Introductions to BesselK

Introduction to the Bessel functions

General

The Bessel functions have been known since the 18th century when mathematicians and scientists started to describe physical processes through differential equations. Many different-looking processes satisfy the same partial differential equations. These equations were named Laplace, d`Alembert (wave), Poisson, Helmholtz, and heat (diffusion) equations. Different methods were used to investigate these equations. The most powerful was the separation of variables method, which in polar coordinates often leads to ordinary differential equations of special structure:

$$w''(z)\, z^2 + w'(z)\, z + \left(z^2 - \nu^2\right) w(z) == 0.$$

This equation with concrete values of the parameter ν appeared in the articles by F. W. Bessel (1816, 1824) who built two partial solutions $w_1(z)$ and $w_2(z)$ of the previous equation in the form of series:

$$w(z) == z^\nu \sum_{j=0}^{\infty} a_j\, z^j + z^{-\nu} \sum_{j=0}^{\infty} b_j\, z^j == z^\nu \left(\sum_{k=0}^{\infty} a_{2k}\, z^{2k} + \sum_{k=0}^{\infty} a_{2k+1}\, z^{2k+1} \right) + z^{-\nu} \left(\sum_{k=0}^{\infty} b_{2k}\, z^{2k} + \sum_{k=0}^{\infty} b_{2k+1}\, z^{2k+1} \right).$$

Substituting the series into the differential equation produces the following solutions:

$$w_1(z) == z^\nu \sum_{k=0}^{\infty} A_k\, z^{2k} \;/;\; A_0 == \frac{2^{-\nu}}{\Gamma(\nu+1)} \bigwedge A_1 == -\frac{2^{-\nu-2}}{\Gamma(\nu+2)} \bigwedge A_k == a_{2k} == \frac{(-1)^k\, 2^{-\nu-2k}}{\Gamma(k+\nu+1)\, k!}$$

$$w_2(z) == z^{-\nu} \sum_{k=0}^{\infty} B_k\, z^{2k} \;/;\; B_0 == \frac{2^{\nu}}{\Gamma(1-\nu)} \bigwedge B_1 == -\frac{2^{\nu-2}}{\Gamma(2-\nu)} \bigwedge B_k == b_{2k} == \frac{(-1)^k\, 2^{\nu-2k}}{\Gamma(k-\nu+1)\, k!}.$$

O. Schlömilch (1857) used the name Bessel functions for these solutions, E. Lommel (1868) considered ν as an arbitrary real parameter, and H. Hankel (1869) considered complex values for ν. The two independent solutions of the differential equation were notated as $J_\nu(z)$ and $J_{-\nu}(z)$.

For integer index ν, the functions $J_\nu(z)$ and $J_{-\nu}(z)$ coincide or have different signs. In such cases, the second linear independent solution of the previous differential equation was introduced by C. G. Neumann (1867) as the limit case of the following special linear combination of the functions $J_\nu(z)$ and $J_{-\nu}(z)$:

$$Y_\nu(z) == \lim_{\mu \to \nu} \frac{\cos(\mu\,\pi)\, J_\mu(z) - J_{-\mu}(z)}{\sin(\mu\,\pi)} \;/;\; \nu \in \mathbb{Z}.$$

J. Watson (1867) introduced the notation Y_ν for this function. Other authors (H. Hankel (1869), H. Weber (1873), and L. Schläfli (1875)) investigated its properties. In particular, the general solution of the previous differential equation for all values of the parameter ν can be presented by the formula:

$$z^2\, w''(z) + z\, w'(z) + \left(z^2 - \nu^2\right) w(z) == 0 \;/;\; w(z) == c_1\, J_\nu(z) + c_2\, Y_\nu(z),$$

where c_1 and c_2 are arbitrary complex constants.

In a similar way, A. B. Basset (1888) and H. M. MacDonald (1899) introduced the modified Bessel functions $I_\nu(z)$ and $K_\nu(z)$, which satisfy the modified Bessel differential equation:

$$z^2\, w''(z) + z\, w'(z) - \left(z^2 + \nu^2\right) w(z) = 0 \;/;\; w(z) = c_1\, I_\nu(z) + c_2\, K_\nu(z).$$

The first differential equation can be converted into the last one by changing the independent variable z to $i\, z$.

Definitions of Bessel functions

The Bessel functions of the first kind $J_\nu(z)$ and $I_\nu(z)$ are defined as sums of the following infinite series:

$$J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+\nu+1)\, k!} \left(\frac{z}{2}\right)^{2k+\nu}$$

$$I_\nu(z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k+\nu+1)\, k!} \left(\frac{z}{2}\right)^{2k+\nu}.$$

These sums are convergent everywhere in the complex z-plane. The Bessel functions of the second kind $K_\nu(z)$ and $Y_\nu(z)$ for noninteger parameter ν are defined as special linear combinations of the last two functions:

$$K_\nu(z) = \frac{\pi \csc(\pi\, \nu)}{2} \left(I_{-\nu}(z) - I_\nu(z)\right) \;/;\; \nu \notin \mathbb{Z}$$

$$Y_\nu(z) = \csc(\pi\, \nu) \left(\cos(\nu\, \pi)\, J_\nu(z) - J_{-\nu}(z)\right) \;/;\; \nu \notin \mathbb{Z}.$$

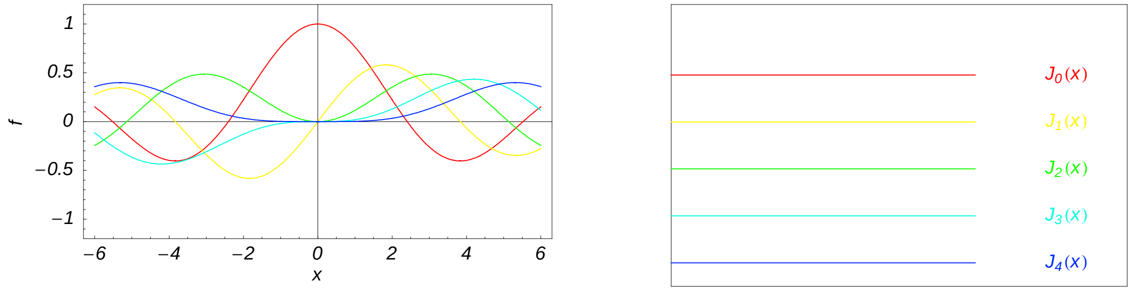
In the case of integer index ν, the right-hand sides of the previous expressions give removable indeterminate values of the type $\frac{0}{0}$. In this case, the Bessel functions $K_\nu(z)$ and $Y_\nu(z)$ are defined through the following limits:

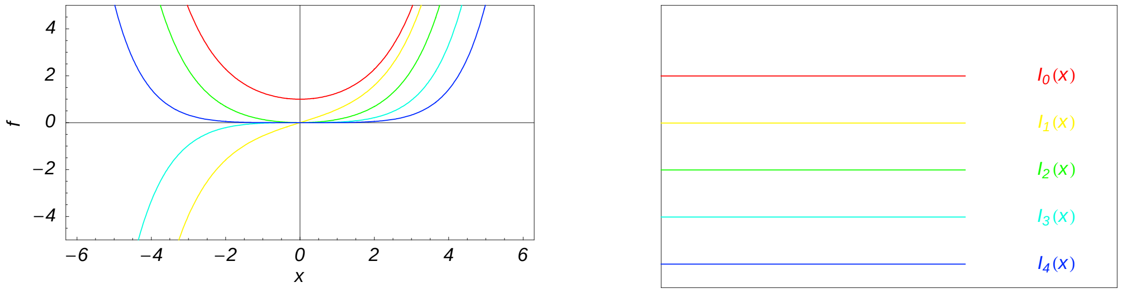
$$K_\nu(z) = \lim_{\mu \to \nu} K_\mu(z) \;/;\; \nu \in \mathbb{Z}$$

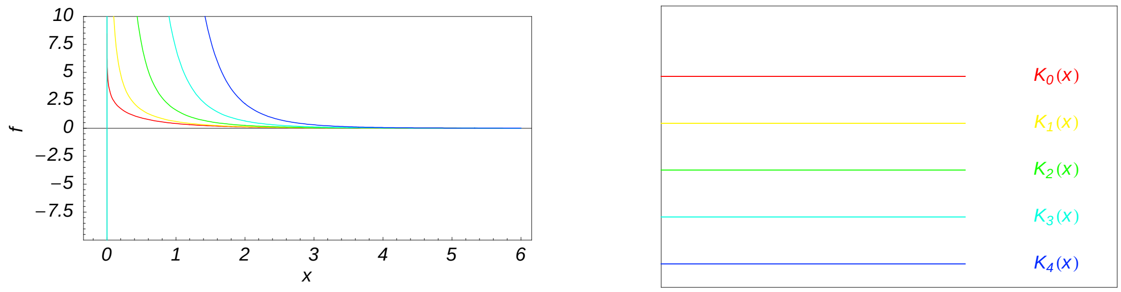
$$Y_\nu(z) = \lim_{\mu \to \nu} Y_\mu(z) \;/;\; \nu \in \mathbb{Z}.$$

