(\hat{\zeta}^{(1,0)}\left(s, a+\frac{1}{3}\right) + \hat{\zeta}^{(1,0)}\left(s, a+\frac{2}{3}\right) + \hat{\zeta}^{(1,0)}(s, a)\right) - \log(3)\, 3^{-s}\left(\hat{\zeta}\left(s, a+\frac{1}{3}\right) + \hat{\zeta}\left(s, a+\frac{2}{3}\right) + \hat{\zeta}(s, a)\right)$$

Argument involving symbolic multiples of variable

For $\hat{\zeta}(s, a)$

10.02.16.0014.01

$$\hat{\zeta}(s, m a) = m^{-s} \sum_{k=0}^{m-1} \hat{\zeta}\left(s, a+\frac{k}{m}\right) \;/;\; m \in \mathbb{N}^+$$

10.02.16.0015.01

$$\hat{\zeta}^{(1,0)}(s, m a) = m^{-s} \sum_{k=0}^{m-1} \hat{\zeta}^{(1,0)}\left(s, a+\frac{k}{m}\right) - \log(m)\, m^{-s} \sum_{k=0}^{m-1} \hat{\zeta}\left(s, a+\frac{k}{m}\right) \;/;\; m \in \mathbb{N}^+$$

Identities

Recurrence identities

Consecutive neighbors

For $\zeta(s, a)$

10.02.17.0001.01

$$\zeta(s, a) = \zeta(s, a+1) + \frac{1}{(a^2)^{s/2}}$$

10.02.17.0002.01

$$\zeta(s, a) = \zeta(s, a-1) - \frac{1}{((a-1)^2)^{s/2}}$$

For $\hat{\zeta}(s, a)$

10.02.17.0003.01

$$\hat{\zeta}(s, a) = \hat{\zeta}(s, a+1) + \frac{1}{a^s}$$

10.02.17.0004.01

$$\hat{\zeta}(s, a) = \left(\hat{\zeta}(s, a-1) - \frac{1}{(a-1)^s}\right)$$

Distant neighbors

For $\zeta(s, a)$

10.02.17.0005.01

$$\zeta(s, a) = \zeta(s, a+n) + \sum_{k=0}^{n-1} \frac{1}{((a+k)^2)^{s/2}} \;/;\; n \in \mathbb{N}$$

10.02.17.0006.01

$$\zeta(s, a) = \zeta(s, a-n) - \sum_{k=0}^{n-1} \frac{1}{((a+k-n)^2)^{s/2}} \;/;\; n \in \mathbb{N}$$

10.02.17.0012.01

$$\zeta(s, a) = \zeta(s, b - \min(0, \operatorname{Re}(b-a))) + \sum_{k=0}^{\operatorname{Re}(b-a)-1} \frac{1}{((a+k)^2)^{s/2}} \;/;\; \operatorname{Re}(b-a) \in \mathbb{Z} \bigwedge \operatorname{Im}(b-a) = 0$$

For $\hat{\zeta}(s, a)$

10.02.17.0007.01

$$\hat{\zeta}(s, a) = \hat{\zeta}(s, a+n) + \sum_{k=0}^{n-1} \frac{1}{(a+k)^s} \text{ /; } n \in \mathbb{N}$$

10.02.17.0008.01

$$\hat{\zeta}(s, a) = z^{-n} \left(\hat{\zeta}(s, a-n) - \sum_{k=0}^{n-1} \frac{z^k}{(a+k-n)^s} \right) \text{ /; } n \in \mathbb{N}$$

10.02.17.0013.01

$$\hat{\zeta}(s, a) = \sum_{k=0}^{\mathrm{Re}(b-a)-1} \frac{1}{(a+k)^s} + \hat{\zeta}(s, b - \min(0, \mathrm{Re}(b-a))) \text{ /; } \mathrm{Re}(b-a) \in \mathbb{Z} \bigwedge \mathrm{Im}(b-a) = 0$$

Functional identities

Major general cases

For $\zeta(s, a)$

10.02.17.0014.01

$$\zeta(s, a) = -\zeta(s, 1-a) + \zeta(s, a + \lceil -\mathrm{Re}(a) \rceil) + \zeta(s, -a - \lfloor -\mathrm{Re}(a) \rfloor) - \delta_{\mathrm{frac}(-\mathrm{Re}(a))} \left(-\mathrm{Im}(a)^2\right)^{-\frac{s}{2}} \text{ /; } a \notin \mathbb{R}$$

Pavlyk 0. (2006)

10.02.17.0009.01

$$\zeta\left(s, \frac{n}{m}\right) = \frac{2\,\Gamma(1-s)}{(2\,\pi\,m)^{1-s}} \sum_{j=1}^{m} \sin\left(\frac{2\,j\,\pi\,n}{m} + \frac{\pi\,s}{2}\right) \zeta\left(1-s, \frac{j}{m}\right) \text{ /; } n \in \mathbb{N}^+ \bigwedge m \in \mathbb{N}^+ \bigwedge n \leq m$$

For $\hat{\zeta}(s, a)$

10.02.17.0010.01

$$\hat{\zeta}(s, a) = 2^{-s} \left(\hat{\zeta}\left(s, \frac{a}{2}\right) + \hat{\zeta}\left(s, \frac{a+1}{2}\right) \right)$$

10.02.17.0011.01

$$\hat{\zeta}(s, a) = q^{-s} \sum_{k=1}^{q} \hat{\zeta}\left(s, \frac{a+k-1}{q}\right) \text{ /; } q \in \mathbb{N}^+$$

Differentiation

Low-order differentiation

With respect to s

For $\zeta(s, a)$

General case

10.02.20.0011.01

$$\zeta^{(1,0)}(s,\,a) = -\frac{1}{2}\sum_{k=0}^{\infty}\frac{\log\left((a+k)^2\right)}{\left((a+k)^2\right)^{s/2}} \;/;\; -a \notin \mathbb{N}$$

10.02.20.0012.01

$$\zeta^{(1,0)}(s,\,a+1) = \zeta^{(1,0)}(s,\,a) + \frac{\log\left(a^2\right)}{2\left(a^2\right)^{s/2}}$$

