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Introductions to Cosh

Introduction to the hyperbolic functions

General

The six well-known hyperbolic functions are the hyperbolic sine $\sinh(z)$, hyperbolic cosine $\cosh(z)$, hyperbolic tangent $\tanh(z)$, hyperbolic cotangent $\coth(z)$, hyperbolic cosecant $\operatorname{csch}(z)$, and hyperbolic secant $\operatorname{sech}(z)$. They are among the most used elementary functions. The hyperbolic functions share many common properties and they have many properties and formulas that are similar to those of the trigonometric functions.

Definitions of the hyperbolic functions

All hyperbolic functions can be defined as simple rational functions of the exponential function of z:

$$\sinh(z) = \frac{e^z - e^{-z}}{2}$$

$$\cosh(z) = \frac{e^z + e^{-z}}{2}$$

$$\tanh(z) = \frac{e^z - e^{-z}}{e^z + e^{-z}}$$

$$\coth(z) = \frac{e^z + e^{-z}}{e^z - e^{-z}}$$

$$\operatorname{csch}(z) = \frac{2}{e^z - e^{-z}}$$

$$\operatorname{sech}(z) = \frac{2}{e^z + e^{-z}}.$$

The functions $\tanh(z)$, $\coth(z)$, $\operatorname{csch}(z)$, and $\operatorname{sech}(z)$ can also be defined through the functions $\sinh(z)$ and $\cosh(z)$ using the following formulas:

$$\tanh(z) = \frac{\sinh(z)}{\cosh(z)}$$

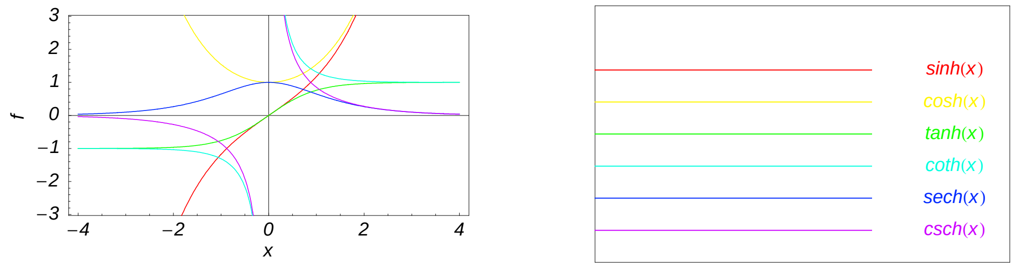
$$\coth(z) = \frac{\cosh(z)}{\sinh(z)}$$

$$\operatorname{csch}(z) = \frac{1}{\sinh(z)}$$

$$\operatorname{sech}(z) = \frac{1}{\cosh(z)}.$$

A quick look at the hyperbolic functions

Here is a quick look at the graphics of the six hyperbolic functions along the real axis.



Connections within the group of hyperbolic functions and with other function groups

Representations through more general functions

The hyperbolic functions are particular cases of more general functions. Among these more general functions, four classes of special functions are of special relevance: Bessel, Jacobi, Mathieu, and hypergeometric functions.

For example, $\sinh(z)$ and $\cosh(z)$ have the following representations through Bessel, Mathieu, and hypergeometric functions:

$$\sinh(z) = -i\sqrt{\frac{\pi i z}{2}}\, J_{1/2}(i z) \quad \sinh(z) = \sqrt{\frac{\pi z}{2}}\, I_{1/2}(z) \quad \sinh(z) = -i\sqrt{\frac{\pi i z}{2}}\, Y_{-1/2}(i z) \quad \sinh(z) = \frac{1}{\sqrt{2\pi}}\left(\sqrt{-z}\, K_{1/2}(-z) - \sqrt{z}\right.$$

$$\cosh(z) = \sqrt{\frac{\pi i z}{2}}\, J_{-1/2}(i z) \quad \cosh(z) = \sqrt{\frac{\pi z}{2}}\, I_{-1/2}(z) \quad \cosh(z) = -\sqrt{\frac{\pi i z}{2}}\, Y_{1/2}(i z) \quad \cosh(z) = \frac{1}{\sqrt{2\pi}}\left(\sqrt{-z}\, K_{1/2}(-z) + \sqrt{z}\right.$$

$$\sinh(z) = -i\,\mathrm{Se}(1, 0, i z) \quad \cosh(z) = \mathrm{Ce}(1, 0, i z)$$

$$\sinh(z) = z\, {}_0F_1\left(; \frac{3}{2}; \frac{z^2}{4}\right) \quad \cosh(z) = {}_0F_1\left(; \frac{1}{2}; \frac{z^2}{4}\right).$$

All hyperbolic functions can be represented as degenerate cases of the corresponding doubly periodic Jacobi elliptic functions when their second parameter is equal to 0 or 1:

$$\sinh(z) = -i\,\mathrm{sd}(i z \mid 0) = -i\,\mathrm{sn}(i z \mid 0) \quad \sinh(z) = \mathrm{sc}(z \mid 1) = \mathrm{sd}(z \mid 1)$$
$$\cosh(z) = \mathrm{cd}(i z \mid 0) = \mathrm{cn}(i z \mid 0) \quad \cosh(z) = \mathrm{nc}(z \mid 1) = \mathrm{nd}(z \mid 1)$$
$$\tanh(z) = -i\,\mathrm{sc}(i z \mid 0) \quad \tanh(z) = \mathrm{sn}(z \mid 1)$$
$$\coth(z) = i\,\mathrm{cs}(i z \mid 0) \quad \coth(z) = \mathrm{ns}(z \mid 1)$$
$$\mathrm{csch}(z) = i\,\mathrm{ds}(i z \mid 0) = i\,\mathrm{ns}(i z \mid 0) \quad \mathrm{csch}(z) = \mathrm{cs}(z \mid 1) = \mathrm{ds}(z \mid 1)$$
$$\mathrm{sech}(z) = \mathrm{dc}(i z \mid 0) = \mathrm{nc}(i z \mid 0) \quad \mathrm{sech}(z) = \mathrm{cn}(z \mid 1) = \mathrm{dn}(z \mid 1).$$

Representations through related equivalent functions

Each of the six hyperbolic functions can be represented through the corresponding trigonometric function:

$$\sinh(z) = -i\sin(i z) \quad \sinh(i z) = i\sin(z)$$
$$\cosh(z) = \cos(i z) \quad \cosh(i z) = \cos(z)$$
$$\tanh(z) = -i\tan(i z) \quad \tanh(i z) = i\tan(z)$$
$$\coth(z) = i\cot(i z) \quad \coth(i z) = -i\cot(z)$$
$$\mathrm{csch}(z) = i\csc(i z) \quad \mathrm{csch}(i z) = -i\csc(z)$$
$$\mathrm{sech}(z) = \sec(i z) \quad \mathrm{sech}(i z) = \sec(z).$$

Relations to inverse functions

Each of the six hyperbolic functions is connected with a corresponding inverse hyperbolic function by two formulas. One direction can be expressed through a simple formula, but the other direction is much more complicated because of the multivalued nature of the inverse function:

$$\sinh\left(\sinh^{-1}(z)\right) = z \qquad \sinh^{-1}(\sinh(z)) = z \;/;\; -\frac{\pi}{2} < \operatorname{Im}(z) < \frac{\pi}{2} \bigvee \left(\operatorname{Im}(z) = -\frac{\pi}{2} \bigwedge \operatorname{Re}(z) \le 0\right) \bigvee \left(\operatorname{Im}(z) = \frac{\pi}{2} \bigwedge \operatorname{Re}(z) \ge 0\right)$$

$$\cosh\left(\cosh^{-1}(z)\right) = z \qquad \cosh^{-1}(\cosh(z)) = z \;/;\; \operatorname{Re}(z) > 0 \land -\pi < \operatorname{Im}(z) \le \pi \lor \operatorname{Re}(z) = 0 \land \operatorname{Im}(z) \ge 0$$

$$\tanh\left(\tanh^{-1}(z)\right) = z \qquad \tanh^{-1}(\tanh(z)) = z \;/;\; -\frac{\pi}{2} < \operatorname{Im}(z) < \frac{\pi}{2} \bigvee \left(\operatorname{Im}(z) = -\frac{\pi}{2} \bigwedge \operatorname{Re}(z) > 0\right) \bigvee \left(\operatorname{Im}(z) = \frac{\pi}{2} \bigwedge \operatorname{Re}(z) < 0\right)$$

$$\coth\left(\coth^{-1}(z)\right) = z \qquad \coth^{-1}(\coth(z)) = z \;/;\; -\frac{\pi}{2} < \operatorname{Im}(z) < \frac{\pi}{2} \bigvee \left(\operatorname{Im}(z) = -\frac{\pi}{2} \bigwedge \operatorname{Re}(z) > 0\right) \bigvee \left(\operatorname{Im}(z) = \frac{\pi}{2} \bigwedge \operatorname{Re}(z) \le 0\right)$$

$$\operatorname{csch}\left(\operatorname{csch}^{-1}(z)\right) = z \qquad \operatorname{csch}^{-1}(\operatorname{csch}(z)) = z \;/;\; -\frac{\pi}{2} < \operatorname{Im}(z) < \frac{\pi}{2} \bigvee \left(\operatorname{Im}(z) = -\frac{\pi}{2} \bigwedge \operatorname{Re}(z) \le 0\right) \bigvee \left(\operatorname{Im}(z) = \frac{\pi}{2} \bigwedge \operatorname{Re}(z) \ge 0\right)$$

$$\operatorname{sech}\left(\operatorname{sech}^{-1}(z)\right) = z \qquad \operatorname{sech}^{-1}(\operatorname{sech}(z)) = z \;/;\; -\pi < \operatorname{Im}(z) \le \pi \land \operatorname{Re}(z) > 0 \lor \operatorname{Re}(z) = 0 \land \operatorname{Im}(z) \ge 0.$$

Representations through other hyperbolic functions

Each of the six hyperbolic functions can be represented through any other function as a rational function of that function with a linear argument. For example, the hyperbolic sine can be representative as a group-defining function because the other five functions can be expressed as:

$$\cosh(z) = -i \sinh\left(\frac{\pi i}{2} - z\right) \qquad \cosh^2(z) = 1 + \sinh^2(z)$$

$$\tanh(z) = \frac{\sinh(z)}{\cosh(z)} = \frac{i \sinh(z)}{\sinh\left(\frac{\pi i}{2} - z\right)} \qquad \tanh^2(z) = \frac{\sinh^2(z)}{1 + \sinh^2(z)}$$

$$\coth(z) = \frac{\cosh(z)}{\sinh(z)} = -\frac{i \sinh\left(\frac{\pi i}{2} - z\right)}{\sinh(z)} \qquad \coth^2(z) = \frac{1 + \sinh^2(z)}{\sinh^2(z)}$$

$$\operatorname{csch}(z) = \frac{1}{\sinh(z)} \qquad \operatorname{csch}^2(z) = \frac{1}{\sinh^2(z)}$$

$$\operatorname{sech}(z) = \frac{1}{\cosh(z)} = \frac{i}{\sinh\left(\frac{\pi i}{2} - z\right)} \qquad \operatorname{sech}^2(z) = \frac{1}{1 + \sinh^2(z)}.$$

All six hyperbolic functions can be transformed into any other function of the group of hyperbolic functions if the argument z is replaced by $p \pi i / 2 + q z$ with $q^2 = 1 \bigwedge p \in \mathbb{Z}$:

$$\sinh(-z - 2\pi i) = -\sinh(z) \qquad \sinh(z - 2\pi i) = \sinh(z)$$

$$\sinh\left(-z - \frac{3\pi i}{2}\right) = i \cosh(z) \qquad \sinh\left(z - \frac{3\pi i}{2}\right) = i \cosh(z)$$

$$\sinh(-z - \pi i) = \sinh(z) \qquad \sinh(z - \pi i) = -\sinh(z)$$

$$\sinh\left(-z - \frac{\pi i}{2}\right) = -i \cosh(z) \qquad \sinh\left(z - \frac{\pi i}{2}\right) = -i \cosh(z)$$

$$\sinh\left(z + \frac{\pi i}{2}\right) = i \cosh(z) \qquad \sinh\left(\frac{\pi i}{2} - z\right) = i \cosh(z)$$

$$\sinh(z + \pi i) = -\sinh(z) \qquad \sinh(\pi i - z) = \sinh(z)$$

$$\sinh\left(z + \frac{3\pi i}{2}\right) = -i \cosh(z) \qquad \sinh\left(\frac{3\pi i}{2} - z\right) = -i \cosh(z)$$

$$\sinh(z + 2\pi i) = \sinh(z) \qquad \sinh(2\pi i - z) = -\sinh(z)$$

$$\cosh(-z-2\pi i)==\cosh(z) \qquad \cosh(z-2\pi i)==\cosh(z)$$
$$\cosh\left(-z-\frac{3\pi i}{2}\right)==-i\sinh(z) \qquad \cosh\left(z-\frac{3\pi i}{2}\right)==i\sinh(z)$$
$$\cosh(-z-\pi i)==-\cosh(z) \qquad \cosh(z-\pi i)==-\cosh(z)$$
$$\cosh\left(-z-\frac{\pi i}{2}\right)==i\sinh(z) \qquad \cosh\left(z-\frac{\pi i}{2}\right)==-i\sinh(z)$$
$$\cosh\left(z+\frac{\pi i}{2}\right)==i\sinh(z) \qquad \cosh\left(\frac{\pi i}{2}-z\right)==-i\sinh(z)$$
$$\cosh(z+\pi i)==-\cosh(z) \qquad \cosh(\pi i-z)==-\cosh(z)$$
$$\cosh\left(z+\frac{3\pi i}{2}\right)==-i\sinh(z) \qquad \cosh\left(\frac{3\pi i}{2}-z\right)==i\sinh(z)$$
$$\cosh(z+2\pi i)==\cosh(z) \qquad \cosh(2\pi i-z)==\cosh(z)$$

$$\tanh(-z-\pi i)==-\tanh(z) \qquad \tanh(z-\pi i)==\tanh(z)$$
$$\tanh\left(-z-\frac{\pi i}{2}\right)==-\coth(z) \qquad \tanh\left(z-\frac{\pi i}{2}\right)==\coth(z)$$
$$\tanh\left(z+\frac{\pi i}{2}\right)==\coth(z) \qquad \tanh\left(\frac{\pi i}{2}-z\right)==-\coth(z)$$
$$\tanh(z+\pi i)==\tanh(z) \qquad \tanh(\pi i-z)==-\tanh(z)$$

$$\coth(-z-\pi i)==-\coth(z) \qquad \coth(z-\pi i)==\coth(z)$$
$$\coth\left(-z-\frac{\pi i}{2}\right)==-\tanh(z) \qquad \coth\left(z-\frac{\pi i}{2}\right)==\tanh(z)$$
$$\coth\left(z+\frac{\pi i}{2}\right)==\tanh(z) \qquad \coth\left(\frac{\pi i}{2}-z\right)==-\tanh(z)$$
$$\coth(z+\pi i)==\coth(z) \qquad \coth(\pi i-z)==-\coth(z)$$

$$\operatorname{csch}(-z-2\pi i)==-\operatorname{csch}(z) \qquad \operatorname{csch}(z-2\pi i)==\operatorname{csch}(z)$$
$$\operatorname{csch}\left(-z-\frac{3\pi i}{2}\right)==-i\operatorname{sech}(z) \qquad \operatorname{csch}\left(z-\frac{3\pi i}{2}\right)==-i\operatorname{sech}(z)$$
$$\operatorname{csch}(-z-\pi i)==\operatorname{csch}(z) \qquad \operatorname{csch}(z-\pi i)==-\operatorname{csch}(z)$$
$$\operatorname{csch}\left(-z-\frac{\pi i}{2}\right)==i\operatorname{sech}(z) \qquad \operatorname{csch}\left(z-\frac{\pi i}{2}\right)==i\operatorname{sech}(z)$$
$$\operatorname{csch}\left(z+\frac{\pi i}{2}\right)==-i\operatorname{sech}(z) \qquad \operatorname{csch}\left(\frac{\pi i}{2}-z\right)==-i\operatorname{sech}(z)$$
$$\operatorname{csch}(z+\pi i)==-\operatorname{csch}(z) \qquad \operatorname{csch}(\pi i-z)==\operatorname{csch}(z)$$
$$\operatorname{csch}\left(z+\frac{3\pi i}{2}\right)==i\operatorname{sech}(z) \qquad \operatorname{csch}\left(\frac{3\pi i}{2}-z\right)==i\operatorname{sech}(z)$$
$$\operatorname{csch}(z+2\pi i)==\operatorname{csch}(z) \qquad \operatorname{csch}(2\pi-z\,i)==-\operatorname{csch}(z)$$

$$\operatorname{sech}(-z-2\pi i)==\operatorname{sech}(z) \qquad \operatorname{sech}(z-2\pi i)==\operatorname{sech}(z)$$
$$\operatorname{sech}\left(-z-\frac{3\pi i}{2}\right)==i\operatorname{csch}(z) \qquad \operatorname{sech}\left(z-\frac{3\pi i}{2}\right)==-i\operatorname{csch}(z)$$
$$\operatorname{sech}(-z-\pi i)==-\operatorname{sech}(z) \qquad \operatorname{sech}(z-\pi i)==-\operatorname{sech}(z)$$
$$\operatorname{sech}\left(-z-\frac{\pi i}{2}\right)==-i\operatorname{csch}(z) \qquad \operatorname{sech}\left(z-\frac{\pi i}{2}\right)==i\operatorname{csch}(z)$$
$$\operatorname{sech}\left(z+\frac{\pi i}{2}\right)==-i\operatorname{csch}(z) \qquad \operatorname{sech}\left(\frac{\pi i}{2}-z\right)==i\operatorname{csch}(z)$$
$$\operatorname{sech}(z+\pi i)==-\operatorname{sech}(z) \qquad \operatorname{sech}(\pi i-z)==-\operatorname{sech}(z)$$
$$\operatorname{sech}\left(z+\frac{3\pi i}{2}\right)==i\operatorname{csch}(z) \qquad \operatorname{sech}\left(\frac{3\pi i}{2}-z\right)==-i\operatorname{csch}(z)$$
$$\operatorname{sech}(z+2\pi i)==\operatorname{sech}(z) \qquad \operatorname{sech}(2\pi i-z)==\operatorname{sech}(z).$$

The best-known properties and formulas for hyperbolic functions

Real values for real arguments

For real values of argument z, the values of all the hyperbolic functions are real (or infinity).

In the points $z = 2\pi n i/m /; n \in \mathbb{Z} \bigwedge m \in \mathbb{Z}$, the values of the hyperbolic functions are algebraic. In several cases, they can even be rational numbers, 1, or i (e.g. $\sinh(\pi i/2) = i$, $\operatorname{sech}(0) = 1$, or $\cosh(\pi i/3) = 1/2$). They can be expressed using only square roots if $n \in \mathbb{Z}$ and m is a product of a power of 2 and distinct Fermat primes {3, 5, 17, 257, …}.

Simple values at zero

All hyperbolic functions has rather simple values for arguments $z == 0$ and $z == \pi i/2$:

$$\sinh(0) == 0 \quad \sinh\left(\frac{\pi i}{2}\right) == i$$
$$\cosh(0) == 1 \quad \cosh\left(\frac{\pi i}{2}\right) == 0$$
$$\tanh(0) == 0 \quad \tanh\left(\frac{\pi i}{2}\right) == \tilde{\infty}$$
$$\coth(0) == \tilde{\infty} \quad \coth\left(\frac{\pi i}{2}\right) == 0$$
$$\operatorname{csch}(0) == \tilde{\infty} \quad \operatorname{csch}\left(\frac{\pi i}{2}\right) == -i$$
$$\operatorname{sech}(0) == 1 \quad \operatorname{sech}\left(\frac{\pi i}{2}\right) == \tilde{\infty}.$$

Analyticity

All hyperbolic functions are defined for all complex values of z, and they are analytical functions of z over the whole complex z-plane and do not have branch cuts or branch points. The two functions $\sinh(z)$ and $\cosh(z)$ are entire functions with an essential singular point at $z = \tilde{\infty}$. All other hyperbolic functions are meromorphic functions with simple poles at points $z == \pi k i /; k \in \mathbb{Z}$ (for $\operatorname{csch}(z)$ and $\coth(z)$) and at points $z == \pi i/2 + \pi k i /; k \in \mathbb{Z}$ (for $\operatorname{sech}(z)$ and $\tanh(z)$).

Periodicity

All hyperbolic functions are periodic functions with a real period ($2\pi i$ or πi):

$$\sinh(z) == \sinh(z + 2\pi i) \quad \sinh(z + 2\pi i k) == \sinh(z) /; k \in \mathbb{Z}$$
$$\cosh(z) == \cosh(z + 2\pi i) \quad \cosh(z + 2\pi i k) == \cosh(z) /; k \in \mathbb{Z}$$
$$\tanh(z) == \tanh(z + \pi i) \quad \tanh(z + \pi i k) == \tanh(z) /; k \in \mathbb{Z}$$
$$\coth(z) == \coth(z + \pi i) \quad \coth(z + \pi i k) == \coth(z) /; k \in \mathbb{Z}$$
$$\operatorname{csch}(z) == \operatorname{csch}(z + 2\pi i) \quad \operatorname{csch}(z + 2\pi i k) == \operatorname{csch}(z) /; k \in \mathbb{Z}$$
$$\operatorname{sech}(z) == \operatorname{sech}(z + 2\pi i) \quad \operatorname{sech}(z + 2\pi i k) == \operatorname{sech}(z) /; k \in \mathbb{Z}.$$

Parity and symmetry

All hyperbolic functions have parity (either odd or even) and mirror symmetry:

$$\sinh(-z) == -\sinh(z) \quad \sinh(\bar{z}) == \overline{\sinh(z)}$$
$$\cosh(-z) == \cosh(z) \quad \cosh(\bar{z}) == \overline{\cosh(z)}$$
$$\tanh(-z) == -\tanh(z) \quad \tanh(\bar{z}) == \overline{\tanh(z)}$$
$$\coth(-z) == -\coth(z) \quad \coth(\bar{z}) == \overline{\coth(z)}$$
$$\operatorname{csch}(-z) == -\operatorname{csch}(z) \quad \operatorname{csch}(\bar{z}) == \overline{\operatorname{csch}(z)}$$
$$\operatorname{sech}(-z) == \operatorname{sech}(z) \quad \operatorname{sech}(\bar{z}) == \overline{\operatorname{sech}(z)}.$$

Simple representations of derivatives

The derivatives of all hyperbolic functions have simple representations that can be expressed through other hyperbolic functions:

$$\frac{\partial \sinh(z)}{\partial z} = \cosh(z) \qquad \frac{\partial \cosh(z)}{\partial z} = \sinh(z) \qquad \frac{\partial \tanh(z)}{\partial z} = \operatorname{sech}^2(z)$$
$$\frac{\partial \coth(z)}{\partial z} = -\operatorname{csch}^2(z) \qquad \frac{\partial \operatorname{csch}(z)}{\partial z} = -\coth(z)\operatorname{csch}(z) \qquad \frac{\partial \operatorname{sech}(z)}{\partial z} = -\operatorname{sech}(z)\tanh(z).$$

Simple differential equations

The solutions of the simplest second-order linear ordinary differential equation with constant coefficients can be represented through sinh(z) and cosh(z). The other hyperbolic functions satisfy first-order nonlinear differential equations:

$$w''(z) - w(z) = 0 \;/;\; w(z) = \cosh(z) \land w(0) = 1 \land w'(0) = 0$$
$$w''(z) - w(z) = 0 \;/;\; w(z) = \sinh(z) \land w(0) = 0 \land w'(0) = 1$$
$$w''(z) - w(z) = 0 \;/;\; w(z) = c_1 \cosh(z) + c_2 \sinh(z).$$

All six hyperbolic functions satisfy first-order nonlinear differential equations:

$$w'(z) - \sqrt{1 + (w(z))^2} = 0 \;/;\; w(z) = \sinh(z) \bigwedge w(0) = 0 \bigwedge |\operatorname{Im}(z)| < \frac{\pi}{2}$$
$$w'(z) - \sqrt{-1 + (w(z))^2} = 0 \;/;\; w(z) = \cosh(z) \bigwedge w(0) = 1 \bigwedge |\operatorname{Im}(z)| < \frac{\pi}{2}$$
$$w'(z) + w(z)^2 - 1 = 0 \;/;\; w(z) = \tanh(z) \land w(0) = 0$$
$$w'(z) + w(z)^2 - 1 = 0 \;/;\; w(z) = \coth(z) \bigwedge w\left(\frac{\pi i}{2}\right) = 0$$
$$w'(z)^2 - w(z)^4 - w(z)^2 = 0 \;/;\; w(z) = \operatorname{csch}(z)$$
$$w'(z)^2 + w(z)^4 - w(z)^2 = 0 \;/;\; w(z) = \operatorname{sech}(z).$$

Applications of hyperbolic functions

Trigonometric functions are intimately related to triangle geometry. Functions like sine and cosine are often introduced as edge lengths of right-angled triangles. Hyperbolic functions occur in the theory of triangles in hyperbolic spaces.

Lobachevsky (1829) and J. Bolyai (1832) independently recognized that Euclid's fifth postulate—saying that for a given line and a point not on the line, there is exactly one line parallel to the first—might be changed and still be a consistent geometry. In the hyperbolic geometry it is allowable for more than one line to be parallel to the first (meaning that the parallel lines will never meet the first, however far they are extended). Translated into triangles, this means that the sum of the three angles is always less than π.

A particularly nice representation of the hyperbolic geometry can be realized in the unit disk of complex numbers (the Poincaré disk model). In this model, points are complex numbers in the unit disk, and the lines are either arcs of circles that meet the boundary of the unit circle orthogonal or diameters of the unit circle.

The distance d between two points (meaning complex numbers) A and B in the Poincaré disk is:

$$d(A, B) = 2\tanh^{-1}\left(\left|\frac{A - B}{1 - \overline{B}A}\right|\right).$$

The attractive feature of the Poincaré disk model is that the hyperbolic angles agree with the Euclidean angles. Formally, the angle α at a point A of two hyperbolic lines $\overline{A\,B}$ and $\overline{A\,C}$ is described by the formula:

$$\cos(\alpha) = \frac{\frac{-A+B}{1-A\,B}\,\overline{\frac{-A+C}{1-A\,C}}}{\left|\frac{-A+B}{1-A\,B}\right|\left|\frac{-A+C}{1-A\,C}\right|}.$$

In the following, the values of the three angles of an hyperbolic triangle at the vertices A, B, and C are denoted through α, β, and γ. The hyperbolic length of the three edges opposite to the angles are denoted a, b, and c.

The cosine rule and the second cosine rule for hyperbolic triangles are:

$$\sinh(b)\sinh(c)\cos(\alpha) == \cosh(b)\cosh(c) - \cosh(a)$$
$$\sinh(a)\sinh(c)\cos(\beta) == \cosh(a)\cosh(c) - \cosh(b)$$
$$\sinh(a)\sinh(b)\cos(\gamma) == \cosh(a)\cosh(b) - \cosh(c)$$

$$\sin(\beta)\sin(\gamma)\cosh(a) == \cos(\beta)\cos(\gamma) + \cos(\alpha)$$
$$\sin(\alpha)\sin(\gamma)\cosh(b) == \cos(\alpha)\cos(\alpha) + \cos(\beta)$$
$$\sin(\alpha)\sin(\beta)\cosh(c) == \cos(\alpha)\cos(\beta) + \cos(\gamma).$$

The sine rule for hyperbolic triangles is:

$$\frac{\sin(\alpha)}{\sinh(a)} == \frac{\sin(\beta)}{\sinh(b)} == \frac{\sin(\gamma)}{\sinh(c)}.$$

For a right-angle triangle, the hyperbolic version of the Pythagorean theorem follows from the preceding formulas (the right angle is taken at vertex A):

$$\cosh(a) == \cosh(b)\cosh(c).$$

Using the series expansion $\cosh(x) \approx 1 + x^2/2$ at small scales the hyperbolic geometry is approximated by the familar Euclidean geometry. The cosine formulas and the sine formulas for hyperbolic triangles with a right angle at vertex A become:

$$\cos(\beta) = \frac{\tanh(c)}{\tanh(a)},\ \sin(\beta) = \frac{\sinh(b)}{\sinh(a)}$$
$$\cos(\gamma) = \frac{\tanh(b)}{\tanh(a)},\ \sin(\gamma) = \frac{\text{sinhh}(c)}{\tanh(a)}.$$

The inscribed circle has the radius:

$$\rho = \sqrt{\tanh^{-1}\left(\frac{\cos^2(\alpha) + \cos^2(\beta) + \cos^2(\gamma) + 2\cos(\alpha)\cos(\beta)\cos(\gamma) - 1}{2\,(1+\cos(\alpha))\,(1+\cos(\beta))\,(1+\cos(\gamma))}\right)}.$$

The circumscribed circle has the radius:

$$\rho = \tanh^{-1}\left(\frac{4\sinh\left(\frac{a}{2}\right)\sinh\left(\frac{b}{2}\right)\sinh\left(\frac{c}{2}\right)}{\sin(\gamma)\sinh(a)\sinh(b)}\right).$$

Other applications

As rational functions of the exponential function, the hyperbolic functions appear virtually everywhere in quantitative sciences. It is impossible to list their numerous applications in teaching, science, engineering, and art.

Introduction to the Hyperbolic Cosine Function

Defining the hyperbolic cosine function

The hyperbolic cosine function is an old mathematical function. It was used in the works of V. Riccati (1757), D. Foncenex (1759), and J. H. Lambert (1768).

This function is easily defined as the half-sum of two exponential functions in the points z and $-z$:

$$\cosh(z) = \frac{e^z + e^{-z}}{2}.$$

After comparison with the famous Euler formula for cosine: $\cos(z) = \frac{e^{iz}+e^{-iz}}{2}$, it is easy to derive the following representation of the hyperbolic cosine through the circular cosine:

$$\cosh(z) = \cos(i z).$$

The previous formula allows the derivation of all properties and formulas for hyperbolic cosine from corresponding properties and formulas for the circular cosine.

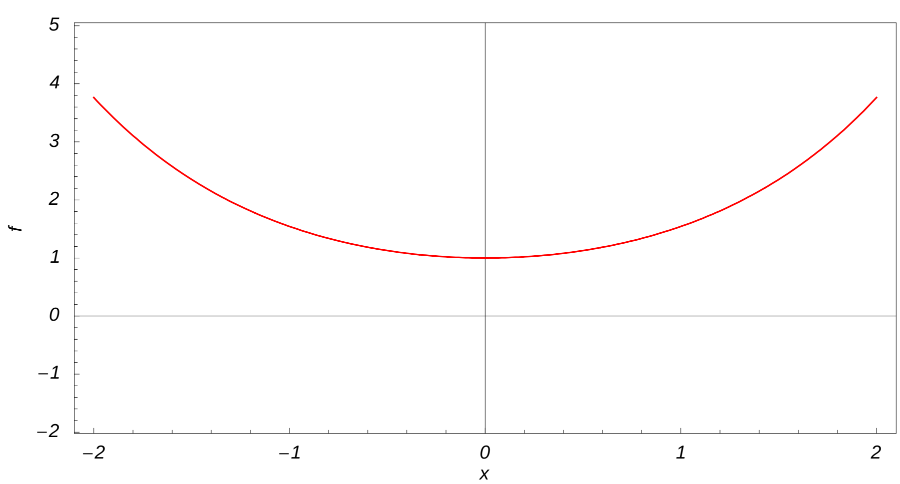
The following formula can sometimes be used as an equivalent definition of the hyperbolic cosine function:

$$\cosh(z) = 1 + \frac{z^2}{2} + \frac{z^4}{24} + \ldots = \sum_{k=0}^{\infty} \frac{z^{2k}}{(2k)!}.$$

This series converges for all finite numbers z.

A quick look at the hyperbolic cosine function

Here is a graphic of the hyperbolic cosine function $f(x) = \cosh(x)$ for real values of its argument x.



Representation using more general functions

The function $\cosh(z)$ is a particular case of more complicated mathematical functions. For example, it is a special case of the generalized hypergeometric function ${}_0F_1(;a;w)$ with the parameter $a == \frac{1}{2}$ at $w == \frac{z^2}{4}$:

$$\cosh(z) == {}_0F_1\left(;\frac{1}{2};\frac{z^2}{4}\right).$$

It is also a particular case of the modified Bessel function $I_\nu(z)$ with the parameter $\nu == -\frac{1}{2}$, multiplied by $\sqrt{\frac{\pi z}{2}}$:

$$\cosh(z) == \sqrt{\frac{\pi z}{2}}\; I_{-\frac{1}{2}}(z).$$

Other Bessel functions can also be expressed through hyperbolic cosine functions for similar values of the parameter:

$$\cosh(z) == \sqrt{\frac{\pi i z}{2}}\; J_{-\frac{1}{2}}(i z) \quad \cosh(z) == -\sqrt{\frac{\pi i z}{2}}\; Y_{\frac{1}{2}}(i z) \quad \cosh(z) == \sqrt{-\frac{z}{2\pi}}\; K_{-\frac{1}{2}}(-z) + \sqrt{\frac{z}{2\pi}}\; K_{-\frac{1}{2}}(z).$$

Struve functions can also degenerate into the hyperbolic cosine function for a similar value of the parameter:

$$\cosh(z) == 1 - \sqrt{\frac{\pi i z}{2}}\; \boldsymbol{H}_{\frac{1}{2}}(i z) \quad \cosh(z) == 1 + \sqrt{\frac{\pi z}{2}}\; \boldsymbol{L}_{\frac{1}{2}}(z).$$

But the function $\cosh(z)$ is also a degenerate case of the doubly periodic Jacobi elliptic functions when their second parameter is equal to 0 or 1:

$$\cosh(z) == \mathrm{cd}(i z \mid 0) == \mathrm{cn}(i z \mid 0)$$
$$\cosh(z) == \mathrm{nc}(z \mid 1) == \mathrm{nd}(z \mid 1).$$

Finally, the function $\cosh(z)$ is the particular case of one more class of functions—the Mathieu functions:

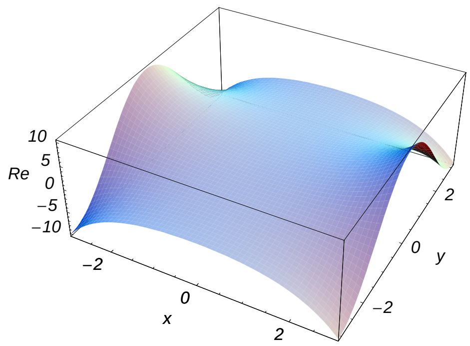
$$\cosh(z) == \mathrm{Ce}(1, 0, i z).$$

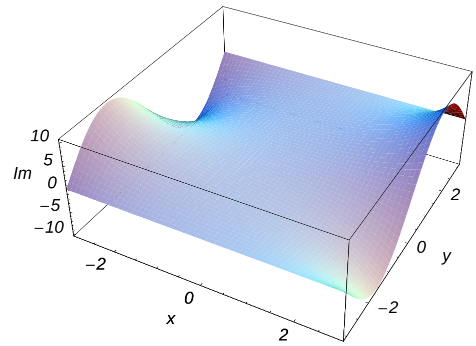
Definition of the hyperbolic cosine for a complex argument

In the complex z-plane, the function $\cosh(z)$ is defined by the same formula that is used for real values:

$$\cosh(z) == \frac{e^z + e^{-z}}{2}.$$

Here are two graphics showing the real and imaginary parts of the hyperbolic cosine function over the complex plane.





