Introductions to Cot

Introduction to the trigonometric functions

General

The six trigonometric functions sine $\sin(z)$, cosine $\cos(z)$, tangent $\tan(z)$, cotangent $\cot(z)$, cosecant $\csc(z)$, and secant $\sec(z)$ are well known and among the most frequently used elementary functions. The most popular functions $\sin(z)$, $\cos(z)$, $\tan(z)$, and $\cot(z)$ are taught worldwide in high school programs because of their natural appearance in problems involving angle measurement and their wide applications in the quantitative sciences.

The trigonometric functions share many common properties.

Definitions of trigonometric functions

All trigonometric functions can be defined as simple rational functions of the exponential function of $i z$:

$$\sin(z) = \frac{e^{i z} - e^{-i z}}{2 i}$$

$$\cos(z) = \frac{e^{i z} + e^{-i z}}{2}$$

$$\tan(z) = -\frac{i \left(e^{i z} - e^{-i z}\right)}{e^{i z} + e^{-i z}}$$

$$\cot(z) = \frac{i \left(e^{i z} + e^{-i z}\right)}{e^{i z} - e^{-i z}}$$

$$\csc(z) = \frac{2 i}{e^{i z} - e^{-i z}}$$

$$\sec(z) = \frac{2}{e^{i z} + e^{-i z}}.$$

The functions $\tan(z)$, $\cot(z)$, $\csc(z)$, and $\sec(z)$ can also be defined through the functions $\sin(z)$ and $\cos(z)$ using the following formulas:

$$\tan(z) = \frac{\sin(z)}{\cos(z)}$$

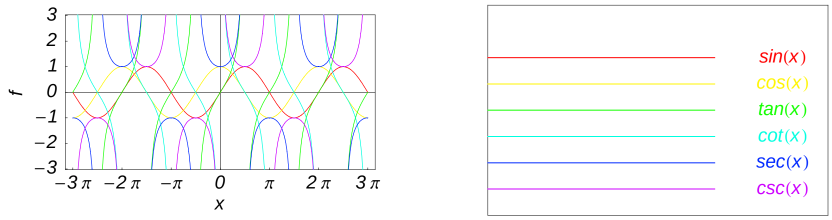
$$\cot(z) = \frac{\cos(z)}{\sin(z)}$$

$$\csc(z) = \frac{1}{\sin(z)}$$

$$\sec(z) = \frac{1}{\cos(z)}.$$

A quick look at the trigonometric functions

Here is a quick look at the graphics for the six trigonometric functions along the real axis.



Connections within the group of trigonometric functions and with other function groups

Representations through more general functions

The trigonometric functions are particular cases of more general functions. Among these more general functions, four different classes of special functions are particularly relevant: Bessel, Jacobi, Mathieu, and hypergeometric functions.

For example, $\sin(z)$ and $\cos(z)$ have the following representations through Bessel, Mathieu, and hypergeometric functions:

$$\sin(z) == \sqrt{\frac{\pi z}{2}}\, J_{1/2}(z) \quad \sin(z) == -i\sqrt{\frac{\pi i z}{2}}\, I_{1/2}(i z) \quad \sin(z) == \sqrt{\frac{\pi z}{2}}\, Y_{-1/2}(z) \quad \sin(z) == \frac{i}{\sqrt{2\pi}}\left(\sqrt{i z}\, K_{1/2}(i z) - \sqrt{-i z}\, K_{1/2}(-i z)\right)$$

$$\cos(z) == \sqrt{\frac{\pi z}{2}}\, J_{-1/2}(z) \quad \cos(z) == \sqrt{\frac{\pi i z}{2}}\, I_{-1/2}(i z) \quad \cos(z) == -\sqrt{\frac{\pi z}{2}}\, Y_{1/2}(z) \quad \cos(z) == \sqrt{\frac{i z}{2\pi}}\, K_{1/2}(i z) + \sqrt{-\frac{i z}{2\pi}}\, K_{1/2}(-i z)$$

$$\sin(z) == \mathrm{Se}(1, 0, z) \quad \cos(z) == \mathrm{Ce}(1, 0, z)$$

$$\sin(z) == z\, {}_0F_1\left(;\, \tfrac{3}{2};\, -\tfrac{z^2}{4}\right) \quad \cos(z) == {}_0F_1\left(;\, \tfrac{1}{2};\, -\tfrac{z^2}{4}\right).$$