10.02.20.0013.01

$$\zeta^{(1,0)}(s,\,a+m) = \zeta^{(1,0)}(s,\,a) + \frac{1}{2}\sum_{k=0}^{m-1}\frac{\log\left((a+k)^2\right)}{\left((a+k)^2\right)^{s/2}} \;/;\; m \in \mathbb{N}^+$$

10.02.20.0014.01

$$\zeta^{(1,0)}(s,\,m) = \zeta'(s) + \frac{1}{2}\sum_{k=0}^{m-2}\frac{\log(k+1)}{(k+1)^s} \;/;\; m \in \mathbb{Z} \land m \geq -1$$

10.02.20.0015.01

$$\zeta^{(2,0)}(s,\,a) = \frac{1}{4}\sum_{k=0}^{\infty}\frac{\log^2\left((a+k)^2\right)}{\left((a+k)^2\right)^{s/2}} \;/;\; -a \notin \mathbb{N}$$

10.02.20.0016.01

$$\zeta^{(2,0)}(s,\,a+1) = \zeta^{(2,0)}(s,\,a) - \frac{\log^2\left(a^2\right)}{4\left(a^2\right)^{s/2}}$$

10.02.20.0017.01

$$\zeta^{(2,0)}(s,\,a+m) = \zeta^{(2,0)}(s,\,a) - \frac{1}{4}\sum_{k=0}^{m-1}\frac{\log^2\left((a+k)^2\right)}{\left((a+k)^2\right)^{s/2}} \;/;\; m \in \mathbb{N}^+$$

10.02.20.0018.01

$$\zeta^{(2,0)}(s,\,m) = \zeta''(s) - \sum_{k=0}^{m-2}\frac{\log^2(k+1)}{(k+1)^s} \;/;\; m \in \mathbb{Z} \land m \geq -1$$

Derivatives at zero

10.02.20.0001.01

$$\zeta^{(1,0)}(0,\,a) = \log\Gamma(a) - \frac{1}{2}\log(2\pi) \;/;\; \mathrm{Re}(a) > 0$$

10.02.20.0019.01

$$\zeta^{(1,0)}(0,\,a) = \log\Gamma(a) - \frac{1}{2}\log(2\pi) - i\,\pi\,\mathrm{Re}(\lfloor a\rfloor)\,(2\,\theta(\mathrm{Im}(a))-1) - \frac{1}{2}\pi\,i\left(1+(-1)^{\lfloor -\mathrm{Re}(a)\rfloor+\lfloor \mathrm{Re}(a)\rfloor}\right)\theta(-\mathrm{Im}(a))\,\theta(-\mathrm{Re}(a))$$

10.02.20.0020.01

$$\zeta^{(1,0)}(0,\,a) = \log\Gamma(a-1) + \log(a-1) - \frac{1}{2}\log(2\pi) + \sum_{k=0}^{\lfloor 1-\mathrm{Re}(a)\rfloor}\left(\log(a+k) - \frac{1}{2}\log\left((a+k)^2\right)\right)$$

10.02.20.0021.01

$$\zeta^{(1,0)}(0, 0) = -\frac{1}{2}\log(2\pi)$$

Derivatives at negarive integer points

10.02.20.0022.01

$$\zeta^{(1,0)}\left(s, \frac{1}{2}\right) = 2^s \log(2)\,\zeta(s) + (2^s - 1)\,\zeta'(s)$$

10.02.20.0023.01

$$\zeta^{(1,0)}(-1, a) = -\frac{1}{12}\left(-6a^2 + 18a - 12(a-1)\log(a-1) + 12\log(A) + 6(a-1)\log(2\pi) - 13\right) + \psi^{(-2)}(a-1) + \left(\sum_{i=0}^{\lfloor -\mathrm{Re}(a)\rfloor}\left((a+i)\log(a+i) - \frac{1}{2}\sqrt{(a+i)^2}\,\log\left((a+i)^2\right)\right)\right)\theta(\lfloor -\mathrm{Re}(a)\rfloor)$$

10.02.20.0024.01

$$\zeta^{(1,0)}(-2, a) = \frac{1}{12\pi^2}\Bigg(2\pi^2\left(a\left(2\gamma a^2 + 3(-6+\gamma)a + \gamma + 18\right) - 9\right) - 3\zeta(3) + \pi^2\Bigg((-6+4\gamma)\lfloor -\mathrm{Re}(a)\rfloor^3 + 3(2a+1)(-3+2\gamma)\lfloor -\mathrm{Re}(a)\rfloor^2 + (6a(a+1)+1)(-3+2\gamma)\lfloor -\mathrm{Re}(a)\rfloor + 6\Bigg((a-1)(\log(2) + 2(a-1)\log(a-1) - 4\log(A) + \log(\pi) - a\log(2\pi)) + 4\psi^{(-3)}(a-1) + 2\sum_{k=0}^{\lfloor -\mathrm{Re}(a)\rfloor}\frac{1}{2}(a+k)^2\left(2\log(a+k) - \log\left((a+k)^2\right)\right) + (-3+2\gamma)\,\zeta(-2, a + \max(0, \lfloor 1 - \mathrm{Re}(a)\rfloor))\Bigg)\Bigg)\Bigg)$$

10.02.20.0025.01

$$\zeta^{(1,0)}(-2, a) = 2\psi^{(-3)}(a-1) + \log(a-1)(a-1)^2 + \frac{1}{12\pi^2}\left(2(a-1)\pi^2\left(2\gamma a^2 - (\log(8) + 3\log(\pi) + \gamma + 9)a + 3(-4\log(A) + \log(2\pi) + 3)\right) - 3\zeta(3)\right) + \left(-\frac{3}{2} + \gamma\right)\zeta(-2, a) + \sum_{k=0}^{\lfloor 1-\mathrm{Re}(a)\rfloor}\frac{1}{2}(a+k)^2\left(2\log(a+k) - \log\left((a+k)^2\right)\right)$$

10.02.20.0026.01

$$\zeta^{(1,0)}(-3, a) = 6\psi^{(-4)}(a-1) + \log(a-1)(a-1)^3 - \frac{1}{2}\log(2\pi)(a-1)^3 - 3\log(A)(a-1)^2 - \frac{3\zeta(3)(a-1)}{4\pi^2} - \frac{11}{720}(120a((a-3)a+3) - 121) + \left(-\frac{11}{6} + \gamma\right)\zeta(-3, a) + \zeta'(-3) + \frac{1}{120}\left(30(a-1)^2a^2 - 1\right)\gamma + \sum_{k=0}^{\lfloor 1-\mathrm{Re}(a)\rfloor}\left((a+k)^3\log(a+k) - \frac{1}{2}\left((a+k)^2\right)^{3/2}\log\left((a+k)^2\right)\right)$$