The best-known properties and formulas for the hyperbolic cosine function

Values in points

The values of the hyperbolic cosine function for special values of its argument can be easily derived from the corresponding values of the circular cosine in special points of the circle:

$$\cosh(0) = 1 \quad \cosh\left(\frac{\pi i}{6}\right) = \frac{\sqrt{3}}{2} \quad \cosh\left(\frac{\pi i}{4}\right) = \frac{1}{\sqrt{2}} \quad \cosh\left(\frac{\pi i}{3}\right) = \frac{1}{2}$$

$$\cosh\left(\frac{\pi i}{2}\right) = 0 \quad \cosh\left(\frac{2\pi i}{3}\right) = -\frac{1}{2} \quad \cosh\left(\frac{3\pi i}{4}\right) = -\frac{1}{\sqrt{2}} \quad \cosh\left(\frac{5\pi i}{6}\right) = -\frac{\sqrt{3}}{2}$$

$$\cosh(\pi i) = -1 \quad \cosh\left(\frac{7\pi i}{6}\right) = -\frac{\sqrt{3}}{2} \quad \cosh\left(\frac{5\pi i}{4}\right) = -\frac{1}{\sqrt{2}} \quad \cosh\left(\frac{4\pi i}{3}\right) = -\frac{1}{2}$$

$$\cosh\left(\frac{3\pi i}{2}\right) = 0 \quad \cosh\left(\frac{5\pi i}{3}\right) = \frac{1}{2} \quad \cosh\left(\frac{7\pi i}{4}\right) = \frac{1}{\sqrt{2}} \quad \cosh\left(\frac{11\pi i}{6}\right) = \frac{\sqrt{3}}{2}$$

$$\cosh(2\pi i) = 1 \quad \cosh(\pi i m) = (-1)^m /; m \in \mathbb{Z} \quad \cosh\left(\pi i\left(\frac{1}{2} + m\right)\right) = 0 /; m \in \mathbb{Z}.$$

The values at infinity can be expressed by the following formulas:

$$\cosh(\infty) = \infty \quad \cosh(-\infty) = \infty.$$

General characteristics

For real values of argument z, the values of $\cosh(z)$ are real.

In the points $z = 2\pi n i / m /; n \in \mathbb{Z} \land m \in \mathbb{Z}$, the values of $\cosh(z)$ are algebraic. In several cases, they can even be rational numbers, 0, or 1. Here are some examples:

$$\cosh(0) = 1 \quad \cosh\left(\frac{\pi i}{3}\right) = \frac{1}{2} \quad \cosh\left(\frac{\pi i}{2}\right) = 0.$$

The values of $\cosh\left(\frac{n \pi i}{m}\right)$ can be expressed using only square roots if $n \in \mathbb{Z}$ and m is a product of a power of 2 and distinct Fermat primes {3, 5, 17, 257, …}.

The function $\cosh(z)$ is an entire analytical function of z that is defined over the whole complex z-plane and does not have branch cuts and branch points. It has an essential singular point at $z = \tilde{\infty}$. It is a periodic function with the period $2\pi i$:

$$\cosh(z + 2\pi i) = \cosh(z)$$
$$\cosh(z) = \cosh(z + 2\pi k i) /; k \in \mathbb{Z}$$
$$\cosh(z) = (-1)^k \cosh(z + \pi k i) /; k \in \mathbb{Z}.$$

The function $\cosh(z)$ is an even function with mirror symmetry:

$$\cosh(-z) = \cosh(z) \quad \cosh(\bar{z}) = \overline{\cosh(z)}.$$

Differentiation

The derivatives of $\cosh(z)$ have simple representations using either the $\sinh(z)$ function or the $\cosh(z)$ function:

$$\frac{\partial \cosh(z)}{\partial z} = \sinh(z)$$
$$\frac{\partial^n \cosh(z)}{\partial z^n} = (-i)^n \cosh\left(z + \frac{i \pi n}{2}\right) /; n \in \mathbb{N}^+.$$

Ordinary differential equation

The function $\cosh(z)$ satisfies the simplest possible linear differential equation with constant coefficients:

$$w''(z) - w(z) = 0 /; w(z) = \text{cohs}(z) \land w(0) = 1 \land w'(0) = 0.$$

The complete solution of this equation can be represented as a linear combination of $\sinh(z)$ and $\cosh(z)$ with arbitrary constant coefficients c_1 and c_2:

$$w''(z) - w(z) = 0 /; w(z) = c_1 \cosh(z) + c_2 \sinh(z).$$

The function $\cosh(z)$ also satisfies first-order nonlinear differential equations:

$$w'(z) - \sqrt{w(z)^2 - 1} = 0 /; w(z) = \cosh(z) \land w(0) = 1.$$

Series representation

The function $\cosh(z)$ has a simple series expansion at the origin that converges in the whole complex z-plane:

$$\cosh(z) = 1 + \frac{z^2}{2} + \frac{z^4}{24} + \ldots = \sum_{k=0}^{\infty} \frac{z^{2k}}{(2k)!}.$$

Product representation

The following famous infinite product representation for $\cosh(z)$ clearly illustrates that $\cosh(z) = 0$ at $z = \pi i k - \frac{\pi i}{2} \bigwedge k \in \mathbb{Z}$:

$$\cosh(z) = \prod_{k=1}^{\infty} \left(\frac{4z^2}{\pi^2 (2k-1)^2} + 1 \right).$$

Indefinite integration

Indefinite integrals of expressions that contain the hyperbolic cosine function can sometimes be expressed using elementary functions. However, special functions are frequently needed to express the results even when the integrands have a simple form (if they can be evaluated in closed form). Here are some examples:

$$\int \cosh(z)\, dz = \sinh(z)$$

$$\int \sqrt{\cosh(z)}\, dz = -2\, i\, E\left(\frac{i\, z}{2} \,\middle|\, 2\right)$$

$$\int z^{\alpha-1} \cosh^{\nu}(a\, z)\, dz = \frac{2^{-\nu}\, (1 - \nu \bmod 2)}{\alpha}\, z^{\alpha} \binom{\nu}{\frac{\nu}{2}} +$$

$$2^{-\nu}\, z^{\alpha} \sum_{j=0}^{\left\lfloor \frac{\nu-1}{2} \right\rfloor} \binom{\nu}{j} \left(\Gamma(\alpha,\, a\,(2\,j-\nu)\,z)\,(a\,(2\,j-\nu)\,z)^{-\alpha} - \Gamma(\alpha,\, a\,(\nu-2\,j)\,z)\,(a\,(\nu-2\,j)\,z)^{-\alpha}\right) /;\ \alpha \neq 0 \bigwedge \nu \in \mathbb{N}^{+}.$$

The last integral cannot be evaluated in closed form using the known classical special functions for arbitrary values of parameters α and ν.

Definite integration

Definite integrals that contain the hyperbolic cosine are sometimes simple, as shown the following example:

$$\int_0^{\infty} \frac{1}{1 + \cosh(t)}\, dt = 1.$$

Some special functions can be used to evaluate more complicated definite integrals. For example, gamma and generalized hypergeometric functions are needed to express the following integrals:

$$\int_0^{\infty} \cosh^{a}(t)\, dt = \frac{\sqrt{\pi}\ \Gamma\left(-\frac{a}{2}\right)}{2\, \Gamma\left(\frac{1-a}{2}\right)} /;\ \mathrm{Re}(\alpha) < 0$$

$$\int_0^{\infty} t^2 \cosh^{a}(t)\, dt = -\frac{2^{1-a}}{a^3}\, {}_4F_3\left(-a,\, -\frac{a}{2},\, -\frac{a}{2},\, -\frac{a}{2};\, 1-\frac{a}{2},\, 1-\frac{a}{2},\, 1-\frac{a}{2};\, -1\right) /;\ \mathrm{Re}(\alpha) < 0.$$

Integral transforms

Numerous formulas for integral transforms from circular cosine functions cannot be easily converted into corresponding formulas with a hyperbolic cosine function because the hyperbolic cosine grows exponentially at infinity. This applies for Fourier cosine and sine transforms, and for Mellin, Hilbert, Hankel, and other transforms.

An exceptional case is the Laplace transform that itself includes the exponential function in the kernel:

$$\mathcal{L}_t[\cosh(t)](z) = \frac{z}{z^2 - 1}.$$

Finite summation

The following finite sums of the hyperbolic cosine can be expressed using the hyperbolic functions:

$$\sum_{k=0}^{n} \cosh(a\,k) = \operatorname{csch}\left(\frac{a}{2}\right) \sinh\left(\frac{a\,(n+1)}{2}\right) \cosh\left(\frac{a\,n}{2}\right)$$

$$\sum_{k=0}^{n} (-1)^k \cosh(a\,k) = \cosh\left(\frac{a+n\,(a+\pi\,i)}{2}\right) \operatorname{sech}\left(\frac{a}{2}\right) \cosh\left(\frac{n\,(a+\pi\,i)}{2}\right)$$

$$\sum_{k=0}^{n} \cosh(a\,k+z) = \operatorname{csch}\left(\frac{a}{2}\right) \sinh\left(\frac{a\,(n+1)}{2}\right) \cosh\left(\frac{a\,n}{2}+z\right)$$

$$\sum_{k=0}^{n} (-1)^k \cosh(a\,k+z) = \cosh\left(\frac{a+n\,(a+\pi\,i)}{2}\right) \operatorname{sech}\left(\frac{a}{2}\right) \cosh\left(\frac{n\,(a+\pi\,i)}{2}+z\right)$$

$$\sum_{k=1}^{n} z^k \cosh(a\,k) = \frac{z}{z^2-2\cosh(a)\,z+1} \left(\cosh(a\,n)\,z^{n+1} - \cosh(n\,a+a)\,z^n - z + \cosh(a)\right).$$

Infinite summation

The following infinite sums can be expressed using elementary functions:

$$\sum_{k=0}^{\infty} \frac{\cosh(a\,k)}{k!} = e^{\cosh(a)} \cosh(\sinh(a))$$

$$\sum_{k=0}^{\infty} \frac{z^k \cosh(a\,k)}{k!} = e^{z\cosh(a)} \cosh(z\sinh(a)).$$

Finite products

The following finite products from the hyperbolic cosine can be expressed using elementary functions:

$$\prod_{k=1}^{n-1} \cosh\left(\frac{i\,\pi\,k}{n}+z\right) = -\frac{2^{1-n}\,i}{\cosh(z)} \sinh\left(n\left(z+\frac{\pi\,i}{2}\right)\right) /; n \in \mathbb{N}^+$$

$$\prod_{k=1}^{n-1} \cosh\left(z+\frac{2\,i\,\pi\,k}{n}\right) = (-2)^{1-n} \operatorname{sech}(z) \left(\cosh(n\,z) - \cos\left(\frac{n\,\pi}{2}\right)\right) /; n \in \mathbb{N}^+.$$

Infinite products

The following infinite product that contains the hyperbolic cosine function can be expressed using the hyperbolic sine function:

$$\prod_{k=1}^{\infty} \cosh\left(\frac{z}{2^k}\right) = \frac{\sinh(z)}{z}.$$

Addition formulas

The hyperbolic cosine of a sum can be represented by the rule: "the hyperbolic cosine of a sum is equal to the sum of the product of the hyperbolic cosines and the product of the hyperbolic sines." A similar rule is valid for the hyperbolic cosine of the difference:

$\cosh(a+b) = \cosh(a)\cosh(b) + \sinh(a)\sinh(b)$
$\cosh(a-b) = \cosh(b)\cosh(a) - \sinh(a)\sinh(b).$

Multiple arguments

In the case of multiple arguments $2z$, $3z$, …, the function $\cosh(nz)$ can be represented as the finite sum that contains powers of the hyperbolic sine and cosine:

$$\cosh(2z) = \cosh^2(z) + \sinh^2(z)$$

$$\cosh(3z) = \cosh^3(z) + 3\sinh^2(z)\cosh(z)$$

$$\cosh(nz) = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n}{2k} \sinh^{2k}(z)\cosh^{n-2k}(z) \;/;\; n \in \mathbb{N}.$$

The function $\cosh(nz)$ can also be represented as the finite sum that contains only the hyperbolic cosine of z:

$$\cosh(nz) = n \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(-1)^k (n-k-1)!\, 2^{n-2k-1}}{k!\,(n-2k)!} \cosh^{n-2k}(z) \;/;\; n \in \mathbb{N}.$$

Half-angle formulas

The hyperbolic cosine of the half-angle can be represented by the following simple formula that is valid in a horizontal strip:

$$\cosh\left(\frac{z}{2}\right) = \sqrt{\frac{\cosh(z)+1}{2}} \;/;\; |\mathrm{Im}(z)| < \pi \lor \mathrm{Im}(z) = -\pi \land \mathrm{Re}(z) \le 0 \lor \mathrm{Im}(z) = \pi \land \mathrm{Re}(z) \ge 0.$$

To make this formula correct for all complex z, a complicated prefactor is needed:

$$\cosh\left(\frac{z}{2}\right) = c(z)\sqrt{\frac{1+\cosh(z)}{2}} \;/;\; c(z) = (-1)^{\left\lfloor \frac{\mathrm{Im}(z)+\pi}{2\pi} \right\rfloor} \left(1 - \left(1 + (-1)^{\left\lfloor \frac{\mathrm{Im}(z)+\pi}{2\pi} \right\rfloor + \left\lfloor -\frac{\mathrm{Im}(z)+\pi}{2\pi} \right\rfloor}\right)\theta(\mathrm{Re}(z))\right),$$

where $c(z)$ contains the unit step, real part, imaginary part, and floor functions.

Sums of two direct functions

The sum of two hyperbolic cosine functions can be described by the rule: "the sum of the hyperbolic cosines is equal to the doubled hyperbolic cosine of the half-difference multiplied by the hyperbolic cosine of the half-sum." A similar rule is valid for the difference of two hyperbolic cosines:

$$\cosh(a) + \cosh(b) = 2\cosh\left(\frac{a-b}{2}\right)\cosh\left(\frac{a+b}{2}\right)$$

$$\cosh(a) - \cosh(b) = 2\sinh\left(\frac{a+b}{2}\right)\sinh\left(\frac{a-b}{2}\right).$$

Products involving the direct function

The product of two hyperbolic cosine functions or the product of the hyperbolic cosine and hyperbolic sine have the following representations:

$$\cosh(a)\cosh(b) = \frac{1}{2}(\cosh(a-b)+\cosh(a+b))$$

$$\cosh(a)\sinh(b) = \frac{1}{2}(\sinh(a+b)-\sinh(a-b)).$$

Powers of the direct function

The integer powers of the hyperbolic cosine functions can be expanded as finite sums of hyperbolic cosine functions with multiple arguments. These sums contain binomial coefficients:

$$\cosh^n(z) = 2^{-n}\binom{n}{\frac{n}{2}}(1-n \bmod 2)+2^{1-n}\sum_{k=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}\binom{n}{k}\cosh((n-2k)z)\ /;\ n\in\mathbb{N}^+.$$

Inequalities

The best-known inequalities for the hyperbolic cosine function are the following:

$$\cosh(x)\geq 1\ /;\ x\in\mathbb{R}$$

$$\cosh(x)\geq|\cos(x)|\ /;\ x\in\mathbb{R}$$

$$|\cosh(z)|\leq\cosh(\operatorname{Re}(z)).$$

Relations with its inverse function

There are simple relations between the function $\cosh(z)$ and its inverse function $\cosh^{-1}(z)$:

$$\cosh\left(\cosh^{-1}(z)\right) = z \quad \cosh^{-1}(\cosh(z)) = z\ /;\ \operatorname{Re}(z)>0\land-\pi<\operatorname{Im}(z)\leq\pi\lor\operatorname{Re}(z)=0\land 0\leq\operatorname{Im}(z)\leq\pi\ .$$

The second formula is valid at least in the right half of the horizontal strip $-\pi<\operatorname{Im}(z)<\pi$. It can be generalized to the full horizontal strip by changing z to $\sqrt{z^2}$ in its right side:

$$\cosh^{-1}(\cosh(z)) = \sqrt{z^2}\ /;\ -\pi<\operatorname{Im}(z)<\pi\lor(\operatorname{Im}(z)=-\pi\land\operatorname{Re}(z)\leq 0)\lor(\operatorname{Im}(z)=\pi\land\operatorname{Re}(z)\geq 0).$$

For the whole complex plane, a much more complicated relation (that contains the unit step, real part, imaginary part, and the floor functions) holds:

$$\cosh^{-1}(\cosh(z)) = \sqrt{z^2}\left(1-\frac{\pi i}{2z}\left(\left(1-(-1)^{\left\lfloor\frac{\operatorname{Im}(z)}{\pi}\right\rfloor}\right)+2\left\lfloor\frac{\operatorname{Im}(z)}{\pi}\right\rfloor\right)\right)+\frac{\pi i}{2}\left(\frac{\sqrt{z^2}}{z}+1\right)\left(1+(-1)^{\left\lfloor\frac{\operatorname{Im}(z)}{2\pi}+\frac{1}{2}\right\rfloor+\left\lfloor-\frac{\operatorname{Im}(z)}{2\pi}-\frac{1}{2}\right\rfloor}\right)\theta(\operatorname{Re}(z))\ /;\ z\neq 0.$$

Representations through other hyperbolic functions

Hyperbolic cosine and sine functions are connected by a very simple formula that contains the linear function in the argument:

$$\cosh(z) = -i\sinh\left(\frac{\pi i}{2}-z\right).$$

Another famous formula, connecting $\cosh(z)$ and $\sinh(z)$, is expressed in the analog of the well-known Pythagorean theorem:

$\cosh^2(z) = \sinh^2(z) + 1$

$\cosh(z) = \sqrt{\sinh^2(z) + 1}$ /; $|\mathrm{Im}(z)| < \frac{\pi}{2} \bigvee \mathrm{Im}(z) = -\frac{\pi}{2} \bigwedge \mathrm{Re}(z) \leq 0 \bigvee \mathrm{Im}(z) = \frac{\pi}{2} \bigwedge \mathrm{Re}(z) \geq 0.$

The restriction on z can be removed, but the formula will get a complicated coefficient $c(z)$ that contains the unit step, real part, imaginary part, and the floor functions and $|c(z)| = 1$:

$\cosh(z) = c(z) \sqrt{\sinh^2(z) + 1}$ /; $c(z) = (-1)^{\left\lfloor \frac{\mathrm{Im}(z)}{\pi} + \frac{1}{2} \right\rfloor} \left(1 - \left(1 + (-1)^{\left\lfloor \frac{\mathrm{Im}(z)}{\pi} + \frac{1}{2} \right\rfloor + \left\lfloor -\frac{\mathrm{Im}(z)}{\pi} - \frac{1}{2} \right\rfloor}\right) \theta(\mathrm{Re}(z))\right).$

The hyperbolic cosine function can also be represented using other hyperbolic functions by the following formulas:

$\cosh(z) = \frac{1+\tanh^2\left(\frac{z}{2}\right)}{1-\tanh^2\left(\frac{z}{2}\right)} \quad \cosh(z) = \frac{\coth^2\left(\frac{z}{2}\right)+1}{\coth^2\left(\frac{z}{2}\right)-1}$

$\cosh(z) = -\frac{i}{\mathrm{csch}\left(\frac{i\pi}{2} - z\right)} \quad \cosh(z) = \frac{1}{\mathrm{sech}(z)}.$

Representation through trigonometric functions

The hyperbolic cosine function has similar representations using related trigonometric functions and formulas:

$\cosh(z) = \cos(i z) \quad \cosh(i z) = \cos(z) \quad \cosh(z) = \sin\left(\frac{\pi}{2} - z\, i\right)$

$\cosh(z) = \frac{1-\tan^2\left(\frac{i z}{2}\right)}{1+\tan^2\left(\frac{i z}{2}\right)} \quad \cosh(z) = \frac{\cot^2\left(\frac{i z}{2}\right)-1}{\cot^2\left(\frac{i z}{2}\right)+1} \quad \cosh(z) = \frac{1}{\csc\left(\frac{\pi}{2} - i z\right)} \quad \cosh(z) = \frac{1}{\sec(i z)}.$

Applications

The hyperbolic cosine function is used throughout mathematics, the exact sciences, and engineering.

Introduction to the Hyperbolic Functions in *Mathematica*

Overview

The following shows how the six hyperbolic functions are realized in *Mathematica*. Examples of evaluating *Mathematica* functions applied to various numeric and exact expressions that involve the hyperbolic functions or return them are shown. These involve numeric and symbolic calculations and plots.

Notations

Mathematica forms of notations

All six hyperbolic functions are represented as built-in functions in *Mathematica*. Following *Mathematica*'s general naming convention, the `StandardForm` function names are simply capitalized versions of the traditional mathematics names. Here is a list `hypFunctions` of the six hyperbolic functions in `StandardForm`.

```
hypFunctions = {Sinh[z], Cosh[z], Tanh[z], Coth[z], Sech[z], Cosh[z]}
```

```
{Sinh[z], Cosh[z], Tanh[z], Coth[z], Sech[z], Cosh[z]}
```

Here is a list `hypFunctions` of the six trigonometric functions in `TraditionalForm`.

```
hypFunctions // TraditionalForm
```

{sinh(*z*), cosh(*z*), tanh(*z*), coth(*z*), sech(*z*), cosh(*z*)}

Additional forms of notations

Mathematica also knows the most popular forms of notations for the hyperbolic functions that are used in other programming languages. Here are three examples: `CForm`, `TeXForm`, and `FortranForm`.

```
hypFunctions /. {z → 2 π z} // CForm
```

```
List(Sinh(2*Pi*z),Cosh(2*Pi*z),Tanh(2*Pi*z),Coth(2*Pi*z),Sech(2*Pi*z),Cosh(2*Pi*z))
```

```
hypFunctions /. {z → 2 π z} // TeXForm
```

```
\{ \sinh (2\,\pi \,z),\cosh (2\,\pi \,z),\tanh (2\,\pi \,z),\coth (2\,\pi \,z),
  \Mfunction{Sech}(2\,\pi \,z),\cosh (2\,\pi \,z)\}
```

```
hypFunctions /. {z → 2 π z} // FortranForm
```

```
List(Sinh(2*Pi*z),Cosh(2*Pi*z),Tanh(2*Pi*z),Coth(2*Pi*z),Sech(2*Pi*z),Cosh(2*Pi*z))
```

Automatic evaluations and transformations

Evaluation for exact, machine-number, and high-precision arguments

For a simple exact argument, *Mathematica* returns an exact result. For instance, for the argument $\pi i/6$, the `Sinh` function evaluates to $i/2$.

$$\text{Sinh}\left[\frac{\pi i}{6}\right]$$

$$\frac{i}{2}$$

$$\{\text{Sinh}[z], \text{Cosh}[z], \text{Tanh}[z], \text{Coth}[z], \text{Csch}[z], \text{Sech}[z]\} \text{ /. } z \to \frac{\pi i}{6}$$

$$\left\{\frac{i}{2}, \frac{\sqrt{3}}{2}, \frac{i}{\sqrt{3}}, -i\sqrt{3}, -2\,i, \frac{2}{\sqrt{3}}\right\}$$

For a generic machine-number argument (a numerical argument with a decimal point and not too many digits), a machine number is returned.

```
Cosh[3.]
```

```
10.0677
```

```
{Sinh[z], Cosh[z], Tanh[z], Coth[z], Csch[z], Sech[z]} /. z → 2.
```

```
{3.62686, 3.7622, 0.964028, 1.03731, 0.275721, 0.265802}
```

The next inputs calculate 100-digit approximations of the six hyperbolic functions at $z = 1$.

```
N[Tanh[1], 40]
```

```
0.7615941559557648881194582826047935904128
```

```
Coth[1] // N[#, 50] &
```

```
1.3130352854993313036361612469308478329120139412405
```

```
N[{Sinh[z], Cosh[z], Tanh[z], Coth[z], Csch[z], Sech[z]} /. z → 1, 100]
```

```
{1.175201193643801456882381850595600815155717981334095870229565413013307567304323895
  607117452089623392,
 1.543080634815243778477905620757061682601529112365863704737402214710769063049223698
  964264726435543036,
 0.761594155955764888119458282604793590412768597257936551596810500121953244576638483
  4589475216736767144,
 1.313035285499331303636161246930847832912013941240452655543152967567084270461874382
  674679241480856303,
 0.850918128239321545133842763287175284181724660910339616990421151729003364321465103
  8997301773288938124,
 0.648054273663885399574977353226150323108489312071942023037865337318717595646712830
  2808547853078928924}
```

Within a second, it is possible to calculate thousands of digits for the hyperbolic functions. The next input calculates 10000 digits for sinh(1), cosh(1), tanh(1), coth(1), sech(1), and csch(1) and analyzes the frequency of the occurrence of the digit *k* in the resulting decimal number.

```
Map[Function[w, {First[#], Length[#]} & /@ Split[Sort[First[RealDigits[w]]]]],
 N[{Sinh[z], Cosh[z], Tanh[z], Coth[z], Csch[z], Sech[z]} /. z → 1, 10 000]]
```

```
{{{0, 980}, {1, 994}, {2, 996}, {3, 1014}, {4, 986}, {5, 1001},
  {6, 1017}, {7, 1020}, {8, 981}, {9, 1011}}, {{0, 1015}, {1, 960}, {2, 997},
  {3, 1037}, {4, 1070}, {5, 1018}, {6, 973}, {7, 997}, {8, 963}, {9, 970}},
 {{0, 971}, {1, 1023}, {2, 1016}, {3, 970}, {4, 949}, {5, 1052}, {6, 981},
  {7, 1056}, {8, 1010}, {9, 972}}, {{0, 975}, {1, 986}, {2, 1023},
  {3, 1004}, {4, 1008}, {5, 977}, {6, 977}, {7, 1036}, {8, 1035}, {9, 979}},
 {{0, 979}, {1, 1030}, {2, 987}, {3, 992}, {4, 1016}, {5, 1030}, {6, 1021},
  {7, 969}, {8, 974}, {9, 1002}}, {{0, 1009}, {1, 971}, {2, 1018},
  {3, 994}, {4, 1011}, {5, 1018}, {6, 958}, {7, 1019}, {8, 1016}, {9, 986}}}
```

Here are 50-digit approximations to the six hyperbolic functions at the complex argument $z = 3 + 5 i$.

```
N[Csch[3 + 5 ⅈ], 100]
```

```
0.0280585164230800759963159842602743697051540123887285931631736730964453318082730911
  1484269546408531396 +
 0.095323634674178402851915930706256451645442166878775479803879772793331583262276221
  3893978444505670174 7 ⅈ
```

```
N[{Sinh[z], Cosh[z], Tanh[z], Coth[z], Csch[z], Sech[z]} /. z → 3 + 5 ⅈ, 50]
```

```
{2.841692295606351943816875395306236435928184163236 0 -
  9.654125476854839136551543634030165992191969121385 3 i,
 2.855815004227387291363901863094609837464360953673 2 -
  9.606383448432581119811156216043416387721859039403 3 i,
 1.004164710694815211920516625931318431185245473573 8 -
  0.002708235836224072132264035368433103592796025912 5751 i,
 0.995845318575854129760420015871648417110265572041 02 +
  0.002685798405758525644653771101281474937897743936 1108 i,
 0.028058516423080075996315984260274369705154012388 729 +
  0.095323634674178402851915930706256451645442166878 775 i,
 0.028433530909971667358833684958646399417265586614 624 +
  0.095644640955286344684316595933099452259073530811 833 i}
```

Mathematica always evaluates mathematical functions with machine precision, if the arguments are machine numbers. In this case, only six digits after the decimal point are shown in the results. The remaining digits are suppressed, but can be displayed using the function InputForm.

```
{Sinh[2.], N[Sinh[2]], N[Sinh[2], 16], N[Sinh[2], 5], N[Sinh[2], 20]}
```

```
{3.62686, 3.62686, 3.62686, 3.62686, 3.6268604078470187677}
```

```
% // InputForm
```

```
{3.6268604078470186, 3.6268604078470186, 3.6268604078470186, 3.6268604078470186,
 3.626860407847018767668213982801262016 44`20}
```

```
Precision[%%]
```

```
16
```

Simplification of the argument

Mathematica uses symmetries and periodicities of all the hyperbolic functions to simplify expressions. Here are some examples.

```
Sinh[-z]
```

```
-Sinh[z]
```

```
Sinh[z + π i]
```

```
-Sinh[z]
```

```
Sinh[z + 2 π i]
```

```
Sinh[z]
```

```
Sinh[z + 34 π i]
```

```
Sinh[z]
```

```
{Sinh[-z], Cosh[-z], Tanh[-z], Coth[-z], Csch[-z], Sech[-z]}
```

```
{-Sinh[z], Cosh[z], -Tanh[z], -Coth[z], -Csch[z], Sech[z]}
```

{Sinh[z + π ⅈ], Cosh[z + π ⅈ], Tanh[z + π ⅈ], Coth[z + π ⅈ], Csch[z + π ⅈ], Sech[z + π ⅈ]}

{-Sinh[z], -Cosh[z], Tanh[z], Coth[z], -Csch[z], -Sech[z]}

{Sinh[z + 2 π ⅈ], Cosh[z + 2 π ⅈ], Tanh[z + 2 π ⅈ], Coth[z + 2 π ⅈ], Csch[z + 2 π ⅈ], Sech[z + 2 π ⅈ]}

{Sinh[z], Cosh[z], Tanh[z], Coth[z], Csch[z], Sech[z]}

{Sinh[z + 342 π ⅈ], Cosh[z + 342 π ⅈ], Tanh[z + 342 π ⅈ], Coth[z + 342 π ⅈ], Csch[z + 342 π ⅈ], Sech[z + 342 π ⅈ]}

{Sinh[z], Cosh[z], Tanh[z], Coth[z], Csch[z], Sech[z]}

Mathematica automatically simplifies the composition of the direct and the inverse hyperbolic functions into the argument.

{Sinh[ArcSinh[z]], Cosh[ArcCosh[z]], Tanh[ArcTanh[z]], Coth[ArcCoth[z]], Csch[ArcCsch[z]], Sech[ArcSech[z]]}

{z, z, z, z, z, z}

Mathematica also automatically simplifies the composition of the direct and any of the inverse hyperbolic functions into algebraic functions of the argument.

{Sinh[ArcSinh[z]], Sinh[ArcCosh[z]], Sinh[ArcTanh[z]], Sinh[ArcCoth[z]], Sinh[ArcCsch[z]], Sinh[ArcSech[z]]}

$$\left\{z, \sqrt{\frac{-1+z}{1+z}}\,(1+z), \frac{z}{\sqrt{1-z^2}}, \frac{1}{\sqrt{1-\frac{1}{z^2}}\,z}, \frac{1}{z}, \frac{\sqrt{\frac{1-z}{1+z}}\,(1+z)}{z}\right\}$$

{Cosh[ArcSinh[z]], Cosh[ArcCosh[z]], Cosh[ArcTanh[z]], Cosh[ArcCoth[z]], Cosh[ArcCsch[z]], Cosh[ArcSech[z]]}

$$\left\{\sqrt{1+z^2}, z, \frac{1}{\sqrt{1-z^2}}, \frac{1}{\sqrt{1-\frac{1}{z^2}}}, \sqrt{1+\frac{1}{z^2}}, \frac{1}{z}\right\}$$

{Tanh[ArcSinh[z]], Tanh[ArcCosh[z]], Tanh[ArcTanh[z]], Tanh[ArcCoth[z]], Tanh[ArcCsch[z]], Tanh[ArcSech[z]]}

$$\left\{\frac{z}{\sqrt{1+z^2}}, \frac{\sqrt{\frac{-1+z}{1+z}}\,(1+z)}{z}, z, \frac{1}{z}, \frac{1}{\sqrt{1+\frac{1}{z^2}}\,z}, \sqrt{\frac{1-z}{1+z}}\,(1+z)\right\}$$

{Coth[ArcSinh[z]], Coth[ArcCosh[z]], Coth[ArcTanh[z]], Coth[ArcCoth[z]], Coth[ArcCsch[z]], Coth[ArcSech[z]]}

$$\left\{\frac{\sqrt{1+z^2}}{z},\ \frac{z}{\sqrt{\frac{-1+z}{1+z}}\ (1+z)},\ \frac{1}{z},\ z,\ \sqrt{1+\frac{1}{z^2}}\ z,\ \frac{1}{\sqrt{\frac{1-z}{1+z}}\ (1+z)}\right\}$$

{Csch[ArcSinh[z]], Csch[ArcCosh[z]], Csch[ArcTanh[z]], Csch[ArcCoth[z]], Csch[ArcCsch[z]], Csch[ArcSech[z]]}

$$\left\{\frac{1}{z},\ \frac{1}{\sqrt{\frac{-1+z}{1+z}}\ (1+z)},\ \frac{\sqrt{1-z^2}}{z},\ \sqrt{1-\frac{1}{z^2}}\ z,\ z,\ \frac{z}{\sqrt{\frac{1-z}{1+z}}\ (1+z)}\right\}$$

{Sech[ArcSinh[z]], Sech[ArcCosh[z]], Sech[ArcTanh[z]], Sech[ArcCoth[z]], Sech[ArcCsch[z]], Sech[ArcSech[z]]}

$$\left\{\frac{1}{\sqrt{1+z^2}},\ \frac{1}{z},\ \sqrt{1-z^2},\ \sqrt{1-\frac{1}{z^2}},\ \frac{1}{\sqrt{1+\frac{1}{z^2}}},\ z\right\}$$

In cases where the argument has the structure $\pi k/2+z$ or $\pi k/2-z$, e $\pi k/2+i z$ or $\pi k/2-i z$ with integer k, trigonometric functions can be automatically transformed into other trigonometric or hyperbolic functions. Here are some examples.

$$\mathbf{Tanh}\left[\frac{\pi i}{2}-z\right]$$

-Coth[z]

Csch[i z]

-i Csc[z]

$$\left\{\mathbf{Sinh}\left[\frac{\pi i}{2}-z\right],\ \mathbf{Cosh}\left[\frac{\pi i}{2}-z\right],\ \mathbf{Tanh}\left[\frac{\pi i}{2}-z\right],\ \mathbf{Coth}\left[\frac{\pi i}{2}-z\right],\ \mathbf{Csch}\left[\frac{\pi i}{2}-z\right],\ \mathbf{Sech}\left[\frac{\pi i}{2}-z\right]\right\}$$

{i Cosh[z], -i Sinh[z], -Coth[z], -Tanh[z], -i Sech[z], i Csch[z]}

{Sinh[i z], Cosh[i z], Tanh[i z], Coth[i z], Csch[i z], Sech[i z]}

{i Sin[z], Cos[z], i Tan[z], -i Cot[z], -i Csc[z], Sec[z]}

Simplification of simple expressions containing hyperbolic functions

Sometimes simple arithmetic operations containing hyperbolic functions can automatically produce other hyperbolic functions.

1 / Sech[z]

Cosh[z]

{1 / Sinh[z], 1 / Cosh[z], 1 / Tanh[z], 1 / Coth[z], 1 / Csch[z], 1 / Sech[z], Sinh[z] / Cosh[z], Cosh[z] / Sinh[z], Sinh[z] / Sinh[π i / 2 - z], Cosh[z] / Sinh[z] ^2}

```
{Csch[z], Sech[z], Coth[z], Tanh[z], Sinh[z],
 Cosh[z], Tanh[z], Coth[z], -ⅈ Tanh[z], Coth[z] Csch[z]}
```

Hyperbolic functions as special cases of more general functions

All hyperbolic functions can be treated as particular cases of some more advanced special functions. For example, sinh(z) and cosh(z) are sometimes the results of auto-simplifications from Bessel, Mathieu, Jacobi, hypergeometric, and Meijer functions (for appropriate values of their parameters).