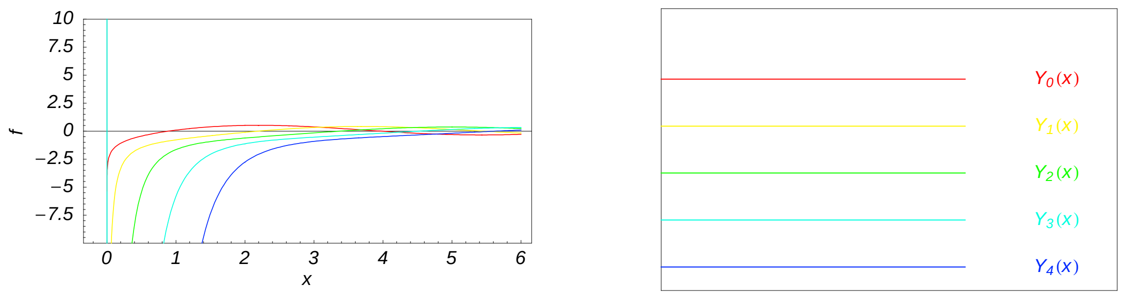
A quick look at the Bessel functions

Here is a quick look at the graphics for the Bessel functions along the real axis.









Connections within the group of Bessel functions and with other function groups

Representations through more general functions

The Bessel functions $J_\nu(z)$, $I_\nu(z)$, $K_\nu(z)$, and $Y_\nu(z)$ are particular cases of more general functions: hypergeometric and Meijer G functions.

In particular, the functions $J_\nu(z)$ and $I_\nu(z)$ can be represented through the regularized hypergeometric functions ${}_0\tilde{F}_1$ (without any restrictions on the parameter ν):

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu {}_0\tilde{F}_1\left(;\nu+1;-\frac{z^2}{4}\right) \quad I_\nu(z) = \left(\frac{z}{2}\right)^\nu {}_0\tilde{F}_1\left(;\nu+1;\frac{z^2}{4}\right).$$

Similar formulas, but with restrictions on the parameter ν, represent $J_\nu(z)$ and $I_\nu(z)$ through the classical hypergeometric function ${}_0F_1$:

$$J_\nu(z) = \frac{1}{\Gamma(\nu+1)}\left(\frac{z}{2}\right)^\nu {}_0F_1\left(;\nu+1;-\frac{z^2}{4}\right) /; -\nu \notin \mathbb{N}^+ \quad I_\nu(z) = \frac{1}{\Gamma(\nu+1)}\left(\frac{z}{2}\right)^\nu {}_0F_1\left(;\nu+1;\frac{z^2}{4}\right) /; -\nu \notin \mathbb{N}^+ .$$

The functions $J_\nu(z)$ and $I_\nu(z)$ can also be represented through the hypergeometric functions ${}_1F_1$ by the following formulas:

$$J_\nu(z) = \frac{z^\nu}{2^\nu e^{i z}\,\Gamma(\nu+1)}\, {}_1F_1\left(\nu+\frac{1}{2};2\nu+1;2 i z\right) \quad I_\nu(z) = \frac{z^\nu}{2^\nu e^{z}\,\Gamma(\nu+1)}\, {}_1F_1\left(\nu+\frac{1}{2};2\nu+1;2z\right)$$

$$J_\nu(z) = \frac{z^\nu}{2^\nu\,\Gamma(\nu+1)} \lim_{a\to\infty} {}_1F_1\left(a;\, \nu+1;\, -\frac{z^2}{4a}\right) \quad I_\nu(z) = \frac{z^\nu}{2^\nu\,\Gamma(\nu+1)} \lim_{a\to\infty} {}_1F_1\left(a;\, \nu+1;\, \frac{z^2}{4a}\right).$$

Similar formulas for other Bessel functions $K_\nu(z)$ and $Y_\nu(z)$ always include restrictions on the parameter, namely $\nu \notin \mathbb{Z}$:

$$K_\nu(z) = \pi \csc(\nu\pi)\left(2^{\nu-1} z^{-\nu}\, {}_0\tilde{F}_1\left(;\, 1-\nu;\, \frac{z^2}{4}\right) - 2^{-\nu-1} z^\nu\, {}_0\tilde{F}_1\left(;\, \nu+1;\, \frac{z^2}{4}\right)\right) /;\, \nu \notin \mathbb{Z}$$

$$K_\nu(z) = 2^{\nu-1}\,\Gamma(\nu)\, z^{-\nu}\, {}_0F_1\left(;\, 1-\nu;\, \frac{z^2}{4}\right) + 2^{-\nu-1}\,\Gamma(-\nu)\, z^\nu\, {}_0F_1\left(;\, \nu+1;\, \frac{z^2}{4}\right) /;\, \nu \notin \mathbb{Z}$$

$$K_\nu(z) = 2^{\nu-1}\,\Gamma(\nu)\, z^{-\nu}\, e^{-z}\, {}_1F_1\left(\frac{1}{2}-\nu;\, 1-2\nu;\, 2z\right) + 2^{-\nu-1}\,\Gamma(-\nu)\, z^\nu\, e^{-z}\, {}_1F_1\left(\frac{1}{2}+\nu;\, 1+2\nu;\, 2z\right) /;\, \nu \notin \mathbb{Z}$$

$$Y_\nu(z) = 2^{-\nu} z^\nu \cot(\nu\pi)\, {}_0\tilde{F}_1\left(;\, \nu+1;\, -\frac{z^2}{4}\right) - 2^\nu z^{-\nu} \csc(\nu\pi)\, {}_0\tilde{F}_1\left(;\, 1-\nu;\, -\frac{z^2}{4}\right) /;\, \nu \notin \mathbb{Z}$$

$$Y_\nu(z) = -\frac{2^\nu z^{-\nu}\,\Gamma(\nu)}{\pi}\, {}_0F_1\left(;\, 1-\nu;\, -\frac{z^2}{4}\right) - \frac{2^{-\nu} z^\nu \cos(\nu\pi)\,\Gamma(-\nu)}{\pi}\, {}_0F_1\left(;\, \nu+1;\, -\frac{z^2}{4}\right) /;\, \nu \notin \mathbb{Z}$$

$$Y_\nu(z) = -\frac{2^{-\nu}\cos(\nu\pi)\,\Gamma(-\nu)\, e^{-iz} z^\nu}{\pi}\, {}_1F_1\left(\nu+\frac{1}{2};\, 2\nu+1;\, 2iz\right) - \frac{2^\nu\,\Gamma(\nu)\, z^{-\nu} e^{-iz}}{\pi}\, {}_1F_1\left(\frac{1}{2}-\nu;\, 1-2\nu;\, 2iz\right) /;\, \nu \notin \mathbb{Z}.$$

In the case of integer ν, the right-hand sides of the preceding six formulas evaluate to removable indeterminate expressions of the type $\infty, -\infty$. The limit of the right-hand sides exists and produces complicated series expansions including logarithmic and polygamma functions. These difficulties can be removed by using the generalized Meijer G function. The generalized Meijer G function allows represention of all four Bessel functions for all values of the parameter ν by the following simple formulas:

$$J_\nu(z) = G_{0,2}^{1,0}\left(\frac{z}{2}, \frac{1}{2} \,\middle|\, \frac{\nu}{2}, -\frac{\nu}{2}\right)$$

$$I_\nu(z) = \pi\, G_{1,3}^{1,0}\left(\frac{z}{2}, \frac{1}{2} \,\middle|\, \begin{matrix} \frac{\nu+1}{2} \\ \frac{\nu}{2}, -\frac{\nu}{2}, \frac{\nu+1}{2} \end{matrix}\right)$$

$$K_\nu(z) = \frac{1}{2}\, G_{0,2}^{2,0}\left(\frac{z}{2}, \frac{1}{2} \,\middle|\, \frac{\nu}{2}, -\frac{\nu}{2}\right)$$

$$Y_\nu(z) = G_{1,3}^{2,0}\left(\frac{z}{2}, \frac{1}{2} \,\middle|\, \begin{matrix} -\frac{1}{2}(\nu+1) \\ \frac{\nu}{2}, -\frac{\nu}{2}, -\frac{1}{2}(\nu+1) \end{matrix}\right).$$