10.02.20.0027.01

$$\zeta^{(1,0)}(-4, a) = 24\,\psi^{(-5)}(a-1) + \log(a-1)\,(a-1)^4 - \frac{1}{2}\log(2\pi)\,(a-1)^4 - 4\log(A)\,(a-1)^3 - \frac{1}{180}\,(375\,a\,((a-3)\,a+3)-386)\,(a-1) + 4\,\zeta'(-3)\,(a-1) + \frac{9\left(\zeta(5) - 2\,(a-1)^2\,\pi^2\,\zeta(3)\right) + (-25+12\,\gamma)\,\pi^4\,\zeta(-4, a)}{12\,\pi^4} + \frac{1}{30}\,a\left(a^2\,(3\,a\,(2\,a-5)+10)-1\right)\gamma + \sum_{k=0}^{\lfloor 1-\operatorname{Re}(a)\rfloor} \frac{1}{2}\,(a+k)^4\left(2\log(a+k) - \log\left((a+k)^2\right)\right)$$

10.02.20.0028.01

$$\zeta^{(1,0)}(-5, a) = 120\,\psi^{(-6)}(a-1) + \log(a-1)\,(a-1)^5 - \frac{1}{2}\log(2\pi)\,(a-1)^5 - 5\log(A)\,(a-1)^4 + 10\,\text{Zeta}'(-3)\,(a-1)^2 - \frac{5\left(2\,(a-1)^2\,\pi^2\,\zeta(3) - 3\,\zeta(5)\right)(a-1)}{4\,\pi^4} + \frac{42\,a\,(a\,(8275 - 822\,a\,((a-5)\,a+10)) - 4220) + 36\,697}{15\,120} + \left(-\frac{137}{60}+\gamma\right)\zeta(-5, a) + \zeta'(-5) + \frac{1}{252}\left(21\,(a-1)^2\,(2\,(a-1)\,a-1)\,a^2+1\right)\gamma + \sum_{k=0}^{\lfloor 1-\operatorname{Re}(a)\rfloor}\left((a+k)^5\log(a+k) - \frac{1}{2}\left((a+k)^2\right)^{5/2}\log\left((a+k)^2\right)\right)$$

10.02.20.0029.01

$$\zeta^{(1,0)}(-n, a) = n!\,\psi^{(-n-1)}(a-1) - \left(\gamma - \frac{\gamma\,(a-1)}{n+1} - \log(a-1) + \psi(n+1) + \sum_{k=0}^{n}\frac{1}{(a-1)^k}\binom{n}{k}\left(\sum_{j=0}^{k}\binom{k}{j}\psi(-j+k+1)\left(\theta(\lfloor -\operatorname{Re}(a)\rfloor)\sum_{i=0}^{\lfloor -\operatorname{Re}(a)\rfloor}(a+i)^{k-j} + \zeta(j-k, a+\max(\lfloor 1-\operatorname{Re}(a)\rfloor, 0))\right)(1-a)^j - \psi(k+1)\,\zeta(-k) - \zeta'(-k)\right)\right)(a-1)^n + \theta(\lfloor -\operatorname{Re}(a)\rfloor)\sum_{i=0}^{\lfloor -\operatorname{Re}(a)\rfloor}\left((a+i)^n\log(a+i) - \frac{1}{2}\left((a+i)^2\right)^{n/2}\log\left((a+i)^2\right)\right) /;\; n \in \mathbb{N}$$

10.02.20.0030.01

$$\zeta^{(1,0)}(-n, 1) = \zeta'(-n) /;\; n \in \mathbb{N}$$

10.02.20.0031.01

$$\zeta^{(1,0)}(-n, m) = n!\,\psi^{(-n-1)}(m-1) - (m-1)^n\left(-\frac{\gamma\,(m-1)}{n+1} - \log(m-1) + \psi(n+1) + \sum_{k=0}^{n}\frac{1}{(m-1)^k}\binom{n}{k}\left(\sum_{j=0}^{k}\binom{k}{j}\psi(-j+k+1)\,\zeta(j-k, m)\,(1-m)^j - \psi(k+1)\,\zeta(-k) - \zeta'(-k)\right) + \gamma\right) /;\; (n \in \mathbb{N} \land m \in \mathbb{Z} \land m \geq 2) \lor \left(n \in \mathbb{N}^+ \land m = 0\right)$$

10.02.20.0032.01

$$\zeta^{(1,0)}(-n,-m)=n!\,\psi^{(-n-1)}(-m-1)-(-1)^n (m+1)^n \left(\gamma+\frac{\gamma (m+1)}{n+1}-\pi\, i-\log(m+1)+\psi(n+1)+\right.$$
$$\left.\sum_{k=0}^{n}\frac{(-1)^k}{(m+1)^k}\binom{n}{k}\left(\sum_{j=0}^{k}\binom{k}{j}\psi(-j+k+1)\left(\sum_{i=0}^{m-1}(i-m)^{k-j}+\zeta(j-k)\right)(m+1)^j-\psi(k+1)\,\zeta(-k)-\zeta'(-k)\right)\right)+$$
$$\sum_{i=0}^{m-1}(i-m)^n\,(i\,\pi+(1-(-1)^n)\log(m-i))\;/;\;n\in\mathbb{N}^+\land m\in\mathbb{N}$$

10.02.20.0033.01

$$\zeta^{(1,0)}\left(1-2n,\frac{p}{q}\right)=\zeta'(1-2n)\,q^{-2n}+\frac{B_{2n}\left(\frac{p}{q}\right)(\psi(2n)-\log(2\pi q))}{2n}+$$
$$\frac{(-1)^{n+1}\pi}{(2\pi q)^{2n}}\sum_{j=1}^{q-1}\sin\left(\frac{2\pi p j}{q}\right)\psi^{(2n-1)}\left(\frac{j}{q}\right)+\frac{(-1)^{n+1}\,2\,(2n-1)!}{(2\pi q)^{2n}}\sum_{j=1}^{q-1}\cos\left(\frac{2\pi p j}{q}\right)\zeta^{(1,0)}\left(2n,\frac{j}{q}\right)-$$
$$\frac{(\psi(2n)-\log(2\pi))\,B_{2n}}{q^{2n}\,2n}\;/;\;n\in\mathbb{N}^+\land p\in\mathbb{Z}\land 0<p<q\land q\in\mathbb{Z}\land q>1$$