BesselI[1/2, z]

$$\frac{\sqrt{\frac{2}{\pi}}\ \mathrm{Sinh}[z]}{\sqrt{z}}$$

MathieuC[1, 0, ⅈ z]

Cosh[z]

JacobiSN[z, 1]

Tanh[z]

{BesselI[1/2, z], MathieuS[1, 0, ⅈ z], JacobiSD[ⅈ z, 0], HypergeometricPFQ[{}, {3/2}, z^2/4], MeijerG[{{}, {}}, {{0}, {-1/2}}, -z^2/4]}

$$\left\{\frac{\sqrt{\frac{2}{\pi}}\ \mathrm{Sinh}[z]}{\sqrt{z}},\ i\,\mathrm{Sinh}[z],\ i\,\mathrm{Sinh}[z],\ \frac{\mathrm{Sinh}\left[\sqrt{z^2}\right]}{\sqrt{z^2}},\ \frac{2\,\mathrm{Sinh}[z]}{\sqrt{\pi}\ z}\right\}$$

{BesselI[-1/2, z], MathieuC[1, 0, ⅈ z], JacobiCD[ⅈ z, 0], Hypergeometric0F1[1/2, z^2/4], MeijerG[{{}, {}}, {{0}, {1/2}}, -z^2/4]}

$$\left\{\frac{\sqrt{\frac{2}{\pi}}\ \mathrm{Cosh}[z]}{\sqrt{z}},\ \mathrm{Cosh}[z],\ \mathrm{Cosh}[z],\ \mathrm{Cosh}\left[\sqrt{z^2}\right],\ \frac{\mathrm{Cosh}[z]}{\sqrt{\pi}}\right\}$$

{JacobiSC[ⅈ z, 0], JacobiNS[z, 1], JacobiNS[ⅈ z, 0], JacobiDC[ⅈ z, 0]}

{ⅈ Tanh[z], Coth[z], -ⅈ Csch[z], Sech[z]}

Equivalence transformations carried out by specialized *Mathematica* functions

General remarks

Automatic evaluation and transformations can sometimes be inconvenient: They act in only one chosen direction and the result can be overly complicated. For example, the expression $i \cosh(z)/2$ is generally preferable to the more complicated $\sinh(\pi i/2 - z)\cosh(\pi i/3)$. *Mathematica* provides automatic transformation of the second expression into the first one. But compact expressions like $\sinh(2z)\cosh(\pi i/16)$ should not be automatically expanded into the more complicated expression $\sinh(z)\cosh(z)\left(2+\left(2+2^{1/2}\right)^{1/2}\right)^{1/2}$. *Mathematica* has special functions that produce these types of expansions. Some of them are demonstrated in the next section.

TrigExpand

The function `TrigExpand` expands out trigonometric and hyperbolic functions. In more detail, it splits up sums and integer multiples that appear in the arguments of trigonometric and hyperbolic functions, and then expands out the products of the trigonometric and hyperbolic functions into sums of powers, using the trigonometric and hyperbolic identities where possible. Here are some examples.

```
TrigExpand[Sinh[x - y]]
```

```
Cosh[y] Sinh[x] - Cosh[x] Sinh[y]
```

```
Cosh[4 z] // TrigExpand
```

```
Cosh[z]^4 + 6 Cosh[z]^2 Sinh[z]^2 + Sinh[z]^4
```

```
TrigExpand[{{Sinh[x + y], Sinh[3 z]},
            {Cosh[x + y], Cosh[3 z]},
            {Tanh[x + y], Tanh[3 z]},
            {Coth[x + y], Coth[3 z]},
            {Csch[x + y], Csch[3 z]},
            {Sech[x + y], Sech[3 z]}}]
```

$$\left\{\left\{\text{Cosh}[y]\,\text{Sinh}[x]+\text{Cosh}[x]\,\text{Sinh}[y],\ 3\,\text{Cosh}[z]^2\,\text{Sinh}[z]+\text{Sinh}[z]^3\right\},\right.$$
$$\left\{\text{Cosh}[x]\,\text{Cosh}[y]+\text{Sinh}[x]\,\text{Sinh}[y],\ \text{Cosh}[z]^3+3\,\text{Cosh}[z]\,\text{Sinh}[z]^2\right\},$$
$$\left\{\frac{\text{Cosh}[y]\,\text{Sinh}[x]}{\text{Cosh}[x]\,\text{Cosh}[y]+\text{Sinh}[x]\,\text{Sinh}[y]}+\frac{\text{Cosh}[x]\,\text{Sinh}[y]}{\text{Cosh}[x]\,\text{Cosh}[y]+\text{Sinh}[x]\,\text{Sinh}[y]},\right.$$
$$\left.\frac{3\,\text{Cosh}[z]^2\,\text{Sinh}[z]}{\text{Cosh}[z]^3+3\,\text{Cosh}[z]\,\text{Sinh}[z]^2}+\frac{\text{Sinh}[z]^3}{\text{Cosh}[z]^3+3\,\text{Cosh}[z]\,\text{Sinh}[z]^2}\right\},$$
$$\left\{\frac{\text{Cosh}[x]\,\text{Cosh}[y]}{\text{Cosh}[y]\,\text{Sinh}[x]+\text{Cosh}[x]\,\text{Sinh}[y]}+\frac{\text{Sinh}[x]\,\text{Sinh}[y]}{\text{Cosh}[y]\,\text{Sinh}[x]+\text{Cosh}[x]\,\text{Sinh}[y]},\right.$$
$$\left.\frac{\text{Cosh}[z]^3}{3\,\text{Cosh}[z]^2\,\text{Sinh}[z]+\text{Sinh}[z]^3}+\frac{3\,\text{Cosh}[z]\,\text{Sinh}[z]^2}{3\,\text{Cosh}[z]^2\,\text{Sinh}[z]+\text{Sinh}[z]^3}\right\},$$
$$\left\{\frac{1}{\text{Cosh}[y]\,\text{Sinh}[x]+\text{Cosh}[x]\,\text{Sinh}[y]},\ \frac{1}{3\,\text{Cosh}[z]^2\,\text{Sinh}[z]+\text{Sinh}[z]^3}\right\},$$
$$\left.\left\{\frac{1}{\text{Cosh}[x]\,\text{Cosh}[y]+\text{Sinh}[x]\,\text{Sinh}[y]},\ \frac{1}{\text{Cosh}[z]^3+3\,\text{Cosh}[z]\,\text{Sinh}[z]^2}\right\}\right\}$$

```
TableForm[(# == TrigExpand[#]) & /@
  Flatten[{{Sinh[x + y], Sinh[3 z]}, {Cosh[x + y], Cosh[3 z]}, {Tanh[x + y], Tanh[3 z]},
    {Coth[x + y], Coth[3 z]}, {Csch[x + y], Csch[3 z]}, {Sech[x + y], Sech[3 z]}}]]
```

```
Sinh[x + y] == Cosh[y] Sinh[x] + Cosh[x] Sinh[y]
Sinh[3 z] == 3 Cosh[z]^2 Sinh[z] + Sinh[z]^3
Cosh[x + y] == Cosh[x] Cosh[y] + Sinh[x] Sinh[y]
Cosh[3 z] == Cosh[z]^3 + 3 Cosh[z] Sinh[z]^2
Tanh[x + y] == Cosh[y] Sinh[x]/(Cosh[x] Cosh[y] + Sinh[x] Sinh[y]) + Cosh[x] Sinh[y]/(Cosh[x] Cosh[y] + Sinh[x] Sinh[y])
Tanh[3 z] == 3 Cosh[z]^2 Sinh[z]/(Cosh[z]^3 + 3 Cosh[z] Sinh[z]^2) + Sinh[z]^3/(Cosh[z]^3 + 3 Cosh[z] Sinh[z]^2)
Coth[x + y] == Cosh[x] Cosh[y]/(Cosh[y] Sinh[x] + Cosh[x] Sinh[y]) + Sinh[x] Sinh[y]/(Cosh[y] Sinh[x] + Cosh[x] Sinh[y])
Coth[3 z] == Cosh[z]^3/(3 Cosh[z]^2 Sinh[z] + Sinh[z]^3) + 3 Cosh[z] Sinh[z]^2/(3 Cosh[z]^2 Sinh[z] + Sinh[z]^3)
Csch[x + y] == 1/(Cosh[y] Sinh[x] + Cosh[x] Sinh[y])
Csch[3 z] == 1/(3 Cosh[z]^2 Sinh[z] + Sinh[z]^3)
Sech[x + y] == 1/(Cosh[x] Cosh[y] + Sinh[x] Sinh[y])
Sech[3 z] == 1/(Cosh[z]^3 + 3 Cosh[z] Sinh[z]^2)
```

TrigFactor

The command `TrigFactor` factors trigonometric and hyperbolic functions. In more detail, it splits up sums and integer multiples that appear in the arguments of trigonometric and hyperbolic functions, and then factors the resulting polynomials in the trigonometric and hyperbolic functions, using the corresponding identities where possible. Here are some examples.

```
TrigFactor[Sinh[x] + ⅈ Cosh[y]]
```

```
(ⅈ Cosh[x/2 - y/2] + Sinh[x/2 - y/2]) (Cosh[x/2 + y/2] - ⅈ Sinh[x/2 + y/2])
```

```
Tanh[x] - Coth[y] // TrigFactor
```

```
-Cosh[x - y] Csch[y] Sech[x]
```

```
TrigFactor[{Sinh[x] + Sinh[y],
            Cosh[x] + Cosh[y],
            Tanh[x] + Tanh[y],
            Coth[x] + Coth[y],
            Csch[x] + Csch[y],
            Sech[x] + Sech[y]}]
```

```
{2 Cosh[x/2 - y/2] Sinh[x/2 + y/2], 2 Cosh[x/2 - y/2] Cosh[x/2 + y/2], Sech[x] Sech[y] Sinh[x + y],
 Csch[x] Csch[y] Sinh[x + y], 1/2 Cosh[x/2 - y/2] Csch[x/2] Csch[y/2] Sech[x/2] Sech[y/2] Sinh[x/2 + y/2],
 2 Cosh[x/2 - y/2] Cosh[x/2 + y/2] / ((Cosh[x/2] - ⅈ Sinh[x/2]) (Cosh[x/2] + ⅈ Sinh[x/2]) (Cosh[y/2] - ⅈ Sinh[y/2]) (Cosh[y/2] + ⅈ Sinh[y/2]))}
```

TrigReduce

The command `TrigReduce` rewrites products and powers of trigonometric and hyperbolic functions in terms of those functions with combined arguments. In more detail, it typically yields a linear expression involving trigonometric and hyperbolic functions with more complicated arguments. `TrigReduce` is approximately opposite to `TrigExpand` and `TrigFactor`. Here are some examples.

```
TrigReduce[Sinh[z]^3]
```

$$\frac{1}{4}(-3\,\text{Sinh}[z]+\text{Sinh}[3\,z])$$

```
Sinh[x] Cosh[y] // TrigReduce
```

$$\frac{1}{2}(\text{Sinh}[x-y]+\text{Sinh}[x+y])$$

```
TrigReduce[{Sinh[z]^2, Cosh[z]^2, Tanh[z]^2, Coth[z]^2, Csch[z]^2, Sech[z]^2}]
```

$$\left\{\frac{1}{2}(-1+\text{Cosh}[2\,z]),\ \frac{1}{2}(1+\text{Cosh}[2\,z]),\ \frac{-1+\text{Cosh}[2\,z]}{1+\text{Cosh}[2\,z]},\ \frac{1+\text{Cosh}[2\,z]}{-1+\text{Cosh}[2\,z]},\ \frac{2}{-1+\text{Cosh}[2\,z]},\ \frac{2}{1+\text{Cosh}[2\,z]}\right\}$$

```
TrigReduce[TrigExpand[{{Sinh[x + y], Sinh[3 z], Sinh[x] Sinh[y]},
                       {Cosh[x + y], Cosh[3 z], Cosh[x] Cosh[y]},
                       {Tanh[x + y], Tanh[3 z], Tanh[x] Tanh[y]},
                       {Coth[x + y], Coth[3 z], Coth[x] Coth[y]},
                       {Csch[x + y], Csch[3 z], Csch[x] Csch[y]},
                       {Sech[x + y], Sech[3 z], Sech[x] Sech[y]}}]]
```

$$\left\{\left\{\text{Sinh}[x+y],\ \text{Sinh}[3\,z],\ \frac{1}{2}(-\text{Cosh}[x-y]+\text{Cosh}[x+y])\right\},\right.$$
$$\left\{\text{Cosh}[x+y],\ \text{Cosh}[3\,z],\ \frac{1}{2}(\text{Cosh}[x-y]+\text{Cosh}[x+y])\right\},$$
$$\left\{\text{Tanh}[x+y],\ \text{Tanh}[3\,z],\ \frac{-\text{Cosh}[x-y]+\text{Cosh}[x+y]}{\text{Cosh}[x-y]+\text{Cosh}[x+y]}\right\},$$
$$\left\{\text{Coth}[x+y],\ \text{Coth}[3\,z],\ \frac{-\text{Cosh}[x-y]-\text{Cosh}[x+y]}{\text{Cosh}[x-y]-\text{Cosh}[x+y]}\right\},$$
$$\left\{\text{Csch}[x+y],\ \text{Csch}[3\,z],\ -\frac{2}{\text{Cosh}[x-y]-\text{Cosh}[x+y]}\right\},$$
$$\left.\left\{\text{Sech}[x+y],\ \text{Sech}[3\,z],\ \frac{2}{\text{Cosh}[x-y]+\text{Cosh}[x+y]}\right\}\right\}$$

```
TrigReduce[TrigFactor[{Sinh[x] + Sinh[y], Cosh[x] + Cosh[y],
   Tanh[x] + Tanh[y], Coth[x] + Coth[y], Csch[x] + Csch[y], Sech[x] + Sech[y]}]]
```

$$\left\{\text{Sinh}[x]+\text{Sinh}[y],\ \text{Cosh}[x]+\text{Cosh}[y],\ \frac{2\,\text{Sinh}[x+y]}{\text{Cosh}[x-y]+\text{Cosh}[x+y]},\right.$$
$$\left.-\frac{2\,\text{Sinh}[x+y]}{\text{Cosh}[x-y]-\text{Cosh}[x+y]},\ -\frac{2\,(\text{Sinh}[x]+\text{Sinh}[y])}{\text{Cosh}[x-y]-\text{Cosh}[x+y]},\ \frac{2\,(\text{Cosh}[x]+\text{Cosh}[y])}{\text{Cosh}[x-y]+\text{Cosh}[x+y]}\right\}$$

TrigToExp

The command TrigToExp converts direct and inverse trigonometric and hyperbolic functions to exponentials or logarithmic functions. It tries, where possible, to give results that do not involve explicit complex numbers. Here are some examples.

```
TrigToExp[Sinh[2 z]]
```

$$-\frac{1}{2}e^{-2z}+\frac{e^{2z}}{2}$$

```
Sinh[z] Tanh[2 z] // TrigToExp
```

$$\frac{(-e^{-z}+e^{z})\left(-e^{-2z}+e^{2z}\right)}{2\left(e^{-2z}+e^{2z}\right)}$$

```
TrigToExp[{Sinh[z], Cosh[z], Tanh[z], Coth[z], Csch[z], Sech[z]}]
```

$$\left\{-\frac{e^{-z}}{2}+\frac{e^{z}}{2},\ \frac{e^{-z}}{2}+\frac{e^{z}}{2},\ \frac{-e^{-z}+e^{z}}{e^{-z}+e^{z}},\ \frac{e^{-z}+e^{z}}{-e^{-z}+e^{z}},\ \frac{2}{-e^{-z}+e^{z}},\ \frac{2}{e^{-z}+e^{z}}\right\}$$

ExpToTrig

The command ExpToTrig converts exponentials to trigonometric or hyperbolic functions. It tries, where possible, to give results that do not involve explicit complex numbers. It is approximately opposite to TrigToExp. Here are some examples.

$$\text{ExpToTrig}\left[e^{x\beta}\right]$$

$$\text{Cosh}[x\beta]+\text{Sinh}[x\beta]$$

$$\frac{e^{x\alpha}-e^{x\beta}}{e^{x\gamma}+e^{x\delta}}\ //\ \text{ExpToTrig}$$

$$\frac{\text{Cosh}[x\alpha]-\text{Cosh}[x\beta]+\text{Sinh}[x\alpha]-\text{Sinh}[x\beta]}{\text{Cosh}[x\gamma]+\text{Cosh}[x\delta]+\text{Sinh}[x\gamma]+\text{Sinh}[x\delta]}$$

```
ExpToTrig[TrigToExp[{Sinh[z], Cosh[z], Tanh[z], Coth[z], Csch[z], Sech[z]}]]
```

```
{Sinh[z], Cosh[z], Tanh[z], Coth[z], Csch[z], Sech[z]}
```

$$\text{ExpToTrig}\left[\left\{\alpha e^{-x\beta}+\alpha e^{x\beta},\ \alpha e^{-x\beta}+\gamma e^{i x\beta}\right\}\right]$$

$$\{2\alpha\,\text{Cosh}[x\beta],\ \gamma\,\text{Cos}[x\beta]+\alpha\,\text{Cosh}[x\beta]+i\gamma\,\text{Sin}[x\beta]-\alpha\,\text{Sinh}[x\beta]\}$$

ComplexExpand

The function ComplexExpand expands expressions assuming that all the occurring variables are real. The value option TargetFunctions is a list of functions from the set {Re, Im, Abs, Arg, Conjugate, Sign}. ComplexExpand tries to give results in terms of the specified functions. Here are some examples.

$$\text{ComplexExpand}[\text{Sinh}[x+i\,y]\ \text{Cosh}[x-i\,y]]$$

$$\text{Cos}[y]^2\,\text{Cosh}[x]\,\text{Sinh}[x]+\text{Cosh}[x]\,\text{Sin}[y]^2\,\text{Sinh}[x]+ i\left(\text{Cos}[y]\,\text{Cosh}[x]^2\,\text{Sin}[y]-\text{Cos}[y]\,\text{Sin}[y]\,\text{Sinh}[x]^2\right)$$

```
Csch[x + ⅈ y] Sech[x - ⅈ y] // ComplexExpand
```

$$-\frac{4\,\text{Cos}[y]^2\,\text{Cosh}[x]\,\text{Sinh}[x]}{(\text{Cos}[2\,y]-\text{Cosh}[2\,x])\,(\text{Cos}[2\,y]+\text{Cosh}[2\,x])}-\frac{4\,\text{Cosh}[x]\,\text{Sin}[y]^2\,\text{Sinh}[x]}{(\text{Cos}[2\,y]-\text{Cosh}[2\,x])\,(\text{Cos}[2\,y]+\text{Cosh}[2\,x])}+\\ i\left(\frac{4\,\text{Cos}[y]\,\text{Cosh}[x]^2\,\text{Sin}[y]}{(\text{Cos}[2\,y]-\text{Cosh}[2\,x])\,(\text{Cos}[2\,y]+\text{Cosh}[2\,x])}-\frac{4\,\text{Cos}[y]\,\text{Sin}[y]\,\text{Sinh}[x]^2}{(\text{Cos}[2\,y]-\text{Cosh}[2\,x])\,(\text{Cos}[2\,y]+\text{Cosh}[2\,x])}\right)$$

```
li1 = {Sinh[x + ⅈ y], Cosh[x + ⅈ y], Tanh[x + ⅈ y], Coth[x + ⅈ y], Csch[x + ⅈ y], Sech[x + ⅈ y]}
```

$$\{\text{Sinh}[x+i\,y],\ \text{Cosh}[x+i\,y],\ \text{Tanh}[x+i\,y],\ \text{Coth}[x+i\,y],\ \text{Csch}[x+i\,y],\ \text{Sech}[x+i\,y]\}$$

```
ComplexExpand[li1]
```

$$\Big\{i\,\text{Cosh}[x]\,\text{Sin}[y]+\text{Cos}[y]\,\text{Sinh}[x],\ \text{Cos}[y]\,\text{Cosh}[x]+i\,\text{Sin}[y]\,\text{Sinh}[x],\\ \frac{i\,\text{Sin}[2\,y]}{\text{Cos}[2\,y]+\text{Cosh}[2\,x]}+\frac{\text{Sinh}[2\,x]}{\text{Cos}[2\,y]+\text{Cosh}[2\,x]},\ \frac{i\,\text{Sin}[2\,y]}{\text{Cos}[2\,y]-\text{Cosh}[2\,x]}-\frac{\text{Sinh}[2\,x]}{\text{Cos}[2\,y]-\text{Cosh}[2\,x]},\\ \frac{2\,i\,\text{Cosh}[x]\,\text{Sin}[y]}{\text{Cos}[2\,y]-\text{Cosh}[2\,x]}-\frac{2\,\text{Cos}[y]\,\text{Sinh}[x]}{\text{Cos}[2\,y]-\text{Cosh}[2\,x]},\ \frac{2\,\text{Cos}[y]\,\text{Cosh}[x]}{\text{Cos}[2\,y]+\text{Cosh}[2\,x]}-\frac{2\,i\,\text{Sin}[y]\,\text{Sinh}[x]}{\text{Cos}[2\,y]+\text{Cosh}[2\,x]}\Big\}$$

```
ComplexExpand[Re[#] & /@ li1, TargetFunctions → {Re, Im}]
```

$$\Big\{\text{Cos}[y]\,\text{Sinh}[x],\ \text{Cos}[y]\,\text{Cosh}[x],\ \frac{\text{Sinh}[2\,x]}{\text{Cos}[2\,y]+\text{Cosh}[2\,x]},\\ -\frac{\text{Sinh}[2\,x]}{\text{Cos}[2\,y]-\text{Cosh}[2\,x]},\ -\frac{2\,\text{Cos}[y]\,\text{Sinh}[x]}{\text{Cos}[2\,y]-\text{Cosh}[2\,x]},\ \frac{2\,\text{Cos}[y]\,\text{Cosh}[x]}{\text{Cos}[2\,y]+\text{Cosh}[2\,x]}\Big\}$$

```
ComplexExpand[Im[#] & /@ li1, TargetFunctions → {Re, Im}]
```

$$\Big\{\text{Cosh}[x]\,\text{Sin}[y],\ \text{Sin}[y]\,\text{Sinh}[x],\ \frac{\text{Sin}[2\,y]}{\text{Cos}[2\,y]+\text{Cosh}[2\,x]},\\ \frac{\text{Sin}[2\,y]}{\text{Cos}[2\,y]-\text{Cosh}[2\,x]},\ \frac{2\,\text{Cosh}[x]\,\text{Sin}[y]}{\text{Cos}[2\,y]-\text{Cosh}[2\,x]},\ -\frac{2\,\text{Sin}[y]\,\text{Sinh}[x]}{\text{Cos}[2\,y]+\text{Cosh}[2\,x]}\Big\}$$

```
ComplexExpand[Abs[#] & /@ li1, TargetFunctions → {Re, Im}]
```

$$\left\{\sqrt{\text{Cosh}[x]^2\,\text{Sin}[y]^2+\text{Cos}[y]^2\,\text{Sinh}[x]^2},\ \sqrt{\text{Cos}[y]^2\,\text{Cosh}[x]^2+\text{Sin}[y]^2\,\text{Sinh}[x]^2},\right.$$

$$\sqrt{\frac{\text{Sin}[2\,y]^2}{(\text{Cos}[2\,y]+\text{Cosh}[2\,x])^2}+\frac{\text{Sinh}[2\,x]^2}{(\text{Cos}[2\,y]+\text{Cosh}[2\,x])^2}},$$

$$\sqrt{\frac{\text{Sin}[2\,y]^2}{(\text{Cos}[2\,y]-\text{Cosh}[2\,x])^2}+\frac{\text{Sinh}[2\,x]^2}{(\text{Cos}[2\,y]-\text{Cosh}[2\,x])^2}},$$

$$\sqrt{\frac{4\,\text{Cosh}[x]^2\,\text{Sin}[y]^2}{(\text{Cos}[2\,y]-\text{Cosh}[2\,x])^2}+\frac{4\,\text{Cos}[y]^2\,\text{Sinh}[x]^2}{(\text{Cos}[2\,y]-\text{Cosh}[2\,x])^2}},$$

$$\left.\sqrt{\frac{4\,\text{Cos}[y]^2\,\text{Cosh}[x]^2}{(\text{Cos}[2\,y]+\text{Cosh}[2\,x])^2}+\frac{4\,\text{Sin}[y]^2\,\text{Sinh}[x]^2}{(\text{Cos}[2\,y]+\text{Cosh}[2\,x])^2}}\right\}$$

% // Simplify[#, {x, y} ∈ Reals] &

$$\left\{\frac{\sqrt{-\text{Cos}[2\,y]+\text{Cosh}[2\,x]}}{\sqrt{2}},\ \frac{\sqrt{\text{Cos}[2\,y]+\text{Cosh}[2\,x]}}{\sqrt{2}},\ \frac{\sqrt{\text{Sin}[2\,y]^2+\text{Sinh}[2\,x]^2}}{\text{Cos}[2\,y]+\text{Cosh}[2\,x]},\right.$$

$$\left.\sqrt{-\frac{\text{Cos}[2\,y]+\text{Cosh}[2\,x]}{\text{Cos}[2\,y]-\text{Cosh}[2\,x]}},\ \frac{\sqrt{2}}{\sqrt{-\text{Cos}[2\,y]+\text{Cosh}[2\,x]}},\ \frac{\sqrt{2}}{\sqrt{\text{Cos}[2\,y]+\text{Cosh}[2\,x]}}\right\}$$

ComplexExpand[Arg[#] & /@ li1, TargetFunctions → {Re, Im}]

$$\{\text{ArcTan}[\text{Cos}[y]\,\text{Sinh}[x],\ \text{Cosh}[x]\,\text{Sin}[y]],\ \text{ArcTan}[\text{Cos}[y]\,\text{Cosh}[x],\ \text{Sin}[y]\,\text{Sinh}[x]],$$

$$\text{ArcTan}\left[\frac{\text{Sinh}[2\,x]}{\text{Cos}[2\,y]+\text{Cosh}[2\,x]},\ \frac{\text{Sin}[2\,y]}{\text{Cos}[2\,y]+\text{Cosh}[2\,x]}\right],$$

$$\text{ArcTan}\left[-\frac{\text{Sinh}[2\,x]}{\text{Cos}[2\,y]-\text{Cosh}[2\,x]},\ \frac{\text{Sin}[2\,y]}{\text{Cos}[2\,y]-\text{Cosh}[2\,x]}\right],$$

$$\text{ArcTan}\left[-\frac{2\,\text{Cos}[y]\,\text{Sinh}[x]}{\text{Cos}[2\,y]-\text{Cosh}[2\,x]},\ \frac{2\,\text{Cosh}[x]\,\text{Sin}[y]}{\text{Cos}[2\,y]-\text{Cosh}[2\,x]}\right],$$

$$\text{ArcTan}\left[\frac{2\,\text{Cos}[y]\,\text{Cosh}[x]}{\text{Cos}[2\,y]+\text{Cosh}[2\,x]},\ -\frac{2\,\text{Sin}[y]\,\text{Sinh}[x]}{\text{Cos}[2\,y]+\text{Cosh}[2\,x]}\right]\}$$

% // Simplify[#, {x, y} ∈ Reals] &

{ArcTan[Cos[y] Sinh[x], Cosh[x] Sin[y]], ArcTan[Cos[y] Cosh[x], Sin[y] Sinh[x]],
ArcTan[Sinh[2 x], Sin[2 y]], ArcTan[Cosh[x] Sinh[x], -Cos[y] Sin[y]],
ArcTan[Cos[y] Sinh[x], -Cosh[x] Sin[y]], ArcTan[Cos[y] Cosh[x], -Sin[y] Sinh[x]]}

ComplexExpand[Conjugate[#] & /@ li1, TargetFunctions → {Re, Im}] // Simplify

$$\left\{-i \operatorname{Cosh}[x] \operatorname{Sin}[y]+\operatorname{Cos}[y] \operatorname{Sinh}[x], \operatorname{Cos}[y] \operatorname{Cosh}[x]-i \operatorname{Sin}[y] \operatorname{Sinh}[x], \frac{-i \operatorname{Sin}[2 y]+\operatorname{Sinh}[2 x]}{\operatorname{Cos}[2 y]+\operatorname{Cosh}[2 x]}, -\frac{i \operatorname{Sin}[2 y]+\operatorname{Sinh}[2 x]}{\operatorname{Cos}[2 y]-\operatorname{Cosh}[2 x]}, \frac{1}{-i \operatorname{Cosh}[x] \operatorname{Sin}[y]+\operatorname{Cos}[y] \operatorname{Sinh}[x]}, \frac{1}{\operatorname{Cos}[y] \operatorname{Cosh}[x]-i \operatorname{Sin}[y] \operatorname{Sinh}[x]}\right\}$$

Simplify

The command `Simplify` performs a sequence of algebraic transformations on an expression, and returns the simplest form it finds. Here are two examples.

```
Simplify[Sinh[2 z] / Sinh[z]]
```

```
2 Cosh[z]
```

```
Sinh[2 z] / Cosh[z] // Simplify
```

```
2 Sinh[z]
```

Here is a large collection of hyperbolic identities. All are written as one large logical conjunction.

```
Simplify[#] & /@ (Cosh[z]^2 - Sinh[z]^2 == 1 ∧
  Sinh[z]^2 == (Cosh[2 z] - 1)/2 ∧ Cosh[z]^2 == (1 + Cosh[2 z])/2 ∧
  Tanh[z]^2 == (Cosh[2 z] - 1)/(Cosh[2 z] + 1) ∧ Coth[z]^2 == (Cosh[2 z] + 1)/(Cosh[2 z] - 1) ∧
  Sinh[2 z] == 2 Sinh[z] Cosh[z] ∧ Cosh[2 z] == Cosh[z]^2 + Sinh[z]^2 == 2 Cosh[z]^2 - 1 ∧
  Sinh[a + b] == Sinh[a] Cosh[b] + Cosh[a] Sinh[b] ∧
    Sinh[a - b] == Sinh[a] Cosh[b] - Cosh[a] Sinh[b] ∧
  Cosh[a + b] == Cosh[a] Cosh[b] + Sinh[a] Sinh[b] ∧
    Cosh[a - b] == Cosh[a] Cosh[b] - Sinh[a] Sinh[b] ∧
  Sinh[a] + Sinh[b] == 2 Sinh[(a + b)/2] Cosh[(a - b)/2] ∧
    Sinh[a] - Sinh[b] == 2 Cosh[(a + b)/2] Sinh[(a - b)/2] ∧
  Cosh[a] + Cosh[b] == 2 Cosh[(a + b)/2] Cosh[(a - b)/2] ∧
    Cosh[a] - Cosh[b] == -2 Sinh[(a + b)/2] Sinh[(b - a)/2] ∧
  Tanh[a] + Tanh[b] == Sinh[a + b]/(Cosh[a] Cosh[b]) ∧ Tanh[a] - Tanh[b] == Sinh[a - b]/(Cosh[a] Cosh[b]) ∧
  A Sinh[z] + B Cosh[z] == A Sqrt[1 - B^2/A^2] Sinh[z + ArcTanh[B/A]] ∧
  Sinh[a] Sinh[b] == (Cosh[a + b] - Cosh[a - b])/2 ∧ Cosh[a] Cosh[b] ==
    (Cosh[a - b] + Cosh[a + b])/2 ∧ Sinh[a] Cosh[b] == (Sinh[a + b] + Sinh[a - b])/2 ∧
  Sinh[z/2]^2 == (Cosh[z] - 1)/2 ∧ Cosh[z/2]^2 == (1 + Cosh[z])/2 ∧
  Tanh[z/2] == (Cosh[z] - 1)/Sinh[z] == Sinh[z]/(1 + Cosh[z]) ∧ Coth[z/2] == Sinh[z]/(Cosh[z] - 1) == (1 + Cosh[z])/Sinh[z])
```

```
True
```

The command `Simplify` has the `Assumption` option. For example, *Mathematica* knows that $\sinh(x) > 0$ for all real positive x, and uses the periodicity of hyperbolic functions for the symbolic integer coefficient k of $k \pi i$.

```
Simplify[Abs[Sinh[x]] > 0, x > 0]
```

```
True
```

```
Abs[Sinh[x]] > 0 // Simplify[#, x > 0] &
```

```
True
```

```
Simplify[{Sinh[z + 2 k π ⅈ], Cosh[z + 2 k π ⅈ], Tanh[z + k π ⅈ],
  Coth[z + k π ⅈ], Csch[z + 2 k π ⅈ], Sech[z + 2 k π ⅈ]}, k ∈ Integers]
```

```
{Sinh[z], Cosh[z], Tanh[z], Coth[z], Csch[z], Sech[z]}
```

```
Simplify[{Sinh[z + k π ⅈ] / Sinh[z], Cosh[z + k π ⅈ] / Cosh[z], Tanh[z + k π ⅈ] / Tanh[z],
  Coth[z + k π ⅈ] / Coth[z], Csch[z + k π ⅈ] / Csch[z], Sech[z + k π ⅈ] / Sech[z]}, k ∈ Integers]
```

$$\{(-1)^k, (-1)^k, 1, 1, (-1)^k, (-1)^k\}$$

Mathematica also knows that the composition of inverse and direct hyperbolic functions produces the value of the inner argument under the appropriate restriction. Here are some examples.

```
Simplify[{ArcSinh[Sinh[z]], ArcTanh[Tanh[z]],
  ArcCoth[Coth[z]], ArcCsch[Csch[z]]}, -π / 2 < Im[z] < π / 2]
```

```
{z, z, z, z}
```

```
Simplify[{ArcCosh[Cosh[z]], ArcSech[Sech[z]]}, -π < Im[z] < π ⋀ Re[z] > 0]
```

```
{z, z}
```

FunctionExpand (and Together)

While the hyperbolic functions auto-evaluate for simple fractions of πi, for more complicated cases they stay as hyperbolic functions to avoid the build up of large expressions. Using the function `FunctionExpand`, such expressions can be transformed into explicit radicals.

$$\mathrm{Cosh}\left[\frac{\pi i}{32}\right]$$

$$\mathrm{Cos}\left[\frac{\pi}{32}\right]$$

$$\mathrm{FunctionExpand}\left[\mathrm{Cosh}\left[\frac{\pi i}{32}\right]\right]$$

$$\frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}$$

$$\mathrm{Coth}\left[\frac{\pi i}{24}\right] \text{ // FunctionExpand}$$

$$-\frac{i\left(\frac{\sqrt{2-\sqrt{2}}}{4}+\frac{1}{4}\sqrt{3\left(2+\sqrt{2}\right)}\right)}{-\frac{1}{4}\sqrt{3\left(2-\sqrt{2}\right)}+\frac{\sqrt{2+\sqrt{2}}}{4}}$$

{Sinh[$\frac{\pi i}{16}$], Cosh[$\frac{\pi i}{16}$], Tanh[$\frac{\pi i}{16}$], Coth[$\frac{\pi i}{16}$], Csch[$\frac{\pi i}{16}$], Sech[$\frac{\pi i}{16}$]}

$$\left\{i \operatorname{Sin}\left[\frac{\pi}{16}\right], \operatorname{Cos}\left[\frac{\pi}{16}\right], i \operatorname{Tan}\left[\frac{\pi}{16}\right], -i \operatorname{Cot}\left[\frac{\pi}{16}\right], -i \operatorname{Csc}\left[\frac{\pi}{16}\right], \operatorname{Sec}\left[\frac{\pi}{16}\right]\right\}$$

FunctionExpand[%]

$$\left\{\frac{1}{2} i \sqrt{2-\sqrt{2+\sqrt{2}}}, \frac{1}{2} \sqrt{2+\sqrt{2+\sqrt{2}}}, i \sqrt{\frac{2-\sqrt{2+\sqrt{2}}}{2+\sqrt{2+\sqrt{2}}}},\right.$$

$$\left.-i \sqrt{\frac{2+\sqrt{2+\sqrt{2}}}{2-\sqrt{2+\sqrt{2}}}}, -\frac{2 i}{\sqrt{2-\sqrt{2+\sqrt{2}}}}, \frac{2}{\sqrt{2+\sqrt{2+\sqrt{2}}}}\right\}$$

{Sinh[$\frac{\pi i}{60}$], Cosh[$\frac{\pi i}{60}$], Tanh[$\frac{\pi i}{60}$], Coth[$\frac{\pi i}{60}$], Csch[$\frac{\pi i}{60}$], Sech[$\frac{\pi i}{60}$]}

$$\left\{i \operatorname{Sin}\left[\frac{\pi}{60}\right], \operatorname{Cos}\left[\frac{\pi}{60}\right], i \operatorname{Tan}\left[\frac{\pi}{60}\right], -i \operatorname{Cot}\left[\frac{\pi}{60}\right], -i \operatorname{Csc}\left[\frac{\pi}{60}\right], \operatorname{Sec}\left[\frac{\pi}{60}\right]\right\}$$