The classical Meijer G function is less convenient because it can lead to additional restrictions:

$$J_\nu(z) = 2^{-\nu} z^\nu\, G_{0,2}^{1,0}\left(\frac{z^2}{4} \,\middle|\, 0, -\nu\right)$$

$$I_\nu(z) = \pi\, 2^{-\nu}\, z^\nu\, G_{1,3}^{1,0}\left(\frac{z^2}{4} \,\middle|\, \begin{matrix} \frac{1}{2} \\ 0, -\nu, \frac{1}{2} \end{matrix}\right)$$

$$K_\nu(z) = \frac{1}{2}\, G_{0,2}^{2,0}\left(\frac{z^2}{4} \,\middle|\, \frac{\nu}{2}, -\frac{\nu}{2}\right) /;\ \mathrm{Re}(z) > 0$$

$$Y_\nu(z) = G_{1,3}^{2,0}\left(\frac{z^2}{4} \,\middle|\, \begin{matrix} -\frac{1}{2}(\nu+1) \\ \frac{\nu}{2}, -\frac{\nu}{2}, -\frac{1}{2}(\nu+1) \end{matrix}\right) /;\ \mathrm{Re}(z) > 0.$$

Representations through other Bessel functions

Each of the Bessel functions can be represented through other Bessel functions:

$$J_\nu(z) = \frac{z^\nu}{(i z)^\nu}\, I_\nu(i z) \quad J_\nu(i z) = \frac{(i z)^\nu}{z^\nu}\, I_\nu(z)$$

$$I_\nu(z) = \frac{z^\nu}{(i z)^\nu}\, J_\nu(i z) \quad I_\nu(i z) = \frac{(i z)^\nu}{z^\nu}\, J_\nu(z)$$

$$J_\nu(z) = \csc(\pi \nu)\, Y_{-\nu}(z) - \cot(\pi \nu)\, Y_\nu(z)$$

$$I_\nu(z) = \frac{z^\nu}{(i z)^\nu}\left(\csc(\pi \nu)\, Y_{-\nu}(i z) - \cot(\pi \nu)\, Y_\nu(i z)\right)$$

$$K_\nu(z) = \frac{\pi}{2}\left(\frac{(i z)^{2\nu} \cos(\pi \nu)}{z^{2\nu}} - 1\right) \csc(\pi \nu)\, I_\nu(z) - \frac{\pi\, (i z)^\nu}{2\, z^\nu}\, Y_\nu(i z) /;\ \nu \notin \mathbb{Z}$$

$$K_\nu(z) = \frac{\pi}{2}\left(\frac{(i z)^\nu \cos(\pi \nu)}{z^\nu} - \frac{z^\nu}{(i z)^\nu}\right) \csc(\pi \nu)\, J_\nu(i z) - \frac{\pi\, (i z)^\nu}{2\, z^\nu}\, Y_\nu(i z) /;\ \nu \notin \mathbb{Z}$$

$$Y_\nu(z) = \frac{z^{-2\nu}}{\pi}\left(-2\, z^\nu\, K_\nu(-i z)\, (-i z)^\nu - \pi\, J_\nu(z) \csc(\pi \nu)\, (-i z)^{2\nu} + \pi\, z^{2\nu}\, J_\nu(z) \cot(\pi \nu)\right) /;\ \nu \notin \mathbb{Z}$$

$$Y_\nu(z) = \frac{(i z)^{-\nu}}{\pi}\, z^{-\nu}\left(\pi \csc(\pi \nu)\left(z^{2\nu} \cos(\pi \nu) - (i z)^{2\nu}\right) I_\nu(i z) - 2\, (i z)^{2\nu}\, K_\nu(i z)\right) /;\ \nu \notin \mathbb{Z}.$$

The best-known properties and formulas for Bessel functions

Real values for real arguments

For real values of parameter ν and positive argument z, the values of all four Bessel functions $J_\nu(z)$, $I_\nu(z)$, $K_\nu(z)$, and $Y_\nu(z)$ are real.

Simple values at zero

The Bessel functions $J_\nu(z)$, $I_\nu(z)$, $K_\nu(z)$, and $Y_\nu(z)$ have rather simple values for the argument $z = 0$:

$$J_0(0) = 1$$

$$I_0(0) = 1$$

$$K_0(0) = \infty$$

$Y_0(0) = -\infty$

$J_\nu(0) = 0 \;/;\; \mathrm{Re}(\nu) > 0$

$I_\nu(0) = 0 \;/;\; \mathrm{Re}(\nu) > 0$

$K_\nu(0) = \tilde{\infty}$

$Y_\nu(0) = \tilde{\infty}.$

Specific values for specialized parameters

In the case of half-integer ν ($\nu = \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \ldots$) all Bessel functions $J_\nu(z)$, $I_\nu(z)$, $K_\nu(z)$ and $Y_\nu(z)$ can be expressed through sine, cosine, or exponential functions multiplied by rational and square root functions. Modulo simple factors, these are the so-called spherical Bessel functions, for example:

$$J_{-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi}}\,\frac{\cos(z)}{\sqrt{z}} \quad I_{-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi}}\,\frac{\cosh(z)}{\sqrt{z}} \quad Y_{-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi}}\,\frac{\sin(z)}{\sqrt{z}}$$

$$J_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi}}\,\frac{\sin(z)}{\sqrt{z}} \quad I_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi}}\,\frac{\sinh(z)}{\sqrt{z}} \quad Y_{\frac{1}{2}}(z) = -\sqrt{\frac{2}{\pi}}\,\frac{\cos(z)}{\sqrt{z}}$$

$$K_{\frac{1}{2}}(z) = K_{-\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2}}\,\frac{e^{-z}}{\sqrt{z}}.$$

The previous formulas are particular cases of the following, more general formulas:

$$J_\nu(z) = \sqrt{\frac{2}{\pi}}\,\frac{1}{\sqrt{z}}\left(\cos\left(\frac{\pi}{2}\left(\nu-\frac{1}{2}\right)-z\right)\sum_{j=0}^{\left\lfloor\frac{2|\nu|-3}{4}\right\rfloor}\frac{(-1)^j\left(2j+|\nu|+\frac{1}{2}\right)!\,(2z)^{-2j-1}}{(2j+1)!\left(-2j+|\nu|-\frac{3}{2}\right)!} - \sin\left(\frac{\pi}{2}\left(\nu-\frac{1}{2}\right)-z\right)\sum_{j=0}^{\left\lfloor\frac{2|\nu|-1}{4}\right\rfloor}\frac{(-1)^j\left(2j+|\nu|-\frac{1}{2}\right)!}{(2j)!\left(-2j+|\nu|-\frac{1}{2}\right)!\,(2z)^{2j}}\right) \;/;\; \nu-\frac{1}{2}\in\mathbb{Z}$$

$$I_\nu(z) = -\frac{1}{\sqrt{z}}\,e^{\frac{\pi i}{2}\left(\frac{1}{2}-\nu\right)}\sqrt{\frac{2}{\pi}}\left(\sinh\left(\frac{\pi i}{2}\left(\frac{1}{2}-\nu\right)-z\right)\sum_{k=0}^{\left\lfloor\frac{2|\nu|-1}{4}\right\rfloor}\frac{\left(|\nu|+2k-\frac{1}{2}\right)!}{(2k)!\left(|\nu|-2k-\frac{1}{2}\right)!\,(2z)^{2k}} + \cosh\left(\frac{\pi i}{2}\left(\frac{1}{2}-\nu\right)-z\right)\sum_{k=0}^{\left\lfloor\frac{2|\nu|-3}{4}\right\rfloor}\frac{\left(|\nu|+2k+\frac{1}{2}\right)!\,(2z)^{-2k-1}}{(2k+1)!\left(|\nu|-2k-\frac{3}{2}\right)!}\right) \;/;\; \nu-\frac{1}{2}\in\mathbb{Z}$$

$$Y_\nu(z) = \sqrt{\frac{2}{\pi}}\,\frac{(-1)^{\nu+\frac{1}{2}}}{\sqrt{z}}\left(\sin\left(\frac{\pi}{2}\left(\nu+\frac{1}{2}\right)+z\right)\sum_{j=0}^{\left\lfloor\frac{2|\nu|-1}{4}\right\rfloor}\frac{(-1)^j\left(|\nu|+2j-\frac{1}{2}\right)!\,(2z)^{-2j}}{(2j)!\left(|\nu|-2j-\frac{1}{2}\right)!} + \cos\left(\frac{\pi}{2}\left(\nu+\frac{1}{2}\right)+z\right)\sum_{j=0}^{\left\lfloor\frac{2|\nu|-3}{4}\right\rfloor}\frac{(-1)^j\left(|\nu|+2j+\frac{1}{2}\right)!\,(2z)^{-2j-1}}{(2j+1)!\left(|\nu|-2j-\frac{3}{2}\right)!}\right) \;/;\; \nu-\frac{1}{2}\in\mathbb{Z}$$

$$K_\nu(z) = \sqrt{\frac{\pi}{2}} \frac{e^{-z}}{\sqrt{z}} \sum_{j=0}^{\left\lfloor |\nu| - \frac{1}{2} \right\rfloor} \frac{\left(j + |\nu| - \frac{1}{2}\right)!}{j! \left(-j + |\nu| - \frac{1}{2}\right)!} (2z)^{-j} \;/;\; \nu - \frac{1}{2} \in \mathbb{Z}.$$