10.02.20.0034.01

$$\zeta^{(1,0)}\left(1-2n,m+\frac{p}{q}\right)=\zeta'(1-2n)\,q^{-2n}+\frac{\psi(2n)-\log(2\pi q)}{2n}B_{2n}\left(\frac{p}{q}\right)+$$
$$\frac{1}{2}\operatorname{sgn}\left(\operatorname{Re}\left(m+\frac{p}{q}\right)\right)\sum_{k=0}^{|m|-1}\frac{\log\left(\left(k+\frac{1}{2}\left(\left(1-2\left(m+\frac{p}{q}\right)\right)\operatorname{sgn}\left(\operatorname{Re}\left(m+\frac{p}{q}\right)\right)+1\right)\right)^2\right)}{\left(\left(k+\frac{1}{2}\left(\left(1-2\left(m+\frac{p}{q}\right)\right)\operatorname{sgn}\left(\operatorname{Re}\left(m+\frac{p}{q}\right)\right)+1\right)\right)^2\right)^{\frac{1}{2}-n}}+$$
$$\frac{(-1)^{n+1}\pi}{(2\pi q)^{2n}}\sum_{j=1}^{q-1}\sin\left(\frac{2\pi p j}{q}\right)\psi^{(2n-1)}\left(\frac{j}{q}\right)+\frac{(-1)^{n+1}\,2\,(2n-1)!}{(2\pi q)^{2n}}\sum_{j=1}^{q-1}\cos\left(\frac{2\pi p j}{q}\right)\zeta^{(1,0)}\left(2n,\frac{j}{q}\right)-\frac{\psi(2n)-\log(2\pi)}{q^{2n}\,2n}B_{2n}\;/;$$
$$m\in\mathbb{Z}\land n\in\mathbb{N}^+\land p\in\mathbb{Z}\land 0<p<q\land q\in\mathbb{Z}\land q>1$$

10.02.20.0035.01

$$\zeta^{(1,0)}\left(1-2n,\frac{1}{6}\right)=\frac{\left(3^{2n-1}-1\right)\log(2)}{6^{2n-1}\,4n}B_{2n}+\frac{\left(2^{2n-1}-1\right)\log(3)}{6^{2n-1}\,4n}B_{2n}-$$
$$\frac{(9^n-1)\left(2^{2n-1}+1\right)\pi}{\sqrt{3}\;6^{2n-1}\,8n}B_{2n}+\frac{\left(2^{2n-1}-1\right)\left(3^{2n-1}-1\right)}{2\,6^{2n-1}}\zeta'(1-2n)-\frac{(-1)^n\left(2^{2n-1}+1\right)}{2\sqrt{3}\;(12\pi)^{2n-1}}\psi^{(2n-1)}\left(\frac{1}{3}\right)\;/;\;n\in\mathbb{N}^+$$

10.02.20.0036.01

$$\zeta^{(1,0)}\left(1-2n,\frac{1}{4}\right)=-\frac{(-1)^n}{4\,(8\pi)^{2n-1}}\psi^{(2n-1)}\left(\frac{1}{4}\right)+\frac{\pi\left(4^n-4^{2n}\right)+\left(2^{2n+1}-8\right)\log(2)}{4^{2n+1}\,n}B_{2n}-\frac{2^{2n-1}-1}{2^{4n-1}}\zeta'(1-2n)\;/;\;n\in\mathbb{N}^+$$

10.02.20.0037.01

$$\zeta^{(1,0)}\left(1-2n,\frac{1}{3}\right)=$$
$$-\frac{9^{-n}\left(\sqrt{3}\;\pi\,(9^n-1)+6\log(3)\right)}{8n}B_{2n}+(-1)^{n-1}\,3^{\frac{1}{2}-2n}\,4^{-n}\,\pi^{1-2n}\,\psi^{(2n-1)}\left(\frac{1}{3}\right)+\frac{1}{2}9^{-n}\,(3-9^n)\,\zeta'(1-2n)\;/;\;n\in\mathbb{N}^+$$

10.02.20.0038.01

$$\zeta^{(1,0)}\left(1-2n, \frac{1}{2}\right) = -\frac{B_{2n}\log(2)}{2^{2n}\,n} - \frac{2^{2n-1}-1}{2^{2n-1}}\,\zeta'(1-2n) \;/;\; n\in\mathbb{N}^+$$

10.02.20.0039.01

$$\zeta^{(1,0)}\left(1-2n, \frac{2}{3}\right) = \frac{9^{-n}\left(\sqrt{3}\,(9^n-1)\,\pi - 6\log(3)\right)}{8n} B_{2n} + (-1)^n\,3^{\frac{1}{2}-2n}\,4^{-n}\,\pi^{1-2n}\,\psi^{(2n-1)}\left(\frac{1}{3}\right) + \frac{1}{2}\,9^{-n}\,(3-9^n)\,\zeta'(1-2n) \;/;\; n\in\mathbb{N}^+$$

10.02.20.0040.01

$$\zeta^{(1,0)}\left(1-2n, \frac{3}{4}\right) = \frac{(-1)^n}{4\,(8\pi)^{2n-1}}\,\psi^{(2n-1)}\left(\frac{1}{4}\right) + \frac{\left(2^{2n+1}-2^3\right)\log(2) - \left(4^n - 4^{2n}\right)\pi}{4^{2n+1}\,n} B_{2n} - \frac{2^{2n-1}-1}{2^{4n-1}}\,\zeta'(1-2n) \;/;\; n\in\mathbb{N}^+$$