On the other hand, all trigonometric functions can be represented as degenerate cases of the corresponding doubly periodic Jacobi elliptic functions when their second parameter is equal to 0 or 1:

$$\begin{array}{ll}
\sin(z) == \mathrm{sd}(z \mid 0) == \mathrm{sn}(z \mid 0) & \sin(z) == -i\, \mathrm{sc}(i z \mid 1) == -i\, \mathrm{sd}(i z \mid 1) \\
\cos(z) == \mathrm{cd}(z \mid 0) == \mathrm{cn}(z \mid 0) & \cos(z) == \mathrm{nc}(i z \mid 1) == \mathrm{nd}(i z \mid 1) \\
\tan(z) == \mathrm{sc}(z \mid 0) & \tan(z) == -i\, \mathrm{sn}(i z \mid 1) \\
\cot(z) == \mathrm{cs}(z \mid 0) & \cot(z) == i\, \mathrm{ns}(i z \mid 1) \\
\csc(z) == \mathrm{ds}(z \mid 0) == \mathrm{ns}(z \mid 0) & \csc(z) == i\, \mathrm{cs}(i z \mid 1) == i\, \mathrm{ds}(i z \mid 1) \\
\sec(z) == \mathrm{dc}(z \mid 0) == \mathrm{nc}(z \mid 0) & \sec(z) == \mathrm{cn}(i z \mid 1) == \mathrm{dn}(i z \mid 1).
\end{array}$$

Representations through related equivalent functions

Each of the six trigonometric functions can be represented through the corresponding hyperbolic function:

$$\begin{array}{ll}
\sin(z) == -i \sinh(i z) & \sin(i z) == i \sinh(z) \\
\cos(z) == \cosh(i z) & \cos(i z) == \cosh(z) \\
\tan(z) == -i \tanh(i z) & \tan(i z) == i \tanh(z) \\
\cot(z) == i \coth(i z) & \cot(i z) == -i \coth(z) \\
\csc(z) == i\, \mathrm{csch}(i z) & \csc(i z) == -i\, \mathrm{csch}(z) \\
\sec(z) == \mathrm{sech}(i z) & \sec(i z) == \mathrm{sech}(z).
\end{array}$$

Relations to inverse functions

Each of the six trigonometric functions is connected with its corresponding inverse trigonometric function by two formulas. One is a simple formula, and the other is much more complicated because of the multivalued nature of the inverse function:

$$\sin(\sin^{-1}(z)) == z \quad \sin^{-1}(\sin(z)) == z /; -\frac{\pi}{2} < \mathrm{Re}(z) < \frac{\pi}{2} \bigvee \mathrm{Re}(z) == -\frac{\pi}{2} \bigwedge \mathrm{Im}(z) \geq 0 \bigvee \mathrm{Re}(z) == \frac{\pi}{2} \bigwedge \mathrm{Im}(z) \leq 0$$
$$\cos(\cos^{-1}(z)) == z \quad \cos^{-1}(\cos(z)) == z /; 0 < \mathrm{Re}(z) < \pi \vee \mathrm{Re}(z) == 0 \wedge \mathrm{Im}(z) \geq 0 \vee \mathrm{Re}(z) == \pi \wedge \mathrm{Im}(z) \leq 0$$
$$\tan(\tan^{-1}(z)) == z \quad \tan^{-1}(\tan(z)) == z /; |\mathrm{Re}(z)| < \frac{\pi}{2} \bigvee \mathrm{Re}(z) == -\frac{\pi}{2} \bigwedge \mathrm{Im}(z) < 0 \bigvee \mathrm{Re}(z) == \frac{\pi}{2} \bigwedge \mathrm{Im}(z) > 0$$
$$\cot(\cot^{-1}(z)) == z \quad \cot^{-1}(\cot(z)) == z /; |\mathrm{Re}(z)| < \frac{\pi}{2} \bigvee \mathrm{Re}(z) == -\frac{\pi}{2} \bigwedge \mathrm{Im}(z) < 0 \bigvee \mathrm{Re}(z) == \frac{\pi}{2} \bigwedge \mathrm{Im}(z) \geq 0$$
$$\csc(\csc^{-1}(z)) == z \quad \csc^{-1}(\csc(z)) == z /; |\mathrm{Re}(z)| < \frac{\pi}{2} \bigvee \mathrm{Re}(z) == -\frac{\pi}{2} \bigwedge \mathrm{Im}(z) \leq 0 \bigvee \mathrm{Re}(z) == \frac{\pi}{2} \bigwedge \mathrm{Im}(z) \geq 0$$
$$\sec(\sec^{-1}(z)) == z \quad \sec^{-1}(\sec(z)) == z /; 0 < \mathrm{Re}(z) < \pi \vee \mathrm{Re}(z) == 0 \wedge \mathrm{Im}(z) \geq 0 \vee \mathrm{Re}(z) == \pi \wedge \mathrm{Im}(z) \leq 0.$$