Together[FunctionExpand[%]]

$$\left\{\frac{1}{16} i \left(-\sqrt{2}-\sqrt{6}+\sqrt{10}+\sqrt{30}+2\sqrt{5+\sqrt{5}}-2\sqrt{3\left(5+\sqrt{5}\right)}\right),\right.$$

$$\frac{1}{16}\left(\sqrt{2}-\sqrt{6}-\sqrt{10}+\sqrt{30}+2\sqrt{5+\sqrt{5}}+2\sqrt{3\left(5+\sqrt{5}\right)}\right),$$

$$-\frac{i\left(1+\sqrt{3}-\sqrt{5}-\sqrt{15}-\sqrt{2\left(5+\sqrt{5}\right)}+\sqrt{6\left(5+\sqrt{5}\right)}\right)}{1-\sqrt{3}-\sqrt{5}+\sqrt{15}+\sqrt{2\left(5+\sqrt{5}\right)}+\sqrt{6\left(5+\sqrt{5}\right)}},$$

$$\frac{i\left(1-\sqrt{3}-\sqrt{5}+\sqrt{15}+\sqrt{2\left(5+\sqrt{5}\right)}+\sqrt{6\left(5+\sqrt{5}\right)}\right)}{1+\sqrt{3}-\sqrt{5}-\sqrt{15}-\sqrt{2\left(5+\sqrt{5}\right)}+\sqrt{6\left(5+\sqrt{5}\right)}},$$

$$-\frac{16 i}{-\sqrt{2}-\sqrt{6}+\sqrt{10}+\sqrt{30}+2\sqrt{5+\sqrt{5}}-2\sqrt{3\left(5+\sqrt{5}\right)}},$$

$$\left.\frac{16}{\sqrt{2}-\sqrt{6}-\sqrt{10}+\sqrt{30}+2\sqrt{5+\sqrt{5}}+2\sqrt{3\left(5+\sqrt{5}\right)}}\right\}$$

If the denominator contains squares of integers other than 2, the results always contain complex numbers (meaning that the imaginary number $i = \sqrt{-1}$ appears unavoidably).

```
{Sinh[π ⅈ/9], Cosh[π ⅈ/9], Tanh[π ⅈ/9], Coth[π ⅈ/9], Csch[π ⅈ/9], Sech[π ⅈ/9]}
```

$$\left\{i \operatorname{Sin}\left[\frac{\pi}{9}\right], \operatorname{Cos}\left[\frac{\pi}{9}\right], i \operatorname{Tan}\left[\frac{\pi}{9}\right], -i \operatorname{Cot}\left[\frac{\pi}{9}\right], -i \operatorname{Csc}\left[\frac{\pi}{9}\right], \operatorname{Sec}\left[\frac{\pi}{9}\right]\right\}$$

```
FunctionExpand[%] // Together
```

$$\Big\{\frac{1}{8}\left(2^{2/3}\left(-1-i\sqrt{3}\right)^{1/3}+i\,2^{2/3}\sqrt{3}\left(-1-i\sqrt{3}\right)^{1/3}-2^{2/3}\left(-1+i\sqrt{3}\right)^{1/3}+i\,2^{2/3}\sqrt{3}\left(-1+i\sqrt{3}\right)^{1/3}\right),$$

$$\frac{1}{8}\left(2^{2/3}\left(-1-i\sqrt{3}\right)^{1/3}+i\,2^{2/3}\sqrt{3}\left(-1-i\sqrt{3}\right)^{1/3}+2^{2/3}\left(-1+i\sqrt{3}\right)^{1/3}-i\,2^{2/3}\sqrt{3}\left(-1+i\sqrt{3}\right)^{1/3}\right),$$

$$\frac{-i\left(-1-i\sqrt{3}\right)^{1/3}+\sqrt{3}\left(-1-i\sqrt{3}\right)^{1/3}+i\left(-1+i\sqrt{3}\right)^{1/3}+\sqrt{3}\left(-1+i\sqrt{3}\right)^{1/3}}{-i\left(-1-i\sqrt{3}\right)^{1/3}+\sqrt{3}\left(-1-i\sqrt{3}\right)^{1/3}-i\left(-1+i\sqrt{3}\right)^{1/3}-\sqrt{3}\left(-1+i\sqrt{3}\right)^{1/3}},$$

$$\frac{-i\left(-1-i\sqrt{3}\right)^{1/3}+\sqrt{3}\left(-1-i\sqrt{3}\right)^{1/3}-i\left(-1+i\sqrt{3}\right)^{1/3}-\sqrt{3}\left(-1+i\sqrt{3}\right)^{1/3}}{-i\left(-1-i\sqrt{3}\right)^{1/3}+\sqrt{3}\left(-1-i\sqrt{3}\right)^{1/3}+i\left(-1+i\sqrt{3}\right)^{1/3}+\sqrt{3}\left(-1+i\sqrt{3}\right)^{1/3}},$$

$$-(8\,i)\Big/\left(-i\,2^{2/3}\left(-1-i\sqrt{3}\right)^{1/3}+2^{2/3}\sqrt{3}\left(-1-i\sqrt{3}\right)^{1/3}+i\,2^{2/3}\left(-1+i\sqrt{3}\right)^{1/3}+2^{2/3}\sqrt{3}\left(-1+i\sqrt{3}\right)^{1/3}\right),$$

$$-(8\,i)\Big/\left(-i\,2^{2/3}\left(-1-i\sqrt{3}\right)^{1/3}+2^{2/3}\sqrt{3}\left(-1-i\sqrt{3}\right)^{1/3}-i\,2^{2/3}\left(-1+i\sqrt{3}\right)^{1/3}-2^{2/3}\sqrt{3}\left(-1+i\sqrt{3}\right)^{1/3}\right)\Big\}$$

Here the function `RootReduce` is used to express the previous algebraic numbers as numbered roots of polynomial equations.

```
RootReduce[Simplify[%]]
```

```
{Root[3 + 36 #1^2 + 96 #1^4 + 64 #1^6 &, 4], Root[-1 - 6 #1 + 8 #1^3 &, 3],
 Root[3 + 27 #1^2 + 33 #1^4 + #1^6 &, 4], Root[1 + 33 #1^2 + 27 #1^4 + 3 #1^6 &, 3],
 Root[64 + 96 #1^2 + 36 #1^4 + 3 #1^6 &, 5], Root[-8 + 6 #1^2 + #1^3 &, 3]}
```

The function `FunctionExpand` also reduces hyperbolic expressions with compound arguments or compositions, including hyperbolic functions, to simpler forms. Here are some examples.

```
FunctionExpand[Coth[Sqrt[-z^2]]]
```

$$-\frac{\sqrt{-z}\ \mathrm{Cot}[z]}{\sqrt{z}}$$

Tanh$\left[\sqrt{i\, z^2}\right]$ // **FunctionExpand**

$$-\frac{(-1)^{3/4}\sqrt{-(-1)^{3/4} z}\ \sqrt{(-1)^{3/4} z}\ \mathrm{Tanh}\left[(-1)^{1/4} z\right]}{z}$$

$\left\{\mathrm{Sinh}\left[\sqrt{z^2}\right], \mathrm{Cosh}\left[\sqrt{z^2}\right], \mathrm{Tanh}\left[\sqrt{z^2}\right], \mathrm{Coth}\left[\sqrt{z^2}\right], \mathrm{Csch}\left[\sqrt{z^2}\right], \mathrm{Sech}\left[\sqrt{z^2}\right]\right\}$ // **FunctionExpand**

$$\left\{\frac{\sqrt{-i z}\ \sqrt{i z}\ \mathrm{Sinh}[z]}{z}, \mathrm{Cosh}[z], \frac{\sqrt{-i z}\ \sqrt{i z}\ \mathrm{Tanh}[z]}{z}, \frac{\sqrt{-i z}\ \sqrt{i z}\ \mathrm{Coth}[z]}{z}, \frac{\sqrt{-i z}\ \sqrt{i z}\ \mathrm{Csch}[z]}{z}, \mathrm{Sech}[z]\right\}$$

Applying `Simplify` to the last expression gives a more compact result.

Simplify[%]

$$\left\{\frac{\sqrt{z^2}\ \mathrm{Sinh}[z]}{z}, \mathrm{Cosh}[z], \frac{\sqrt{z^2}\ \mathrm{Tanh}[z]}{z}, \frac{\sqrt{z^2}\ \mathrm{Coth}[z]}{z}, \frac{\sqrt{z^2}\ \mathrm{Csch}[z]}{z}, \mathrm{Sech}[z]\right\}$$

Here are some similar examples.

Sinh[2 ArcTanh[z]] // FunctionExpand

$$\frac{2 z}{1 - z^2}$$

Cosh$\left[\frac{\mathrm{ArcCoth}[z]}{2}\right]$ // **FunctionExpand**

$$\frac{\sqrt{1 + \frac{\sqrt{-i z}\ \sqrt{i z}}{\sqrt{(-1+z)(1+z)}}}}{\sqrt{2}}$$

{Sinh[2 ArcSinh[z]], Cosh[2 ArcCosh[z]], Tanh[2 ArcTanh[z]], Coth[2 ArcCoth[z]], Csch[2 ArcCsch[z]], Sech[2 ArcSech[z]]} // FunctionExpand

$$\left\{2 z \sqrt{i(-i + z)}\ \sqrt{-i(i + z)}, z^2 + (-1 + z)(1 + z), -\frac{2(-1 + z) z (1 + z)}{\left(1 - z^2\right)\left(1 + z^2\right)}, \frac{1}{2}\left(1 - \frac{1}{z^2}\right) z \left(\frac{1}{(-1 + z)(1 + z)} + \frac{z^2}{(-1 + z)(1 + z)}\right), \frac{\sqrt{-z}\ z^{3/2}}{2\sqrt{-1 - z^2}}, \frac{z^2}{2 - z^2}\right\}$$

$$\left\{\text{Sinh}\left[\frac{\text{ArcSinh}[z]}{2}\right], \text{Cosh}\left[\frac{\text{ArcCosh}[z]}{2}\right], \text{Tanh}\left[\frac{\text{ArcTanh}[z]}{2}\right], \text{Coth}\left[\frac{\text{ArcCoth}[z]}{2}\right], \text{Csch}\left[\frac{\text{ArcCsch}[z]}{2}\right], \text{Sech}\left[\frac{\text{ArcSech}[z]}{2}\right]\right\} \text{ // FunctionExpand}$$

$$\left\{\frac{z\sqrt{-1+\sqrt{i(-i+z)}\sqrt{-i(i+z)}}}{\sqrt{2}\sqrt{-iz}\sqrt{iz}}, \frac{\sqrt{1+z}}{\sqrt{2}}, \frac{z}{1+\sqrt{1-z}\sqrt{1+z}}, z\left(1+\frac{\sqrt{(-1+z)(1+z)}}{\sqrt{-iz}\sqrt{iz}}\right), \frac{\sqrt{2}\sqrt{-\frac{i}{z}}\sqrt{\frac{i}{z}}\,z}{\sqrt{-1+\frac{\sqrt{-1-z^2}}{\sqrt{-z}\sqrt{z}}}}, \frac{\sqrt{2}\sqrt{-z}}{\sqrt{-1-z}}\right\}$$

$$\text{Simplify}[\%]$$

$$\left\{\frac{z\sqrt{-1+\sqrt{1+z^2}}}{\sqrt{2}\sqrt{z^2}}, \frac{\sqrt{1+z}}{\sqrt{2}}, \frac{z}{1+\sqrt{1-z^2}}, z+\frac{\sqrt{z^2}\sqrt{-1+z^2}}{z}, \frac{\sqrt{2}\sqrt{\frac{1}{z^2}}\,z}{\sqrt{-1+\sqrt{1+\frac{1}{z^2}}}}, \frac{\sqrt{2}}{\sqrt{1+\frac{1}{z}}}\right\}$$

FullSimplify

The function `FullSimplify` tries a wider range of transformations than the function `Simplify` and returns the simplest form it finds. Here are some examples that contrast the results of applying these functions to the same expressions.

$$\text{Cosh}\left[\frac{1}{2}\text{Log}[1-iz]-\frac{1}{2}\text{Log}[1+iz]\right] \text{ // Simplify}$$

$$\text{Cosh}\left[\frac{1}{2}(\text{Log}[1-iz]-\text{Log}[1+iz])\right]$$

$$\% \text{ // FullSimplify}$$

$$\frac{1}{\sqrt{1+z^2}}$$

$$\left\{\text{Sinh}\left[-\text{Log}\left[iz+\sqrt{1-z^2}\right]\right], \text{Cosh}\left[-\text{Log}\left[iz+\sqrt{1-z^2}\right]\right], \text{Tanh}\left[-\text{Log}\left[iz+\sqrt{1-z^2}\right]\right], \text{Coth}\left[-\text{Log}\left[iz+\sqrt{1-z^2}\right]\right], \text{Csch}\left[-\text{Log}\left[iz+\sqrt{1-z^2}\right]\right], \text{Sech}\left[-\text{Log}\left[iz+\sqrt{1-z^2}\right]\right]\right\} \text{ // Simplify}$$

$$\left\{-\mathrm{i}\, z,\ \frac{1-z^2+\mathrm{i}\, z\sqrt{1-z^2}}{\mathrm{i}\, z+\sqrt{1-z^2}},\ -\frac{-1+\left(\mathrm{i}\, z+\sqrt{1-z^2}\right)^2}{1+\left(\mathrm{i}\, z+\sqrt{1-z^2}\right)^2},\ -\frac{1+\left(\mathrm{i}\, z+\sqrt{1-z^2}\right)^2}{-1+\left(\mathrm{i}\, z+\sqrt{1-z^2}\right)^2},\ \frac{\mathrm{i}}{z},\ \frac{2\left(\mathrm{i}\, z+\sqrt{1-z^2}\right)}{1+\left(\mathrm{i}\, z+\sqrt{1-z^2}\right)^2}\right\}$$

```
{Sinh[-Log[ⅈ z + Sqrt[1 - z^2]]], Cosh[-Log[ⅈ z + Sqrt[1 - z^2]]],
  Tanh[-Log[ⅈ z + Sqrt[1 - z^2]]], Coth[-Log[ⅈ z + Sqrt[1 - z^2]]],
  Csch[-Log[ⅈ z + Sqrt[1 - z^2]]], Sech[-Log[ⅈ z + Sqrt[1 - z^2]]]} // FullSimplify
```

$$\left\{-\mathrm{i}\, z,\ \sqrt{1-z^2},\ -\frac{\mathrm{i}\, z}{\sqrt{1-z^2}},\ \frac{\mathrm{i}\sqrt{1-z^2}}{z},\ \frac{\mathrm{i}}{z},\ \frac{1}{\sqrt{1-z^2}}\right\}$$

Operations performed by specialized *Mathematica* functions

Series expansions

Calculating the series expansion of hyperbolic functions to hundreds of terms can be done in seconds. Here are some examples.

```
Series[Sinh[z], {z, 0, 5}]
```

$$z+\frac{z^3}{6}+\frac{z^5}{120}+O[z]^6$$

```
Normal[%]
```

$$z+\frac{z^3}{6}+\frac{z^5}{120}$$

```
Series[{Sinh[z], Cosh[z], Tanh[z], Coth[z], Csch[z], Sech[z]}, {z, 0, 3}]
```

$$\left\{z+\frac{z^3}{6}+O[z]^4,\ 1+\frac{z^2}{2}+O[z]^4,\ z-\frac{z^3}{3}+O[z]^4,\ \frac{1}{z}+\frac{z}{3}-\frac{z^3}{45}+O[z]^4,\ \frac{1}{z}-\frac{z}{6}+\frac{7 z^3}{360}+O[z]^4,\ 1-\frac{z^2}{2}+O[z]^4\right\}$$

```
Series[Coth[z], {z, 0, 100}] // Timing
```

$$\left\{0.79\ \text{Second},\ \frac{1}{z}+\frac{z}{3}-\frac{z^3}{45}+\frac{2 z^5}{945}-\frac{z^7}{4725}+\frac{2 z^9}{93\,555}-\frac{1382 z^{11}}{638\,512\,875}+\frac{4 z^{13}}{18\,243\,225}-\frac{3617 z^{15}}{162\,820\,783\,125}+\frac{87\,734 z^{17}}{38\,979\,295\,480\,125}-\frac{349\,222 z^{19}}{1\,531\,329\,465\,290\,625}+\right.$$

$$\frac{310\,732\,z^{21}}{13\,447\,856\,940\,643\,125}-\frac{472\,728\,182\,z^{23}}{201\,919\,571\,963\,756\,521\,875}+\frac{2\,631\,724\,z^{25}}{11\,094\,481\,976\,030\,578\,125}-$$

$$\frac{13\,571\,120\,588\,z^{27}}{564\,653\,660\,170\,076\,273\,671\,875}+\frac{13\,785\,346\,041\,608\,z^{29}}{5\,660\,878\,804\,669\,082\,674\,070\,015\,625}-$$

$$\frac{7\,709\,321\,041\,217\,z^{31}}{31\,245\,110\,285\,511\,170\,603\,633\,203\,125}+\frac{303\,257\,395\,102\,z^{33}}{12\,130\,454\,581\,433\,748\,587\,292\,890\,625}-$$

$$\frac{52\,630\,543\,106\,106\,954\,746\,z^{35}}{20\,777\,977\,561\,866\,588\,586\,487\,628\,662\,044\,921\,875}+\frac{616\,840\,823\,966\,644\,z^{37}}{2\,403\,467\,618\,492\,375\,776\,343\,276\,883\,984\,375}-$$

$$\frac{522\,165\,436\,992\,898\,244\,102\,z^{39}}{20\,080\,431\,172\,289\,638\,826\,798\,401\,128\,390\,556\,640\,625}+$$

$$\frac{6\,080\,390\,575\,672\,283\,210\,764\,z^{41}}{2\,307\,789\,189\,818\,960\,127\,712\,594\,427\,864\,667\,427\,734\,375}-$$

$$\frac{10\,121\,188\,937\,927\,645\,176\,372\,z^{43}}{37\,913\,679\,547\,025\,773\,526\,706\,908\,457\,776\,679\,169\,921\,875}+$$

$$\frac{207\,461\,256\,206\,578\,143\,748\,856\,z^{45}}{7\,670\,102\,214\,448\,301\,053\,033\,358\,480\,610\,212\,529\,462\,890\,625}-$$

$$\frac{11\,218\,806\,737\,995\,635\,372\,498\,255\,094\,z^{47}}{4\,093\,648\,603\,384\,274\,996\,519\,698\,921\,478\,879\,580\,162\,286\,669\,921\,875}+$$

$$\frac{79\,209\,152\,838\,572\,743\,713\,996\,404\,z^{49}}{285\,258\,771\,457\,546\,764\,463\,363\,635\,252\,374\,414\,183\,254\,365\,234\,375}-$$

$$\frac{246\,512\,528\,657\,073\,833\,030\,130\,766\,724\,z^{51}}{8\,761\,982\,491\,474\,419\,367\,550\,817\,114\,626\,909\,562\,924\,278\,968\,505\,859\,375}+$$

$$\frac{233\,199\,709\,079\,078\,899\,371\,344\,990\,501\,528\,z^{53}}{81\,807\,125\,729\,900\,063\,867\,074\,959\,072\,425\,603\,825\,198\,823\,017\,351\,806\,640\,625}-$$

$$\frac{1\,416\,795\,959\,607\,558\,144\,963\,094\,708\,378\,988\,z^{55}}{4\,905\,352\,087\,939\,496\,310\,826\,487\,207\,538\,302\,184\,255\,342\,959\,123\,162\,841\,796\,875}+$$

$$\frac{23\,305\,824\,372\,104\,839\,134\,357\,731\,308\,699\,592\,z^{57}}{796\,392\,368\,980\,577\,121\,745\,974\,726\,570\,063\,253\,238\,310\,542\,073\,919\,837\,646\,484\,375}-$$

$$\frac{9\,721\,865\,123\,870\,044\,576\,322\,439\,952\,638\,561\,968\,331\,928\,z^{59}}{3\,278\,777\,586\,273\,629\,598\,615\,520\,165\,380\,455\,583\,231\,003\,564\,645\,636\,125\,000\,418\,914\,794\,921\,875}+$$

$$\frac{6\,348\,689\,256\,302\,894\,731\,330\,601\,216\,724\,328\,336\,z^{61}}{21\,132\,271\,510\,899\,613\,925\,529\,439\,369\,536\,628\,424\,678\,570\,233\,931\,462\,891\,949\,462\,890\,625}-$$

$$\frac{106\,783\,830\,147\,866\,529\,886\,385\,444\,979\,142\,647\,942\,017\,z^{63}}{3\,508\,062\,732\,166\,890\,409\,707\,514\,582\,539\,928\,001\,638\,766\,051\,683\,792\,497\,378\,070\,587\,158\,203\,125}+$$

$$\left(267\,745\,458\,568\,424\,664\,373\,021\,714\,282\,169\,516\,771\,254\,382\,z^{65}\right)\Big/$$
$$86\,812\,790\,293\,146\,213\,360\,651\,966\,604\,262\,937\,105\,495\,141\,563\,588\,806\,888\,204\,273\,501\,373\,291\,015\,625-\left(250\,471\,004\,320\,250\,327\,955\,196\,022\,920\,428\,000\,776\,938\,z^{67}\right)\Big/$$
$$801\,528\,196\,428\,242\,695\,121\,010\,267\,455\,843\,804\,062\,822\,357\,897\,831\,858\,125\,102\,407\,684\,326\,171\,875+\left(172\,043\,582\,552\,384\,800\,434\,637\,321\,986\,040\,823\,829\,878\,646\,884\,z^{69}\right)\Big/$$
$$5\,433\,748\,964\,547\,053\,581\,149\,916\,185\,708\,338\,218\,048\,392\,402\,830\,337\,634\,114\,958\,370\,880\,742\,156\,982\,421\,875-\left(11\,655\,909\,923\,339\,888\,220\,876\,554\,489\,282\,134\,730\,564\,976\,603\,688\,520\,858\,z^{71}\right)\Big/$$
$$3\,633\,348\,205\,269\,879\,230\,856\,840\,004\,304\,821\,536\,968\,049\,780\,112\,803\,650\,817\,771\,432\,558\,560\,793$$

458 452 606 201 171 875 +
($3\,692\,153\,220\,456\,342\,488\,035\,683\,646\,645\,690\,290\,452\,790\,030\,604\, z^{73}$) /
11 359 005 221 796 317 918 049 302 062 760 294 302 183 889 391 189 419 445 133 951 612 582 060 536
346 435 546 875 − ($5\,190\,545\,015\,986\,394\,254\,249\,936\,008\,544\,252\,611\,445\,319\,542\,919\,116\, z^{75}$) /
157 606 197 452 423 911 112 934 066 120 799 083 442 801 465 302 753 194 801 233 578 624 576 089
941 806 793 212 890 625 +
($255\,290\,071\,123\,323\,586\,643\,187\,098\,799\,718\,199\,072\,122\,692\,536\,861\,835\,992\, z^{77}$) /
76 505 736 228 426 953 173 738 238 352 183 101 801 688 392 812 244 485 181 277 127 930 109 049 138
257 655 704 498 291 015 625 −
($9\,207\,568\,598\,958\,915\,293\,871\,149\,938\,038\,093\,699\,588\,515\,745\,502\,577\,839\,313\,734\, z^{79}$) /
27 233 582 984 369 795 892 070 228 410 001 578 355 986 013 571 390 071 723 225 259 349 721 067 988
068 852 863 296 604 156 494 140 625 +
($163\,611\,136\,505\,867\,886\,519\,332\,147\,296\,221\,453\,678\,803\,514\,884\,902\,772\,183\,572\, z^{81}$) /
4 776 089 171 877 348 057 451 105 924 101 750 653 118 402 745 283 825 543 113 171 217 116 857 704
024 700 607 798 175 811 767 578 125 −
($8\,098\,304\,783\,741\,161\,440\,924\,524\,640\,446\,924\,039\,959\,669\,564\,792\,363\,509\,124\,335\,729\,908\, z^{83}$) /
2 333 207 846 470 426 678 843 707 227 616 712 214 909 162 634 745 895 349 325 948 586 531 533 393
530 725 143 500 144 033 328 342 437 744 140 625 +
($122\,923\,650\,124\,219\,284\,385\,832\,157\,660\,699\,813\,260\,991\,755\,656\,444\,452\,420\,836\,648\, z^{85}$) /
349 538 086 043 843 717 584 559 187 055 386 621 548 470 304 913 596 772 372 737 435 524 697 231
069 047 713 981 709 496 784 210 205 078 125 −
($476\,882\,359\,517\,824\,548\,362\,004\,154\,188\,840\,670\,307\,545\,554\,753\,464\,961\,562\,516\,323\,845\,108\, z^{87}$) /
13 383 510 964 174 348 021 497 060 628 653 950 829 663 288 548 327 870 152 944 013 988 358 928 114
528 962 242 087 062 453 152 690 410 614 013 671 875 +
($1\,886\,491\,646\,433\,732\,479\,814\,597\,361\,998\,744\,134\,040\,407\,919\,471\,435\,385\,970\,472\,345\,164\,676\,056$
z^{89}) /
522 532 651 330 971 490 226 753 590 247 329 744 050 384 290 675 644 135 735 656 667 608 610 471
400 391 047 234 539 824 350 830 981 313 610 076 904 296 875 −
($450\,638\,590\,680\,882\,618\,431\,105\,331\,665\,591\,912\,924\,988\,342\,163\,281\,788\,877\,675\,244\,114\,763\,912$
z^{91}) /
1 231 931 818 039 911 948 327 467 370 123 161 265 684 460 571 086 659 079 080 437 659 781 065 743
269 173 212 919 832 661 978 537 311 246 395 111 083 984 375 +
($415\,596\,189\,473\,955\,564\,121\,634\,614\,268\,323\,814\,113\,534\,779\,643\,471\,190\,276\,158\,333\,713\,923\,216$
z^{93}) /
11 213 200 675 690 943 223 287 032 785 929 540 201 272 600 687 465 377 745 332 153 847 964 679 254
692 602 138 023 498 144 562 090 675 557 613 372 802 734 375 −
($423\,200\,899\,194\,533\,026\,195\,195\,456\,219\,648\,467\,346\,087\,908\,778\,120\,468\,301\,277\,466\,840\,101\,336$
$699\,974\,518\, z^{95}$) /
112 694 926 530 960 148 011 367 752 417 874 063 473 378 698 369 880 587 800 838 274 234 349 237
591 647 453 413 782 021 538 312 594 164 677 406 144 702 434 539 794 921 875 +
($5\,543\,531\,483\,502\,489\,438\,698\,050\,411\,951\,314\,743\,456\,505\,773\,755\,468\,368\,087\,670\,306\,121\,873\,229$
$244\, z^{97}$) /
14 569 479 835 935 377 894 165 191 004 250 040 526 616 509 162 234 077 285 176 247 476 968 227 225
810 918 346 966 001 491 701 692 846 112 140 419 483 184 814 453 125 −
($378\,392\,151\,276\,488\,501\,180\,909\,732\,277\,974\,887\,490\,811\,366\,132\,267\,744\,533\,542\,784\,817\,245\,581$
$660\,788\,990\,844\, z^{99}$) /
9 815 205 420 757 514 710 108 178 059 369 553 458 327 392 260 750 404 049 930 407 987 933 582 359
080 767 225 644 716 670 683 512 153 512 547 802 166 033 089 160 919 189 453 125 + $O[z]^{101}$]

Mathematica comes with the add-on package `DiscreteMath`RSolve`` that allows finding the general terms of series for many functions. After loading this package, and using the package function `SeriesTerm`, the following n^{th} term for odd hyperbolic functions can be evaluated.

```
<< DiscreteMath`RSolve`
```

```
SeriesTerm[{Sinh[z], Cosh[z], Tanh[z], Coth[z], Csch[z], Sech[z]}, {z, 0, n}] z^n
```

$$\left\{z^n \text{If}\left[\text{Odd}[n], \frac{1}{n!}, 0\right], z^n \text{If}\left[\text{Even}[n], \frac{1}{n!}, 0\right], z^n \text{If}\left[\text{Odd}[n], \frac{2^{1+n}\left(-1+2^{1+n}\right) \text{BernoulliB}[1+n]}{(1+n)!}, 0\right], \frac{2^{1+n} z^n \text{BernoulliB}[1+n]}{(1+n)!}, \frac{2^{1+n} z^n \text{BernoulliB}\left[1+n, \frac{1}{2}\right]}{(1+n)!}, \frac{z^n \text{EulerE}[n]}{n!}\right\}$$

Here is a quick check of the last result.

This series should be evaluated to {sinh(z), cosh(z), tanh(z), coth(z), csch(z), sech(z)}, which can be concluded from the following relation.

```
Sum[#, {n, 0, 100}] & /@ % -
 Series[{Sinh[z], Cosh[z], Tanh[z], Coth[z], Csch[z], Sech[z]}, {z, 0, 100}]
```

$$\left\{O[z]^{101}, O[z]^{101}, O[z]^{101}, -\frac{1}{z} - 1 + O[z]^{101}, -\frac{1}{z} + O[z]^{101}, O[z]^{101}\right\}$$

Differentiation

Mathematica can evaluate derivatives of hyperbolic functions of an arbitrary positive integer order.

```
D[Sinh[z], z]
```

```
Cosh[z]
```

```
Sinh[z] // D[#, z] &
```

```
Cosh[z]
```

∂_z `{Sinh[z], Cosh[z], Tanh[z], Coth[z], Csch[z], Sech[z]}`

$$\left\{\text{Cosh}[z], \text{Sinh}[z], \text{Sech}[z]^2, -\text{Csch}[z]^2, -\text{Coth}[z]\,\text{Csch}[z], -\text{Sech}[z]\,\text{Tanh}[z]\right\}$$

$\partial_{\{z,2\}}$ `{Sinh[z], Cosh[z], Tanh[z], Coth[z], Csch[z], Sech[z]}`

$$\left\{\text{Sinh}[z], \text{Cosh}[z], -2\,\text{Sech}[z]^2\,\text{Tanh}[z], 2\,\text{Coth}[z]\,\text{Csch}[z]^2, \text{Coth}[z]^2\,\text{Csch}[z] + \text{Csch}[z]^3, -\text{Sech}[z]^3 + \text{Sech}[z]\,\text{Tanh}[z]^2\right\}$$

```
Table[D[{Sinh[z], Cosh[z], Tanh[z], Coth[z], Csch[z], Sech[z]}, {z, n}], {n, 4}]
```

$$\{\{\text{Cosh}[z], \text{Sinh}[z], \text{Sech}[z]^2, -\text{Csch}[z]^2, -\text{Coth}[z]\,\text{Csch}[z], -\text{Sech}[z]\,\text{Tanh}[z]\},$$
$$\{\text{Sinh}[z], \text{Cosh}[z], -2\,\text{Sech}[z]^2\,\text{Tanh}[z], 2\,\text{Coth}[z]\,\text{Csch}[z]^2, \text{Coth}[z]^2\,\text{Csch}[z] + \text{Csch}[z]^3, -\text{Sech}[z]^3 + \text{Sech}[z]\,\text{Tanh}[z]^2\},$$
$$\{\text{Cosh}[z], \text{Sinh}[z], -2\,\text{Sech}[z]^4 + 4\,\text{Sech}[z]^2\,\text{Tanh}[z]^2, -4\,\text{Coth}[z]^2\,\text{Csch}[z]^2 - 2\,\text{Csch}[z]^4, -\text{Coth}[z]^3\,\text{Csch}[z] - 5\,\text{Coth}[z]\,\text{Csch}[z]^3, 5\,\text{Sech}[z]^3\,\text{Tanh}[z] - \text{Sech}[z]\,\text{Tanh}[z]^3\},$$
$$\{\text{Sinh}[z], \text{Cosh}[z], 16\,\text{Sech}[z]^4\,\text{Tanh}[z] - 8\,\text{Sech}[z]^2\,\text{Tanh}[z]^3, 8\,\text{Coth}[z]^3\,\text{Csch}[z]^2 + 16\,\text{Coth}[z]\,\text{Csch}[z]^4, \text{Coth}[z]^4\,\text{Csch}[z] + 18\,\text{Coth}[z]^2\,\text{Csch}[z]^3 + 5\,\text{Csch}[z]^5, 5\,\text{Sech}[z]^5 - 18\,\text{Sech}[z]^3\,\text{Tanh}[z]^2 + \text{Sech}[z]\,\text{Tanh}[z]^4\}\}$$

Finite summation

Mathematica can calculate finite sums that contain hyperbolic functions. Here are two examples.

```
Sum[Sinh[a k], {k, 0, n}]
```

$$\frac{-1 + e^{a+a n}}{2\,(-1 + e^a)} - \frac{e^{-a n}\,(-1 + e^{a+a n})}{2\,(-1 + e^a)}$$

$$\sum_{k=0}^{n} (-1)^k\,\text{Sinh}[a\,k]$$

$$-\frac{e^a + (-e^{-a})^n}{2\,(1 + e^a)} + \frac{1 + e^a\,(-e^a)^n}{2\,(1 + e^a)}$$

Infinite summation

Mathematica can calculate infinite sums that contain hyperbolic functions. Here are some examples.

$$\sum_{k=1}^{\infty} z^k\,\text{Sinh}[k\,x]$$

$$-\frac{z}{2\,(e^x - z)} - \frac{e^x\,z}{2\,(-1 + e^x\,z)}$$

$$\sum_{k=1}^{\infty} \frac{\text{Sinh}[k\,x]}{k\,!}$$

$$\frac{1}{2}\left(1 - e^{e^{-x}}\right) + \frac{1}{2}\left(-1 + e^{e^x}\right)$$

$$\sum_{k=1}^{\infty} \frac{\text{Cosh}[k\,x]}{k}$$

$$-\frac{1}{2}\,\text{Log}[1 - e^{-x}] - \frac{1}{2}\,\text{Log}[1 - e^x]$$

Finite products

Mathematica can calculate some finite symbolic products that contain the hyperbolic functions. Here are two examples.

$$\prod_{k=1}^{n-1} \text{Sinh}\left[\frac{\pi k i}{n}\right]$$

$$\left(\frac{i}{2}\right)^{-1+n} n$$

$$\prod_{k=1}^{n-1} \text{Cosh}\left[z + \frac{\pi k i}{n}\right]$$

$$-(-1)^n 2^{1-n} \text{Sech}[z] \text{Sin}\left[\frac{1}{2} n (\pi + 2 i z)\right]$$

Infinite products

Mathematica can calculate infinite products that contain hyperbolic functions. Here are some examples.

$$\prod_{k=1}^{\infty} \text{Exp}\left[z^k \text{Sinh}[k x]\right]$$

$$e^{-\frac{\left(-1+e^{2 x}\right) z}{2 \left(e^x - z\right) \left(-1+e^x z\right)}}$$

$$\prod_{k=1}^{\infty} \text{Exp}\left[\frac{\text{Cosh}[k x]}{k!}\right]$$

$$e^{\frac{1}{2} \left(-2+e^{e^{-x}}+e^{e^x}\right)}$$

Indefinite integration

Mathematica can calculate a huge set of doable indefinite integrals that contain hyperbolic functions. Here are some examples.