10.02.20.0041.01

$$\zeta^{(1,0)}\left(1-2n, \frac{5}{6}\right) = \frac{\left(3^{2n-1}-1\right)\log(2)}{6^{2n-1}\,4n} B_{2n} + \frac{\left(2^{2n-1}-1\right)\log(3)}{6^{2n-1}\,4n} B_{2n} + \frac{(9^n-1)\left(2^{2n-1}+1\right)\pi}{\sqrt{3}\;6^{2n-1}\,8n} B_{2n} + \frac{\left(2^{2n-1}-1\right)\left(3^{2n-1}-1\right)}{2\,6^{2n-1}}\,\zeta'(1-2n) + \frac{(-1)^n\left(2^{2n-1}+1\right)}{2\sqrt{3}\,(12\pi)^{2n-1}}\,\psi^{(2n-1)}\left(\frac{1}{3}\right) \;/;\; n\in\mathbb{N}^+$$

10.02.20.0002.01

$$\zeta^{(1,0)}\left(-1, \frac{1}{4}\right) == \frac{\log(\text{Glaisher})}{8} - \frac{1}{96} + \frac{C}{4\pi}$$

10.02.20.0042.01

$$\zeta^{(1,0)}(-1, 1) == \frac{1}{12} - \log(A)$$

10.02.20.0043.01

$$\zeta^{(1,0)}(-n, a) + (-1)^n\,\zeta^{(1,0)}(-n, 1-a) = \frac{\pi i}{n+1} B_{n+1}(a) + \frac{e^{-\frac{\pi i n}{2}}\,n!}{(2\pi)^n}\,\text{Li}_{n+1}\left(e^{2\pi i a}\right) \;/;\; -\frac{\pi}{2} < \arg(a) \le \frac{\pi}{2} \bigwedge -\frac{\pi}{2} < \arg(1-a) \le \frac{\pi}{2} \bigwedge n\in\mathbb{N}^+$$

10.02.20.0044.01

$$\zeta^{(1,0)}(s, a) = \zeta^{(1,0)}(s, b - \min(0, \text{Re}(b-a))) - \frac{1}{2}\sum_{k=0}^{\text{Re}(b-a)-1} \frac{\log\left((a+k)^2\right)}{\left((a+k)^2\right)^{s/2}} \;/;\; \text{Re}(b-a)\in\mathbb{Z} \land \text{Im}(b-a) = 0$$

10.02.20.0045.01

$$\zeta^{(1,0)}(s, a) = \zeta^{(1,0)}(s, a - \lfloor\text{Re}(a)\rfloor) + \frac{1}{2}\,\text{sgn}(\text{Re}(a)) \sum_{k=0}^{|\lfloor\text{Re}(a)\rfloor|-1} \frac{\log\left(\left(k + \frac{1}{2}(1-2a)\,\text{sgn}(\text{Re}(a)) + \frac{1}{2}\right)^2\right)}{\left(\left(k + \frac{1}{2}(1-2a)\,\text{sgn}(\text{Re}(a)) + \frac{1}{2}\right)^2\right)^{s/2}} \;/;\; \neg\, a\in\mathbb{Z}$$

For $\hat{\zeta}(s, a)$

10.02.20.0003.01

$$\frac{\partial \hat{\zeta}(s, a)}{\partial s} == -\sum_{k=0}^{\infty} \frac{\log(a+k)}{(a+k)^s} \;/;\; \text{Re}(s) > 1$$

10.02.20.0046.01

$$\frac{\partial\hat{\zeta}(s,a+1)}{\partial s} == \frac{\partial\hat{\zeta}(s,a)}{\partial s} + \frac{\log(a)}{a^s}$$

10.02.20.0047.01

$$\frac{\partial\hat{\zeta}(s,a+m)}{\partial s} == \frac{\partial\hat{\zeta}(s,a)}{\partial s} + \sum_{k=0}^{m-1}\frac{\log(a+k)}{(a+k)^s} \;/;\; m\in\mathbb{N}^+$$

10.02.20.0048.01

$$\frac{\partial\hat{\zeta}(s,a)}{\partial s} = \frac{\partial\hat{\zeta}(s,b-\min(0,\mathrm{Re}(b-a)))}{\partial s} - \sum_{k=0}^{\mathrm{Re}(b-a)-1}\frac{\log(a+k)}{(a+k)^s} \;/;\; \mathrm{Re}(b-a)\in\mathbb{Z}\bigwedge\mathrm{Im}(b-a)=0$$

10.02.20.0004.01

$$\frac{\partial^2\hat{\zeta}(s,a)}{\partial s^2} == \sum_{k=0}^{\infty}\frac{\log^2(a+k)}{(a+k)^s} \;/;\; \mathrm{Re}(s)>1$$

10.02.20.0049.01

$$\frac{\partial^2\hat{\zeta}(s,a+1)}{\partial s^2} == \frac{\partial^2\hat{\zeta}(s,a)}{\partial s^2} - \frac{\log^2(a)}{a^s}$$

10.02.20.0050.01

$$\frac{\partial^2\hat{\zeta}(s,a+m)}{\partial s^2} == \frac{\partial^2\hat{\zeta}(s,a)}{\partial s^2} - \sum_{k=0}^{m-1}\frac{\log^2(a+k)}{(a+k)^s} \;/;\; m\in\mathbb{N}^+$$

With respect to *a*

For $\zeta(s, a)$

10.02.20.0051.01

$$\zeta^{(0,1)}(s,a) = -s\sum_{k=0}^{\infty}\frac{a+k}{\left((a+k)^2\right)^{\frac{s}{2}+1}} \;/;\; -a\notin\mathbb{N}$$

10.02.20.0052.01

$$\zeta^{(0,1)}(s,a) = s\,(\zeta(s+1,a) - 2\,\zeta(s+1,a+\max(\lfloor -\mathrm{Re}(a)\rfloor+1,0))) \;/;\; -a\notin\mathbb{N}$$

10.02.20.0053.01

$$\zeta^{(0,2)}(s,a) = s\,(s+1)\sum_{k=0}^{\infty}\frac{1}{\left((a+k)^2\right)^{\frac{s}{2}+1}} \;/;\; -a\notin\mathbb{N}$$

10.02.20.0054.01

$$\zeta^{(0,2)}(s,a) = s\,(s+1)\,\zeta(s+2,a) \;/;\; -a\notin\mathbb{N}$$

For $\hat{\zeta}(s, a)$

10.02.20.0005.01

$$\frac{\partial\hat{\zeta}(s,a)}{\partial a} == -s\,\hat{\zeta}(s+1,a)$$

10.02.20.0006.01

$$\frac{\partial^2 \hat{\zeta}(s, a)}{\partial a^2} = s\,(s+1)\,\hat{\zeta}(s+2, a)$$