Representations through other trigonometric functions

Each of the six trigonometric functions can be represented by any other trigonometric function as a rational function of that function with linear arguments. For example, the sine function can be representative as a group-defining function because the other five functions can be expressed as follows:

$$\cos(z) == \sin\left(\frac{\pi}{2} - z\right) \qquad \cos^2(z) == 1 - \sin^2(z)$$
$$\tan(z) == \frac{\sin(z)}{\cos(z)} == \frac{\sin(z)}{\sin\left(\frac{\pi}{2} - z\right)} \qquad \tan^2(z) == \frac{\sin^2(z)}{1 - \sin^2(z)}$$
$$\cot(z) == \frac{\cos(z)}{\sin(z)} == \frac{\sin\left(\frac{\pi}{2} - z\right)}{\sin(z)} \qquad \cot^2(z) == \frac{1 - \sin^2(z)}{\sin^2(z)}$$
$$\csc(z) == \frac{1}{\sin(z)} \qquad \csc^2(z) == \frac{1}{\sin^2(z)}$$
$$\sec(z) == \frac{1}{\cos(z)} == \frac{1}{\sin\left(\frac{\pi}{2} - z\right)} \qquad \sec^2(z) == \frac{1}{1 - \sin^2(z)}.$$

All six trigonometric functions can be transformed into any other trigonometric function of this group if the argument z is replaced by $p\,\pi/2 + q\,z$ with $q^2 == 1 \bigwedge p \in \mathbb{Z}$:

$$\sin(-z - 2\pi) == -\sin(z) \qquad \sin(z - 2\pi) == \sin(z)$$
$$\sin\left(-z - \frac{3\pi}{2}\right) == \cos(z) \qquad \sin\left(z - \frac{3\pi}{2}\right) == \cos(z)$$
$$\sin(-z - \pi) == \sin(z) \qquad \sin(z - \pi) == -\sin(z)$$
$$\sin\left(-z - \frac{\pi}{2}\right) == -\cos(z) \qquad \sin\left(z - \frac{\pi}{2}\right) == -\cos(z)$$
$$\sin\left(z + \frac{\pi}{2}\right) == \cos(z) \qquad \sin\left(\frac{\pi}{2} - z\right) == \cos(z)$$
$$\sin(z + \pi) == -\sin(z) \qquad \sin(\pi - z) == \sin(z)$$
$$\sin\left(z + \frac{3\pi}{2}\right) == -\cos(z) \qquad \sin\left(\frac{3\pi}{2} - z\right) == -\cos(z)$$
$$\sin(z + 2\pi) == \sin(z) \qquad \sin(2\pi - z) == -\sin(z)$$

$$\cos(-z-2\pi) == \cos(z) \quad \cos(z-2\pi) == \cos(z)$$
$$\cos\left(-z-\frac{3\pi}{2}\right) == \sin(z) \quad \cos\left(z-\frac{3\pi}{2}\right) == -\sin(z)$$
$$\cos(-z-\pi) == -\cos(z) \quad \cos(z-\pi) == -\cos(z)$$
$$\cos\left(-z-\frac{\pi}{2}\right) == -\sin(z) \quad \cos\left(z-\frac{\pi}{2}\right) == \sin(z)$$
$$\cos\left(z+\frac{\pi}{2}\right) == -\sin(z) \quad \cos\left(\frac{\pi}{2}-z\right) == \sin(z)$$
$$\cos(z+\pi) == -\cos(z) \quad \cos(\pi-z) == -\cos(z)$$
$$\cos\left(z+\frac{3\pi}{2}\right) == \sin(z) \quad \cos\left(\frac{3\pi}{2}-z\right) == -\sin(z)$$
$$\cos(z+2\pi) == \cos(z) \quad \cos(2\pi-z) == \cos(z)$$