$$\int \text{Sinh}[7 z] \, dz$$

$$\frac{1}{7} \text{Cosh}[7 z]$$

$$\int \left\{\left\{\text{Sinh}[z], \text{Sinh}[z]^a\right\}, \{\text{Cosh}[z], \text{Cosh}[z]^a\}, \{\text{Tanh}[z], \text{Tanh}[z]^a\}, \{\text{Coth}[z], \text{Coth}[z]^a\}, \{\text{Csch}[z], \text{Csch}[z]^a\}, \{\text{Sech}[z], \text{Sech}[z]^a\}\right\} dz$$

$$\Big\{\Big\{\text{Cosh}[z],\; -\text{Cosh}[z]\,\text{Hypergeometric2F1}\Big[\frac{1}{2},\frac{1-a}{2},\frac{3}{2},\text{Cosh}[z]^2\Big]\,\text{Sinh}[z]^{1+a}\,\big(-\text{Sinh}[z]^2\big)^{\frac{1}{2}(-1-a)}\Big\},$$

$$\Big\{\text{Sinh}[z],\; -\frac{\text{Cosh}[z]^{1+a}\,\text{Hypergeometric2F1}\Big[\frac{1+a}{2},\frac{1}{2},\frac{3+a}{2},\text{Cosh}[z]^2\Big]\,\text{Sinh}[z]}{(1+a)\sqrt{-\text{Sinh}[z]^2}}\Big\},$$

$$\Big\{\text{Log}[\text{Cosh}[z]],\; \frac{\text{Hypergeometric2F1}\Big[\frac{1+a}{2},1,1+\frac{1+a}{2},\text{Tanh}[z]^2\Big]\,\text{Tanh}[z]^{1+a}}{1+a}\Big\},$$

$$\Big\{\text{Log}[\text{Sinh}[z]],\; \frac{\text{Coth}[z]^{1+a}\,\text{Hypergeometric2F1}\Big[\frac{1+a}{2},1,1+\frac{1+a}{2},\text{Coth}[z]^2\Big]}{1+a}\Big\},$$

$$\Big\{-\text{Log}\Big[\text{Cosh}\Big[\frac{z}{2}\Big]\Big]+\text{Log}\Big[\text{Sinh}\Big[\frac{z}{2}\Big]\Big],\; -\text{Cosh}[z]\,\text{Csch}[z]^{-1+a}\,\text{Hypergeometric2F1}\Big[\frac{1}{2},\frac{1+a}{2},\frac{3}{2},\text{Cosh}[z]^2\Big]\,\big(-\text{Sinh}[z]^2\big)^{\frac{1}{2}(-1+a)}\Big\},$$

$$\Big\{2\,\text{ArcTan}\Big[\text{Tanh}\Big[\frac{z}{2}\Big]\Big],\; -\frac{\text{Hypergeometric2F1}\Big[\frac{1-a}{2},\frac{1}{2},\frac{3-a}{2},\text{Cosh}[z]^2\Big]\,\text{Sech}[z]^{-1+a}\,\text{Sinh}[z]}{(1-a)\sqrt{-\text{Sinh}[z]^2}}\Big\}\Big\}$$

Definite integration

Mathematica can calculate wide classes of definite integrals that contain hyperbolic functions. Here are some examples.

$$\int_0^{\pi/2}\sqrt[3]{\text{Sinh}[z]}\,dz$$

$$-\frac{(-1)^{1/3}\sqrt{\pi}\,\text{Gamma}\big[\frac{2}{3}\big]}{2\,\text{Gamma}\big[\frac{7}{6}\big]}+(-1)^{1/3}\,\text{Cosh}\Big[\frac{\pi}{2}\Big]\,\text{Hypergeometric2F1}\Big[\frac{1}{2},\frac{1}{3},\frac{3}{2},\text{Cosh}\Big[\frac{\pi}{2}\Big]^2\Big]$$

$$\int_1^{\pi/2}\Big\{\sqrt{\text{Sinh}[z]},\sqrt{\text{Cosh}[z]},\sqrt{\text{Tanh}[z]},\sqrt{\text{Coth}[z]},\sqrt{\text{Csch}[z]},\sqrt{\text{Sech}[z]}\Big\}\,dz$$

$$\Big\{2\,(-1)^{1/4}\,\text{EllipticE}\Big[\Big(\frac{1}{4}-\frac{i}{4}\Big)\pi,\,2\Big]-2\,(-1)^{1/4}\,\text{EllipticE}\Big[\frac{1}{4}\,(-2\,i+\pi),\,2\Big],$$

$$2\,i\,\text{EllipticE}\Big[\frac{i}{2},\,2\Big]-2\,i\,\text{EllipticE}\Big[\frac{i\,\pi}{4},\,2\Big],$$

$$\frac{1}{2}\Bigg(i\,\text{Log}\Big[1-i\sqrt{\frac{-1+e^2}{1+e^2}}\Big]-i\,\text{Log}\Big[1+i\sqrt{\frac{-1+e^2}{1+e^2}}\Big]-$$

$$i\,\text{Log}\Big[1-i\sqrt{\frac{-1+e^\pi}{1+e^\pi}}\Big]+i\,\text{Log}\Big[1+i\sqrt{\frac{-1+e^\pi}{1+e^\pi}}\Big]+\text{Log}\Big[1-\sqrt{\text{Tanh}[1]}\Big]-$$

$$\text{Log}\Big[1+\sqrt{\text{Tanh}[1]}\Big]-\text{Log}\Big[1-\sqrt{\text{Tanh}\Big[\frac{\pi}{2}\Big]}\Big]+\text{Log}\Big[1+\sqrt{\text{Tanh}\Big[\frac{\pi}{2}\Big]}\Big]\Bigg),$$

$$\frac{1}{2}\,i\Bigg(\text{Log}\Big[1-i\sqrt{\frac{1+e^2}{-1+e^2}}\Big]-\text{Log}\Big[1+i\sqrt{\frac{1+e^2}{-1+e^2}}\Big]-\text{Log}\Big[1-i\sqrt{\frac{1+e^\pi}{-1+e^\pi}}\Big]+$$

$$\text{Log}\Big[1+i\sqrt{\frac{1+e^\pi}{-1+e^\pi}}\Big]-i\,\text{Log}\Big[-1+\sqrt{\text{Coth}[1]}\Big]+i\,\text{Log}\Big[1+\sqrt{\text{Coth}[1]}\Big]+$$

$$i\,\text{Log}\Big[-1+\sqrt{\text{Coth}\Big[\frac{\pi}{2}\Big]}\Big]-i\,\text{Log}\Big[1+\sqrt{\text{Coth}\Big[\frac{\pi}{2}\Big]}\Big]\Bigg),$$

$$2\,(-1)^{3/4}\,\text{EllipticF}\Big[\Big(\frac{1}{4}-\frac{i}{4}\Big)\pi,\,2\Big]-2\,(-1)^{3/4}\,\text{EllipticF}\Big[\frac{1}{4}\,(-2\,i+\pi),\,2\Big],$$

$$2\,i\,\text{EllipticF}\Big[\frac{i}{2},\,2\Big]-2\,i\,\text{EllipticF}\Big[\frac{i\,\pi}{4},\,2\Big]\Big\}$$

$$\int_1^{\frac{\pi}{4}}\Big\{\big\{\text{Sinh}[z],\,\text{Sinh}[z]^a\big\},\,\{\text{Cosh}[z],\,\text{Cosh}[z]^a\},\,\{\text{Tanh}[z],\,\text{Tanh}[z]^a\},$$

$$\{\text{Coth}[z],\,\text{Coth}[z]^a\},\,\{\text{Csch}[z],\,\text{Csch}[z]^a\},\,\{\text{Sech}[z],\,\text{Sech}[z]^a\}\Big\}\,dz$$

$$\Bigg\{\Big\{-\text{Cosh}[1]+\text{Cosh}\Big[\frac{\pi}{4}\Big],$$
$$(-1)^{-\frac{1}{2}-\frac{a}{2}}\,\text{Cosh}[1]\,\text{Hypergeometric2F1}\Big[\frac{1}{2},\frac{1-a}{2},\frac{3}{2},\text{Cosh}[1]^2\Big]\,\text{Sinh}[1]^{1+2\left(-\frac{1}{2}-\frac{a}{2}\right)+a}-$$
$$(-1)^{-\frac{1}{2}-\frac{a}{2}}\,\text{Cosh}\Big[\frac{\pi}{4}\Big]\,\text{Hypergeometric2F1}\Big[\frac{1}{2},\frac{1-a}{2},\frac{3}{2},\text{Cosh}\Big[\frac{\pi}{4}\Big]^2\Big]\,\text{Sinh}\Big[\frac{\pi}{4}\Big]^{1+2\left(-\frac{1}{2}-\frac{a}{2}\right)+a}\Big\},$$
$$\Big\{-\text{Sinh}[1]+\text{Sinh}\Big[\frac{\pi}{4}\Big],-\frac{i\,\text{Cosh}[1]^{1+a}\,\text{Hypergeometric2F1}\Big[\frac{1+a}{2},\frac{1}{2},\frac{3+a}{2},\text{Cosh}[1]^2\Big]}{1+a}+$$
$$\frac{i\,\text{Cosh}\big[\frac{\pi}{4}\big]^{1+a}\,\text{Hypergeometric2F1}\Big[\frac{1+a}{2},\frac{1}{2},\frac{3+a}{2},\text{Cosh}\big[\frac{\pi}{4}\big]^2\Big]}{1+a}\Big\},$$
$$\Big\{-\text{Log}[\text{Cosh}[1]]+\text{Log}\Big[\text{Cosh}\Big[\frac{\pi}{4}\Big]\Big],-\frac{\text{Hypergeometric2F1}\Big[\frac{1+a}{2},1,1+\frac{1+a}{2},\text{Tanh}[1]^2\Big]\,\text{Tanh}[1]^{1+a}}{1+a}+$$
$$\frac{\text{Hypergeometric2F1}\Big[\frac{1+a}{2},1,1+\frac{1+a}{2},\text{Tanh}\big[\frac{\pi}{4}\big]^2\Big]\,\text{Tanh}\big[\frac{\pi}{4}\big]^{1+a}}{1+a}\Big\},$$
$$\Big\{-\text{Log}[\text{Sinh}[1]]+\text{Log}\Big[\text{Sinh}\Big[\frac{\pi}{4}\Big]\Big],-\frac{\text{Coth}[1]^{1+a}\,\text{Hypergeometric2F1}\Big[\frac{1+a}{2},1,1+\frac{1+a}{2},\text{Coth}[1]^2\Big]}{1+a}+$$
$$\frac{\text{Coth}\big[\frac{\pi}{4}\big]^{1+a}\,\text{Hypergeometric2F1}\Big[\frac{1+a}{2},1,1+\frac{1+a}{2},\text{Coth}\big[\frac{\pi}{4}\big]^2\Big]}{1+a}\Big\},$$
$$\Big\{\text{Log}\Big[\text{Cosh}\Big[\frac{1}{2}\Big]\Big]-\text{Log}\Big[\text{Cosh}\Big[\frac{\pi}{8}\Big]\Big]-\text{Log}\Big[\text{Sinh}\Big[\frac{1}{2}\Big]\Big]+\text{Log}\Big[\text{Sinh}\Big[\frac{\pi}{8}\Big]\Big],$$
$$(-1)^{-\frac{1}{2}+\frac{a}{2}}\,\text{Cosh}[1]\,\text{Csch}[1]^{-1+a}\,\text{Hypergeometric2F1}\Big[\frac{1}{2},\frac{1+a}{2},\frac{3}{2},\text{Cosh}[1]^2\Big]\,\text{Sinh}[1]^{2\left(-\frac{1}{2}+\frac{a}{2}\right)}-$$
$$(-1)^{-\frac{1}{2}+\frac{a}{2}}\,\text{Cosh}\Big[\frac{\pi}{4}\Big]\,\text{Csch}\Big[\frac{\pi}{4}\Big]^{-1+a}\,\text{Hypergeometric2F1}\Big[\frac{1}{2},\frac{1+a}{2},\frac{3}{2},\text{Cosh}\Big[\frac{\pi}{4}\Big]^2\Big]\,\text{Sinh}\Big[\frac{\pi}{4}\Big]^{2\left(-\frac{1}{2}+\frac{a}{2}\right)}\Big\},$$
$$\Big\{-2\,\text{ArcTan}\Big[\text{Tanh}\Big[\frac{1}{2}\Big]\Big]+2\,\text{ArcTan}\Big[\text{Tanh}\Big[\frac{\pi}{8}\Big]\Big],$$
$$\frac{i\,\text{Hypergeometric2F1}\Big[\frac{1-a}{2},\frac{1}{2},\frac{3-a}{2},\text{Cosh}[1]^2\Big]\,\text{Sech}[1]^{-1+a}}{-1+a}-$$
$$\frac{i\,\text{Hypergeometric2F1}\Big[\frac{1-a}{2},\frac{1}{2},\frac{3-a}{2},\text{Cosh}\big[\frac{\pi}{4}\big]^2\Big]\,\text{Sech}\big[\frac{\pi}{4}\big]^{-1+a}}{-1+a}\Big\}\Bigg\}$$

$$\int_0^\infty\Big\{\frac{1}{a+b\,\text{Sinh}[z]},\frac{1}{a+b\,\text{Cosh}[z]},$$
$$\frac{1}{a+b\,\text{Tanh}[z]},\frac{1}{a+b\,\text{Coth}[z]},\frac{1}{a+b\,\text{Csch}[z]},\frac{1}{a+b\,\text{Sech}[z]}\Big\}\,dz$$

$$\left\{-\frac{1}{\sqrt{-a^2-b^2}}\left(i\left(\text{Log}\left[1-\frac{i\,a}{\sqrt{-a^2-b^2}}\right]-\text{Log}\left[1+\frac{i\,a}{\sqrt{-a^2-b^2}}\right]+\text{Log}\left[\frac{i\,a-i\,b+\sqrt{-a^2-b^2}}{\sqrt{-a^2-b^2}}\right]-\text{Log}\left[\frac{-i\,a+i\,b+\sqrt{-a^2-b^2}}{\sqrt{-a^2-b^2}}\right]\right)\right),\right.$$

$$-\frac{1}{\sqrt{-a^2+b^2}}\left(i\left(\text{Log}\left[1-\frac{i\,a}{\sqrt{-a^2+b^2}}\right]-\text{Log}\left[1+\frac{i\,a}{\sqrt{-a^2+b^2}}\right]-\text{Log}\left[\frac{-i\,a-i\,b+\sqrt{-a^2+b^2}}{\sqrt{-a^2+b^2}}\right]+\text{Log}\left[\frac{i\,a+i\,b+\sqrt{-a^2+b^2}}{\sqrt{-a^2+b^2}}\right]\right)\right),$$

$$\frac{b\,(\text{Log}[2\,a]-\text{Log}[a+b])}{a^2-b^2},\ \frac{b\,(\text{Log}[-a-b]-\text{Log}[-2\,b])}{-a^2+b^2},$$

$$\frac{1}{a\sqrt{-a^2-b^2}}\left(i\,b\left(\text{Log}\left[\frac{-i\,b+\sqrt{-a^2-b^2}}{\sqrt{-a^2-b^2}}\right]-\text{Log}\left[\frac{i\,a-i\,b+\sqrt{-a^2-b^2}}{\sqrt{-a^2-b^2}}\right]-\text{Log}\left[\frac{i\,b+\sqrt{-a^2-b^2}}{\sqrt{-a^2-b^2}}\right]+\text{Log}\left[\frac{-i\,a+i\,b+\sqrt{-a^2-b^2}}{\sqrt{-a^2-b^2}}\right]\right)\right),$$

$$\left.\frac{1}{a\sqrt{a^2-b^2}}\left(i\,b\left(\text{Log}\left[1-\frac{i\,b}{\sqrt{a^2-b^2}}\right]-\text{Log}\left[1+\frac{i\,b}{\sqrt{a^2-b^2}}\right]-\text{Log}\left[\frac{-i\,a-i\,b+\sqrt{a^2-b^2}}{\sqrt{a^2-b^2}}\right]+\text{Log}\left[\frac{i\,a+i\,b+\sqrt{a^2-b^2}}{\sqrt{a^2-b^2}}\right]\right)\right)\right\}$$

Limit operation

Mathematica can calculate limits that contain hyperbolic functions. Here are some examples.

$$\textbf{Limit}\left[\frac{\textbf{Sinh}[\textbf{z}]}{\textbf{z}}+\textbf{Cosh}[\textbf{z}]^3,\ \textbf{z}\to\textbf{0}\right]$$

2

$$\textbf{Limit}\left[\left(\frac{\textbf{Tanh}[\textbf{x}]}{\textbf{x}}\right)^{\frac{1}{x^2}},\ \textbf{x}\to\textbf{0}\right]$$

$$\frac{1}{e^{1/3}}$$

```
Limit[Sinh[Sqrt[z^2]]/z, z -> 0, Direction -> 1]
```

```
-1
```

```
Limit[Sinh[Sqrt[z^2]]/z, z -> 0, Direction -> -1]
```

```
1
```

Solving equations

The next input solves equations that contain hyperbolic functions. The message indicates that the multivalued functions are used to express the result and that some solutions might be absent.

```
Solve[Tanh[z]^2 + 3 Sinh[z + Pi/6] == 4, z]
```

```
Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found.
```

```
{{z -> -ArcSech[
    3 (-2 Root[1 + 6 #1^2 - 6 Sinh[π/6] #1^3 + 9 #1^4 + 9 Cosh[π/6]^2 #1^4 - 18 Sinh[π/6] #1^5 - 9 Cosh[π/6]^2
            #1^6 + 9 Sinh[π/6]^2 #1^6 &, 1] - 3 Root[1 + 6 #1^2 - 6 Sinh[π/6] #1^3 + 9 #1^4 +
            9 Cosh[π/6]^2 #1^4 - 18 Sinh[π/6] #1^5 - 9 Cosh[π/6]^2 #1^6 + 9 Sinh[π/6]^2 #1^6 &, 1]^3 -
       3 Cosh[π/6]^2 Root[1 + 6 #1^2 - 6 Sinh[π/6] #1^3 + 9 #1^4 + 9 Cosh[π/6]^2 #1^4 -
            18 Sinh[π/6] #1^5 - 9 Cosh[π/6]^2 #1^6 + 9 Sinh[π/6]^2 #1^6 &, 1]^3 +
       3 Cosh[π/6]^2 Root[1 + 6 #1^2 - 6 Sinh[π/6] #1^3 + 9 #1^4 + 9 Cosh[π/6]^2 #1^4 -
            18 Sinh[π/6] #1^5 - 9 Cosh[π/6]^2 #1^6 + 9 Sinh[π/6]^2 #1^6 &, 1]^5 +
       2 Root[1 + 6 #1^2 - 6 Sinh[π/6] #1^3 + 9 #1^4 + 9 Cosh[π/6]^2 #1^4 - 18 Sinh[π/6] #1^5 -
            9 Cosh[π/6]^2 #1^6 + 9 Sinh[π/6]^2 #1^6 &, 1]^2 Sinh[π/6] +
       6 Root[1 + 6 #1^2 - 6 Sinh[π/6] #1^3 + 9 #1^4 + 9 Cosh[π/6]^2 #1^4 - 18 Sinh[π/6] #1^5 -
            9 Cosh[π/6]^2 #1^6 + 9 Sinh[π/6]^2 #1^6 &, 1]^4 Sinh[π/6] -
       3 Root[1 + 6 #1^2 - 6 Sinh[π/6] #1^3 + 9 #1^4 + 9 Cosh[π/6]^2 #1^4 - 18 Sinh[π/6] #1^5 -
```

$$9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^6+9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2 \#1^6\;\&,\,1\right]^5 \text{Sinh}\left[\frac{\pi}{6}\right]^2\Bigg)\Bigg]\Bigg\},$$

$$\Bigg\{z\to\text{ArcSech}\Bigg[3\,\Bigg(-2\,\text{Root}\left[1+6\,\#1^2-6\,\text{Sinh}\left[\frac{\pi}{6}\right]\#1^3+9\,\#1^4+9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\#1^4-\right.$$

$$\left.18\,\text{Sinh}\left[\frac{\pi}{6}\right]\#1^5-9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\#1^6+9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\#1^6\;\&,\,2\right]-$$

$$3\,\text{Root}\left[1+6\,\#1^2-6\,\text{Sinh}\left[\frac{\pi}{6}\right]\#1^3+9\,\#1^4+9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\#1^4-18\,\text{Sinh}\left[\frac{\pi}{6}\right]\#1^5-\right.$$

$$\left.9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\#1^6+9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\#1^6\;\&,\,2\right]^3-$$

$$3\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\text{Root}\left[1+6\,\#1^2-6\,\text{Sinh}\left[\frac{\pi}{6}\right]\#1^3+9\,\#1^4+9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\#1^4-\right.$$

$$\left.18\,\text{Sinh}\left[\frac{\pi}{6}\right]\#1^5-9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\#1^6+9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\#1^6\;\&,\,2\right]^3+$$

$$3\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\text{Root}\left[1+6\,\#1^2-6\,\text{Sinh}\left[\frac{\pi}{6}\right]\#1^3+9\,\#1^4+9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\#1^4-\right.$$

$$\left.18\,\text{Sinh}\left[\frac{\pi}{6}\right]\#1^5-9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\#1^6+9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\#1^6\;\&,\,2\right]^5+$$

$$2\,\text{Root}\left[1+6\,\#1^2-6\,\text{Sinh}\left[\frac{\pi}{6}\right]\#1^3+9\,\#1^4+9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\#1^4-18\,\text{Sinh}\left[\frac{\pi}{6}\right]\#1^5-\right.$$

$$\left.9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\#1^6+9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\#1^6\;\&,\,2\right]^2\text{Sinh}\left[\frac{\pi}{6}\right]+$$

$$6\,\text{Root}\left[1+6\,\#1^2-6\,\text{Sinh}\left[\frac{\pi}{6}\right]\#1^3+9\,\#1^4+9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\#1^4-18\,\text{Sinh}\left[\frac{\pi}{6}\right]\#1^5-\right.$$

$$\left.9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\#1^6+9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\#1^6\;\&,\,2\right]^4\text{Sinh}\left[\frac{\pi}{6}\right]-$$

$$3\,\text{Root}\left[1+6\,\#1^2-6\,\text{Sinh}\left[\frac{\pi}{6}\right]\#1^3+9\,\#1^4+9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\#1^4-18\,\text{Sinh}\left[\frac{\pi}{6}\right]\#1^5-\right.$$

$$\left.9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\#1^6+9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\#1^6\;\&,\,2\right]^5\text{Sinh}\left[\frac{\pi}{6}\right]^2\Bigg)\Bigg]\Bigg\},$$

$$\Bigg\{z\to\text{ArcSech}\Bigg[3\,\Bigg(-2\,\text{Root}\left[1+6\,\#1^2-6\,\text{Sinh}\left[\frac{\pi}{6}\right]\#1^3+9\,\#1^4+9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\#1^4-\right.$$

$$\left.18\,\text{Sinh}\left[\frac{\pi}{6}\right]\#1^5-9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\#1^6+9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\#1^6\;\&,\,3\right]-$$

$$3\,\text{Root}\left[1+6\,\#1^2-6\,\text{Sinh}\left[\frac{\pi}{6}\right]\#1^3+9\,\#1^4+9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\#1^4-18\,\text{Sinh}\left[\frac{\pi}{6}\right]\#1^5-\right.$$

$$\left.9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\#1^6+9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\#1^6\;\&,\,3\right]^3-$$

$$3\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\text{Root}\left[1+6\,\#1^2-6\,\text{Sinh}\left[\frac{\pi}{6}\right]\#1^3+9\,\#1^4+9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\#1^4-\right.$$

$$18\,\text{Sinh}\left[\frac{\pi}{6}\right] \#1^5 - 9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^6 + 9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2 \#1^6 \,\&,\, 3\Big]^3 +$$

$$3\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 \text{Root}\Big[1 + 6\,\#1^2 - 6\,\text{Sinh}\left[\frac{\pi}{6}\right] \#1^3 + 9\,\#1^4 + 9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^4 -$$

$$18\,\text{Sinh}\left[\frac{\pi}{6}\right] \#1^5 - 9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^6 + 9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2 \#1^6 \,\&,\, 3\Big]^5 +$$

$$2\,\text{Root}\Big[1 + 6\,\#1^2 - 6\,\text{Sinh}\left[\frac{\pi}{6}\right] \#1^3 + 9\,\#1^4 + 9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^4 - 18\,\text{Sinh}\left[\frac{\pi}{6}\right] \#1^5 -$$

$$9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^6 + 9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2 \#1^6 \,\&,\, 3\Big]^2 \text{Sinh}\left[\frac{\pi}{6}\right] +$$

$$6\,\text{Root}\Big[1 + 6\,\#1^2 - 6\,\text{Sinh}\left[\frac{\pi}{6}\right] \#1^3 + 9\,\#1^4 + 9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^4 - 18\,\text{Sinh}\left[\frac{\pi}{6}\right] \#1^5 -$$

$$9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^6 + 9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2 \#1^6 \,\&,\, 3\Big]^4 \text{Sinh}\left[\frac{\pi}{6}\right] -$$

$$3\,\text{Root}\Big[1 + 6\,\#1^2 - 6\,\text{Sinh}\left[\frac{\pi}{6}\right] \#1^3 + 9\,\#1^4 + 9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^4 - 18\,\text{Sinh}\left[\frac{\pi}{6}\right] \#1^5 -$$

$$9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^6 + 9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2 \#1^6 \,\&,\, 3\Big]^5 \text{Sinh}\left[\frac{\pi}{6}\right]^2 \Big)\Big]\Big\},$$

$$\Big\{z \to \text{ArcSech}\Big[3\,\Big(-2\,\text{Root}\Big[1 + 6\,\#1^2 - 6\,\text{Sinh}\left[\frac{\pi}{6}\right] \#1^3 + 9\,\#1^4 + 9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^4 -$$

$$18\,\text{Sinh}\left[\frac{\pi}{6}\right] \#1^5 - 9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^6 + 9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2 \#1^6 \,\&,\, 4\Big] -$$

$$3\,\text{Root}\Big[1 + 6\,\#1^2 - 6\,\text{Sinh}\left[\frac{\pi}{6}\right] \#1^3 + 9\,\#1^4 + 9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^4 - 18\,\text{Sinh}\left[\frac{\pi}{6}\right] \#1^5 -$$

$$9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^6 + 9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2 \#1^6 \,\&,\, 4\Big]^3 -$$

$$3\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 \text{Root}\Big[1 + 6\,\#1^2 - 6\,\text{Sinh}\left[\frac{\pi}{6}\right] \#1^3 + 9\,\#1^4 + 9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^4 -$$

$$18\,\text{Sinh}\left[\frac{\pi}{6}\right] \#1^5 - 9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^6 + 9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2 \#1^6 \,\&,\, 4\Big]^3 +$$

$$3\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 \text{Root}\Big[1 + 6\,\#1^2 - 6\,\text{Sinh}\left[\frac{\pi}{6}\right] \#1^3 + 9\,\#1^4 + 9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^4 -$$

$$18\,\text{Sinh}\left[\frac{\pi}{6}\right] \#1^5 - 9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^6 + 9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2 \#1^6 \,\&,\, 4\Big]^5 +$$

$$2\,\text{Root}\Big[1 + 6\,\#1^2 - 6\,\text{Sinh}\left[\frac{\pi}{6}\right] \#1^3 + 9\,\#1^4 + 9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^4 - 18\,\text{Sinh}\left[\frac{\pi}{6}\right] \#1^5 -$$

$$9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^6 + 9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2 \#1^6 \,\&,\, 4\Big]^2 \text{Sinh}\left[\frac{\pi}{6}\right] +$$

$$6\,\text{Root}\Big[1 + 6\,\#1^2 - 6\,\text{Sinh}\left[\frac{\pi}{6}\right] \#1^3 + 9\,\#1^4 + 9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^4 - 18\,\text{Sinh}\left[\frac{\pi}{6}\right] \#1^5 -$$

$$9\,\text{Cosh}[\tfrac{\pi}{6}]^2\,\#1^6+9\,\text{Sinh}[\tfrac{\pi}{6}]^2\,\#1^6\,\&,\,4]^4\,\text{Sinh}[\tfrac{\pi}{6}]-$$

$$3\,\text{Root}[1+6\,\#1^2-6\,\text{Sinh}[\tfrac{\pi}{6}]\,\#1^3+9\,\#1^4+9\,\text{Cosh}[\tfrac{\pi}{6}]^2\,\#1^4-18\,\text{Sinh}[\tfrac{\pi}{6}]\,\#1^5-$$

$$9\,\text{Cosh}[\tfrac{\pi}{6}]^2\,\#1^6+9\,\text{Sinh}[\tfrac{\pi}{6}]^2\,\#1^6\,\&,\,4]^5\,\text{Sinh}[\tfrac{\pi}{6}]^2)]]\},$$

$$\{z\to-\text{ArcSech}[3\,(-2\,\text{Root}[1+6\,\#1^2-6\,\text{Sinh}[\tfrac{\pi}{6}]\,\#1^3+9\,\#1^4+9\,\text{Cosh}[\tfrac{\pi}{6}]^2\,\#1^4-$$

$$18\,\text{Sinh}[\tfrac{\pi}{6}]\,\#1^5-9\,\text{Cosh}[\tfrac{\pi}{6}]^2\,\#1^6+9\,\text{Sinh}[\tfrac{\pi}{6}]^2\,\#1^6\,\&,\,5]-$$

$$3\,\text{Root}[1+6\,\#1^2-6\,\text{Sinh}[\tfrac{\pi}{6}]\,\#1^3+9\,\#1^4+9\,\text{Cosh}[\tfrac{\pi}{6}]^2\,\#1^4-18\,\text{Sinh}[\tfrac{\pi}{6}]\,\#1^5-$$

$$9\,\text{Cosh}[\tfrac{\pi}{6}]^2\,\#1^6+9\,\text{Sinh}[\tfrac{\pi}{6}]^2\,\#1^6\,\&,\,5]^3-$$

$$3\,\text{Cosh}[\tfrac{\pi}{6}]^2\,\text{Root}[1+6\,\#1^2-6\,\text{Sinh}[\tfrac{\pi}{6}]\,\#1^3+9\,\#1^4+9\,\text{Cosh}[\tfrac{\pi}{6}]^2\,\#1^4-$$

$$18\,\text{Sinh}[\tfrac{\pi}{6}]\,\#1^5-9\,\text{Cosh}[\tfrac{\pi}{6}]^2\,\#1^6+9\,\text{Sinh}[\tfrac{\pi}{6}]^2\,\#1^6\,\&,\,5]^3+$$

$$3\,\text{Cosh}[\tfrac{\pi}{6}]^2\,\text{Root}[1+6\,\#1^2-6\,\text{Sinh}[\tfrac{\pi}{6}]\,\#1^3+9\,\#1^4+9\,\text{Cosh}[\tfrac{\pi}{6}]^2\,\#1^4-$$

$$18\,\text{Sinh}[\tfrac{\pi}{6}]\,\#1^5-9\,\text{Cosh}[\tfrac{\pi}{6}]^2\,\#1^6+9\,\text{Sinh}[\tfrac{\pi}{6}]^2\,\#1^6\,\&,\,5]^5+$$

$$2\,\text{Root}[1+6\,\#1^2-6\,\text{Sinh}[\tfrac{\pi}{6}]\,\#1^3+9\,\#1^4+9\,\text{Cosh}[\tfrac{\pi}{6}]^2\,\#1^4-18\,\text{Sinh}[\tfrac{\pi}{6}]\,\#1^5-$$

$$9\,\text{Cosh}[\tfrac{\pi}{6}]^2\,\#1^6+9\,\text{Sinh}[\tfrac{\pi}{6}]^2\,\#1^6\,\&,\,5]^2\,\text{Sinh}[\tfrac{\pi}{6}]+$$

$$6\,\text{Root}[1+6\,\#1^2-6\,\text{Sinh}[\tfrac{\pi}{6}]\,\#1^3+9\,\#1^4+9\,\text{Cosh}[\tfrac{\pi}{6}]^2\,\#1^4-18\,\text{Sinh}[\tfrac{\pi}{6}]\,\#1^5-$$

$$9\,\text{Cosh}[\tfrac{\pi}{6}]^2\,\#1^6+9\,\text{Sinh}[\tfrac{\pi}{6}]^2\,\#1^6\,\&,\,5]^4\,\text{Sinh}[\tfrac{\pi}{6}]-$$

$$3\,\text{Root}[1+6\,\#1^2-6\,\text{Sinh}[\tfrac{\pi}{6}]\,\#1^3+9\,\#1^4+9\,\text{Cosh}[\tfrac{\pi}{6}]^2\,\#1^4-18\,\text{Sinh}[\tfrac{\pi}{6}]\,\#1^5-$$

$$9\,\text{Cosh}[\tfrac{\pi}{6}]^2\,\#1^6+9\,\text{Sinh}[\tfrac{\pi}{6}]^2\,\#1^6\,\&,\,5]^5\,\text{Sinh}[\tfrac{\pi}{6}]^2)]]\},$$

$$\{z\to-\text{ArcSech}[3\,(-2\,\text{Root}[1+6\,\#1^2-6\,\text{Sinh}[\tfrac{\pi}{6}]\,\#1^3+9\,\#1^4+9\,\text{Cosh}[\tfrac{\pi}{6}]^2\,\#1^4-$$

$$18\,\text{Sinh}[\tfrac{\pi}{6}]\,\#1^5-9\,\text{Cosh}[\tfrac{\pi}{6}]^2\,\#1^6+9\,\text{Sinh}[\tfrac{\pi}{6}]^2\,\#1^6\,\&,\,6]-$$

$$3\,\text{Root}[1+6\,\#1^2-6\,\text{Sinh}[\tfrac{\pi}{6}]\,\#1^3+9\,\#1^4+9\,\text{Cosh}[\tfrac{\pi}{6}]^2\,\#1^4-18\,\text{Sinh}[\tfrac{\pi}{6}]\,\#1^5-$$

$$9\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^6 + 9\,\mathrm{Sinh}\left[\frac{\pi}{6}\right]^2 \#1^6\,\&,\,6\Big]^3 -$$

$$3\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2 \mathrm{Root}\Big[1 + 6\,\#1^2 - 6\,\mathrm{Sinh}\left[\frac{\pi}{6}\right] \#1^3 + 9\,\#1^4 + 9\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^4 -$$

$$18\,\mathrm{Sinh}\left[\frac{\pi}{6}\right] \#1^5 - 9\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^6 + 9\,\mathrm{Sinh}\left[\frac{\pi}{6}\right]^2 \#1^6\,\&,\,6\Big]^3 +$$

$$3\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2 \mathrm{Root}\Big[1 + 6\,\#1^2 - 6\,\mathrm{Sinh}\left[\frac{\pi}{6}\right] \#1^3 + 9\,\#1^4 + 9\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^4 -$$

$$18\,\mathrm{Sinh}\left[\frac{\pi}{6}\right] \#1^5 - 9\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^6 + 9\,\mathrm{Sinh}\left[\frac{\pi}{6}\right]^2 \#1^6\,\&,\,6\Big]^5 +$$

$$2\,\mathrm{Root}\Big[1 + 6\,\#1^2 - 6\,\mathrm{Sinh}\left[\frac{\pi}{6}\right] \#1^3 + 9\,\#1^4 + 9\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^4 - 18\,\mathrm{Sinh}\left[\frac{\pi}{6}\right] \#1^5 -$$

$$9\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^6 + 9\,\mathrm{Sinh}\left[\frac{\pi}{6}\right]^2 \#1^6\,\&,\,6\Big]^2 \mathrm{Sinh}\left[\frac{\pi}{6}\right] +$$

$$6\,\mathrm{Root}\Big[1 + 6\,\#1^2 - 6\,\mathrm{Sinh}\left[\frac{\pi}{6}\right] \#1^3 + 9\,\#1^4 + 9\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^4 - 18\,\mathrm{Sinh}\left[\frac{\pi}{6}\right] \#1^5 -$$

$$9\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^6 + 9\,\mathrm{Sinh}\left[\frac{\pi}{6}\right]^2 \#1^6\,\&,\,6\Big]^4 \mathrm{Sinh}\left[\frac{\pi}{6}\right] -$$

$$3\,\mathrm{Root}\Big[1 + 6\,\#1^2 - 6\,\mathrm{Sinh}\left[\frac{\pi}{6}\right] \#1^3 + 9\,\#1^4 + 9\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^4 - 18\,\mathrm{Sinh}\left[\frac{\pi}{6}\right] \#1^5 -$$

$$9\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^6 + 9\,\mathrm{Sinh}\left[\frac{\pi}{6}\right]^2 \#1^6\,\&,\,6\Big]^5 \mathrm{Sinh}\left[\frac{\pi}{6}\right]^2\Big)\Big]\Big]\Big\}\Big\}$$