It can be shown that for other values of the parameters ν, the Bessel functions cannot be represented through elementary functions. But for values ν equal to $\pm \frac{1}{3}, \pm \frac{4}{3}, \pm \frac{7}{3}, \ldots$, and $\pm \frac{2}{3}, \pm \frac{5}{3}, \pm \frac{8}{3}, \ldots$, all Bessel functions can be converted into other known special functions, the Airy functions and their derivatives, for example:

$$J_{-\frac{1}{3}}(z) = \frac{1}{2^{2/3} \sqrt[3]{3} \sqrt[3]{z}} \left(3 \operatorname{Ai}\left(-\left(\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right)\right) + \sqrt{3} \operatorname{Bi}\left(-\left(\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right)\right)\right)$$

$$J_{\frac{1}{3}}(z) = \frac{1}{2^{2/3} \sqrt[3]{3} \sqrt[3]{z}} \left(3 \operatorname{Ai}\left(-\left(\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right)\right) - \sqrt{3} \operatorname{Bi}\left(-\left(\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right)\right)\right)$$

$$I_{-\frac{1}{3}}(z) = \frac{1}{2^{2/3} \sqrt[3]{3} \sqrt[3]{z}} \left(3 \operatorname{Ai}\left(\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right) + \sqrt{3} \operatorname{Bi}\left(\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right)\right)$$

$$I_{\frac{1}{3}}(z) = \frac{1}{2^{2/3} \sqrt[3]{3} \sqrt[3]{z}} \left(\sqrt{3} \operatorname{Bi}\left(\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right) - 3 \operatorname{Ai}\left(\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right)\right)$$

$$Y_{-\frac{1}{3}}(z) = \frac{1}{2^{2/3} \sqrt[3]{3} \sqrt[3]{z}} \left(\sqrt{3} \operatorname{Ai}\left(-\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right) - 3 \operatorname{Bi}\left(-\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right)\right)$$

$$Y_{\frac{1}{3}}(z) = -\frac{1}{2^{2/3} \sqrt[3]{3} \sqrt[3]{z}} \left(\sqrt{3} \operatorname{Ai}\left(-\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right) + 3 \operatorname{Bi}\left(-\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right)\right)$$

$$K_{\pm\frac{1}{3}}(z) = \frac{\sqrt[3]{2} \sqrt[6]{3} \pi}{\sqrt[3]{z}} \operatorname{Ai}\left(\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right).$$

Analyticity

All four Bessel functions $J_\nu(z)$, $I_\nu(z)$, $K_\nu(z)$, and $Y_\nu(z)$ are defined for all complex values of the parameter ν and variable z, and they are analytical functions of ν and z over the whole complex ν- and z-planes.

Poles and essential singularities

For fixed ν, the functions $J_\nu(z)$, $I_\nu(z)$, $K_\nu(z)$, and $Y_\nu(z)$ have an essential singularity at $z = \tilde{\infty}$. At the same time, the point $z = \tilde{\infty}$ is a branch point (except in the case of integer ν for the two functions $J_\nu(z)$ and $I_\nu(z)$).

For fixed integer ν, the functions $J_\nu(z)$ and $I_\nu(z)$ are entire functions of z.

For fixed z, the functions $J_\nu(z)$, $I_\nu(z)$, $K_\nu(z)$, and $Y_\nu(z)$ are entire functions of ν and have only one essential singular point at $\nu = \tilde{\infty}$.

Branch points and branch cuts

For fixed noninteger ν, the functions $J_\nu(z)$, $I_\nu(z)$, $K_\nu(z)$, and $Y_\nu(z)$ have two branch points: $z == 0$, $z == \tilde{\infty}$, and one straight line branch cut between them.

For fixed integer ν, only the functions $Y_\nu(z)$ and $K_\nu(z)$ have two branch points: $z == 0$, $z == \tilde{\infty}$, and one straight line branch cut between them.

For cases where the functions $J_\nu(z)$, $I_\nu(z)$, $K_\nu(z)$, and $Y_\nu(z)$ have branch cuts, the branch cuts are single-valued functions on the z-plane cut along the interval $(-\infty, 0)$, where they are continuous from above:

$$\lim_{\epsilon\to+0} J_\nu(x + i\,\epsilon) == J_\nu(x) \;/;\; x < 0$$

$$\lim_{\epsilon\to+0} I_\nu(x + i\,\epsilon) == I_\nu(x) \;/;\; x < 0$$

$$\lim_{\epsilon\to+0} Y_\nu(x + i\,\epsilon) == Y_\nu(x) \;/;\; x < 0$$

$$\lim_{\epsilon\to+0} K_\nu(x + i\,\epsilon) == K_\nu(x) \;/;\; x < 0.$$

These functions have discontinuities that are described by the following formulas:

$$\lim_{\epsilon\to+0} J_\nu(x - i\,\epsilon) == e^{-2 i \pi \nu} J_\nu(x) \;/;\; x < 0$$

$$\lim_{\epsilon\to+0} I_\nu(x - i\,\epsilon) == e^{-2 i \pi \nu} I_\nu(x) \;/;\; x < 0$$

$$\lim_{\epsilon\to+0} Y_\nu(x - i\,\epsilon) == \cot(\nu\,\pi)\, e^{-2 \pi i \nu} J_\nu(x) - \csc(\nu\,\pi)\, e^{2 \pi i \nu} J_{-\nu}(x) \;/;\; \nu \notin \mathbb{Z} \bigwedge x < 0$$

$$\lim_{\epsilon\to+0} Y_\nu(x - i\,\epsilon) == Y_\nu(x) - 4\, i\, J_\nu(x) \;/;\; \nu \in \mathbb{Z} \land x < 0$$

$$\lim_{\epsilon\to+0} K_\nu(x - i\,\epsilon) == \frac{\pi \csc(\nu\,\pi)}{2} \left(e^{2 \pi i \nu} I_{-\nu}(x) - e^{-2 \pi i \nu} I_\nu(x)\right) /;\; \nu \notin \mathbb{Z} \bigwedge x < 0$$

$$\lim_{\epsilon\to+0} K_\nu(x - i\,\epsilon) == (-1)^\nu\, 2\, i\, \pi\, I_\nu(x) + K_\nu(x) \;/;\; \nu \in \mathbb{Z} \land x < 0.$$

Periodicity

All Bessel functions $J_\nu(z)$, $I_\nu(z)$, $K_\nu(z)$, and $Y_\nu(z)$ do not have periodicity.