$$\tan(-z-\pi) == -\tan(z) \quad \tan(z-\pi) == \tan(z)$$
$$\tan\left(-z-\frac{\pi}{2}\right) == \cot(z) \quad \tan\left(z-\frac{\pi}{2}\right) == -\cot(z)$$
$$\tan\left(z+\frac{\pi}{2}\right) == -\cot(z) \quad \tan\left(\frac{\pi}{2}-z\right) == \cot(z)$$
$$\tan(z+\pi) == \tan(z) \quad \tan(\pi-z) == -\tan(z)$$

$$\cot(-z-\pi) == -\cot(z) \quad \cot(z-\pi) == \cot(z)$$
$$\cot\left(-z-\frac{\pi}{2}\right) == \tan(z) \quad \cot\left(z-\frac{\pi}{2}\right) == -\tan(z)$$
$$\cot\left(z+\frac{\pi}{2}\right) == -\tan(z) \quad \cot\left(\frac{\pi}{2}-z\right) == \tan(z)$$
$$\cot(z+\pi) == \cot(z) \quad \cot(\pi-z) == -\cot(z)$$

$$\csc(-z-2\pi) == -\csc(z) \quad \csc(z-2\pi) == \csc(z)$$
$$\csc\left(-z-\frac{3\pi}{2}\right) == \sec(z) \quad \csc\left(z-\frac{3\pi}{2}\right) == \sec(z)$$
$$\csc(-z-\pi) == \csc(z) \quad \csc(z-\pi) == -\csc(z)$$
$$\csc\left(-z-\frac{\pi}{2}\right) == -\sec(z) \quad \csc\left(z-\frac{\pi}{2}\right) == -\sec(z)$$
$$\csc\left(z+\frac{\pi}{2}\right) == \sec(z) \quad \csc\left(\frac{\pi}{2}-z\right) == \sec(z)$$
$$\csc(z+\pi) == -\csc(z) \quad \csc(\pi-z) == \csc(z)$$
$$\csc\left(z+\frac{3\pi}{2}\right) == -\sec(z) \quad \csc\left(\frac{3\pi}{2}-z\right) == -\sec(z)$$
$$\csc(z+2\pi) == \csc(z) \quad \csc(2\pi-z) == -\csc(z)$$

$$\sec(-z-2\pi) == \sec(z) \quad \sec(z-2\pi) == \sec(z)$$
$$\sec\left(-z-\frac{3\pi}{2}\right) == \csc(z) \quad \sec\left(z-\frac{3\pi}{2}\right) == -\csc(z)$$
$$\sec(-z-\pi) == -\sec(z) \quad \sec(z-\pi) == -\sec(z)$$
$$\sec\left(-z-\frac{\pi}{2}\right) == -\csc(z) \quad \sec\left(z-\frac{\pi}{2}\right) == \csc(z)$$
$$\sec\left(z+\frac{\pi}{2}\right) == -\csc(z) \quad \sec\left(\frac{\pi}{2}-z\right) == \csc(z)$$
$$\sec(z+\pi) == -\sec(z) \quad \sec(\pi-z) == -\sec(z)$$
$$\sec\left(z+\frac{3\pi}{2}\right) == \csc(z) \quad \sec\left(\frac{3\pi}{2}-z\right) == -\csc(z)$$
$$\sec(z+2\pi) == \sec(z) \quad \sec(2\pi-z) == \sec(z).$$

The best-known properties and formulas for trigonometric functions

Real values for real arguments

For real values of argument z, the values of all the trigonometric functions are real (or infinity).