Complete solutions can be obtained by using the function `Reduce`.

```
Reduce[Sinh[x] == a, x] // InputForm
```

```
// InputForm = C[1] ∈ Integers &&
  (x == I * Pi - ArcSinh[a] + (2 * I) * Pi * C[1] || x == ArcSinh[a] + (2 * I) * Pi * C[1])
```

```
Reduce[Cosh[x] == a, x] // InputForm
```

```
// InputForm =
C[1] ∈ Integers && (x == -ArcCosh[a] + (2 * I) * Pi * C[1] || x == ArcCosh[a] + (2 * I) * Pi * C[1])
```

```
Reduce[Tanh[x] == a, x] // InputForm
```

```
// InputForm = C[1] ∈ Integers && -1 + a^2 ≠ 0 && x == ArcTanh[a] + I * Pi * C[1]
```

```
Reduce[Coth[x] == a, x] // InputForm
```

```
// InputForm = C[1] ∈ Integers && -1 + a^2 ≠ 0 && x == ArcCoth[a] + I * Pi * C[1]
```

```
Reduce[Csch[x] == a, x] // InputForm
```

```
// InputForm = C[1] ∈ Integers && a ≠ 0 &&
  (x == I * Pi - ArcSinh[a^(-1)] + (2 * I) * Pi * C[1] || x == ArcSinh[a^(-1)] + (2 * I) * Pi * C[1])
```

```
Reduce[Sech[x] == a, x] // InputForm
```

```
// InputForm = C[1] ∈ Integers && a ≠ 0 &&
  (x == -ArcCosh[a^(-1)] + (2*I)*Pi*C[1] || x == ArcCosh[a^(-1)] + (2*I)*Pi*C[1])
```

Solving differential equations

Here are differential equations whose linear-independent solutions are hyperbolic functions. The solutions of the simplest second-order linear ordinary differential equation with constant coefficients can be represented through sinh(z) and cosh(z).

```
{DSolve[w''[z] - w[z] == 0, w[z], z],
  DSolve[w'[z] + w[z]^2 - 1 == 0, w[z], z]} // (ExpToTrig //@ #) &
```

$$\left\{\{\{w[z] \to C[1]\,\mathrm{Cosh}[z] + C[2]\,\mathrm{Cosh}[z] + C[1]\,\mathrm{Sinh}[z] - C[2]\,\mathrm{Sinh}[z]\}\},\ \left\{\left\{w[z] \to \frac{\mathrm{Cosh}[2\,z] + \mathrm{Cosh}[2\,C[1]] + \mathrm{Sinh}[2\,z] + \mathrm{Sinh}[2\,C[1]]}{\mathrm{Cosh}[2\,z] - \mathrm{Cosh}[2\,C[1]] + \mathrm{Sinh}[2\,z] - \mathrm{Sinh}[2\,C[1]]}\right\}\right\}\right\}$$

All hyperbolic functions satisfy first-order nonlinear differential equations. In carrying out the algorithm to solve the nonlinear differential equation, *Mathematica* has to solve a transcendental equation. In doing so, the generically multivariate inverse of a function is encountered, and a message is issued that a solution branch is potentially missed.

$$\mathrm{DSolve}\left[\left\{w'[z] == \sqrt{1 + w[z]^2},\ w[0] == 0\right\},\ w[z],\ z\right]$$

```
Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found.
```

```
{{w[z] → Sinh[z]}}
```

$$\mathrm{DSolve}\left[\left\{w'[z] == \sqrt{-1 + w[z]^2},\ w[0] == 1\right\},\ w[z],\ z\right]\ //\ \mathrm{FullSimplify}$$

```
Solve::ifun : Inverse functions are being used by Solve, so some
   solutions may not be found; use Reduce for complete solution information. More…
```

```
{{w[z] → Cosh[z]}}
```

$$\mathrm{DSolve}\left[\left\{w'[z] + w[z]^2 - 1 == 0,\ w[0] == 0\right\},\ w[z],\ z\right]\ //\ \mathrm{FullSimplify}$$

```
Solve::ifun : Inverse functions are being used by Solve, so some
   solutions may not be found; use Reduce for complete solution information. More…
```

```
{{w[z] → Tanh[z]}}
```

$$\mathrm{DSolve}\left[\left\{w'[z] - w[z]^2 + 1 == 0,\ w\left[\frac{\pi\, i}{2}\right] == 0\right\},\ w[z],\ z\right]\ //\ \mathrm{FullSimplify}$$

```
Solve::ifun : Inverse functions are being used by Solve, so some
   solutions may not be found; use Reduce for complete solution information. More…
```

```
{{w[z] → -Coth[z]}}
```

Integral transforms

Mathematica supports the main integral transforms like direct and inverse Fourier, Laplace, and Z transforms that can give results containing classical or generalized functions. Here are some transforms of hyperbolic functions.

```
LaplaceTransform[Sinh[t], t, s]
```

$$\frac{1}{-1+s^2}$$

```
LaplaceTransform[Cosh[t], t, s]
```

$$\frac{s}{-1+s^2}$$

```
FourierTransform[Csch[t], t, s]
```

$$i\sqrt{\frac{\pi}{2}}\,\mathrm{Tanh}\left[\frac{\pi s}{2}\right]$$

```
FourierTransform[Sech[t], t, s]
```

$$\sqrt{\frac{\pi}{2}}\,\mathrm{Sech}\left[\frac{\pi s}{2}\right]$$

Plotting

Mathematica has built-in functions for 2D and 3D graphics. Here are some examples.

$$\mathrm{Plot}\left[\mathrm{Sin}\left[\mathrm{Sinh}\left[\sum_{k=0}^{5} z^k\right]\right], \{z, -3/2, 4/5\}, \mathrm{PlotRange} \to \mathrm{All}, \mathrm{PlotPoints} \to 120\right];$$

```
Plot3D[Re[Tanh[x + i y]], {x, -2, 2}, {y, -2, 2},
  PlotPoints → 240, PlotRange → {-5, 5},
  ClipFill → None, Mesh → False, AxesLabel → {"x", "y", None}];
```

$$\mathrm{ContourPlot}\left[\mathrm{Arg}\left[\mathrm{Sech}\left[\frac{1}{x+i\,y}\right]\right], \left\{x, -\frac{1}{4}, \frac{1}{4}\right\}, \left\{y, -\frac{1}{3}, \frac{1}{3}\right\},\right.$$

```
  PlotPoints → 400, PlotRange → {-π, π}, FrameLabel → {"x", "y", None, None},
  ColorFunction → Hue, ContourLines → False, Contours → 200];
```

Introduction to the Hyperbolic Cosine Function in *Mathematica*

Overview

The following shows how the hyperbolic cosine function is realized in *Mathematica*. Examples of evaluating *Mathematica* functions applied to various numeric and exact expressions that involve the hyperbolic cosine function or return it are shown. These involve numeric and symbolic calculations and plots.

Notations

***Mathematica* forms of notations**

Following *Mathematica*'s general naming convention, function names in `StandardForm` are just the capitalized versions of their traditional mathematics names. This shows the hyperbolic cosine function in `StandardForm`.

```
Cosh[z]
```

```
Cosh[z]
```

This shows the hyperbolic cosine function in `TraditionalForm`.

```
% // TraditionalForm
```

cosh(z)

Additional forms of notations

Mathematica also knows the most popular forms of notations for the hyperbolic cosine function that are used in other programming languages. Here are three examples: `CForm`, `TeXForm`, and `FortranForm`.

```
{CForm[Cosh[2 π z]], TeXForm[Cosh[2 π z]], FortranForm[Cosh[2 π z]]}
```

```
{Cosh(2*Pi*z), \cosh(2\, \pi \, z), Cosh(2*Pi*z)}
```

Automatic evaluations and transformations

Evaluation for exact, machine-number, and high-precision arguments

For the exact argument $z = \pi i / 4$, *Mathematica* returns an exact result.

$$\text{Cosh}\left[\frac{i\pi}{4}\right]$$

$$\frac{1}{\sqrt{2}}$$

$$\text{Cosh}[z] \text{ /. } z \to \frac{\pi i}{4}$$

$$\frac{1}{\sqrt{2}}$$

For a machine-number argument (a numerical argument with a decimal point and not too many digits), a machine number is also returned.

```
Cosh[5.]
```

```
74.2099
```

```
Cosh[z] /. z → 3.
```

```
10.0677
```

The next inputs calculate 100-digit approximations at $z = 1$ and $z = 2$.

```
N[Cosh[z] /. z → 1, 100]
```

```
1.543080634815243778477905620757061682601529112365863704737402214710769063049223698⋱
 64264726435543036
```

```
N[Cosh[2], 100]
```

```
3.762195691083631459562213477773746108293973558230711602777643347588323585090272726⋱
 60705303784889422
```

```
Cosh[2] // N[#, 100] &
```

```
3.762195691083631459562213477773746108293973558230711602777643347588323585090272726⋱
 60705303784889422
```

It is possible to calculate thousands of digits for the hyperbolic cosine function in less then a second. The next input calculates 10000 digits for cosh(1) and analyzes the frequency of the digit *k* in the resulting decimal number.

```
Map[Function[w, {First[#], Length[#]} & /@ Split[Sort[First[RealDigits[w]]]]] ,
 N[{Cosh[z]} /. z → 1, 10 000]]
```

```
{{{0, 1015}, {1, 960}, {2, 997}, {3, 1037},
  {4, 1070}, {5, 1018}, {6, 973}, {7, 997}, {8, 963}, {9, 970}}}
```

Here is a 50-digit approximation of the hyperbolic cosine function at the complex argument $z = 3 + 2\,i$.

```
N[Cosh[3 + 2 ⅈ], 50]
```

```
-4.1896256909688072301325550196159737286219454041279 +
 9.1092278937553365979791972627788621213326202389202 ⅈ
```

```
{N[Cosh[z] /. z → 3 + 2 ⅈ, 50], Cosh[3 + 2 ⅈ] // N[#, 50] &}
```

```
{-4.1896256909688072301325550196159737286219454041279 +
  9.1092278937553365979791972627788621213326202389202 ⅈ,
 -4.1896256909688072301325550196159737286219454041279 +
  9.1092278937553365979791972627788621213326202389202 ⅈ}
```

Mathematica automatically evaluates mathematical functions with machine precision, if the arguments of the function are machine-number elements. In this case only six digits after the decimal point are shown in the results. The remaining digits are suppressed, but can be displayed using the function `InputForm`.

```
{Cosh[2.], N[Cosh[2]], N[Cosh[2], 16], N[Cosh[2], 5], N[Cosh[2], 20]}
```

```
{3.7622, 3.7622, 3.7622, 3.7622, 3.7621956910836314596}
```

```
% // InputForm
```

```
{3.7621956910836314, 3.7621956910836314, 3.7621956910836314, 3.7621956910836314,
 3.76219569108363145956221347777374561721`20}
```

Simplification of the argument

Mathematica knows the symmetry and periodicity of the hyperbolic cosine function. Here are some examples.

```
Cosh[-3]
```

```
Cosh[3]
```

```
{Cosh[-z], Cosh[z + π ⅈ], Cosh[z + 2 π ⅈ], Cosh[z + 42 π ⅈ], Cosh[-z + 21 π ⅈ]}
```

```
{Cosh[z], -Cosh[z], Cosh[z], Cosh[z], -Cosh[z]}
```

Mathematica automatically simplifies the composition of the direct and the inverse hyperbolic cosine functions into its argument.

```
Cosh[ArcCosh[z]]
```

```
z
```

Mathematica also automatically simplifies the composition of the direct and any of the inverse hyperbolic functions into algebraic functions of the argument.

```
{Cosh[ArcSinh[z]], Cosh[ArcCosh[z]], Cosh[ArcTanh[z]],
 Cosh[ArcCoth[z]], Cosh[ArcCsch[z]], Cosh[ArcSech[z]]}
```

$$\left\{\sqrt{1+z^2},\ z,\ \frac{1}{\sqrt{1-z^2}},\ \frac{1}{\sqrt{1-\frac{1}{z^2}}},\ \sqrt{1+\frac{1}{z^2}},\ \frac{1}{z}\right\}$$

If the argument has the structure $\pi i k/2+z$ or $\pi i k/2-z$, and $\pi i k/2+i z$ or $\pi i k/2-i z$ with integer k, the hyperbolic cosine function can be automatically transformed into hyperbolic or trigonometric sine or cosine functions.

$$\text{Cosh}\left[\frac{\pi i}{2}-4\right]$$

```
-ⅈ Sinh[4]
```

$$\left\{\text{Cosh}\left[\frac{\pi i}{2}-z\right],\ \text{Cosh}\left[\frac{\pi i}{2}+z\right],\ \text{Cosh}\left[-\frac{\pi i}{2}-z\right],\ \text{Cosh}\left[-\frac{\pi i}{2}+z\right],\ \text{Cosh}[\pi i-z],\ \text{Cosh}[\pi i+z]\right\}$$

```
{-ⅈ Sinh[z], ⅈ Sinh[z], ⅈ Sinh[z], -ⅈ Sinh[z], -Cosh[z], -Cosh[z]}
```

```
Cosh[ⅈ 5]
```

```
Cos[5]
```

$$\left\{\text{Cosh}[i z],\ \text{Cosh}\left[\frac{\pi i}{2}-i z\right],\ \text{Cosh}\left[\frac{\pi i}{2}+i z\right],\ \text{Cosh}[\pi i-i z],\ \text{Cosh}[\pi i+i z]\right\}$$

```
{Cos[z], Sin[z], -Sin[z], -Cos[z], -Cos[z]}
```

Simplification of simple expressions containing the cosine function

Sometimes simple arithmetic operations containing the hyperbolic cosine function can automatically produce other hyperbolic functions.

```
1 / Cosh[4]
```

```
Sech[4]
```

```
{1 / Cosh[z], 1 / Cosh[π ⅈ / 2 - z], Cosh[π ⅈ / 2 - z] / Cosh[z],
 Cosh[z] / Cosh[π ⅈ / 2 - z], 1 / Cosh[π ⅈ / 2 - z], Cosh[π ⅈ / 2 - z] / Cosh[z]^2}
```

```
{Sech[z], ⅈ Csch[z], -ⅈ Tanh[z], ⅈ Coth[z], ⅈ Csch[z], -ⅈ Sech[z] Tanh[z]}
```

The hyperbolic cosine function arising as special cases from more general functions

The hyperbolic cosine function can be treated as a particular case of other more general special functions. For example, $\cosh(z)$ can appear automatically from Bessel, Mathieu, Jacobi, hypergeometric, and Meijer functions for appropriate values of their parameters.

```
{BesselI[-1/2, z], MathieuC[1, 0, ⅈ z], JacobiNC[z, 1],
 HypergeometricPFQ[{}, {1/2}, z^2/4], MeijerG[{{}, {}}, {{0}, {1/2}}, -z^2/4]}
```

$$\left\{\frac{\sqrt{\frac{2}{\pi}}\,\text{Cosh}[z]}{\sqrt{z}},\ \text{Cosh}[z],\ \text{Cosh}[z],\ \text{Cosh}\left[\sqrt{z^2}\right],\ \frac{\text{Cosh}[z]}{\sqrt{\pi}}\right\}$$

Equivalence transformations carried out by specialized *Mathematica* functions

General remarks

Almost everybody prefers using $\cosh(z)/2$ instead of $\sin(\pi/2 - i\,z)\cosh(\pi\,i/3)$. *Mathematica* automatically transforms the second expression into the first one. The automatic application of transformation rules to mathematical expressions can give overly complicated results. Compact expressions like $\cosh(2\,z)\cosh(\pi\,i/16)$ should not be automatically expanded into the more complicated expression $\left(\cosh^2(z) - 1/2\right)\left(2 + \left(2 + 2^{1/2}\right)^{1/2}\right)^{1/2}$. *Mathematica* has special functions that produce such expansions. Some are demonstrated in the next section.

TrigExpand

The function `TrigExpand` expands out trigonometric and hyperbolic functions. In more detail, it splits up sums and integer multiples that appear in the arguments of trigonometric and hyperbolic functions, and then expands out the products of trigonometric and hyperbolic functions into sums of powers, using trigonometric and hyperbolic identities where possible. Here are some examples.

```
TrigExpand[Cosh[x - y]]
```

```
Cosh[x] Cosh[y] - Sinh[x] Sinh[y]
```

```
Cosh[4 z] // TrigExpand
```

```
Cosh[z]^4 + 6 Cosh[z]^2 Sinh[z]^2 + Sinh[z]^4
```

```
Cosh[2 z]^2 // TrigExpand
```

$$\frac{1}{2} + \frac{\text{Cosh}[z]^4}{2} + 3\,\text{Cosh}[z]^2\,\text{Sinh}[z]^2 + \frac{\text{Sinh}[z]^4}{2}$$

```
TrigExpand[{Cosh[x + y + z], Cosh[3 z]}]
```

```
{Cosh[x] Cosh[y] Cosh[z] + Cosh[z] Sinh[x] Sinh[y] +
  Cosh[y] Sinh[x] Sinh[z] + Cosh[x] Sinh[y] Sinh[z], Cosh[z]^3 + 3 Cosh[z] Sinh[z]^2}
```

TrigFactor

The function `TrigFactor` factors trigonometric and hyperbolic functions. In more detail, it splits up sums and integer multiples that appear in arguments of trigonometric and hyperbolic functions, and then factors the resulting polynomials into trigonometric and hyperbolic functions, using trigonometric and hyperbolic identities where possible. Here are some examples.

TrigFactor[Cosh[x] + Cosh[y]]

$$2\,\mathrm{Cosh}\left[\frac{x}{2}-\frac{y}{2}\right]\,\mathrm{Cosh}\left[\frac{x}{2}+\frac{y}{2}\right]$$

Cosh[x] - Cosh[y] // TrigFactor

$$2\,\mathrm{Sinh}\left[\frac{x}{2}-\frac{y}{2}\right]\,\mathrm{Sinh}\left[\frac{x}{2}+\frac{y}{2}\right]$$

TrigReduce

The function `TrigReduce` rewrites the products and powers of trigonometric and hyperbolic functions in terms of trigonometric and hyperbolic functions with combined arguments. In more detail, it typically yields a linear expression that involves trigonometric and hyperbolic functions with more complicated arguments. `TrigReduce` is approximately inverse to `TrigExpand` and `TrigFactor`. Here are some examples.

TrigReduce[Cosh[x] Cosh[y]]

$$\frac{1}{2}\,(\mathrm{Cosh}[x-y]+\mathrm{Cosh}[x+y])$$

Sinh[x] Cosh[y] // TrigReduce

$$\frac{1}{2}\,(\mathrm{Sinh}[x-y]+\mathrm{Sinh}[x+y])$$

Table[TrigReduce[Cosh[z]^n], {n, 2, 5}]

$$\left\{\frac{1}{2}\,(1+\mathrm{Cosh}[2\,z]),\ \frac{1}{4}\,(3\,\mathrm{Cosh}[z]+\mathrm{Cosh}[3\,z]),\ \frac{1}{8}\,(3+4\,\mathrm{Cosh}[2\,z]+\mathrm{Cosh}[4\,z]),\ \frac{1}{16}\,(10\,\mathrm{Cosh}[z]+5\,\mathrm{Cosh}[3\,z]+\mathrm{Cosh}[5\,z])\right\}$$

TrigReduce[TrigExpand[{Cosh[x + y + z], Cosh[3 z], Cosh[x] Cosh[y]}]]

$$\left\{\mathrm{Cosh}[x+y+z],\ \mathrm{Cosh}[3\,z],\ \frac{1}{2}\,(\mathrm{Cosh}[x-y]+\mathrm{Cosh}[x+y])\right\}$$

TrigFactor[Cosh[x] + Cosh[y]] // TrigReduce

```
Cosh[x] + Cosh[y]
```

TrigToExp

The function `TrigToExp` converts trigonometric and hyperbolic functions to exponentials. It tries, where possible, to give results that do not involve explicit complex numbers. Here are some examples.

```
TrigToExp[Cosh[z]]
```

$$\frac{e^{-z}}{2} + \frac{e^{z}}{2}$$

```
Cosh[a z] + Cosh[b z] // TrigToExp
```

$$\frac{e^{-a z}}{2} + \frac{e^{a z}}{2} + \frac{e^{-b z}}{2} + \frac{e^{b z}}{2}$$

ExpToTrig

The function `ExpToTrig` converts exponentials to trigonometric and hyperbolic functions. It is approximately inverse to `TrigToExp`. Here are some examples.

```
ExpToTrig[TrigToExp[Cosh[z]]]
```

```
Cosh[z]
```

$\{\alpha\, e^{-x \beta} + \alpha\, e^{x \beta},\ \alpha\, e^{-x \beta} + \gamma\, e^{x \beta}\}$ `// ExpToTrig`

```
{2 α Cosh[x β], α Cosh[x β] + γ Cosh[x β] - α Sinh[x β] + γ Sinh[x β]}
```

ComplexExpand

The function `ComplexExpand` expands expressions assuming that all the variables are real. The value option `TargetFunctions` is a list of functions from the set `{Re, Im, Abs, Arg, Conjugate, Sign}`. `ComplexExpand` tries to give results in terms of the functions specified. Here are some examples.

```
ComplexExpand[Cosh[x + ⅈ y]]
```

```
Cos[y] Cosh[x] + ⅈ Sin[y] Sinh[x]
```

```
Cosh[x + ⅈ y] + Cosh[x - ⅈ y] // ComplexExpand
```

```
2 Cos[y] Cosh[x]
```

```
ComplexExpand[Re[Cosh[x + ⅈ y]], TargetFunctions → {Re, Im}]
```

```
Cos[y] Cosh[x]
```

```
ComplexExpand[Im[Cosh[x + ⅈ y]], TargetFunctions → {Re, Im}]
```

```
Sin[y] Sinh[x]
```

```
ComplexExpand[Abs[Cosh[x + ⅈ y]], TargetFunctions → {Re, Im}]
```

$$\sqrt{\mathrm{Cos}[y]^2\, \mathrm{Cosh}[x]^2 + \mathrm{Sin}[y]^2\, \mathrm{Sinh}[x]^2}$$

```
ComplexExpand[Abs[Cosh[x + ⅈ y]], TargetFunctions → {Re, Im}] //
  Simplify[#, {x, y} ∈ Reals] &
```

$$\frac{\sqrt{\text{Cos}[2\,y] + \text{Cosh}[2\,x]}}{\sqrt{2}}$$

```
ComplexExpand[Re[Cosh[x + ⅈ y]] + Im[Cosh[x + ⅈ y]], TargetFunctions → {Re, Im}]
```

```
Cos[y] Cosh[x] + Sin[y] Sinh[x]
```

```
ComplexExpand[Arg[Cosh[x + ⅈ y]], TargetFunctions → {Re, Im}]
```

```
ArcTan[Cos[y] Cosh[x], Sin[y] Sinh[x]]
```

```
ComplexExpand[Arg[Cosh[x + ⅈ y]], TargetFunctions → {Re, Im}] //
 Simplify[#, {x, y} ∈ Reals] &
```

```
ArcTan[Cos[y] Cosh[x], Sin[y] Sinh[x]]
```

```
ComplexExpand[Conjugate[Cosh[x + ⅈ y]], TargetFunctions → {Re, Im}] // Simplify
```

```
Cos[y] Cosh[x] - ⅈ Sin[y] Sinh[x]
```

Simplify

The function `Simplify` performs a sequence of algebraic transformations on its argument, and returns the simplest form it finds. Here are some examples.

```
Cosh[2 ArcSinh[z]] / (1 + 2 z^2) // Simplify
```

```
1
```

```
{Simplify[Cosh[2 z] - Cosh[z]^2], Cosh[2 z] - Sinh[z]^2 // Simplify}
```

```
{Sinh[z]^2, Cosh[z]^2}
```

Here is a large collection of hyperbolic identities. Each is written as a logical conjunction.

$$\text{Simplify}[\#]\ \&\ /@\ \left(\text{Cosh}[z]^2 - \text{Sinh}[z]^2 == 1 \bigwedge \text{Cosh}[z]^2 == \frac{\text{Cosh}[2z]+1}{2} \bigwedge \text{Cosh}[2z] == \text{Cosh}[z]^2 + \text{Sinh}[z]^2 \bigwedge \text{Cosh}[a+b] == \text{Cosh}[a]\,\text{Cosh}[b] + \text{Sinh}[a]\,\text{Sinh}[b] \bigwedge \text{Cosh}[a-b] == \text{Cosh}[a]\,\text{Cosh}[b] - \text{Sinh}[a]\,\text{Sinh}[b] \bigwedge \text{Cosh}[a] + \text{Cosh}[b] == 2\,\text{Cosh}\left[\frac{a-b}{2}\right]\text{Cosh}\left[\frac{a+b}{2}\right] \bigwedge \text{Cosh}[a] - \text{Cosh}[b] == 2\,\text{Sinh}\left[\frac{a+b}{2}\right]\text{Sinh}\left[\frac{a-b}{2}\right] \bigwedge A\,\text{Sinh}[z] + B\,\text{Cosh}[z] == A\sqrt{1-\frac{B^2}{A^2}}\ \text{Sinh}\left[z + \text{ArcTanh}\left[\frac{B}{A}\right]\right] \bigwedge \text{Cosh}[a]\,\text{Cosh}[b] == \frac{\text{Cosh}[a+b]+\text{Cosh}[a-b]}{2} \bigwedge \text{Sinh}[a]\,\text{Cosh}[b] == \frac{\text{Sinh}[a+b]+\text{Sinh}[a-b]}{2} \bigwedge \text{Cosh}[a]^2 - \text{Cosh}[b]^2 == \text{Sinh}[a-b]\,\text{Sinh}[a+b]\right)$$

```
True
```

The function `Simplify` has the `Assumption` option. For example, *Mathematica* knows that $\cosh(x) \geq 1$ for all real x, and recognizes the periodicity of hyperbolic functions for the symbolic integer coefficient k of $k \pi i$.

```
Simplify[Cosh[x] ≥ 1, x ∈ Reals]
```

```
True
```

```
Cosh[x] ≥ 0 // Simplify[#, x ≥ 0] &
```

```
True
```

```
Simplify[{Cosh[z + 2 k π ⅈ], Cosh[z + k π ⅈ] / Cosh[z]}, k ∈ Integers]
```

$\{\text{Cosh}[z],\ (-1)^k\}$

Mathematica also knows that the composition of the inverse and direct hyperbolic functions gives the value of the internal argument under the corresponding restriction.

```
ArcCosh[Cosh[z]]
```

```
ArcCosh[Cosh[z]]
```

```
Simplify[ArcCosh[Cosh[z]], Re[z] > 0 ∧ -π < Im[z] < π]
```

z

FunctionExpand (and Together)

While the hyperbolic cosine function auto-evaluates for simple fractions of $i\pi$, for more complicated cases it stays as a hyperbolic cosine function to avoid the build up of large expressions. Using the function `FunctionExpand`, the hyperbolic cosine function can sometimes be transformed into explicit radicals. Here are some examples.

$$\left\{\text{Cosh}\left[\frac{\pi i}{16}\right], \text{Cosh}\left[\frac{\pi i}{60}\right]\right\}$$

$$\left\{\text{Cos}\left[\frac{\pi}{16}\right], \text{Cos}\left[\frac{\pi}{60}\right]\right\}$$

FunctionExpand[%]

$$\left\{\frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2}}}, -\frac{-\frac{1}{8}\sqrt{3}\left(-1+\sqrt{5}\right)-\frac{1}{4}\sqrt{\frac{1}{2}\left(5+\sqrt{5}\right)}}{\sqrt{2}}-\frac{\frac{1}{8}\left(-1+\sqrt{5}\right)-\frac{1}{4}\sqrt{\frac{3}{2}\left(5+\sqrt{5}\right)}}{\sqrt{2}}\right\}$$

Together[%]

$$\left\{\frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2}}}, \frac{1}{16}\left(\sqrt{2}-\sqrt{6}-\sqrt{10}+\sqrt{30}+2\sqrt{5+\sqrt{5}}+2\sqrt{3\left(5+\sqrt{5}\right)}\right)\right\}$$

If the denominator contains squares of integers other than 2, the results always contain complex numbers (meaning that the imaginary number $i = \sqrt{-1}$ appears unavoidably).

$$\left\{\text{Cosh}\left[\frac{\pi i}{9}\right]\right\}$$

$$\left\{\text{Cos}\left[\frac{\pi}{9}\right]\right\}$$

FunctionExpand[%] // Together

$$\left\{\frac{1}{8}\left(2^{2/3}\left(-1-i\sqrt{3}\right)^{1/3}+ i\,2^{2/3}\sqrt{3}\left(-1-i\sqrt{3}\right)^{1/3}+2^{2/3}\left(-1+i\sqrt{3}\right)^{1/3}-i\,2^{2/3}\sqrt{3}\left(-1+i\sqrt{3}\right)^{1/3}\right)\right\}$$

The function `RootReduce` allows for writing the last algebraic numbers as roots of polynomial equations.

RootReduce[Simplify[%]]

$$\left\{\text{Root}\left[-1-6\,\#1+8\,\#1^3\,\&, 3\right]\right\}$$

The function `FunctionExpand` also reduces hyperbolic expressions with compound arguments or compositions, including inverse hyperbolic functions, to simpler ones. Here are some examples.

$$\left\{\text{Cosh}\left[\sqrt{z^2}\right], \text{Cosh}\left[\frac{\text{ArcCosh}[z]}{2}\right], \text{Cosh}[3\,\text{ArcCosh}[z]]\right\} \text{ // FunctionExpand}$$

$$\left\{\text{Cosh}[z], \frac{\sqrt{1+z}}{\sqrt{2}}, z^3+3\,(-1+z)\,z\,(1+z)\right\}$$

Applying `Simplify` to the last expression gives a more compact result.

```
Simplify[%]
```

$$\left\{\text{Cosh}[z], \frac{\sqrt{1+z}}{\sqrt{2}}, z\,\left(-3+4\,z^2\right)\right\}$$

FullSimplify

The function `FullSimplify` tries a wider range of transformations than `Simplify` and returns the simplest form it finds. Here are some examples that compare the results of applying the functions `Simplify` and `FullSimplify` to the same expressions.

$$\text{set1} = \left\{\text{Cosh}\left[\text{Log}\left[z+\sqrt{1+z^2}\right]\right], \text{Cosh}\left[\text{Log}\left[z+\sqrt{-1+z}\,\sqrt{1+z}\right]\right], \text{Cosh}\left[-\frac{1}{2}\text{Log}[1-z]+\frac{1}{2}\text{Log}[1+z]\right], \text{Cosh}\left[-\frac{1}{2}\text{Log}\left[1-\frac{1}{z}\right]+\frac{1}{2}\text{Log}\left[1+\frac{1}{z}\right]\right], \text{Cosh}\left[\text{Log}\left[\sqrt{1+\frac{1}{z^2}}+\frac{1}{z}\right]\right], \text{Cosh}\left[\text{Log}\left[\sqrt{-1+\frac{1}{z}}\,\sqrt{1+\frac{1}{z}}+\frac{1}{z}\right]\right]\right\}$$

$$\left\{\frac{1+\left(z+\sqrt{1+z^2}\right)^2}{2\left(z+\sqrt{1+z^2}\right)}, \frac{1+\left(z+\sqrt{-1+z}\,\sqrt{1+z}\right)^2}{2\left(z+\sqrt{-1+z}\,\sqrt{1+z}\right)}, \text{Cosh}\left[\frac{1}{2}\text{Log}[1-z]-\frac{1}{2}\text{Log}[1+z]\right], \text{Cosh}\left[\frac{1}{2}\text{Log}\left[1-\frac{1}{z}\right]-\frac{1}{2}\text{Log}\left[1+\frac{1}{z}\right]\right], \frac{1+\left(\sqrt{1+\frac{1}{z^2}}+\frac{1}{z}\right)^2}{2\left(\sqrt{1+\frac{1}{z^2}}+\frac{1}{z}\right)}, \frac{1+\left(\sqrt{-1+\frac{1}{z}}\,\sqrt{1+\frac{1}{z}}+\frac{1}{z}\right)^2}{2\left(\sqrt{-1+\frac{1}{z}}\,\sqrt{1+\frac{1}{z}}+\frac{1}{z}\right)}\right\}$$

```
set1 // Simplify
```

$$\left\{\frac{1+z^2+z\sqrt{1+z^2}}{z+\sqrt{1+z^2}}, z, \text{Cosh}\left[\frac{1}{2}\,(\text{Log}[1-z]-\text{Log}[1+z])\right], \text{Cosh}\left[\frac{1}{2}\left(-\text{Log}\left[1+\frac{1}{z}\right]+\text{Log}\left[\frac{-1+z}{z}\right]\right)\right], \frac{1+\sqrt{1+\frac{1}{z^2}}\,z+z^2}{z\left(1+\sqrt{1+\frac{1}{z^2}}\,z\right)}, \frac{1}{z}\right\}$$

```
set1 // FullSimplify
```

$$\left\{\sqrt{1+z^2},\ z,\ \frac{1}{\sqrt{1-z^2}},\ \text{Cosh}\left[\frac{1}{2}\left(-\text{Log}\left[1+\frac{1}{z}\right]+\text{Log}\left[\frac{-1+z}{z}\right]\right)\right],\ \sqrt{1+\frac{1}{z^2}},\ \frac{1}{z}\right\}$$

Operations carried out by specialized *Mathematica* functions

Series expansions

Calculating the series expansion of a hyperbolic cosine function to hundreds of terms can be done in seconds.

```
Series[Cosh[z], {z, 0, 3}]
```

$$1+\frac{z^2}{2}+O[z]^4$$

```
Normal[%]
```

$$1+\frac{z^2}{2}$$

```
Series[Cosh[z], {z, 0, 100}] // Timing
```

$$\left\{0.02\ \text{Second},\ 1+\frac{z^2}{2}+\frac{z^4}{24}+\frac{z^6}{720}+\frac{z^8}{40320}+\frac{z^{10}}{3628800}+\right.$$
$$\frac{z^{12}}{479001600}+\frac{z^{14}}{87178291200}+\frac{z^{16}}{20922789888000}+\frac{z^{18}}{6402373705728000}+$$
$$\frac{z^{20}}{2432902008176640000}+\frac{z^{22}}{1124000727777607680000}+\frac{z^{24}}{620448401733239439360000}+$$
$$\frac{z^{26}}{403291461126605635584000000}+\frac{z^{28}}{304888344611713860501504000000}+$$
$$\frac{z^{30}}{265252859812191058636308480000000}+\frac{z^{32}}{263130836933693530167218012160000000}+$$
$$\frac{z^{34}}{295232799039604140847618609643520000000}+$$
$$\frac{z^{36}}{371993326789901217467999448150835200000000}+$$
$$\frac{z^{38}}{523022617466601111760007224100074291200000000}+$$
$$\frac{z^{40}}{815915283247897734345611269596115894272000000000}+$$
$$\frac{z^{42}}{1405006117752879898543142606244511569936384000000000}+$$
$$\frac{z^{44}}{2658271574788448768043625811014615890319638528000000000}+$$
$$\frac{z^{46}}{5502622159812088949850305428800254892961651752960000000000}+$$

$$\frac{z^{48}}{12\,413\,915\,592\,536\,072\,670\,862\,289\,047\,373\,375\,038\,521\,486\,354\,677\,760\,000\,000\,000}+$$