Symbolic differentiation

With respect to *s*

For $\zeta(s, a)$

10.02.20.0055.01

$$\frac{\partial^n \zeta(s, a)}{\partial s^n} = \frac{(-1)^n}{2^n} \sum_{k=0}^{\infty} \frac{\log^n\left((a+k)^2\right)}{\left((a+k)^2\right)^{s/2}} \;/;\; -a \notin \mathbb{N} \bigwedge \mathrm{Re}(s) > 1 \bigwedge n \in \mathbb{N}$$

10.02.20.0056.01

$$\frac{\partial^n \zeta(s, a+1)}{\partial s^n} = \frac{\partial^n \zeta(s, a)}{\partial s^n} + \frac{(-1)^{n-1} \log^n\left(a^2\right)}{2^n \left(a^2\right)^{s/2}} \;/;\; n \in \mathbb{N}$$

10.02.20.0057.01

$$\frac{\partial^n \zeta(s, a+m)}{\partial s^n} = \frac{\partial^n \zeta(s, a)}{\partial s^n} + \frac{(-1)^{n-1}}{2^n} \sum_{k=0}^{m-1} \frac{\log^n\left((a+k)^2\right)}{\left((a+k)^2\right)^{s/2}} \;/;\; m \in \mathbb{N}^+ \bigwedge n \in \mathbb{N}$$

10.02.20.0058.01

$$\frac{\partial^n \zeta(s, 1)}{\partial s^n} = \frac{\partial^n \zeta(s)}{\partial s^n} \;/;\; n \in \mathbb{N}^+$$

10.02.20.0059.01

$$\frac{\partial^n \zeta(s, m)}{\partial s^n} = \frac{\partial^n \zeta(s)}{\partial s^n} + (-1)^{n-1} \sum_{k=0}^{m-2} \frac{\log^n(k+1)}{(k+1)^s} \;/;\; m \in \mathbb{N} \bigwedge n \in \mathbb{N}^+$$

10.02.20.0060.01

$$\zeta^{(n,0)}\left(s, \frac{1}{2}\right) = \sum_{k=0}^{n} \binom{n}{k} 2^s \log^{n-k}(2)\, \zeta^{(k)}(s) - \zeta^{(n)}(s) \;/;\; n \in \mathbb{N}^+$$

10.02.20.0061.01

$$\frac{\partial^n \zeta(s, a)}{\partial s^n} = \frac{\partial^n \zeta(s, b - \min(0, \mathrm{Re}(b-a)))}{\partial s^n} + \frac{(-1)^n}{2^n} \sum_{k=0}^{\mathrm{Re}(b-a)-1} \frac{\log^n\left((a+k)^2\right)}{\left((a+k)^2\right)^{s/2}} \;/;\; \mathrm{Re}(b-a) \in \mathbb{Z} \bigwedge \mathrm{Im}(b-a) = 0 \bigwedge n \in \mathbb{N}^+$$

For $\hat{\zeta}(s, a)$

10.02.20.0007.02

$$\frac{\partial^n \hat{\zeta}(s, a)}{\partial s^n} = (-1)^n \sum_{k=0}^{\infty} \frac{\log^n(a+k)}{(a+k)^s} \;/;\; \mathrm{Re}(s) > 1 \land n \in \mathbb{N}$$

10.02.20.0062.01

$$\frac{\partial^n \hat{\zeta}(s, a+1)}{\partial s^n} = \frac{\partial^n \hat{\zeta}(s, a)}{\partial s^n} + \frac{(-1)^{n-1} \log^n(a)}{a^s} \;/;\; n \in \mathbb{N}$$

10.02.20.0063.01

$$\frac{\partial^n \hat{\zeta}(s, a+m)}{\partial s^n} == \frac{\partial^n \hat{\zeta}(s, a)}{\partial s^n} + (-1)^{n-1} \sum_{k=0}^{m-1} \frac{\log^n(a+k)}{(a+k)^s} /; m \in \mathbb{N}^+ \bigwedge n \in \mathbb{N}$$

10.02.20.0064.01

$$\frac{\partial^n \hat{\zeta}(s, a)}{\partial s^n} = \frac{\partial^n \hat{\zeta}(s, b - \min(0, \mathrm{Re}(b-a)))}{\partial s^n} + (-1)^n \sum_{k=0}^{\mathrm{Re}(b-a)-1} \frac{\log^n(a+k)}{(a+k)^s} /; \mathrm{Re}(b-a) \in \mathbb{Z} \bigwedge \mathrm{Im}(b-a) = 0 \bigwedge n \in \mathbb{N}^+$$

With respect to *a*

For $\zeta(s, a)$

10.02.20.0065.01

$$\frac{\partial^n \zeta(s, a)}{\partial a^n} = (1-n-s)_n \sum_{k=0}^{\infty} \frac{1}{(a+k)^n \left((a+k)^2\right)^{s/2}} /; -a \notin \mathbb{N} \bigwedge n \in \mathbb{N}$$

10.02.20.0066.01

$$\frac{\partial^n \zeta(s, a)}{\partial a^n} = (1-n-s)_n \left(\sum_{k=0}^{\lfloor -\mathrm{Re}(a) \rfloor} \frac{1}{(a+k)^n \left((a+k)^2\right)^{s/2}} + \zeta(n+s, a + \max(\lfloor -\mathrm{Re}(a) \rfloor + 1, 0)) \right) /; -a \notin \mathbb{N} \bigwedge n \in \mathbb{N}$$

For $\hat{\zeta}(s, a)$

10.02.20.0008.02

$$\frac{\partial^n \hat{\zeta}(s, a)}{\partial a^n} == (1-n-s)_n \, \hat{\zeta}(n+s, a) /; n \in \mathbb{N}$$

Fractional integro-differentiation

With respect to *s*

For $\hat{\zeta}(s, a)$

10.02.20.0009.01

$$\frac{\partial^\alpha \hat{\zeta}(s, a)}{\partial s^\alpha} == s^{-\alpha} \sum_{k=0}^{\infty} \frac{(-s \log(a+k))^\alpha \, Q(-\alpha, 0, -s \log(a+k))}{(a+k)^s} /; \mathrm{Re}(s) > 1$$