Parity and symmetry

All Bessel functions $J_\nu(z)$, $I_\nu(z)$, $K_\nu(z)$, and $Y_\nu(z)$ have mirror symmetry (ignoring the interval (-∞, 0)):

$$J_{\bar{\nu}}(\bar{z}) == \overline{J_\nu(z)} \;/;\; z \notin (-\infty, 0) \quad Y_{\bar{\nu}}(\bar{z}) == \overline{Y_\nu(z)} \;/;\; z \notin (-\infty, 0)$$

$$I_{\bar{\nu}}(\bar{z}) == \overline{I_\nu(z)} \;/;\; z \notin (-\infty, 0) \quad K_{\bar{\nu}}(\bar{z}) == \overline{K_\nu(z)} \;/;\; z \notin (-\infty, 0)\;.$$

The two Bessel functions of the first kind have special parity (either odd or even) in each variable:

$$J_\nu(-z) == (-z)^\nu z^{-\nu} J_\nu(z) \quad J_{-n}(z) == (-1)^n J_n(z) \;/;\; n \in \mathbb{Z}$$

$$I_\nu(-z) == (-z)^\nu z^{-\nu} I_\nu(z) \quad I_{-n}(z) == I_n(z) \;/;\; n \in \mathbb{Z}.$$

The two Bessel functions of the second kind have special parity (either odd or even) only in their parameter:

$$Y_{-n}(z) = (-1)^n Y_n(z) /; n \in \mathbb{Z}$$

$$K_{-\nu}(z) = K_\nu(z).$$

Series representations

The Bessel functions $J_\nu(z)$, $I_\nu(z)$, $K_\nu(z)$, and $Y_\nu(z)$ have the following series expansions (which converge in the whole complex z-plane):

$$J_\nu(z) \propto \frac{1}{\Gamma(\nu+1)} \left(\frac{z}{2}\right)^\nu \left(1 - \frac{z^2}{4(\nu+1)} + \frac{z^4}{32(\nu+1)(\nu+2)} - \ldots\right) /; (z \to 0)$$

$$J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+\nu+1)\, k!} \left(\frac{z}{2}\right)^{2k+\nu}$$

$$I_\nu(z) \propto \frac{1}{\Gamma(\nu+1)} \left(\frac{z}{2}\right)^\nu \left(1 + \frac{z^2}{4(\nu+1)} + \frac{z^4}{32(\nu+1)(\nu+2)} + \ldots\right) /; (z \to 0)$$

$$I_\nu(z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k+\nu+1)\, k!} \left(\frac{z}{2}\right)^{2k+\nu}$$

$$K_\nu(z) \propto \frac{1}{2} \left(\Gamma(\nu) \left(\frac{z}{2}\right)^{-\nu} \left(1 + \frac{z^2}{4(1-\nu)} + \frac{z^4}{32(1-\nu)(2-\nu)} + \ldots\right) + \Gamma(-\nu) \left(\frac{z}{2}\right)^\nu \left(1 + \frac{z^2}{4(\nu+1)} + \frac{z^4}{32(\nu+1)(\nu+2)} + \ldots\right)\right) /; (z \to 0) \land \nu \notin \mathbb{Z}$$

$$K_\nu(z) = \frac{\pi \csc(\pi\nu)}{2} \left(\sum_{k=0}^{\infty} \frac{1}{\Gamma(k-\nu+1)\, k!} \left(\frac{z}{2}\right)^{2k-\nu} - \sum_{k=0}^{\infty} \frac{1}{\Gamma(k+\nu+1)\, k!} \left(\frac{z}{2}\right)^{2k+\nu}\right) /; \nu \notin \mathbb{Z}$$

$$Y_\nu(z) = \frac{\cos(\pi\nu)\,\Gamma(-\nu)}{\pi} \left(\frac{z}{2}\right)^\nu \left(1 - \frac{z^2}{4(\nu+1)} + \frac{z^4}{32(\nu+1)(\nu+2)} - \ldots\right) - \frac{\Gamma(\nu)}{\pi} \left(\frac{z}{2}\right)^{-\nu} \left(1 - \frac{z^2}{4(1-\nu)} + \frac{z^4}{32(1-\nu)(2-\nu)} - \ldots\right) /; (z \to 0) \land \nu \notin \mathbb{Z}$$

$$Y_\nu(z) = \csc(\pi\nu) \left(\cos(\nu\pi) \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+\nu+1)\, k!} \left(\frac{z}{2}\right)^{2k+\nu} - \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k-\nu+1)\, k!} \left(\frac{z}{2}\right)^{2k-\nu}\right) /; \nu \notin \mathbb{Z}.$$

The last four formulas have restrictions that do not allow their right sides to become indeterminate expressions for integer ν.

In such cases, evaluation of the limit from the right sides leads to much more complicated representations, for example:

$$K_0(z) \propto \left(-\gamma + \frac{1}{4}(1-\gamma) z^2 + \frac{1}{128}(3-2\gamma) z^4 + \ldots\right) - \log\left(\frac{z}{2}\right) \left(1 + \frac{z^2}{4} + \frac{z^4}{64} + \ldots\right) /; (z \to 0)$$

$$K_1(z) \propto \frac{1}{z} + \frac{z}{4} \left(2\gamma - 1 + \frac{1}{8}\left(2\gamma - \frac{5}{2}\right) z^2 + \frac{1}{192}\left(2\gamma - \frac{10}{3}\right) z^4 + \ldots\right) + \frac{z}{2} \log\left(\frac{z}{2}\right) \left(1 + \frac{z^2}{8} + \frac{z^4}{192} + \ldots\right) /; (z \to 0)$$

$$K_n(z) = (-1)^{n-1} \log\left(\frac{z}{2}\right) \left(\frac{z}{2}\right)^n \sum_{k=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2k}}{k!\,(k+n)!} +$$
$$\frac{1}{2} \left(\frac{z}{2}\right)^{-n} \sum_{k=0}^{n-1} \frac{(-1)^k (n-k-1)!}{k!} \left(\frac{z}{2}\right)^{2k} + \frac{(-1)^n}{2} \left(\frac{z}{2}\right)^n \sum_{k=0}^{\infty} \frac{\psi(k+1)+\psi(k+n+1)}{k!\,(k+n)!} \left(\frac{z}{2}\right)^{2k} /; n \in \mathbb{N}$$

$$Y_0(z) = \frac{2}{\pi} \log\left(\frac{z}{2}\right) \left(1 - \frac{z^2}{4} + \frac{z^4}{64} + \ldots\right) - \frac{1}{\pi}\left(-2\gamma + \frac{1}{4}(-2+2\gamma) z^2 + \frac{1}{64}(3-2\gamma) z^4 + \ldots\right) /; (z \to 0)$$

$$Y_1(z) \propto \frac{z}{\pi} \log\left(\frac{z}{2}\right) \left(1 - \frac{z^2}{8} + \frac{z^4}{192} + \ldots\right) - \frac{2}{\pi z} - \frac{z}{2\pi}\left(-2\gamma + 1 + \frac{1}{8}\left(-\frac{5}{2} + 2\gamma\right) z^2 + \frac{1}{192}\left(\frac{10}{3} - 2\gamma\right) z^4 + \ldots\right) /; (z \to 0)$$

$$Y_n(z) = -\frac{1}{\pi} \left(\frac{z}{2}\right)^{-n} \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!} \left(\frac{z}{2}\right)^{2k} +$$
$$\frac{2}{\pi} \log\left(\frac{z}{2}\right) \left(\frac{z}{2}\right)^n \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(k+n)!} \left(\frac{z}{2}\right)^{2k} - \frac{1}{\pi} \left(\frac{z}{2}\right)^n \sum_{k=0}^{\infty} \frac{(-1)^k (\psi(k+1)+\psi(k+n+1))}{k!\,(k+n)!} \left(\frac{z}{2}\right)^{2k} /; n \in \mathbb{N}.$$

Interestingly, closed-form expressions for the truncated version of the Taylor series at the origin can be expressed through the generalized hypergeometric function ${}_1F_2$ and the Meijer G function, for example:

$$J_\nu(z) = F_\infty(z, \nu) /; \left(\left(F_n(z, \nu) = \sum_{k=0}^{n} \frac{(-1)^k \left(\frac{z}{2}\right)^{2k+\nu}}{\Gamma(k+\nu+1)\,k!} = J_\nu(z) + \frac{(-1)^n 2^{-2n-\nu-2} z^{2n+\nu+2}}{\Gamma(n+\nu+2)\,(n+1)!}\, {}_1F_2\left(1; n+2, n+\nu+2; -\frac{z^2}{4}\right)\right) \bigwedge n \in \mathbb{N}\right)$$

$$K_\nu(z) = F_\infty(z, \nu) /;$$
$$\left(\left(F_m(z, \nu) = \frac{1}{2}\left(\Gamma(\nu) \left(\frac{z}{2}\right)^{-\nu} \sum_{k=0}^{m} \frac{\left(\frac{z}{2}\right)^{2k}}{(1-\nu)_k\, k!} + \Gamma(-\nu) \left(\frac{z}{2}\right)^{\nu} \sum_{k=0}^{m} \frac{\left(\frac{z}{2}\right)^{2k}}{(\nu+1)_k\, k!}\right) = K_\nu(z) + \frac{\pi}{\sin(\nu\pi)\,(m+1)!}\left(\frac{2^{-2m-\nu-3} z^{2m+\nu+2}}{\Gamma(m+\nu+2)}\right.\right.\right.$$
$$\left.\left.\left. {}_1F_2\left(1; m+2, m+\nu+2; \frac{z^2}{4}\right) - \frac{2^{-2m+\nu-3} z^{2m-\nu+2}}{\Gamma(m-\nu+2)}\, {}_1F_2\left(1; m+2, m-\nu+2; \frac{z^2}{4}\right)\right)\right) \bigwedge m \in \mathbb{N}\right) /; \nu \notin \mathbb{Z}$$