$z^{50}\Big/30\,414\,093\,201\,713\,378\,043\,612\,608\,166\,064\,768\,844\,377\,641\,568\,960\,512\,000\,000\,000\,000 + z^{52}\Big/$
$80\,658\,175\,170\,943\,878\,571\,660\,636\,856\,403\,766\,975\,289\,505\,440\,883\,277\,824\,000\,000\,000\,000 + z^{54}\Big/$
$230\,843\,697\,339\,241\,380\,472\,092\,742\,683\,027\,581\,083\,278\,564\,571\,807\,941\,132\,288\,000\,000\,000\,000 +$
$z^{56}\Big/$
$710\,998\,587\,804\,863\,451\,854\,045\,647\,463\,724\,949\,736\,497\,978\,881\,168\,458\,687\,447\,040\,000\,000\,000\,000$
$+$
$z^{58}\Big/$
$2\,350\,561\,331\,282\,878\,571\,829\,474\,910\,515\,074\,683\,828\,862\,318\,181\,142\,924\,420\,699\,914\,240\,000\,000\,000\,000 +$
$z^{60}\Big/$
$8\,320\,987\,112\,741\,390\,144\,276\,341\,183\,223\,364\,380\,754\,172\,606\,361\,245\,952\,449\,277\,696\,409\,600\,000\,000\,000\,000 + z^{62}\Big/$
$31\,469\,973\,260\,387\,937\,525\,653\,122\,354\,950\,764\,088\,012\,280\,797\,258\,232\,192\,163\,168\,247\,821\,107\,200\,000\,000\,000\,000 + z^{64}\Big/$
$126\,886\,932\,185\,884\,164\,103\,433\,389\,335\,161\,480\,802\,865\,516\,174\,545\,192\,198\,801\,894\,375\,214\,704\,230\,400\,000\,000\,000\,000 + z^{66}\Big/$
$544\,344\,939\,077\,443\,064\,003\,729\,240\,247\,842\,752\,644\,293\,064\,388\,798\,874\,532\,860\,126\,869\,671\,081\,148\,416\,000\,000\,000\,000\,000 + z^{68}\Big/$
$2\,480\,035\,542\,436\,830\,599\,600\,990\,418\,569\,171\,581\,047\,399\,201\,355\,367\,672\,371\,710\,738\,018\,221\,445\,712\,183\,296\,000\,000\,000\,000\,000 + z^{70}\Big/$
$11\,978\,571\,669\,969\,891\,796\,072\,783\,721\,689\,098\,736\,458\,938\,142\,546\,425\,857\,555\,362\,864\,628\,009\,582\,789\,845\,319\,680\,000\,000\,000\,000\,000 + z^{72}\Big/$
$61\,234\,458\,376\,886\,086\,861\,524\,070\,385\,274\,672\,740\,778\,091\,784\,697\,328\,983\,823\,014\,963\,978\,384\,987\,221\,689\,274\,204\,160\,000\,000\,000\,000\,000 + z^{74}\Big/$
$330\,788\,544\,151\,938\,641\,225\,953\,028\,221\,253\,782\,145\,683\,251\,820\,934\,971\,170\,611\,926\,835\,411\,235\,700\,971\,565\,459\,250\,872\,320\,000\,000\,000\,000\,000 + z^{76}\Big/$
$1\,885\,494\,701\,666\,050\,254\,987\,932\,260\,861\,146\,558\,230\,394\,535\,379\,329\,335\,672\,487\,982\,961\,844\,043\,495\,537\,923\,117\,729\,972\,224\,000\,000\,000\,000\,000\,000 + z^{78}\Big/$
$11\,324\,281\,178\,206\,297\,831\,457\,521\,158\,732\,046\,228\,731\,749\,579\,488\,251\,990\,048\,962\,825\,668\,835\,325\,234\,200\,766\,245\,086\,213\,177\,344\,000\,000\,000\,000\,000\,000 + z^{80}\Big/$
$71\,569\,457\,046\,263\,802\,294\,811\,533\,723\,186\,532\,165\,584\,657\,342\,365\,752\,577\,109\,445\,058\,227\,039\,255\,480\,148\,842\,668\,944\,867\,280\,814\,080\,000\,000\,000\,000\,000\,000 + z^{82}\Big/$
$475\,364\,333\,701\,284\,174\,842\,138\,206\,989\,404\,946\,643\,813\,294\,067\,993\,328\,617\,160\,934\,076\,743\,994\,734\,899\,148\,613\,007\,131\,808\,479\,167\,119\,360\,000\,000\,000\,000\,000\,000 + z^{84}\Big/$
$3\,314\,240\,134\,565\,353\,266\,999\,387\,579\,130\,131\,288\,000\,666\,286\,242\,049\,487\,118\,846\,032\,383\,059\,131\,291\,716\,864\,129\,885\,722\,968\,716\,753\,156\,177\,920\,000\,000\,000\,000\,000\,000 + z^{86}\Big/$
$24\,227\,095\,383\,672\,732\,381\,765\,523\,203\,441\,259\,715\,284\,870\,552\,429\,381\,750\,838\,764\,496\,720\,162\,249\,742\,450\,276\,789\,464\,634\,901\,319\,465\,571\,660\,595\,200\,000\,000\,000\,000\,000\,000 + z^{88}\Big/$
$185\,482\,642\,257\,398\,439\,114\,796\,845\,645\,546\,284\,380\,220\,968\,949\,399\,346\,684\,421\,580\,986\,889\,562\,184\,028\,199\,319\,100\,141\,244\,804\,501\,828\,416\,633\,516\,851\,200\,000\,000\,000\,000\,000\,000 + z^{90}\Big/$
$1\,485\,715\,964\,481\,761\,497\,309\,522\,733\,620\,825\,737\,885\,569\,961\,284\,688\,766\,942\,216\,863\,704\,985\,393\,094\,065\,876\,545\,992\,131\,370\,884\,059\,645\,617\,234\,469\,978\,112\,000\,000\,000\,000\,000\,000\,000 + z^{92}\Big/$
$12\,438\,414\,054\,641\,307\,255\,475\,324\,325\,873\,553\,077\,577\,991\,715\,875\,414\,356\,840\,239\,582\,938\,137\,710\,983\,519\,518\,443\,046\,123\,837\,041\,347\,353\,107\,486\,982\,656\,753\,664\,000\,000\,000\,000\,000\,000\,000 +$
$z^{94}\Big/$

```
108 736 615 665 674 308 027 365 285 256 786 601 004 186 803 580 182 872 307 497 374 434 045 199 
 869 417 927 630 229 109 214 583 415 458 560 865 651 202 385 340 530 688 000 000 000 000 000 000 
 000 + z^96 /
991 677 934 870 949 689 209 571 401 541 893 801 158 183 648 651 267 795 444 376 054 838 492 222 
 809 091 499 987 689 476 037 000 748 982 075 094 738 965 754 305 639 874 560 000 000 000 000 000 
 000 000 + z^98 /
9 426 890 448 883 247 745 626 185 743 057 242 473 809 693 764 078 951 663 494 238 777 294 707 070 
 023 223 798 882 976 159 207 729 119 823 605 850 588 608 460 429 412 647 567 360 000 000 000 000 
 000 000 000 + z^100 /
93 326 215 443 944 152 681 699 238 856 266 700 490 715 968 264 381 621 468 592 963 895 217 599 993 
 229 915 608 941 463 976 156 518 286 253 697 920 827 223 758 251 185 210 916 864 000 000 000 000 
 000 000 000 000 + O[z]^101 }
```

Mathematica comes with the add-on package `DiscreteMath`RSolve`` that allows finding the general terms of the series for many functions. After loading this package, and using the package function `SeriesTerm`, the following n^{th} term of $\cosh(z)$ can be evaluated.

```
<< DiscreteMath`RSolve`
```

```
SeriesTerm[Cosh[z], {z, 0, n}] z^n
```

$$z^n \text{If}\left[\text{Even}[n], \frac{1}{n!}, 0\right]$$

This result can be verified as follows. The previous expression is not zero for $n == 2\,k$ and the corresponding sum has the following value.

$$\text{Sum}\left[\frac{1}{(2\,k)!} z^{2\,k}, \{k, 0, \infty\}\right]$$

$$\text{Cosh}\left[\sqrt{z^2}\right]$$

```
FunctionExpand[%]
```

```
Cosh[z]
```

Differentiation

Mathematica can evaluate derivatives of the hyperbolic cosine function of an arbitrary positive integer order.

$$\partial_z \text{Cosh}[z]$$

```
Sinh[z]
```

$$\partial_{\{z,2\}} \text{Cosh}[z]$$

```
Cosh[z]
```

```
Table[D[Cosh[z], {z, n}], {n, 10}]
```

```
{Sinh[z], Cosh[z], Sinh[z], Cosh[z],
 Sinh[z], Cosh[z], Sinh[z], Cosh[z], Sinh[z], Cosh[z]}
```

Finite summation

Mathematica can calculate finite symbolic sums that contain the hyperbolic cosine function. Here are some examples.

$$\sum_{k=1}^{n} \text{Cosh}[a\,k]$$

$$\frac{e^a\,(-1+e^{a\,n})}{2\,(-1+e^a)} + \frac{e^{-a\,n}\,(-1+e^{a\,n})}{2\,(-1+e^a)}$$

$$\sum_{k=1}^{n} (-1)^k\,\text{Cosh}[a\,k]$$

$$\frac{-1+(-e^{-a})^n}{2\,(1+e^a)} + \frac{e^a\,(-1+(-e^a)^n)}{2\,(1+e^a)}$$

Infinite summation

Mathematica can calculate infinite sums including the hyperbolic cosine function. Here are some examples.

$$\sum_{k=1}^{\infty} z^k\,\text{Cosh}[k\,x]$$

$$\frac{z}{2\,(e^x-z)} - \frac{e^x\,z}{2\,(-1+e^x\,z)}$$

$$\sum_{k=1}^{\infty} \frac{\text{Cosh}[k\,x]}{k\,!}$$

$$\frac{1}{2}\left(-1+e^{e^{-x}}\right) + \frac{1}{2}\left(-1+e^{e^x}\right)$$

$$\sum_{k=1}^{\infty} \frac{\text{Cosh}[k\,x]}{k}$$

$$-\frac{1}{2}\,\text{Log}[1-e^{-x}] - \frac{1}{2}\,\text{Log}[1-e^x]$$

Finite products

Mathematica can calculate some finite symbolic products that contain the hyperbolic cosine function. Here is an example.

$$\prod_{k=1}^{n-1} \text{Cosh}\left[\frac{k\,\pi\,i}{n}\right]$$

$$-(-1)^n\,2^{1-n}\,\text{Sin}\left[\frac{n\,\pi}{2}\right]$$

$$\prod_{k=1}^{n-1} \text{Cosh}\left[z+\frac{\pi\,k\,i}{n}\right]$$

$$-(-1)^n\, 2^{1-n}\, \text{Sech}[z]\, \text{Sin}\left[\frac{1}{2}\, n\, (\pi + 2\, i\, z)\right]$$

Indefinite integration

Mathematica can calculate a huge set of doable indefinite integrals that contain the hyperbolic cosine function. Here are some examples.

$$\int \text{Cosh}[z]\, dz$$

$$\text{Sinh}[z]$$

$$\int \text{Cosh}[z]^a\, dz$$

$$-\frac{\text{Cosh}[z]^{1+a}\, \text{Hypergeometric2F1}\left[\frac{1+a}{2}, \frac{1}{2}, \frac{3+a}{2}, \text{Cosh}[z]^2\right] \text{Sinh}[z]}{(1+a)\sqrt{-\text{Sinh}[z]^2}}$$

Definite integration

Mathematica can calculate wide classes of definite integrals that contain the hyperbolic cosine function. Here are some examples.

$$\int_a^b \frac{\text{Cosh}[z]}{z}\, dz$$

$$-\text{CoshIntegral}[a] + \text{CoshIntegral}[b]$$

$$\int_0^\infty e^{a\, t}\, \text{Cosh}[b\, t]\, dt$$

$$\text{If}\left[0 > \text{Re}[a+b]\, \&\&\, \text{Re}[-a+b] > 0\, \&\&\, \text{Re}[a] < 0,\ \frac{a}{-a^2+b^2},\ \int_0^\infty e^{a\, t}\, \text{Cosh}[b\, t]\, dt\right]$$

Limit operation

Mathematica can calculate limits that contain the hyperbolic cosine function. Here are some examples.

$$\text{Limit}\left[\frac{\text{Cosh}[2\, z] - 1}{z^2},\ z \to 0\right]$$

$$2$$

$$\text{Limit}\left[\frac{\text{Cosh}\left[\sqrt{z^2}\right] - 1}{z^2},\ z \to 0\right]$$

$$\frac{1}{2}$$

$$\text{Limit}\left[\frac{\text{Cosh}\left[\left(z^2\right)^{1/4}\right] - 1}{z},\ z \to 0,\ \text{Direction} \to 1\right]$$

$-\frac{1}{2}$

$$\text{Limit}\left[\frac{\text{Cosh}\left[\left(z^2\right)^{1/4}\right]-1}{z}, z\to 0, \text{Direction}\to -1\right]$$

$\frac{1}{2}$

Solving equations

The next inputs solve two equations that contain the hyperbolic cosine function. Because of the multivalued nature of the inverse hyperbolic cosine function, a message is printed indicating that only some of the possible solutions are returned.

```
Solve[Cosh[z]^2 + 3 Cosh[z + Pi/6] == 4, z]
```

```
Solve::ifun : Inverse functions are being used by Solve, so some
   solutions may not be found; use Reduce for complete solution information. More…
```

$$\Bigg\{\Bigg\{z\to \text{ArcCosh}\Bigg[-\frac{3}{2}\text{Cosh}\left[\frac{\pi}{6}\right]-\frac{1}{2}\sqrt{\Bigg(8+9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2+\frac{1}{3}\left(-8+9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2-9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\right)+}$$

$$\left(432\,\text{Cosh}\left[\frac{\pi}{6}\right]^2+\left(-8+9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2-9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\right)^2+12\left(16+9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\right)\right)\Big/$$

$$\Bigg(3\Bigg(\frac{1}{2}\Bigg(15\,552\,\text{Cosh}\left[\frac{\pi}{6}\right]^2+1296\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\left(-8+9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2-9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\right)+$$

$$2\left(-8+9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2-9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\right)^3+972\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\left(16+9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\right)-$$

$$72\left(-8+9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2-9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\right)\left(16+9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\right)+$$

$$\sqrt{\Bigg(-80\,621\,568\,\text{Sinh}\left[\frac{\pi}{6}\right]^4-10\,077\,696\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\text{Sinh}\left[\frac{\pi}{6}\right]^4+}$$

$$65\,190\,096\,\text{Cosh}\left[\frac{\pi}{6}\right]^4\text{Sinh}\left[\frac{\pi}{6}\right]^4+25\,509\,168\,\text{Cosh}\left[\frac{\pi}{6}\right]^6\text{Sinh}\left[\frac{\pi}{6}\right]^4-$$

$$166\,281\,984\,\text{Sinh}\left[\frac{\pi}{6}\right]^6+124\,711\,488\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\text{Sinh}\left[\frac{\pi}{6}\right]^6-$$

$$76\,527\,504\,\text{Cosh}\left[\frac{\pi}{6}\right]^4\text{Sinh}\left[\frac{\pi}{6}\right]^6-113\,374\,080\,\text{Sinh}\left[\frac{\pi}{6}\right]^8+$$

$$76\,527\,504\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\text{Sinh}\left[\frac{\pi}{6}\right]^8-25\,509\,168\,\text{Sinh}\left[\frac{\pi}{6}\right]^{10}\Bigg)}\Bigg)\Bigg)^{1/3}\Bigg)+$$

$$\frac{1}{3}\Bigg(\frac{1}{2}\Bigg(15\,552\,\text{Cosh}\left[\frac{\pi}{6}\right]^2+1296\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\left(-8+9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2-9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\right)+$$

$$2\ \Big(-8+9\,\text{Cosh}\big[\tfrac{\pi}{6}\big]^{2}-9\,\text{Sinh}\big[\tfrac{\pi}{6}\big]^{2}\Big)^{3}+972\,\text{Cosh}\big[\tfrac{\pi}{6}\big]^{2}\ \Big(16+9\,\text{Sinh}\big[\tfrac{\pi}{6}\big]^{2}\Big)-$$

$$72\ \Big(-8+9\,\text{Cosh}\big[\tfrac{\pi}{6}\big]^{2}-9\,\text{Sinh}\big[\tfrac{\pi}{6}\big]^{2}\Big)\ \Big(16+9\,\text{Sinh}\big[\tfrac{\pi}{6}\big]^{2}\Big)+$$

$$\sqrt{\Big(-80\,621\,568\,\text{Sinh}\big[\tfrac{\pi}{6}\big]^{4}-10\,077\,696\,\text{Cosh}\big[\tfrac{\pi}{6}\big]^{2}\,\text{Sinh}\big[\tfrac{\pi}{6}\big]^{4}+}$$

$$65\,190\,096\,\text{Cosh}\big[\tfrac{\pi}{6}\big]^{4}\,\text{Sinh}\big[\tfrac{\pi}{6}\big]^{4}+25\,509\,168\,\text{Cosh}\big[\tfrac{\pi}{6}\big]^{6}\,\text{Sinh}\big[\tfrac{\pi}{6}\big]^{4}-$$

$$166\,281\,984\,\text{Sinh}\big[\tfrac{\pi}{6}\big]^{6}+124\,711\,488\,\text{Cosh}\big[\tfrac{\pi}{6}\big]^{2}\,\text{Sinh}\big[\tfrac{\pi}{6}\big]^{6}-$$

$$76\,527\,504\,\text{Cosh}\big[\tfrac{\pi}{6}\big]^{4}\,\text{Sinh}\big[\tfrac{\pi}{6}\big]^{6}-113\,374\,080\,\text{Sinh}\big[\tfrac{\pi}{6}\big]^{8}+$$

$$76\,527\,504\,\text{Cosh}\big[\tfrac{\pi}{6}\big]^{2}\,\text{Sinh}\big[\tfrac{\pi}{6}\big]^{8}-25\,509\,168\,\text{Sinh}\big[\tfrac{\pi}{6}\big]^{10}\Big)\Big)\Big)^{1/3}\Big)-$$

$$\frac{1}{2}\sqrt{\Big(8+9\,\text{Cosh}\big[\tfrac{\pi}{6}\big]^{2}+9\,\text{Sinh}\big[\tfrac{\pi}{6}\big]^{2}-\frac{1}{3}\ \Big(-8+9\,\text{Cosh}\big[\tfrac{\pi}{6}\big]^{2}-9\,\text{Sinh}\big[\tfrac{\pi}{6}\big]^{2}\Big)-}$$

$$\Big(432\,\text{Cosh}\big[\tfrac{\pi}{6}\big]^{2}+\Big(-8+9\,\text{Cosh}\big[\tfrac{\pi}{6}\big]^{2}-9\,\text{Sinh}\big[\tfrac{\pi}{6}\big]^{2}\Big)^{2}+12\ \Big(16+9\,\text{Sinh}\big[\tfrac{\pi}{6}\big]^{2}\Big)\Big)\Big/$$

$$\Big(3\ \Big(\frac{1}{2}\ \Big(15\,552\,\text{Cosh}\big[\tfrac{\pi}{6}\big]^{2}+1296\,\text{Cosh}\big[\tfrac{\pi}{6}\big]^{2}\ \Big(-8+9\,\text{Cosh}\big[\tfrac{\pi}{6}\big]^{2}-9\,\text{Sinh}\big[\tfrac{\pi}{6}\big]^{2}\Big)+$$

$$2\ \Big(-8+9\,\text{Cosh}\big[\tfrac{\pi}{6}\big]^{2}-9\,\text{Sinh}\big[\tfrac{\pi}{6}\big]^{2}\Big)^{3}+972\,\text{Cosh}\big[\tfrac{\pi}{6}\big]^{2}\ \Big(16+9\,\text{Sinh}\big[\tfrac{\pi}{6}\big]^{2}\Big)-$$

$$72\ \Big(-8+9\,\text{Cosh}\big[\tfrac{\pi}{6}\big]^{2}-9\,\text{Sinh}\big[\tfrac{\pi}{6}\big]^{2}\Big)\ \Big(16+9\,\text{Sinh}\big[\tfrac{\pi}{6}\big]^{2}\Big)+$$

$$\sqrt{\Big(-80\,621\,568\,\text{Sinh}\big[\tfrac{\pi}{6}\big]^{4}-10\,077\,696\,\text{Cosh}\big[\tfrac{\pi}{6}\big]^{2}\,\text{Sinh}\big[\tfrac{\pi}{6}\big]^{4}+}$$

$$65\,190\,096\,\text{Cosh}\big[\tfrac{\pi}{6}\big]^{4}\,\text{Sinh}\big[\tfrac{\pi}{6}\big]^{4}+25\,509\,168\,\text{Cosh}\big[\tfrac{\pi}{6}\big]^{6}\,\text{Sinh}\big[\tfrac{\pi}{6}\big]^{4}-$$

$$166\,281\,984\,\text{Sinh}\big[\tfrac{\pi}{6}\big]^{6}+124\,711\,488\,\text{Cosh}\big[\tfrac{\pi}{6}\big]^{2}\,\text{Sinh}\big[\tfrac{\pi}{6}\big]^{6}-$$

$$76\,527\,504\,\text{Cosh}\big[\tfrac{\pi}{6}\big]^{4}\,\text{Sinh}\big[\tfrac{\pi}{6}\big]^{6}-113\,374\,080\,\text{Sinh}\big[\tfrac{\pi}{6}\big]^{8}+$$

$$76\,527\,504\,\text{Cosh}\big[\tfrac{\pi}{6}\big]^{2}\,\text{Sinh}\big[\tfrac{\pi}{6}\big]^{8}-25\,509\,168\,\text{Sinh}\big[\tfrac{\pi}{6}\big]^{10}\Big)\Big)\Big)^{1/3}\Big)-$$

$$\frac{1}{3}\ \Big(\frac{1}{2}\ \Big(15\,552\,\text{Cosh}\big[\tfrac{\pi}{6}\big]^{2}+1296\,\text{Cosh}\big[\tfrac{\pi}{6}\big]^{2}\ \Big(-8+9\,\text{Cosh}\big[\tfrac{\pi}{6}\big]^{2}-9\,\text{Sinh}\big[\tfrac{\pi}{6}\big]^{2}\Big)+$$

$$2\,\left(-8+9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2-9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\right)^3+972\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\,\left(16+9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\right)-$$

$$72\,\left(-8+9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2-9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\right)\,\left(16+9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\right)+$$

$$\sqrt{\left(-80\,621\,568\,\text{Sinh}\left[\frac{\pi}{6}\right]^4-10\,077\,696\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\,\text{Sinh}\left[\frac{\pi}{6}\right]^4+\right.}$$

$$65\,190\,096\,\text{Cosh}\left[\frac{\pi}{6}\right]^4\,\text{Sinh}\left[\frac{\pi}{6}\right]^4+25\,509\,168\,\text{Cosh}\left[\frac{\pi}{6}\right]^6\,\text{Sinh}\left[\frac{\pi}{6}\right]^4-$$

$$166\,281\,984\,\text{Sinh}\left[\frac{\pi}{6}\right]^6+124\,711\,488\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\,\text{Sinh}\left[\frac{\pi}{6}\right]^6-$$

$$76\,527\,504\,\text{Cosh}\left[\frac{\pi}{6}\right]^4\,\text{Sinh}\left[\frac{\pi}{6}\right]^6-113\,374\,080\,\text{Sinh}\left[\frac{\pi}{6}\right]^8+$$

$$\left.76\,527\,504\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\,\text{Sinh}\left[\frac{\pi}{6}\right]^8-25\,509\,168\,\text{Sinh}\left[\frac{\pi}{6}\right]^{10}\right)\Big)\Big)^{1/3}-$$

$$\left(192\,\text{Cosh}\left[\frac{\pi}{6}\right]-216\,\text{Cosh}\left[\frac{\pi}{6}\right]^3+24\,\text{Cosh}\left[\frac{\pi}{6}\right]\,\left(-8+9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2-9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\right)\right)\Big/$$

$$\Bigg(4\,\sqrt{\Bigg(8+9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2+\frac{1}{3}\,\left(-8+9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2-9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\right)+}$$

$$\left(432\,\text{Cosh}\left[\frac{\pi}{6}\right]^2+\left(-8+9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2-9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\right)^2+12\,\left(16+9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\right)\right)\Big/$$

$$\Bigg(3\,\Bigg(\frac{1}{2}\,\Bigg(15\,552\,\text{Cosh}\left[\frac{\pi}{6}\right]^2+1296\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\,\left(-8+9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2-9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\right)+$$

$$2\,\left(-8+9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2-9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\right)^3+972\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\,\left(16+9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\right)-$$

$$72\,\left(-8+9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2-9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\right)\,\left(16+9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\right)+$$

$$\sqrt{\left(-80\,621\,568\,\text{Sinh}\left[\frac{\pi}{6}\right]^4-10\,077\,696\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\,\text{Sinh}\left[\frac{\pi}{6}\right]^4+\right.}$$

$$65\,190\,096\,\text{Cosh}\left[\frac{\pi}{6}\right]^4\,\text{Sinh}\left[\frac{\pi}{6}\right]^4+25\,509\,168\,\text{Cosh}\left[\frac{\pi}{6}\right]^6\,\text{Sinh}\left[\frac{\pi}{6}\right]^4-$$

$$166\,281\,984\,\text{Sinh}\left[\frac{\pi}{6}\right]^6+124\,711\,488\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\,\text{Sinh}\left[\frac{\pi}{6}\right]^6-$$

$$76\,527\,504\,\text{Cosh}\left[\frac{\pi}{6}\right]^4\,\text{Sinh}\left[\frac{\pi}{6}\right]^6-113\,374\,080\,\text{Sinh}\left[\frac{\pi}{6}\right]^8+$$

$$\left.76\,527\,504\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\,\text{Sinh}\left[\frac{\pi}{6}\right]^8-25\,509\,168\,\text{Sinh}\left[\frac{\pi}{6}\right]^{10}\right)\Big)\Big)^{1/3}\Bigg)+\frac{1}{3}$$

$$\left(\frac{1}{2}\left(15\,552\,\text{Cosh}\left[\frac{\pi}{6}\right]^2+1296\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\left(-8+9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2-9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\right)+\right.\right.$$

$$2\left(-8+9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2-9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\right)^3+972\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\left(16+9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\right)-$$

$$72\left(-8+9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2-9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\right)\left(16+9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\right)+$$

$$\sqrt{}\left(\left(-80\,621\,568\,\text{Sinh}\left[\frac{\pi}{6}\right]^4-10\,077\,696\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\text{Sinh}\left[\frac{\pi}{6}\right]^4+\right.\right.$$

$$65\,190\,096\,\text{Cosh}\left[\frac{\pi}{6}\right]^4\text{Sinh}\left[\frac{\pi}{6}\right]^4+25\,509\,168\,\text{Cosh}\left[\frac{\pi}{6}\right]^6\text{Sinh}\left[\frac{\pi}{6}\right]^4-$$

$$166\,281\,984\,\text{Sinh}\left[\frac{\pi}{6}\right]^6+124\,711\,488\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\text{Sinh}\left[\frac{\pi}{6}\right]^6-$$

$$76\,527\,504\,\text{Cosh}\left[\frac{\pi}{6}\right]^4\text{Sinh}\left[\frac{\pi}{6}\right]^6-113\,374\,080\,\text{Sinh}\left[\frac{\pi}{6}\right]^8+$$

$$\left.\left.\left.\left.76\,527\,504\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\text{Sinh}\left[\frac{\pi}{6}\right]^8-25\,509\,168\,\text{Sinh}\left[\frac{\pi}{6}\right]^{10}\right)\right)\right)^{1/3}\right)\Bigg)\Bigg)\Bigg]\Bigg\},$$

$$\left\{z\to-\text{ArcCosh}\left[-\frac{3}{2}\,\text{Cosh}\left[\frac{\pi}{6}\right]-\frac{1}{2}\sqrt{}\left(8+9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2+\frac{1}{3}\left(-8+9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2-9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\right)+\right.\right.\right.$$

$$\left(432\,\text{Cosh}\left[\frac{\pi}{6}\right]^2+\left(-8+9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2-9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\right)^2+12\left(16+9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\right)\right)\Big/$$

$$\left(3\left(\frac{1}{2}\left(15\,552\,\text{Cosh}\left[\frac{\pi}{6}\right]^2+1296\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\left(-8+9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2-9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\right)+\right.\right.\right.$$

$$2\left(-8+9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2-9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\right)^3+972\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\left(16+9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\right)-$$

$$72\left(-8+9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2-9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\right)\left(16+9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\right)+$$

$$\sqrt{}\left(\left(-80\,621\,568\,\text{Sinh}\left[\frac{\pi}{6}\right]^4-10\,077\,696\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\text{Sinh}\left[\frac{\pi}{6}\right]^4+\right.\right.$$

$$65\,190\,096\,\text{Cosh}\left[\frac{\pi}{6}\right]^4\text{Sinh}\left[\frac{\pi}{6}\right]^4+25\,509\,168\,\text{Cosh}\left[\frac{\pi}{6}\right]^6\text{Sinh}\left[\frac{\pi}{6}\right]^4-$$

$$166\,281\,984\,\text{Sinh}\left[\frac{\pi}{6}\right]^6+124\,711\,488\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\text{Sinh}\left[\frac{\pi}{6}\right]^6-$$

$$76\,527\,504\,\text{Cosh}\left[\frac{\pi}{6}\right]^4\text{Sinh}\left[\frac{\pi}{6}\right]^6-113\,374\,080\,\text{Sinh}\left[\frac{\pi}{6}\right]^8+$$

$$\left.\left.\left.\left.76\,527\,504\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\text{Sinh}\left[\frac{\pi}{6}\right]^8-25\,509\,168\,\text{Sinh}\left[\frac{\pi}{6}\right]^{10}\right)\right)\right)^{1/3}\right)+$$

$$\frac{1}{3}\ \Bigg(\frac{1}{2}\ \Bigg(15\,552\ \text{Cosh}\Big[\frac{\pi}{6}\Big]^2 + 1296\ \text{Cosh}\Big[\frac{\pi}{6}\Big]^2\ \Big(-8 + 9\ \text{Cosh}\Big[\frac{\pi}{6}\Big]^2 - 9\ \text{Sinh}\Big[\frac{\pi}{6}\Big]^2\Big) +$$

$$2\ \Big(-8 + 9\ \text{Cosh}\Big[\frac{\pi}{6}\Big]^2 - 9\ \text{Sinh}\Big[\frac{\pi}{6}\Big]^2\Big)^3 + 972\ \text{Cosh}\Big[\frac{\pi}{6}\Big]^2\ \Big(16 + 9\ \text{Sinh}\Big[\frac{\pi}{6}\Big]^2\Big) -$$

$$72\ \Big(-8 + 9\ \text{Cosh}\Big[\frac{\pi}{6}\Big]^2 - 9\ \text{Sinh}\Big[\frac{\pi}{6}\Big]^2\Big)\ \Big(16 + 9\ \text{Sinh}\Big[\frac{\pi}{6}\Big]^2\Big) +$$

$$\sqrt{\Big(-80\,621\,568\ \text{Sinh}\Big[\frac{\pi}{6}\Big]^4 - 10\,077\,696\ \text{Cosh}\Big[\frac{\pi}{6}\Big]^2\ \text{Sinh}\Big[\frac{\pi}{6}\Big]^4 +}$$

$$65\,190\,096\ \text{Cosh}\Big[\frac{\pi}{6}\Big]^4\ \text{Sinh}\Big[\frac{\pi}{6}\Big]^4 + 25\,509\,168\ \text{Cosh}\Big[\frac{\pi}{6}\Big]^6\ \text{Sinh}\Big[\frac{\pi}{6}\Big]^4 -$$

$$166\,281\,984\ \text{Sinh}\Big[\frac{\pi}{6}\Big]^6 + 124\,711\,488\ \text{Cosh}\Big[\frac{\pi}{6}\Big]^2\ \text{Sinh}\Big[\frac{\pi}{6}\Big]^6 -$$

$$76\,527\,504\ \text{Cosh}\Big[\frac{\pi}{6}\Big]^4\ \text{Sinh}\Big[\frac{\pi}{6}\Big]^6 - 113\,374\,080\ \text{Sinh}\Big[\frac{\pi}{6}\Big]^8 +$$

$$76\,527\,504\ \text{Cosh}\Big[\frac{\pi}{6}\Big]^2\ \text{Sinh}\Big[\frac{\pi}{6}\Big]^8 - 25\,509\,168\ \text{Sinh}\Big[\frac{\pi}{6}\Big]^{10}\Big)\Bigg)\Bigg)^{1/3}\Bigg) +$$

$$\frac{1}{2}\ \sqrt{\Bigg(8 + 9\ \text{Cosh}\Big[\frac{\pi}{6}\Big]^2 + 9\ \text{Sinh}\Big[\frac{\pi}{6}\Big]^2 - \frac{1}{3}\ \Big(-8 + 9\ \text{Cosh}\Big[\frac{\pi}{6}\Big]^2 - 9\ \text{Sinh}\Big[\frac{\pi}{6}\Big]^2\Big) -}$$

$$\Bigg(432\ \text{Cosh}\Big[\frac{\pi}{6}\Big]^2 + \Big(-8 + 9\ \text{Cosh}\Big[\frac{\pi}{6}\Big]^2 - 9\ \text{Sinh}\Big[\frac{\pi}{6}\Big]^2\Big)^2 + 12\ \Big(16 + 9\ \text{Sinh}\Big[\frac{\pi}{6}\Big]^2\Big)\Bigg)\ \Big/$$

$$\Bigg(3\ \Bigg(\frac{1}{2}\ \Bigg(15\,552\ \text{Cosh}\Big[\frac{\pi}{6}\Big]^2 + 1296\ \text{Cosh}\Big[\frac{\pi}{6}\Big]^2\ \Big(-8 + 9\ \text{Cosh}\Big[\frac{\pi}{6}\Big]^2 - 9\ \text{Sinh}\Big[\frac{\pi}{6}\Big]^2\Big) +$$

$$2\ \Big(-8 + 9\ \text{Cosh}\Big[\frac{\pi}{6}\Big]^2 - 9\ \text{Sinh}\Big[\frac{\pi}{6}\Big]^2\Big)^3 + 972\ \text{Cosh}\Big[\frac{\pi}{6}\Big]^2\ \Big(16 + 9\ \text{Sinh}\Big[\frac{\pi}{6}\Big]^2\Big) -$$

$$72\ \Big(-8 + 9\ \text{Cosh}\Big[\frac{\pi}{6}\Big]^2 - 9\ \text{Sinh}\Big[\frac{\pi}{6}\Big]^2\Big)\ \Big(16 + 9\ \text{Sinh}\Big[\frac{\pi}{6}\Big]^2\Big) +$$

$$\sqrt{\Big(-80\,621\,568\ \text{Sinh}\Big[\frac{\pi}{6}\Big]^4 - 10\,077\,696\ \text{Cosh}\Big[\frac{\pi}{6}\Big]^2\ \text{Sinh}\Big[\frac{\pi}{6}\Big]^4 +}$$

$$65\,190\,096\ \text{Cosh}\Big[\frac{\pi}{6}\Big]^4\ \text{Sinh}\Big[\frac{\pi}{6}\Big]^4 + 25\,509\,168\ \text{Cosh}\Big[\frac{\pi}{6}\Big]^6\ \text{Sinh}\Big[\frac{\pi}{6}\Big]^4 -$$

$$166\,281\,984\ \text{Sinh}\Big[\frac{\pi}{6}\Big]^6 + 124\,711\,488\ \text{Cosh}\Big[\frac{\pi}{6}\Big]^2\ \text{Sinh}\Big[\frac{\pi}{6}\Big]^6 -$$

$$76\,527\,504\ \text{Cosh}\Big[\frac{\pi}{6}\Big]^4\ \text{Sinh}\Big[\frac{\pi}{6}\Big]^6 - 113\,374\,080\ \text{Sinh}\Big[\frac{\pi}{6}\Big]^8 +$$

$$76\,527\,504\ \text{Cosh}\Big[\frac{\pi}{6}\Big]^2\ \text{Sinh}\Big[\frac{\pi}{6}\Big]^8 - 25\,509\,168\ \text{Sinh}\Big[\frac{\pi}{6}\Big]^{10}\Big)\Bigg)\Bigg)^{1/3}\Bigg) -$$