With respect to *a*

For $\zeta(s, a)$

10.02.20.0010.01

$$\frac{\partial^\alpha \zeta(s, a)}{\partial a^\alpha} == \frac{\Gamma(1-s)\, a^{-s-\alpha}}{\Gamma(1-s-\alpha)} + a^{-\alpha} \sum_{k=1}^{\infty} k^{-s} \, {}_2\tilde{F}_1\left(1, s; 1-\alpha; -\frac{a}{k}\right) /; \mathrm{Re}(s) > 1 \land \mathrm{Re}(a) > 0$$

Integration

Indefinite integration

Involving only one direct function with respect to *s*

For $\hat{\zeta}(s, a)$

10.02.21.0001.01

$$\int \hat{\zeta}(s, a)\, ds = -\sum_{k=0}^{\infty} \frac{(a+k)^{-s}}{\log(a+k)} \;/;\; \operatorname{Re}(s) > 1 \land \operatorname{Re}(a) > 0$$

10.02.21.0002.01

$$\int s^{\alpha-1}\, \hat{\zeta}(s, a)\, ds = -s^{\alpha} \sum_{k=0}^{\infty} \frac{\Gamma(\alpha, s\log(a+k))}{(s\log(a+k))^{\alpha}} \;/;\; \operatorname{Re}(s) > 1 \land \operatorname{Re}(a) > 0$$

Involving only one direct function with respect to *a*

For $\hat{\zeta}(s, a)$

10.02.21.0003.01

$$\int \hat{\zeta}(s, a)\, da = \frac{1}{1-s}\, \zeta(s-1, a)$$

10.02.21.0004.01

$$\int a^{\alpha-1}\, \hat{\zeta}(s, a)\, da = \frac{a^{\alpha}}{\alpha - s} \sum_{k=0}^{\infty} \frac{1}{(a+k)^{s} \left(\frac{a+k}{a}\right)^{-s}}\, {}_2F_1\!\left(s-\alpha, s; s-\alpha+1; -\frac{k}{a}\right)$$

Summation

Finite summation

Finite summation

For $\zeta(s, a)$

10.02.23.0001.01

$$\sum_{k=1}^{q} \zeta\!\left(s, \frac{k}{q}\right) = q^{s}\, \zeta(s) \;/;\; q \in \mathbb{N}^{+}$$

For $\hat{\zeta}(s, a)$

10.02.23.0002.01

$$\sum_{k=1}^{q} \hat{\zeta}\!\left(s, \frac{a+k}{q}\right) = q^{s}\, \hat{\zeta}(s, a+1) \;/;\; q \in \mathbb{N}^{+}$$

Infinite summation

Infinite summation

For $\zeta(s, a)$

10.02.23.0003.01

$$\sum_{k=1}^{\infty} \frac{\zeta(n, k+1)}{k} = \frac{n}{2}\,\zeta(n+1) - \frac{1}{2}\sum_{k=1}^{n-2} \zeta(n-k)\,\zeta(k+1) \;/;\; n-1 \in \mathbb{N}^+$$

10.02.23.0004.01

$$\sum_{k=0}^{\infty} \frac{(-1)^k\,(a)_k}{k!}\,\zeta(a+k, x)\,y^k = \zeta(a, x+y) \;/;\; \left|\frac{y}{x}\right| < 1$$

10.02.23.0005.01

$$\sum_{k=0}^{\infty} \frac{z^k\,\zeta(k+2, z+1)}{a+k} = -\Gamma(a-1)\,\psi^{(1-a)}(z)\,z^{-a} + \frac{\psi(a)+\gamma-\log(z)}{z^2} + \frac{\psi(z)}{(a-1)\,z} \;/;\; |z| < 1$$

10.02.23.0006.01

$$\sum_{k=0}^{\infty} \frac{z^k}{k+n}\,\zeta(k+2, a) = (n-2)!\;z^{-n}\sum_{j=0}^{n-2} \frac{z^j}{j!\,(n-j-2)!}\left(\psi(n-j-1)\,\zeta(j-n+2, a-z) + \zeta^{(1,0)}(j-n+2, a-z)\right) - \left(\psi(n-1)\,\zeta(2-n, a) + \zeta^{(1,0)}(2-n, a)\right)z^{-n} + \frac{\psi(a)+\gamma}{(n-1)\,z} \;/;\; \mathrm{Re}(a) > 0 \land n \in \mathbb{Z} \land n > 1$$

10.02.23.0007.01

$$\sum_{k=0}^{\infty} \frac{(-1)^k\,(k+1)!\,z^k}{(a)_k}\,\zeta\!\left(k+2, \frac{1}{2}\right) = \frac{\Gamma(a)}{2\,z^2}\left(2^{1-a}\,z^{3-a}\left(8\,\psi^{(2-a)}(2\,z) - 2^a\,\psi^{(2-a)}(z)\right) + \frac{2\,\gamma\,z + (a-2)\log(4)}{\Gamma(a-1)}\right)$$

Operations

Limit operation

Limit operation

For $\hat{\zeta}(s, a)$

10.02.25.0001.01

$$\lim_{s\to 1}\left(\hat{\zeta}(s, a) - \frac{1}{s-1}\right) = -\psi(a)$$

Representations through more general functions

Through hypergeometric functions

Involving ${}_p\tilde{F}_q$

For $\hat{\zeta}(s, a)$

10.02.26.0001.01

$$\hat{\zeta}\,(s, a) = a^{-s}\,\Gamma\,(a+1)^s\;{}_{s+1}\tilde{F}_s(1, a_1, a_2, \ldots, a_s; a_1+1, a_2+1, \ldots, a_s+1; 1) \;/;\; a_1 = a_2 = \ldots = a_s = a \bigwedge -a \notin \mathbb{N} \bigwedge s \in \mathbb{N}^+$$

Involving ${}_pF_q$

For $\hat{\zeta}(s, a)$

10.02.26.0002.01

$$\hat{\zeta}(s, a) = a^{-s} {}_{s+1}F_s(1, a_1, a_2, \ldots, a_s; a_1 + 1, a_2 + 1, \ldots, a_s + 1; 1) /; a_1 = a_2 = \ldots = a_s = a \land a \notin \mathbb{N} \land s \in \mathbb{N}^+$$