$$K_n(z) = F_\infty(z, n) /;$$
$$\left(\left(F_m(z, n) = \frac{(-1)^n}{2} \left(\frac{z}{2}\right)^n \sum_{k=0}^{m} \frac{\psi(k+1)+\psi(k+n+1)}{k!\,(k+n)!} \left(\frac{z}{2}\right)^{2k} + (-1)^{n-1} \log\left(\frac{z}{2}\right) I_n(z) + \frac{1}{2} \sum_{k=0}^{n-1} \frac{(-1)^k (-k+n-1)!}{k!} \left(\frac{z}{2}\right)^{2k-n} =\right.\right.$$
$$z^n \left(z^2\right)^{-\frac{n}{2}} K_n\left(\sqrt{z^2}\right) - \log\left(\frac{z^2}{4}\right) \frac{(-1)^n 2^{-2m-n-3} z^{2(m+1)+n}}{(m+1)!\,(m+n+1)!}\, {}_1F_2\left(1; m+2, m+n+2; \frac{z^2}{4}\right) +$$
$$\left.\left.(-1)^{n-1} I_n(z) \log\left(\frac{z}{2}\right) + \frac{1}{2}(-1)^n \log\left(\frac{z^2}{4}\right) I_n(z) - \frac{(-1)^n}{2} \left(\frac{z}{2}\right)^n G_{2,4}^{2,2}\left(\frac{z^2}{4} \,\middle|\, \begin{matrix} m+1, m+1 \\ m+1, m+1, 0, -n \end{matrix}\right)\right) \bigwedge n \in \mathbb{N}\right).$$

Asymptotic series expansions

The asymptotic behavior of the Bessel functions $J_\nu(z)$, $I_\nu(z)$, $K_\nu(z)$, and $Y_\nu(z)$ can be described by the following formulas (which show only the main terms):

$$J_\nu(z) \propto \frac{\sqrt{2}}{\sqrt{\pi}} z^\nu \left(z^2\right)^{-\frac{2\nu+1}{4}} \left(\cos\left(\sqrt{z^2} - \frac{\pi(2\nu+1)}{4}\right)\left(1+O\left(\frac{1}{z^2}\right)\right) + \frac{1-4\nu^2}{8\sqrt{z^2}} \sin\left(\sqrt{z^2} - \frac{\pi(2\nu+1)}{4}\right)\left(1+O\left(\frac{1}{z^2}\right)\right)\right) /; (|z| \to \infty)$$

$$I_\nu(z) \propto \frac{1}{\sqrt{2\pi}} z^\nu \left(-z^2\right)^{-\frac{2\nu+1}{4}} \left(\exp\left(-i\left(\frac{(2\nu+1)\pi}{4} - \sqrt{-z^2}\right)\right)\left(1+O\left(\frac{1}{z}\right)\right) + \exp\left(i\left(\frac{(2\nu+1)\pi}{4} - \sqrt{-z^2}\right)\right)\left(1+O\left(\frac{1}{z}\right)\right)\right) /; (|z| \to \infty)$$

$$K_\nu(z) \propto \sqrt{\frac{\pi}{2}} \frac{e^{-z}}{\sqrt{z}} \left(1+O\left(\frac{1}{z}\right)\right) /; (|z| \to \infty)$$

$$Y_\nu(z) \propto \sqrt{\frac{2}{\pi}} z^{-\nu} \left(z^2\right)^{-\frac{2\nu+1}{4}} \csc(\pi\nu) \left(\left(z^{2\nu} \cos(\pi\nu) \cos\left(\sqrt{z^2} - \frac{(1+2\nu)\pi}{4}\right) - \left(z^2\right)^\nu \cos\left(\sqrt{z^2} - \frac{(1-2\nu)\pi}{4}\right)\right)\left(1+O\left(\frac{1}{z^2}\right)\right) - \frac{4\nu^2-1}{8\sqrt{z^2}} \left(z^{2\nu}\cos(\pi\nu)\sin\left(\sqrt{z^2} - \frac{(1+2\nu)\pi}{4}\right) - \left(z^2\right)^\nu \sin\left(\sqrt{z^2} - \frac{(1-2\nu)\pi}{4}\right)\right)\left(1+O\left(\frac{1}{z^2}\right)\right)\right) /; (|z| \to \infty).$$

The previous formulas are valid for any direction approaching the point z to infinity ($|z| \to \infty$). In particular cases, when $|\mathrm{Arg}(z)| < \frac{\pi}{2}$ or $|\mathrm{Arg}(z)| < \pi$, the second and fourth formulas can be simplified to the following forms:

$$I_\nu(z) \propto \frac{e^z}{\sqrt{2\pi z}} \left(1+O\left(\frac{1}{z}\right)\right) /; |\mathrm{Arg}(z)| < \frac{\pi}{2} \bigwedge (|z| \to \infty)$$

$$Y_\nu(z) \propto \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{z}} \left(\frac{4\nu^2-1}{8z} \cos\left(z - \frac{(1+2\nu)\pi}{4}\right)\left(1+O\left(\frac{1}{z^2}\right)\right) + \sin\left(z - \frac{(1+2\nu)\pi}{4}\right)\left(1+O\left(\frac{1}{z^2}\right)\right)\right) /; |\mathrm{Arg}(z)| < \pi \bigwedge (|z| \to \infty).$$

Integral representations

The Bessel functions $J_\nu(z)$, $I_\nu(z)$, $K_\nu(z)$, and $Y_\nu(z)$ have simple integral representations through the cosine (or the hyperbolic cosine or exponential function) and power functions in the integrand:

$$J_\nu(z) = \frac{2^{1-\nu} z^\nu}{\sqrt{\pi}\, \Gamma\left(\nu+\frac{1}{2}\right)} \int_0^1 \left(1-t^2\right)^{\nu-\frac{1}{2}} \cos(z\,t)\, dt /; \mathrm{Re}(\nu) > -\frac{1}{2}$$

$$I_\nu(z) = \frac{2^{1-\nu} z^\nu}{\sqrt{\pi}\, \Gamma\left(\nu+\frac{1}{2}\right)} \int_0^1 \left(1-t^2\right)^{\nu-\frac{1}{2}} \cosh(z\,t)\, dt /; \mathrm{Re}(\nu) > -\frac{1}{2}$$

$$K_\nu(z) = \frac{\sqrt{\pi}\, z^\nu}{2^\nu\, \Gamma\left(\nu+\frac{1}{2}\right)} \int_1^\infty e^{-z\,t} \left(t^2-1\right)^{\nu-\frac{1}{2}} dt /; \mathrm{Re}(\nu) > -\frac{1}{2} \bigwedge \mathrm{Re}(z) > 0$$

$$Y_\nu(z) = -\frac{2^{\nu+1} z^{-\nu}}{\sqrt{\pi}\, \Gamma\left(\frac{1}{2}-\nu\right)} \int_1^\infty \left(t^2-1\right)^{-\nu-\frac{1}{2}} \cos(z\,t)\, dt /; |\mathrm{Re}(\nu)| < \frac{1}{2} \bigwedge z > 0.$$