$$\frac{1}{3}\Bigg(\frac{1}{2}\Bigg(15\,552\operatorname{Cosh}[\tfrac{\pi}{6}]^2+1296\operatorname{Cosh}[\tfrac{\pi}{6}]^2\,(-8+9\operatorname{Cosh}[\tfrac{\pi}{6}]^2-9\operatorname{Sinh}[\tfrac{\pi}{6}]^2)+$$

$$2\,(-8+9\operatorname{Cosh}[\tfrac{\pi}{6}]^2-9\operatorname{Sinh}[\tfrac{\pi}{6}]^2)^3+972\operatorname{Cosh}[\tfrac{\pi}{6}]^2\,(16+9\operatorname{Sinh}[\tfrac{\pi}{6}]^2)-$$

$$72\,(-8+9\operatorname{Cosh}[\tfrac{\pi}{6}]^2-9\operatorname{Sinh}[\tfrac{\pi}{6}]^2)\,(16+9\operatorname{Sinh}[\tfrac{\pi}{6}]^2)+$$

$$\sqrt{}\Big(-80\,621\,568\operatorname{Sinh}[\tfrac{\pi}{6}]^4-10\,077\,696\operatorname{Cosh}[\tfrac{\pi}{6}]^2\operatorname{Sinh}[\tfrac{\pi}{6}]^4+$$

$$65\,190\,096\operatorname{Cosh}[\tfrac{\pi}{6}]^4\operatorname{Sinh}[\tfrac{\pi}{6}]^4+25\,509\,168\operatorname{Cosh}[\tfrac{\pi}{6}]^6\operatorname{Sinh}[\tfrac{\pi}{6}]^4-$$

$$166\,281\,984\operatorname{Sinh}[\tfrac{\pi}{6}]^6+124\,711\,488\operatorname{Cosh}[\tfrac{\pi}{6}]^2\operatorname{Sinh}[\tfrac{\pi}{6}]^6-$$

$$76\,527\,504\operatorname{Cosh}[\tfrac{\pi}{6}]^4\operatorname{Sinh}[\tfrac{\pi}{6}]^6-113\,374\,080\operatorname{Sinh}[\tfrac{\pi}{6}]^8+$$

$$76\,527\,504\operatorname{Cosh}[\tfrac{\pi}{6}]^2\operatorname{Sinh}[\tfrac{\pi}{6}]^8-25\,509\,168\operatorname{Sinh}[\tfrac{\pi}{6}]^{10}\Big)\Bigg)\Bigg)^{1/3}-$$

$$\Big(192\operatorname{Cosh}[\tfrac{\pi}{6}]-216\operatorname{Cosh}[\tfrac{\pi}{6}]^3+24\operatorname{Cosh}[\tfrac{\pi}{6}]\,(-8+9\operatorname{Cosh}[\tfrac{\pi}{6}]^2-9\operatorname{Sinh}[\tfrac{\pi}{6}]^2)\Big)\Big/$$

$$\Bigg(4\sqrt{}\Bigg(8+9\operatorname{Sinh}[\tfrac{\pi}{6}]^2+\frac{1}{3}\,(-8+9\operatorname{Cosh}[\tfrac{\pi}{6}]^2-9\operatorname{Sinh}[\tfrac{\pi}{6}]^2)+$$

$$\Big(432\operatorname{Cosh}[\tfrac{\pi}{6}]^2+(-8+9\operatorname{Cosh}[\tfrac{\pi}{6}]^2-9\operatorname{Sinh}[\tfrac{\pi}{6}]^2)^2+12\,(16+9\operatorname{Sinh}[\tfrac{\pi}{6}]^2)\Big)\Big/$$

$$\Bigg(3\,\Bigg(\frac{1}{2}\Bigg(15\,552\operatorname{Cosh}[\tfrac{\pi}{6}]^2+1296\operatorname{Cosh}[\tfrac{\pi}{6}]^2\,(-8+9\operatorname{Cosh}[\tfrac{\pi}{6}]^2-9\operatorname{Sinh}[\tfrac{\pi}{6}]^2)+$$

$$2\,(-8+9\operatorname{Cosh}[\tfrac{\pi}{6}]^2-9\operatorname{Sinh}[\tfrac{\pi}{6}]^2)^3+972\operatorname{Cosh}[\tfrac{\pi}{6}]^2\,(16+9\operatorname{Sinh}[\tfrac{\pi}{6}]^2)-$$

$$72\,(-8+9\operatorname{Cosh}[\tfrac{\pi}{6}]^2-9\operatorname{Sinh}[\tfrac{\pi}{6}]^2)\,(16+9\operatorname{Sinh}[\tfrac{\pi}{6}]^2)+$$

$$\sqrt{}\Big(-80\,621\,568\operatorname{Sinh}[\tfrac{\pi}{6}]^4-10\,077\,696\operatorname{Cosh}[\tfrac{\pi}{6}]^2\operatorname{Sinh}[\tfrac{\pi}{6}]^4+$$

$$65\,190\,096\operatorname{Cosh}[\tfrac{\pi}{6}]^4\operatorname{Sinh}[\tfrac{\pi}{6}]^4+25\,509\,168\operatorname{Cosh}[\tfrac{\pi}{6}]^6\operatorname{Sinh}[\tfrac{\pi}{6}]^4-$$

$$166\,281\,984\operatorname{Sinh}[\tfrac{\pi}{6}]^6+124\,711\,488\operatorname{Cosh}[\tfrac{\pi}{6}]^2\operatorname{Sinh}[\tfrac{\pi}{6}]^6-$$

$$76\,527\,504\operatorname{Cosh}[\tfrac{\pi}{6}]^4\operatorname{Sinh}[\tfrac{\pi}{6}]^6-113\,374\,080\operatorname{Sinh}[\tfrac{\pi}{6}]^8+$$

$$76\,527\,504\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\,\text{Sinh}\left[\frac{\pi}{6}\right]^8 - 25\,509\,168\,\text{Sinh}\left[\frac{\pi}{6}\right]^{10}\Big)\Big)\Big)^{1/3}\Big) +$$

$$\frac{1}{3}\,\Big(\frac{1}{2}\,\Big(15\,552\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 + 1296\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\,\left(-8 + 9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 - 9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\right) +$$

$$2\,\left(-8 + 9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 - 9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\right)^3 + 972\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\,\left(16 + 9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\right) -$$

$$72\,\left(-8 + 9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 - 9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\right)\,\left(16 + 9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\right) +$$

$$\sqrt{\Big(}-80\,621\,568\,\text{Sinh}\left[\frac{\pi}{6}\right]^4 - 10\,077\,696\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\,\text{Sinh}\left[\frac{\pi}{6}\right]^4 +$$

$$65\,190\,096\,\text{Cosh}\left[\frac{\pi}{6}\right]^4\,\text{Sinh}\left[\frac{\pi}{6}\right]^4 + 25\,509\,168\,\text{Cosh}\left[\frac{\pi}{6}\right]^6\,\text{Sinh}\left[\frac{\pi}{6}\right]^4 -$$

$$166\,281\,984\,\text{Sinh}\left[\frac{\pi}{6}\right]^6 + 124\,711\,488\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\,\text{Sinh}\left[\frac{\pi}{6}\right]^6 -$$

$$76\,527\,504\,\text{Cosh}\left[\frac{\pi}{6}\right]^4\,\text{Sinh}\left[\frac{\pi}{6}\right]^6 - 113\,374\,080\,\text{Sinh}\left[\frac{\pi}{6}\right]^8 +$$

$$76\,527\,504\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\,\text{Sinh}\left[\frac{\pi}{6}\right]^8 - 25\,509\,168\,\text{Sinh}\left[\frac{\pi}{6}\right]^{10}\Big)\Big)\Big)^{1/3}\Big)\Big)\Big)\Big]\Big\},$$

$$\Big\{z \to -\text{ArcCosh}\Big[-\frac{3}{2}\,\text{Cosh}\left[\frac{\pi}{6}\right] + \frac{1}{2}\,\sqrt{\Big(}8 + 9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2 + \frac{1}{3}\,\left(-8 + 9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 - 9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\right) +$$

$$\Big(432\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 + \left(-8 + 9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 - 9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\right)^2 + 12\,\left(16 + 9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\right)\Big)\Big/$$

$$\Big(3\,\Big(\frac{1}{2}\,\Big(15\,552\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 + 1296\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\,\left(-8 + 9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 - 9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\right) +$$

$$2\,\left(-8 + 9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 - 9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\right)^3 + 972\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\,\left(16 + 9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\right) -$$

$$72\,\left(-8 + 9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 - 9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\right)\,\left(16 + 9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\right) +$$

$$\sqrt{\Big(}-80\,621\,568\,\text{Sinh}\left[\frac{\pi}{6}\right]^4 - 10\,077\,696\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\,\text{Sinh}\left[\frac{\pi}{6}\right]^4 +$$

$$65\,190\,096\,\text{Cosh}\left[\frac{\pi}{6}\right]^4\,\text{Sinh}\left[\frac{\pi}{6}\right]^4 + 25\,509\,168\,\text{Cosh}\left[\frac{\pi}{6}\right]^6\,\text{Sinh}\left[\frac{\pi}{6}\right]^4 -$$

$$166\,281\,984\,\text{Sinh}\left[\frac{\pi}{6}\right]^6 + 124\,711\,488\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\,\text{Sinh}\left[\frac{\pi}{6}\right]^6 -$$

$$76\,527\,504\,\text{Cosh}\left[\frac{\pi}{6}\right]^4\,\text{Sinh}\left[\frac{\pi}{6}\right]^6 - 113\,374\,080\,\text{Sinh}\left[\frac{\pi}{6}\right]^8 +$$

$$76\,527\,504\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2 \mathrm{Sinh}\left[\frac{\pi}{6}\right]^8 - 25\,509\,168\,\mathrm{Sinh}\left[\frac{\pi}{6}\right]^{10}\Bigg)\Bigg)\Bigg)^{1/3}\Bigg) +$$

$$\frac{1}{3}\left(\frac{1}{2}\left(15\,552\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2 + 1296\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2 \left(-8 + 9\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2 - 9\,\mathrm{Sinh}\left[\frac{\pi}{6}\right]^2\right) +\right.\right.$$

$$2\left(-8 + 9\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2 - 9\,\mathrm{Sinh}\left[\frac{\pi}{6}\right]^2\right)^3 + 972\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2 \left(16 + 9\,\mathrm{Sinh}\left[\frac{\pi}{6}\right]^2\right) -$$

$$72\left(-8 + 9\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2 - 9\,\mathrm{Sinh}\left[\frac{\pi}{6}\right]^2\right)\left(16 + 9\,\mathrm{Sinh}\left[\frac{\pi}{6}\right]^2\right) +$$

$$\sqrt{\Bigg(-80\,621\,568\,\mathrm{Sinh}\left[\frac{\pi}{6}\right]^4 - 10\,077\,696\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2 \mathrm{Sinh}\left[\frac{\pi}{6}\right]^4 +}$$

$$65\,190\,096\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^4 \mathrm{Sinh}\left[\frac{\pi}{6}\right]^4 + 25\,509\,168\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^6 \mathrm{Sinh}\left[\frac{\pi}{6}\right]^4 -$$

$$166\,281\,984\,\mathrm{Sinh}\left[\frac{\pi}{6}\right]^6 + 124\,711\,488\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2 \mathrm{Sinh}\left[\frac{\pi}{6}\right]^6 -$$

$$76\,527\,504\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^4 \mathrm{Sinh}\left[\frac{\pi}{6}\right]^6 - 113\,374\,080\,\mathrm{Sinh}\left[\frac{\pi}{6}\right]^8 +$$

$$76\,527\,504\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2 \mathrm{Sinh}\left[\frac{\pi}{6}\right]^8 - 25\,509\,168\,\mathrm{Sinh}\left[\frac{\pi}{6}\right]^{10}\Bigg)\Bigg)\Bigg)^{1/3}\Bigg) -$$

$$\frac{1}{2}\sqrt{\Bigg(8 + 9\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2 + 9\,\mathrm{Sinh}\left[\frac{\pi}{6}\right]^2 - \frac{1}{3}\left(-8 + 9\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2 - 9\,\mathrm{Sinh}\left[\frac{\pi}{6}\right]^2\right) -}$$

$$\left(432\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2 + \left(-8 + 9\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2 - 9\,\mathrm{Sinh}\left[\frac{\pi}{6}\right]^2\right)^2 + 12\left(16 + 9\,\mathrm{Sinh}\left[\frac{\pi}{6}\right]^2\right)\right) \Big/$$

$$\left(3\left(\frac{1}{2}\left(15\,552\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2 + 1296\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2 \left(-8 + 9\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2 - 9\,\mathrm{Sinh}\left[\frac{\pi}{6}\right]^2\right) +\right.\right.\right.$$

$$2\left(-8 + 9\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2 - 9\,\mathrm{Sinh}\left[\frac{\pi}{6}\right]^2\right)^3 + 972\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2 \left(16 + 9\,\mathrm{Sinh}\left[\frac{\pi}{6}\right]^2\right) -$$

$$72\left(-8 + 9\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2 - 9\,\mathrm{Sinh}\left[\frac{\pi}{6}\right]^2\right)\left(16 + 9\,\mathrm{Sinh}\left[\frac{\pi}{6}\right]^2\right) +$$

$$\sqrt{\Bigg(-80\,621\,568\,\mathrm{Sinh}\left[\frac{\pi}{6}\right]^4 - 10\,077\,696\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2 \mathrm{Sinh}\left[\frac{\pi}{6}\right]^4 +}$$

$$65\,190\,096\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^4 \mathrm{Sinh}\left[\frac{\pi}{6}\right]^4 + 25\,509\,168\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^6 \mathrm{Sinh}\left[\frac{\pi}{6}\right]^4 -$$

$$166\,281\,984\,\mathrm{Sinh}\left[\frac{\pi}{6}\right]^6 + 124\,711\,488\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2 \mathrm{Sinh}\left[\frac{\pi}{6}\right]^6 -$$

$$76\,527\,504\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^4 \mathrm{Sinh}\left[\frac{\pi}{6}\right]^6 - 113\,374\,080\,\mathrm{Sinh}\left[\frac{\pi}{6}\right]^8 +$$

$$76\,527\,504\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2\mathrm{Sinh}\left[\frac{\pi}{6}\right]^8-25\,509\,168\,\mathrm{Sinh}\left[\frac{\pi}{6}\right]^{10}\Big)\Big)\Big)^{1/3}\Big)-$$

$$\frac{1}{3}\left(\frac{1}{2}\left(15\,552\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2+1296\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2\left(-8+9\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2-9\,\mathrm{Sinh}\left[\frac{\pi}{6}\right]^2\right)+\right.\right.$$

$$2\left(-8+9\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2-9\,\mathrm{Sinh}\left[\frac{\pi}{6}\right]^2\right)^3+972\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2\left(16+9\,\mathrm{Sinh}\left[\frac{\pi}{6}\right]^2\right)-$$

$$72\left(-8+9\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2-9\,\mathrm{Sinh}\left[\frac{\pi}{6}\right]^2\right)\left(16+9\,\mathrm{Sinh}\left[\frac{\pi}{6}\right]^2\right)+$$

$$\sqrt{}\Big(-80\,621\,568\,\mathrm{Sinh}\left[\frac{\pi}{6}\right]^4-10\,077\,696\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2\mathrm{Sinh}\left[\frac{\pi}{6}\right]^4+$$

$$65\,190\,096\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^4\mathrm{Sinh}\left[\frac{\pi}{6}\right]^4+25\,509\,168\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^6\mathrm{Sinh}\left[\frac{\pi}{6}\right]^4-$$

$$166\,281\,984\,\mathrm{Sinh}\left[\frac{\pi}{6}\right]^6+124\,711\,488\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2\mathrm{Sinh}\left[\frac{\pi}{6}\right]^6-$$

$$76\,527\,504\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^4\mathrm{Sinh}\left[\frac{\pi}{6}\right]^6-113\,374\,080\,\mathrm{Sinh}\left[\frac{\pi}{6}\right]^8+$$

$$76\,527\,504\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2\mathrm{Sinh}\left[\frac{\pi}{6}\right]^8-25\,509\,168\,\mathrm{Sinh}\left[\frac{\pi}{6}\right]^{10}\Big)\Big)\Big)^{1/3}+$$

$$\left(192\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]-216\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^3+24\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]\left(-8+9\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2-9\,\mathrm{Sinh}\left[\frac{\pi}{6}\right]^2\right)\right)\Big/$$

$$\Big(4\sqrt{}\Big(8+9\,\mathrm{Sinh}\left[\frac{\pi}{6}\right]^2+\frac{1}{3}\left(-8+9\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2-9\,\mathrm{Sinh}\left[\frac{\pi}{6}\right]^2\right)+$$

$$\left(432\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2+\left(-8+9\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2-9\,\mathrm{Sinh}\left[\frac{\pi}{6}\right]^2\right)^2+12\left(16+9\,\mathrm{Sinh}\left[\frac{\pi}{6}\right]^2\right)\right)\Big/$$

$$\Big(3\Big(\frac{1}{2}\Big(15\,552\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2+1296\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2\left(-8+9\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2-9\,\mathrm{Sinh}\left[\frac{\pi}{6}\right]^2\right)+$$

$$2\left(-8+9\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2-9\,\mathrm{Sinh}\left[\frac{\pi}{6}\right]^2\right)^3+972\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2\left(16+9\,\mathrm{Sinh}\left[\frac{\pi}{6}\right]^2\right)-$$

$$72\left(-8+9\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2-9\,\mathrm{Sinh}\left[\frac{\pi}{6}\right]^2\right)\left(16+9\,\mathrm{Sinh}\left[\frac{\pi}{6}\right]^2\right)+$$

$$\sqrt{}\Big(-80\,621\,568\,\mathrm{Sinh}\left[\frac{\pi}{6}\right]^4-10\,077\,696\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2\mathrm{Sinh}\left[\frac{\pi}{6}\right]^4+$$

$$65\,190\,096\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^4\mathrm{Sinh}\left[\frac{\pi}{6}\right]^4+25\,509\,168\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^6\mathrm{Sinh}\left[\frac{\pi}{6}\right]^4-$$

$$166\,281\,984\,\mathrm{Sinh}\left[\frac{\pi}{6}\right]^6+124\,711\,488\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2\mathrm{Sinh}\left[\frac{\pi}{6}\right]^6-$$

$$76\,527\,504\,\text{Cosh}\left[\frac{\pi}{6}\right]^4\,\text{Sinh}\left[\frac{\pi}{6}\right]^6 - 113\,374\,080\,\text{Sinh}\left[\frac{\pi}{6}\right]^8 +$$

$$76\,527\,504\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\,\text{Sinh}\left[\frac{\pi}{6}\right]^8 - 25\,509\,168\,\text{Sinh}\left[\frac{\pi}{6}\right]^{10}\Big)\Big)\Big)^{1/3}\Big) +$$

$$\frac{1}{3}\,\Big(\frac{1}{2}\,\Big(15\,552\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 + 1296\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\,\left(-8 + 9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 - 9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\right) +$$

$$2\,\left(-8 + 9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 - 9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\right)^3 + 972\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\,\left(16 + 9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\right) -$$

$$72\,\left(-8 + 9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 - 9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\right)\,\left(16 + 9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\right) +$$

$$\sqrt{}\Big(\Big(-80\,621\,568\,\text{Sinh}\left[\frac{\pi}{6}\right]^4 - 10\,077\,696\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\,\text{Sinh}\left[\frac{\pi}{6}\right]^4 +$$

$$65\,190\,096\,\text{Cosh}\left[\frac{\pi}{6}\right]^4\,\text{Sinh}\left[\frac{\pi}{6}\right]^4 + 25\,509\,168\,\text{Cosh}\left[\frac{\pi}{6}\right]^6\,\text{Sinh}\left[\frac{\pi}{6}\right]^4 -$$

$$166\,281\,984\,\text{Sinh}\left[\frac{\pi}{6}\right]^6 + 124\,711\,488\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\,\text{Sinh}\left[\frac{\pi}{6}\right]^6 -$$

$$76\,527\,504\,\text{Cosh}\left[\frac{\pi}{6}\right]^4\,\text{Sinh}\left[\frac{\pi}{6}\right]^6 - 113\,374\,080\,\text{Sinh}\left[\frac{\pi}{6}\right]^8 +$$

$$76\,527\,504\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\,\text{Sinh}\left[\frac{\pi}{6}\right]^8 - 25\,509\,168\,\text{Sinh}\left[\frac{\pi}{6}\right]^{10}\Big)\Big)\Big)^{1/3}\Big)\Big)\Big)\Big]\Big\},$$

$$\Big\{z \to -\text{ArcCosh}\Big[-\frac{3}{2}\,\text{Cosh}\left[\frac{\pi}{6}\right] + \frac{1}{2}\,\sqrt{}\Big(\Big(8 + 9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2 + \frac{1}{3}\,\left(-8 + 9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 - 9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\right) +$$

$$\left(432\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 + \left(-8 + 9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 - 9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\right)^2 + 12\,\left(16 + 9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\right)\right)\Big/$$

$$\Big(3\,\Big(\frac{1}{2}\,\Big(15\,552\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 + 1296\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\,\left(-8 + 9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 - 9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\right) +$$

$$2\,\left(-8 + 9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 - 9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\right)^3 + 972\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\,\left(16 + 9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\right) -$$

$$72\,\left(-8 + 9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 - 9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\right)\,\left(16 + 9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\right) +$$

$$\sqrt{}\Big(\Big(-80\,621\,568\,\text{Sinh}\left[\frac{\pi}{6}\right]^4 - 10\,077\,696\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\,\text{Sinh}\left[\frac{\pi}{6}\right]^4 +$$

$$65\,190\,096\,\text{Cosh}\left[\frac{\pi}{6}\right]^4\,\text{Sinh}\left[\frac{\pi}{6}\right]^4 + 25\,509\,168\,\text{Cosh}\left[\frac{\pi}{6}\right]^6\,\text{Sinh}\left[\frac{\pi}{6}\right]^4 -$$

$$166\,281\,984\,\text{Sinh}\left[\frac{\pi}{6}\right]^6 + 124\,711\,488\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\,\text{Sinh}\left[\frac{\pi}{6}\right]^6 -$$

$$76\,527\,504\,\text{Cosh}\left[\frac{\pi}{6}\right]^4\text{Sinh}\left[\frac{\pi}{6}\right]^6-113\,374\,080\,\text{Sinh}\left[\frac{\pi}{6}\right]^8+$$

$$76\,527\,504\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\text{Sinh}\left[\frac{\pi}{6}\right]^8-25\,509\,168\,\text{Sinh}\left[\frac{\pi}{6}\right]^{10}\Bigg)\Bigg)\Bigg)^{1/3}\Bigg)+$$

$$\frac{1}{3}\left(\frac{1}{2}\left(15\,552\,\text{Cosh}\left[\frac{\pi}{6}\right]^2+1296\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\left(-8+9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2-9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\right)+\right.\right.$$

$$2\left(-8+9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2-9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\right)^3+972\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\left(16+9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\right)-$$

$$72\left(-8+9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2-9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\right)\left(16+9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\right)+$$

$$\sqrt{}\Bigg(-80\,621\,568\,\text{Sinh}\left[\frac{\pi}{6}\right]^4-10\,077\,696\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\text{Sinh}\left[\frac{\pi}{6}\right]^4+$$

$$65\,190\,096\,\text{Cosh}\left[\frac{\pi}{6}\right]^4\text{Sinh}\left[\frac{\pi}{6}\right]^4+25\,509\,168\,\text{Cosh}\left[\frac{\pi}{6}\right]^6\text{Sinh}\left[\frac{\pi}{6}\right]^4-$$

$$166\,281\,984\,\text{Sinh}\left[\frac{\pi}{6}\right]^6+124\,711\,488\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\text{Sinh}\left[\frac{\pi}{6}\right]^6-$$

$$76\,527\,504\,\text{Cosh}\left[\frac{\pi}{6}\right]^4\text{Sinh}\left[\frac{\pi}{6}\right]^6-113\,374\,080\,\text{Sinh}\left[\frac{\pi}{6}\right]^8+$$

$$76\,527\,504\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\text{Sinh}\left[\frac{\pi}{6}\right]^8-25\,509\,168\,\text{Sinh}\left[\frac{\pi}{6}\right]^{10}\Bigg)\Bigg)\Bigg)^{1/3}\Bigg)+$$

$$\frac{1}{2}\sqrt{}\Bigg(8+9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2+9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2-\frac{1}{3}\left(-8+9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2-9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\right)-$$

$$\left(432\,\text{Cosh}\left[\frac{\pi}{6}\right]^2+\left(-8+9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2-9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\right)^2+12\left(16+9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\right)\right)\Big/$$

$$\Bigg(3\left(\frac{1}{2}\left(15\,552\,\text{Cosh}\left[\frac{\pi}{6}\right]^2+1296\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\left(-8+9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2-9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\right)+\right.\right.$$

$$2\left(-8+9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2-9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\right)^3+972\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\left(16+9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\right)-$$

$$72\left(-8+9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2-9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\right)\left(16+9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\right)+$$

$$\sqrt{}\Bigg(-80\,621\,568\,\text{Sinh}\left[\frac{\pi}{6}\right]^4-10\,077\,696\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\text{Sinh}\left[\frac{\pi}{6}\right]^4+$$

$$65\,190\,096\,\text{Cosh}\left[\frac{\pi}{6}\right]^4\text{Sinh}\left[\frac{\pi}{6}\right]^4+25\,509\,168\,\text{Cosh}\left[\frac{\pi}{6}\right]^6\text{Sinh}\left[\frac{\pi}{6}\right]^4-$$

$$166\,281\,984\,\text{Sinh}\left[\frac{\pi}{6}\right]^6+124\,711\,488\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\text{Sinh}\left[\frac{\pi}{6}\right]^6-$$

$$76\,527\,504\,\text{Cosh}\left[\frac{\pi}{6}\right]^4\,\text{Sinh}\left[\frac{\pi}{6}\right]^6 - 113\,374\,080\,\text{Sinh}\left[\frac{\pi}{6}\right]^8 +$$

$$76\,527\,504\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\,\text{Sinh}\left[\frac{\pi}{6}\right]^8 - 25\,509\,168\,\text{Sinh}\left[\frac{\pi}{6}\right]^{10}\Big)\Big)\Big)^{1/3}\Big) -$$

$$\frac{1}{3}\left(\frac{1}{2}\left(15\,552\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 + 1296\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\left(-8 + 9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 - 9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\right) +\right.\right.$$

$$2\left(-8 + 9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 - 9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\right)^3 + 972\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\left(16 + 9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\right) -$$

$$72\left(-8 + 9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 - 9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\right)\left(16 + 9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\right) +$$

$$\sqrt{\Big(-80\,621\,568\,\text{Sinh}\left[\frac{\pi}{6}\right]^4 - 10\,077\,696\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\,\text{Sinh}\left[\frac{\pi}{6}\right]^4 +}$$

$$65\,190\,096\,\text{Cosh}\left[\frac{\pi}{6}\right]^4\,\text{Sinh}\left[\frac{\pi}{6}\right]^4 + 25\,509\,168\,\text{Cosh}\left[\frac{\pi}{6}\right]^6\,\text{Sinh}\left[\frac{\pi}{6}\right]^4 -$$

$$166\,281\,984\,\text{Sinh}\left[\frac{\pi}{6}\right]^6 + 124\,711\,488\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\,\text{Sinh}\left[\frac{\pi}{6}\right]^6 -$$

$$76\,527\,504\,\text{Cosh}\left[\frac{\pi}{6}\right]^4\,\text{Sinh}\left[\frac{\pi}{6}\right]^6 - 113\,374\,080\,\text{Sinh}\left[\frac{\pi}{6}\right]^8 +$$

$$76\,527\,504\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\,\text{Sinh}\left[\frac{\pi}{6}\right]^8 - 25\,509\,168\,\text{Sinh}\left[\frac{\pi}{6}\right]^{10}\Big)\Big)\Big)^{1/3} +$$

$$\left(192\,\text{Cosh}\left[\frac{\pi}{6}\right] - 216\,\text{Cosh}\left[\frac{\pi}{6}\right]^3 + 24\,\text{Cosh}\left[\frac{\pi}{6}\right]\left(-8 + 9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 - 9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\right)\right) \Big/$$

$$\Big(4\sqrt{\Big(8 + 9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2 + \frac{1}{3}\left(-8 + 9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 - 9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\right) +}$$

$$\left(432\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 + \left(-8 + 9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 - 9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\right)^2 + 12\left(16 + 9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\right)\right) \Big/$$

$$\Big(3\left(\frac{1}{2}\left(15\,552\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 + 1296\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\left(-8 + 9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 - 9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\right) +\right.\right.$$

$$2\left(-8 + 9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 - 9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\right)^3 + 972\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\left(16 + 9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\right) -$$

$$72\left(-8 + 9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 - 9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\right)\left(16 + 9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\right) +$$

$$\sqrt{\Big(-80\,621\,568\,\text{Sinh}\left[\frac{\pi}{6}\right]^4 - 10\,077\,696\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\,\text{Sinh}\left[\frac{\pi}{6}\right]^4 +}$$

$$65\,190\,096\,\text{Cosh}\left[\frac{\pi}{6}\right]^4\,\text{Sinh}\left[\frac{\pi}{6}\right]^4 + 25\,509\,168\,\text{Cosh}\left[\frac{\pi}{6}\right]^6\,\text{Sinh}\left[\frac{\pi}{6}\right]^4 -$$

$$166\,281\,984\,\text{Sinh}\left[\frac{\pi}{6}\right]^6 + 124\,711\,488\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\,\text{Sinh}\left[\frac{\pi}{6}\right]^6 -$$

$$76\,527\,504\,\text{Cosh}\left[\frac{\pi}{6}\right]^4\,\text{Sinh}\left[\frac{\pi}{6}\right]^6 - 113\,374\,080\,\text{Sinh}\left[\frac{\pi}{6}\right]^8 +$$

$$76\,527\,504\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\,\text{Sinh}\left[\frac{\pi}{6}\right]^8 - 25\,509\,168\,\text{Sinh}\left[\frac{\pi}{6}\right]^{10}\Big)\Big)\Big)^{1/3}\Big) +$$

$$\frac{1}{3}\Big(\frac{1}{2}\Big(15\,552\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 + 1296\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\Big(-8 + 9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 - 9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\Big) +$$

$$2\Big(-8 + 9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 - 9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\Big)^3 + 972\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\Big(16 + 9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\Big) -$$

$$72\Big(-8 + 9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 - 9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\Big)\Big(16 + 9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2\Big) +$$

$$\sqrt{\Big(}\Big(-80\,621\,568\,\text{Sinh}\left[\frac{\pi}{6}\right]^4 - 10\,077\,696\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\,\text{Sinh}\left[\frac{\pi}{6}\right]^4 +$$

$$65\,190\,096\,\text{Cosh}\left[\frac{\pi}{6}\right]^4\,\text{Sinh}\left[\frac{\pi}{6}\right]^4 + 25\,509\,168\,\text{Cosh}\left[\frac{\pi}{6}\right]^6\,\text{Sinh}\left[\frac{\pi}{6}\right]^4 -$$

$$166\,281\,984\,\text{Sinh}\left[\frac{\pi}{6}\right]^6 + 124\,711\,488\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\,\text{Sinh}\left[\frac{\pi}{6}\right]^6 -$$

$$76\,527\,504\,\text{Cosh}\left[\frac{\pi}{6}\right]^4\,\text{Sinh}\left[\frac{\pi}{6}\right]^6 - 113\,374\,080\,\text{Sinh}\left[\frac{\pi}{6}\right]^8 +$$

$$76\,527\,504\,\text{Cosh}\left[\frac{\pi}{6}\right]^2\,\text{Sinh}\left[\frac{\pi}{6}\right]^8 - 25\,509\,168\,\text{Sinh}\left[\frac{\pi}{6}\right]^{10}\Big)\Big)\Big)^{1/3}\Big)\Big)\Big)\Big]\Big\}\Big\}$$

```
Solve[Cosh[x] == a, x]
```

```
Solve::ifun : Inverse functions are being used by Solve, so some
   solutions may not be found; use Reduce for complete solution information. More…
```

```
{{x → -ArcCosh[a]}, {x → ArcCosh[a]}}
```

A complete solution of the previous equation can be obtained using the function `Reduce`.

```
Reduce[Cosh[x] == a, x] // InputForm
```

```
C[1] ∈ Integers && (x == -ArcCosh[a] + (2*I)*Pi*C[1] ||
  x == ArcCosh[a] + (2*I)*Pi*C[1])
```

Solving differential equations

Here are differential equations whose linear independent solutions include the hyperbolic cosine function. The solutions of the simplest second-order linear ordinary differential equation with constant coefficients can be represented using sinh(z) and cosh(z).

```
DSolve[w''[z] - w[z] == 0, w[z], z] // (ExpToTrig //@ #) &
```

```
{{w[z] → C[1] Cosh[z] + C[2] Cosh[z] + C[1] Sinh[z] - C[2] Sinh[z]}}
```

```
DSolve[4 w[z] - 5 w''[z] + w^(4)[z] == 0, w[z], z] /. Exp[x_] :> Cosh[x] + Sinh[x]
```

{{w → Function[{z}, C[3] Cos[z] + C[1] Cos[2 z] + C[4] Sin[z] + C[2] Sin[2 z]]}}

Integral transforms

Mathematica supports the main integral transforms like direct and inverse Laplace and Fourier transforms that can give results that contain classical or generalized functions.

```
LaplaceTransform[Cosh[t], t, s]
```

$$\frac{s}{-1+s^2}$$

```
FourierTransform[1/Cosh[t], t, s]
```

$$\sqrt{\frac{\pi}{2}}\ \mathrm{Sech}\left[\frac{\pi s}{2}\right]$$

```
FourierSinTransform[1/Cosh[t], t, s]
```

$$\frac{1}{2\sqrt{2\pi}}\left(i\left(\mathrm{HarmonicNumber}\left[-\frac{3}{4}-\frac{i s}{4}\right]-\mathrm{HarmonicNumber}\left[-\frac{1}{4}-\frac{i s}{4}\right]+\mathrm{HarmonicNumber}\left[\frac{1}{4} i (i+s)\right]-\mathrm{HarmonicNumber}\left[\frac{1}{4} i (3 i+s)\right]\right)\right)$$

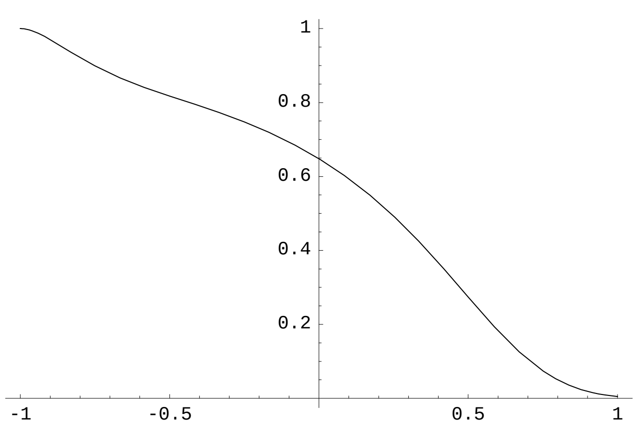
```
FourierCosTransform[1/Cosh[t], t, s]
```

$$\sqrt{\frac{\pi}{2}}\ \mathrm{Sech}\left[\frac{\pi s}{2}\right]$$

Plotting

Mathematica has built-in functions for 2D and 3D graphics. Here are some examples.

```
Plot[1/Cosh[Sum[z^k, {k, 0, 5}]], {z, -1, 1}];
```



```
Plot3D[Re[Cosh[x + ⅈ y]], {x, -π, π}, {y, 0, π},
  PlotPoints → 240, PlotRange → {-5, 5},
  ClipFill → None, Mesh → False, AxesLabel → {"x", "y", None}];
```

```
ContourPlot[Arg[Cosh[1/(x + ⅈ y)]], {x, -1/2, 1/2}, {y, -1/2, 1/2},
  PlotPoints → 400, PlotRange → {-π, π}, FrameLabel → {"x", "y", None, None},
  ColorFunction → Hue, ContourLines → False, Contours → 200];
```

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