Through Meijer *G*

Classical cases for the direct function itself

For $\hat{\zeta}(s, a)$

10.02.26.0003.01

$$\hat{\zeta}(s, a) = G_{s+1,s+1}^{1,s+1}\left(-1 \,\middle|\, \begin{array}{c} 0, 1-a, \ldots, 1-a \\ 0, -a, \ldots, -a \end{array}\right) /; s \in \mathbb{N}^+$$

Through other functions

Through other functions

For $\zeta(s, a)$

10.02.26.0004.01

$$\zeta(s, a) = \Phi(1, s, a)$$

For $\hat{\zeta}(s, a)$

10.02.26.0005.01

$$\hat{\zeta}(s, a) = \hat{\Phi}(1, s, a)$$

For $\tilde{\zeta}(s, a)$

10.02.26.0006.01

$$\tilde{\zeta}(s, a) = \tilde{\Phi}(1, s, a)$$

Representations through equivalent functions

Interrelations

10.02.27.0004.01

$$\zeta(s, a) = \left(\frac{(\lfloor -\mathrm{Re}(a) \rfloor + \lfloor \mathrm{Re}(a) \rfloor + 1)\,\theta(\mathrm{Im}(a))}{\left((a + \lfloor -\mathrm{Re}(a) \rfloor)^2\right)^{s/2}} + \hat{\zeta}(s, a + \lfloor -\mathrm{Re}(a) \rfloor + 1)\right)\left(1 - e^{(2\,\theta(\mathrm{Im}(a)) - 1)\,\pi\, i\, s}\right)\theta(-\mathrm{Re}(a)) + \left(\left(e^{(2\,\theta(\mathrm{Im}(a)) - 1)\,\pi\, i\, s} - 1\right)\theta(-\mathrm{Re}(a)) + 1\right)\hat{\zeta}(s, a)$$

10.02.27.0005.01

$$\zeta(s, a) = \hat{\zeta}(s, a) \;/;\; -\frac{\pi}{2} < \arg(a) \leq \frac{\pi}{2}$$

10.02.27.0006.01

$$\zeta(2\,n, a) = \hat{\zeta}(2\,n, a) \;/;\; n \in \mathbb{Z}$$

10.02.27.0007.01

$$\zeta(s, a) = \zeta(s, a + \lceil -\mathrm{Re}(a) \rceil) + \zeta(s, -a - \lfloor -\mathrm{Re}(a) \rfloor) - \hat{\zeta}(s, 1-a) - \delta_{\mathrm{frac}(-\mathrm{Re}(a))} \left(-\mathrm{Im}(a)^2\right)^{-\frac{s}{2}} \;/;\; -\frac{\pi}{2} < \arg(1-a) \leq \frac{\pi}{2} \bigwedge a \notin \mathbb{R}$$

Pavlyk 0. (2006)

10.02.27.0008.01

$$\hat{\zeta}(s, a) = \zeta(s, a)\, e^{-\theta(\lfloor -\mathrm{Re}(a) \rfloor)(2\,\theta(\mathrm{Im}(a))-1)\,\pi\, i\, s} + \theta(\lfloor -\mathrm{Re}(a) \rfloor)\left(1 - e^{-(2\,\theta(\mathrm{Im}(a))-1)\,\pi\, i\, s}\right)\left(\frac{\theta(\mathrm{Im}(a))\,(\lfloor -\mathrm{Re}(a) \rfloor + \lfloor \mathrm{Re}(a) \rfloor + 1)}{\left((a + \lfloor -\mathrm{Re}(a) \rfloor)^2\right)^{s/2}} + \zeta(s, a + \lfloor -\mathrm{Re}(a) \rfloor + 1)\right)$$

With related functions

With related functions

For $\zeta(s, a)$

10.02.27.0001.01

$$\hat{\zeta}(s, a) == \zeta(s) - H_{a-1}^{(s)}$$

10.02.27.0009.01

$$\zeta(s, a) == \zeta(s) - H_{a-1}^{(s)} \;/;\; -\frac{\pi}{2} < \arg(a) \leq \frac{\pi}{2}$$

10.02.27.0010.01

$$\zeta(s, a) = \left(\left(e^{(2\,\theta(\mathrm{Im}(a))-1)\,\pi\, i\, s} - 1\right)\theta(-\mathrm{Re}(a)) + 1\right)\left(\zeta(s) - H_{a-1}^{(s)}\right) + \left(-H_{a+\lfloor -\mathrm{Re}(a) \rfloor}^{(s)} + \frac{(\lfloor -\mathrm{Re}(a) \rfloor + \lfloor \mathrm{Re}(a) \rfloor + 1)\,\theta(\mathrm{Im}(a))}{\left((a + \lfloor -\mathrm{Re}(a) \rfloor)^2\right)^{s/2}} + \zeta(s)\right)\left(1 - e^{(2\,\theta(\mathrm{Im}(a))-1)\,\pi\, i\, s}\right)\theta(-\mathrm{Re}(a))$$

10.02.27.0002.01

$$\zeta(n, a) == \frac{(-1)^n}{(n-1)!}\,\psi^{(n-1)}(a) \;/;\; n - 1 \in \mathbb{N}^+ \bigwedge \mathrm{Re}(a) > 0$$

10.02.27.0011.01

$$\zeta(n, a) = \frac{(-1)^n\,\psi^{(n-1)}(a + (\lfloor -\mathrm{Re}(a) \rfloor + 1)\,\theta(\lfloor -\mathrm{Re}(a) \rfloor + 1))}{(n-1)!} + \sum_{k=0}^{\lfloor -\mathrm{Re}(a) \rfloor} \frac{1}{\left((a+k)^2\right)^{n/2}} \;/;\; n - 1 \in \mathbb{N}^+$$

10.02.27.0003.01

$$\zeta\left(s, \frac{m}{n}\right) == \frac{1}{n} \sum_{k=1}^{n} n^s\, \mathrm{Li}_s\left(e^{\frac{2\pi i k}{n}}\right) e^{-\frac{2\pi i k m}{n}} \;/;\; m \in \mathbb{N}^+ \bigwedge n \in \mathbb{N}^+ \bigwedge m \leq n$$

History

– H. Mellin (1899)
– E. Lindelöf (1899)

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