In the points $z = 2\pi n/m$ /; $n \in \mathbb{Z} \bigwedge m \in \mathbb{Z}$, the values of trigonometric functions are algebraic. In several cases they can even be rational numbers or integers (like $\sin(\pi/2) = 1$ or $\sin(\pi/6) = 1/2$). The values of trigonometric functions can be expressed using only square roots if $n \in \mathbb{Z}$ and m is a product of a power of 2 and distinct Fermat primes $\{3, 5, 17, 257, \ldots\}$.

Simple values at zero

All trigonometric functions have rather simple values for arguments $z == 0$ and $z == \pi/2$:

$$\sin(0) == 0 \quad \sin\left(\frac{\pi}{2}\right) == 1$$
$$\cos(0) == 1 \quad \cos\left(\frac{\pi}{2}\right) == 0$$
$$\tan(0) == 0 \quad \tan\left(\frac{\pi}{2}\right) == \tilde{\infty}$$
$$\cot(0) == \tilde{\infty} \quad \cot\left(\frac{\pi}{2}\right) == 0$$
$$\csc(0) == \tilde{\infty} \quad \csc\left(\frac{\pi}{2}\right) == 1$$
$$\sec(0) == 1 \quad \sec\left(\frac{\pi}{2}\right) == \tilde{\infty}.$$

Analyticity

All trigonometric functions are defined for all complex values of z, and they are analytical functions of z over the whole complex z-plane and do not have branch cuts or branch points. The two functions $\sin(z)$ and $\cos(z)$ are entire functions with an essential singular point at $z = \tilde{\infty}$. All other trigonometric functions are meromorphic functions with simple poles at points $z == \pi k$ /; $k \in \mathbb{Z}$ for $\csc(z)$ and $\cot(z)$, and at points $z == \pi/2 + \pi k$ /; $k \in \mathbb{Z}$ for $\sec(z)$ and $\tan(z)$.

Periodicity

All trigonometric functions are periodic functions with a real period (2π or π):

$$\sin(z) == \sin(z + 2\pi) \quad \sin(z + 2\pi k) == \sin(z) \text{ /; } k \in \mathbb{Z}$$
$$\cos(z) == \cos(z + 2\pi) \quad \cos(z + 2\pi k) == \cos(z) \text{ /; } k \in \mathbb{Z}$$
$$\tan(z) == \tan(z + \pi) \quad \tan(z + \pi k) == \tan(z) \text{ /; } k \in \mathbb{Z}$$
$$\cot(z) == \cot(z + \pi) \quad \cot(z + \pi k) == \cot(z) \text{ /; } k \in \mathbb{Z}$$
$$\csc(z) == \csc(z + 2\pi) \quad \csc(z + 2\pi k) == \csc(z) \text{ /; } k \in \mathbb{Z}$$
$$\sec(z) == \sec(z + 2\pi) \quad \sec(z + 2\pi k) == \sec(z) \text{ /; } k \in \mathbb{Z}.$$

Parity and symmetry

All trigonometric functions have parity (either odd or even) and mirror symmetry:

$$\sin(-z) == -\sin(z) \quad \sin(\bar{z}) == \overline{\sin(z)}$$
$$\cos(-z) == \cos(z) \quad \cos(\bar{z}) == \overline{\cos(z)}$$
$$\tan(-z) == -\tan(z) \quad \tan(\bar{z}) == \overline{\tan(z)}$$
$$\cot(-z) == -\cot(z) \quad \cot(\bar{z}) == \overline{\cot(z)}$$
$$\csc(-z) == -\csc(z) \quad \csc(\bar{z}) == \overline{\csc(z)}$$
$$\sec(-z) == \sec(z) \quad \sec(\bar{z}) == \overline{\sec(z)}.$$

Simple representations of derivatives

The derivatives of all trigonometric functions have simple representations that can be expressed through other trigonometric functions:

$$\frac{\partial \sin(z)}{\partial z} == \cos(z) \qquad \frac{\partial \cos(z)}{\partial z} == -\sin(z) \qquad \frac{\partial \tan(z)}{\partial z} == \sec^2(z)$$

$$\frac{\partial \cot(z)}{\partial z} == -\csc^2(z) \qquad \frac{\partial \csc(z)}{\partial z} == -\cot(z)\csc(z) \qquad \frac{\partial \sec(z)}{\partial z} == \sec(z)\tan(z).$$