Transformations

The argument of the Bessel functions $J_\nu(z)$, $I_\nu(z)$, $K_\nu(z)$, and $Y_\nu(z)$ sometimes can be simplified through formulas that remove square roots from the arguments. For the Bessel functions of the second kind $K_\nu(z)$ and $Y_\nu(z)$ with integer index ν, this operation is realized by special formulas that include logarithms:

$$J_\nu\left(\sqrt{z^2}\right) = z^{-\nu} \left(z^2\right)^{\nu/2} J_\nu(z)$$

$$I_\nu\left(\sqrt{z^2}\right) = z^{-\nu} \left(z^2\right)^{\nu/2} I_\nu(z)$$

$$K_\nu\left(\sqrt{z^2}\right) = z^\nu \left(z^2\right)^{-\frac{\nu}{2}} K_\nu(z) - \frac{\pi \csc(\pi \nu)}{2} \left(z^{-\nu} \left(z^2\right)^{\nu/2} - z^\nu \left(z^2\right)^{-\frac{\nu}{2}}\right) I_\nu(z) \;/;\; \nu \notin \mathbb{Z}$$

$$K_\nu\left(\sqrt{z^2}\right) = \left(\frac{\sqrt{z^2}}{z}\right)^\nu \left(K_\nu(z) - (-1)^\nu \left(\log\left(\sqrt{z^2}\right) - \log(z)\right) I_\nu(z)\right) /;\; \nu \in \mathbb{Z}$$

$$Y_\nu\left(\sqrt{z^2}\right) = z^\nu \left(z^2\right)^{-\frac{\nu}{2}} Y_\nu(z) + \cot(\pi \nu) \left(z^{-\nu} \left(z^2\right)^{\nu/2} - z^\nu \left(z^2\right)^{-\frac{\nu}{2}}\right) J_\nu(z) \;/;\; \nu \notin \mathbb{Z}$$

$$Y_\nu\left(\sqrt{z^2}\right) = \left(\frac{\sqrt{z^2}}{z}\right)^\nu \left(Y_\nu(z) + \frac{2 \left(\log\left(\sqrt{z^2}\right) - \log(z)\right)}{\pi} J_\nu(z)\right) /;\; \nu \in \mathbb{Z}.$$

If the argument of a Bessel function includes an explicit minus sign, the following formulas produce Bessel functions without the minus sign argument:

$$J_\nu(-z) = (-z)^\nu z^{-\nu} J_\nu(z)$$

$$I_\nu(-z) = (-z)^\nu z^{-\nu} I_\nu(z)$$

$$K_\nu(-z) = z^\nu K_\nu(z) (-z)^{-\nu} + \frac{\pi}{2} ((-z)^{-\nu} z^\nu - (-z)^\nu z^{-\nu}) I_\nu(z) \csc(\pi \nu) \;/;\; \nu \notin \mathbb{Z}$$

$$K_\nu(-z) = (-1)^\nu K_\nu(z) + (\log(z) - \log(-z)) I_\nu(z) \;/;\; \nu \in \mathbb{Z}$$

$$Y_\nu(-z) = z^\nu Y_\nu(z) (-z)^{-\nu} + ((-z)^\nu z^{-\nu} - (-z)^{-\nu} z^\nu) J_\nu(z) \cot(\pi \nu) \;/;\; \nu \notin \mathbb{Z}$$

$$Y_\nu(-z) = (-1)^\nu \left(Y_\nu(z) - \frac{2}{\pi} (\log(z) - \log(-z)) J_\nu(z)\right) /;\; \nu \in \mathbb{Z}.$$

If the arguments of the Bessel functions include sums, the following formulas hold:

$$J_\nu(z_1 + z_2) = \sum_{k=-\infty}^{\infty} J_{\nu-k}(z_1) J_k(z_2) \;/;\; \left|\frac{z_2}{z_1}\right| < 1 \bigvee \nu \in \mathbb{Z}$$

$$I_\nu(z_1 + z_2) = \sum_{k=-\infty}^{\infty} I_{\nu-k}(z_1) I_k(z_2) \;/;\; \left|\frac{z_2}{z_1}\right| < 1 \bigvee \nu \in \mathbb{Z}$$

$$K_\nu(z_1 + z_2) = \sum_{k=-\infty}^{\infty} (-1)^k K_{\nu-k}(z_1) I_k(z_2) \;/;\; \left|\frac{z_2}{z_1}\right| < 1$$

$$Y_\nu(z_1 + z_2) = \sum_{k=-\infty}^{\infty} Y_{\nu-k}(z_1)\, J_k(z_2) \;/;\; \left|\frac{z_2}{z_1}\right| < 1.$$

If arguments of the Bessel functions include products, the following formulas hold:

$$J_\nu(z_1\, z_2) = z_1^\nu \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(z_1^2 - 1\right)^k J_{k+\nu}(z_2) \left(\frac{z_2}{2}\right)^k$$

$$I_\nu(z_1\, z_2) = z_1^\nu \sum_{k=0}^{\infty} \frac{\left(z_1^2 - 1\right)^k}{k!} I_{k+\nu}(z_2) \left(\frac{z_2}{2}\right)^k$$

$$K_\nu(z_1\, z_2) = z_1^\nu \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(z_1^2 - 1\right)^k K_{k+\nu}(z_2) \left(\frac{z_2}{2}\right)^k \;/;\; \left|z_1^2 - 1\right| < 1$$

$$Y_\nu(z_1\, z_2) = z_1^\nu \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(z_1^2 - 1\right)^k Y_{k+\nu}(z_2) \left(\frac{z_2}{2}\right)^k \;/;\; \left|z_1^2 - 1\right| < 1.$$

Identities

The Bessel functions $J_\nu(z)$, $I_\nu(z)$, $K_\nu(z)$, and $Y_\nu(z)$ satisfy the following recurrence identities:

$$J_\nu(z) = \frac{2(\nu+1)}{z} J_{\nu+1}(z) - J_{\nu+2}(z)$$

$$J_\nu(z) = \frac{2(\nu-1)}{z} J_{\nu-1}(z) - J_{\nu-2}(z)$$

$$I_\nu(z) = \frac{2(\nu+1)}{z} I_{\nu+1}(z) + I_{\nu+2}(z)$$

$$I_\nu(z) = I_{\nu-2}(z) - \frac{2(\nu-1)}{z} I_{\nu-1}(z)$$

$$K_\nu(z) = K_{\nu+2}(z) - \frac{2(\nu+1)}{z} K_{\nu+1}(z)$$

$$K_\nu(z) = K_{\nu-2}(z) + \frac{2(\nu-1)}{z} K_{\nu-1}(z)$$

$$Y_\nu(z) = \frac{2(\nu+1)}{z} Y_{\nu+1}(z) - Y_{\nu+2}(z)$$

$$Y_\nu(z) = \frac{2(\nu-1)}{z} Y_{\nu-1}(z) - Y_{\nu-2}(z).$$

The last eight identities can be generalized to the following recurrence identities with jump length n:

$$J_\nu(z) = C_n(\nu, z)\, J_{\nu+n}(z) - C_{n-1}(\nu, z)\, J_{\nu+n+1}(z) \;/;$$
$$C_0(\nu, z) = 1 \bigwedge C_1(\nu, z) = \frac{2(\nu+1)}{z} \bigwedge C_n(\nu, z) = \frac{2(n+\nu)}{z} C_{n-1}(\nu, z) - C_{n-2}(\nu, z) \bigwedge n \in \mathbb{N}^+$$

$$J_\nu(z) = C_n(\nu, z)\, J_{\nu-n}(z) - C_{n-1}(\nu, z)\, J_{\nu-n-1}(z) \;\; /;$$
$$C_0(\nu, z) = 1 \bigwedge C_1(\nu, z) = \frac{2(\nu-1)}{z} \bigwedge C_n(\nu, z) = \frac{2(-n+\nu)}{z} C_{n-1}(\nu, z) - C_{n-2}(\nu, z) \bigwedge n \in \mathbb{N}^+$$

$$I_\nu(z) = C_n(\nu, z)\, I_{\nu+n}(z) + C_{n-1}(\nu, z)\, I_{\nu+n+1}(z) \;\; /;$$
$$C_0(\nu, z) = 1 \bigwedge C_1(\nu, z) = \frac{2(\nu+1)}{z} \bigwedge C_n(\nu, z) = \frac{2(n+\nu)}{z} C_{n-1}(\nu, z) + C_{n-2}(\nu, z) \bigwedge n \in \mathbb{N}^+$$

$$I_\nu(z) = C_n(\nu, z)\, I_{\nu-n}(z) + C_{n-1}(\nu, z)\, I_{\nu-n-1}(z) \;\; /;$$
$$C_0(\nu, z) = 1 \bigwedge C_1(\nu, z) = -\frac{2(\nu-1)}{z} \bigwedge C_n(\nu, z) = -\frac{2(-n+\nu)}{z} C_{n-1}(\nu, z) + C_{n-2}(\nu, z) \bigwedge n \in \mathbb{N}^+$$

$$K_\nu(z) = C_n(\nu, z)\, K_{\nu+n}(z) + C_{n-1}(\nu, z)\, K_{\nu+n+1}(z) \;\; /;$$
$$C_0(\nu, z) = 1 \bigwedge C_1(\nu, z) = -\frac{2(\nu+1)}{z} \bigwedge C_n(\nu, z) = -\frac{2(n+\nu)}{z} C_{n-1}(\nu, z) + C_{n-2}(\nu, z) \bigwedge n \in \mathbb{N}^+$$