Simple differential equations

The solutions of the simplest second-order linear ordinary differential equation with constant coefficients can be represented through $\sin(z)$ and $\cos(z)$:

$$w''(z) + w(z) == 0 \;/;\; w(z) == \cos(z) \bigwedge w(0) == 1 \bigwedge w'(0) == 0$$

$$w''(z) + w(z) == 0 \;/;\; w(z) == \sin(z) \bigwedge w(0) == 0 \bigwedge w'(0) == 1$$

$$w''(z) + w(z) == 0 \;/;\; w(z) == c_1 \cos(z) + c_2 \sin(z).$$

All six trigonometric functions satisfy first-order nonlinear differential equations:

$$w'(z) - \sqrt{1 - (w(z))^2} == 0 \;/;\; w(z) == \sin(z) \bigwedge w(0) == 0 \bigwedge |\mathrm{Re}(z)| < \frac{\pi}{2}$$

$$w'(z) - \sqrt{1 - (w(z))^2} == 0 \;/;\; w(z) == \cos(z) \bigwedge w(0) == 1 \bigwedge |\mathrm{Re}(z)| < \frac{\pi}{2}$$

$$w'(z) - w(z)^2 - 1 == 0 \;/;\; w(z) == \tan(z) \wedge w(0) == 0$$

$$w'(z) + w(z)^2 + 1 == 0 \;/;\; w(z) == \cot(z) \bigwedge w\left(\frac{\pi}{2}\right) == 0$$

$$w'(z)^2 - w(z)^4 + w(z)^2 == 0 \;/;\; w(z) == \csc(z)$$

$$w'(z)^2 - w(z)^4 + w(z)^2 == 0 \;/;\; w(z) == \sec(z).$$

Applications of trigonometric functions

Triangle theorems

The prime application of the trigonometric functions are triangle theorems. In a triangle, a, b, and c represent the lengths of the sides opposite to the angles, Δ the area , R the circumradius, and r the inradius. Then the following identities hold:

$$\alpha + \beta + \gamma == \pi$$

$$\frac{\sin(\alpha)}{a} == \frac{\sin(\beta)}{b} == \frac{\sin(\gamma)}{c}$$

$$\sin(\alpha)\sin(\beta)\sin(\gamma) == \frac{\Delta}{2R^2} \qquad \sin(\alpha) == \frac{2\Delta}{b\,c}$$

$$\cos(\alpha) == \frac{b^2 + c^2 - a^2}{2\,b\,c} \qquad \cot(\alpha) == \frac{b^2 + c^2 - a^2}{4\Delta}$$

$$\sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right)\sin\left(\frac{\gamma}{2}\right) == \frac{r}{4R} \qquad \cos(\alpha) + \cos(\beta) + \cos(\gamma) == 1 + \frac{r}{R}$$

$$\cot(\alpha) + \cot(\beta) + \cot(\gamma) == \frac{a^2 + b^2 + c^2}{4\,\Delta}$$

$$\tan(\alpha) + \tan(\beta) + \tan(\gamma) == \tan(\alpha)\tan(\beta)\tan(\gamma)$$

$$\cot(\alpha)\cot(\beta) + \cot(\alpha)\cot(\gamma) + \cot(\beta)\cot(\gamma) == 1$$

$$\cos^2(\alpha) + \cos^2(\beta) + \cos^2(\gamma) == 1 - 2\cos(\alpha)\cos(\beta)\cos(\gamma)$$

$$\frac{\tan\left(\frac{\alpha}{2}\right)\tan\left(\frac{\beta}{2}\right)}{\tan\left(\frac{\alpha}{2}\right) + \tan\left(\frac{\beta}{2}\right)} == \frac{r}{c}.$$

For a right-angle triangle the following relations hold:

$$\sin(\alpha) == \frac{a}{c} \;/;\; \gamma == \frac{\pi}{2} \qquad \cos(\alpha) == \frac{b}{c} \;/;\; \gamma == \frac{\pi}{2}$$

$$\tan(\alpha) == \frac{a}{b} \;/;\; \gamma == \frac{\pi}{2} \qquad \cot(\alpha) == \frac{b}{a} \;/;\; \gamma == \frac{\pi}{2}$$

$$\csc(\alpha) == \frac{c}{a} \;/;\; \gamma == \frac{\pi}{2} \qquad \sec(\alpha) == \frac{c}{b} \;/;\; \gamma == \frac{\pi}{2}.$$

Other applications

Because the trigonometric functions appear virtually everywhere in quantitative sciences, it is impossible to list their numerous applications in teaching, science, engineering, and art.