$$K_\nu(z) = C_n(\nu, z)\, K_{\nu-n}(z) + C_{n-1}(\nu, z)\, K_{\nu-n-1}(z) \;\; /;$$
$$C_0(\nu, z) = 1 \bigwedge C_1(\nu, z) = \frac{2(\nu-1)}{z} \bigwedge C_n(\nu, z) = \frac{2(-n+\nu)}{z} C_{n-1}(\nu, z) + C_{n-2}(\nu, z) \bigwedge n \in \mathbb{N}^+$$

$$Y_\nu(z) = C_n(\nu, z)\, Y_{\nu+n}(z) - C_{n-1}(\nu, z)\, Y_{\nu+n+1}(z) \; /;$$
$$C_0(\nu, z) = 1 \bigwedge C_1(\nu, z) = \frac{2(\nu+1)}{z} \bigwedge C_n(\nu, z) = \frac{2(n+\nu)}{z} C_{n-1}(\nu, z) - C_{n-2}(\nu, z) \bigwedge n \in \mathbb{N}^+$$

$$Y_\nu(z) = C_n(\nu, z)\, Y_{\nu-n}(z) - C_{n-1}(\nu, z)\, Y_{\nu-n-1}(z) \;\; /;$$
$$C_0(\nu, z) = 1 \bigwedge C_1(\nu, z) = \frac{2(\nu-1)}{z} \bigwedge C_n(\nu, z) = \frac{2(-n+\nu)}{z} C_{n-1}(\nu, z) - C_{n-2}(\nu, z) \bigwedge n \in \mathbb{N}^+.$$

Simple representations of derivatives

The derivatives of all the four Bessel functions $J_\nu(z)$, $I_\nu(z)$, $K_\nu(z)$, and $Y_\nu(z)$ have rather simple and symmetrical representations that can be expressed through other Bessel functions with different indices:

$$\frac{\partial J_\nu(z)}{\partial z} = \frac{1}{2}\left(J_{\nu-1}(z) - J_{\nu+1}(z)\right)$$

$$\frac{\partial I_\nu(z)}{\partial z} = \frac{1}{2}\left(I_{\nu-1}(z) + I_{\nu+1}(z)\right)$$

$$\frac{\partial Y_\nu(z)}{\partial z} = \frac{1}{2}\left(Y_{\nu-1}(z) - Y_{\nu+1}(z)\right)$$

$$\frac{\partial K_\nu(z)}{\partial z} = -\frac{1}{2}\left(K_{\nu-1}(z) + K_{\nu+1}(z)\right).$$

But these derivatives can be represented in other forms, for example:

$$\frac{\partial J_\nu(z)}{\partial z} = J_{\nu-1}(z) - \frac{\nu}{z} J_\nu(z) = \frac{\nu}{z} J_\nu(z) - J_{\nu+1}(z)$$

$$\frac{\partial I_\nu(z)}{\partial z} = I_{\nu-1}(z) - \frac{\nu}{z} I_\nu(z) = \frac{\nu}{z} I_\nu(z) + I_{\nu+1}(z)$$

$$\frac{\partial Y_\nu(z)}{\partial z} == Y_{\nu-1}(z) - \frac{\nu}{z} Y_\nu(z) == \frac{\nu}{z} Y_\nu(z) - Y_{\nu+1}(z)$$

$$\frac{\partial K_\nu(z)}{\partial z} == -K_{\nu-1}(z) - \frac{\nu}{z} K_\nu(z) == \frac{\nu}{z} K_\nu(z) - K_{\nu+1}(z).$$

The symbolic n^{th}-order derivatives have more complicated representations through the regularized hypergeometric function ${}_2\tilde{F}_3$ or generalized Meijer G function:

$$\frac{\partial^n J_\nu(z)}{\partial z^n} == 2^{n-2\nu} \sqrt{\pi}\; z^{\nu-n}\, \Gamma(\nu+1)\, {}_2\tilde{F}_3\!\left(\frac{\nu+1}{2}, \frac{\nu+2}{2}; \frac{\nu-n+1}{2}, \frac{\nu-n+2}{2}, \nu+1; -\frac{z^2}{4}\right) /; n \in \mathbb{N}$$

$$\frac{\partial^n I_\nu(z)}{\partial z^n} == 2^{n-2\nu} \sqrt{\pi}\; z^{\nu-n}\, \Gamma(\nu+1)\, {}_2\tilde{F}_3\!\left(\frac{\nu+1}{2}, \frac{\nu+2}{2}; \frac{1-n+\nu}{2}, \frac{2-n+\nu}{2}, \nu+1; \frac{z^2}{4}\right) /; n \in \mathbb{N}$$

$$\frac{\partial^n K_\nu(z)}{\partial z^n} == \frac{1}{2} G_{2,4}^{2,2}\!\left(\frac{z}{2}, \frac{1}{2} \,\middle|\, \begin{array}{c} \frac{1-n}{2}, -\frac{n}{2} \\ \frac{\nu-n}{2}, -\frac{\nu+n}{2}, \frac{1}{2}, 0 \end{array}\right) /; n \in \mathbb{N}$$

$$\frac{\partial^n Y_\nu(z)}{\partial z^n} == G_{3,5}^{2,2}\!\left(\frac{z}{2}, \frac{1}{2} \,\middle|\, \begin{array}{c} \frac{1-n}{2}, -\frac{n}{2}, -\frac{n+\nu+1}{2} \\ \frac{\nu-n}{2}, -\frac{n+\nu}{2}, \frac{1}{2}, 0, -\frac{n+\nu+1}{2} \end{array}\right) /; n \in \mathbb{N}.$$

Differential equations

The Bessel functions $J_\nu(z)$, $Y_\nu(z)$, $I_\nu(z)$, and $K_\nu(z)$ appeared as special solutions of two linear second-order differential equations (the so-called Bessel equation):

$$z^2 w''(z) + z\, w'(z) + \left(z^2 - \nu^2\right) w(z) == 0 /; w(z) == c_1 J_\nu(z) + c_2 Y_\nu(z)$$

$$z^2 w''(z) + z\, w'(z) - \left(z^2 + \nu^2\right) w(z) == 0 /; w(z) == c_1 I_\nu(z) + c_2 K_\nu(z),$$

where c_1 and c_2 are arbitrary constants.

Zeros

When ν is real, the functions $J_\nu(z)$ and $\left(\frac{\partial J_\nu(z)}{\partial z}\right)$ each have an infinite number of real zeros, all of which are simple with the possible exception of the zero $z = 0$:

$$J_\nu(z) == 0 /; z == z_k \bigwedge k \in \mathbb{N} \bigwedge \nu \in \mathbb{R} \bigwedge \operatorname{Re}(z_k) == z_k$$

$$\frac{\partial J_\nu(z)}{\partial z} == 0 /; z == z_k \bigwedge k \in \mathbb{N} \bigwedge \nu \in \mathbb{R}.$$

When $\nu \geq -1$, the zeros of $J_\nu(z)$ are all real. If $\nu < -1$ and ν is not an integer, the number of complex zeros of $J_\nu(z)$ is $2\lfloor -\nu \rfloor$; if $\lfloor -\nu \rfloor$ is odd, two of these zeros lie on the imaginary axis.

If $\nu > 0$, all zeros of $\left(\frac{\partial J_\nu(z)}{\partial z}\right)$ are real.

The function $K_\nu(z)$ has no zeros in the region $|\operatorname{Arg}(z)| \leq \frac{\pi}{2}$ for any real ν.

When ν is real, the functions $Y_\nu(z)$ and $\left(\frac{\partial Y_\nu(z)}{\partial z}\right)$ each have an infinite number of real zeros, all of which are simple with the possible exception of the zero $z = 0$:

$$Y_\nu(z) \equiv 0 \;/;\; z \equiv z_k \bigwedge k \in \mathbb{N} \bigwedge \nu \in \mathbb{R} \bigwedge \operatorname{Re}(z_k) \equiv z_k$$

$$\frac{\partial Y_\nu(z)}{\partial z} \equiv 0 \;/;\; z \equiv z_k \bigwedge k \in \mathbb{N} \bigwedge \nu \in \mathbb{R}.$$

Applications of Bessel functions

Applications of Bessel functions include mechanics, electrodynamics, electroengineering, solid state physics, and celestial mechanics.

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