Introduction to the Cotangent Function

Defining the cotangent function

The cotangent function is an old mathematical function. It was mentioned in 1620 by E. Gunter who invented the notation of "cotangens". Later on J. Keill (1726) and L. Euler (1748) used this function and its notation cot in their investigations.

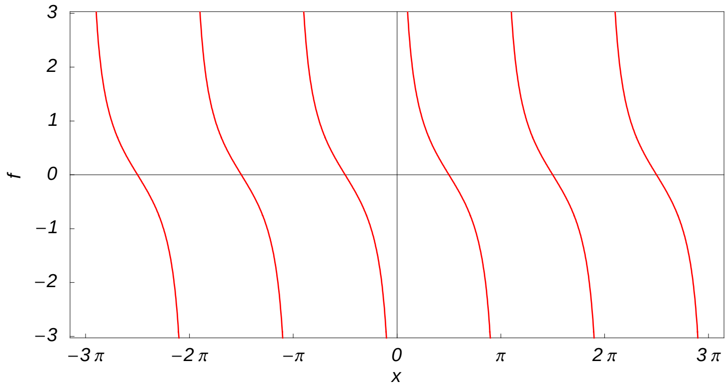
The classical definition of the cotangent function for real arguments is: "the cotangent of an angle α in a right-angle triangle is the ratio of the length of the adjacent leg to the length to the opposite leg." This description of $\cot(\alpha)$ is valid for $0 < \alpha < \pi/2$ when the triangle is nondegenerate. This approach to the cotangent can be expanded to arbitrary real values of α if consideration is given to the arbitrary point $\{x, y\}$ in the x,y-Cartesian plane and $\cot(\alpha)$ is defined as the ratio x/y assuming that α is the value of the angle between the positive direction of the x-axis and the direction from the origin to the point $\{x, y\}$.

Comparing the cotangent definition with the definitions of the sine and cosine functions shows that the following formula can also be used as a definition of the cotangent function:

$$\cot(z) == \frac{\cos(z)}{\sin(z)}.$$

A quick look at the cotangent function

Here is a graphic of the cotangent function $f(x) = \cot(x)$ for real values of its argument x.



Representation through more general functions

The cotangent function cot(z) can be represented using more general mathematical functions. As the ratio of the cosine and sine functions that are particular cases of the generalized hypergeometric, Bessel, Struve, and Mathieu functions, the cotangent function can also be represented as ratios of those special functions. But these representations are not very useful. It is more useful to write the cotangent function as particular cases of one special function. That can be done using doubly periodic Jacobi elliptic functions that degenerate into the cotangent function when their second parameter is equal to 0 or 1.

$$\cot(z) = \operatorname{cs}(z \mid 0) = \operatorname{sc}\left(\frac{\pi}{2} - z \,\middle|\, 0\right) = i \operatorname{ns}(i z \mid 1) = -i \operatorname{sn}\left(\frac{\pi i}{2} - i z \,\middle|\, 1\right).$$

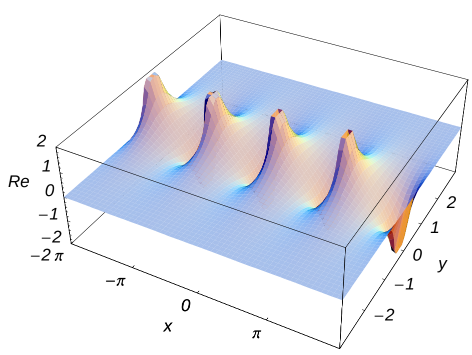
Definition of the cotangent function for a complex argument

In the complex z-plane, the function cot(z) is defined using cos(z) and sin(z) or the exponential function e^w in the points $i z$ and $-i z$ through the formula:

$$\cot(z) = \frac{\cos(z)}{\sin(z)} = \frac{i\left(e^{i z} + e^{-i z}\right)}{e^{i z} - e^{-i z}}.$$

In the points $z = \pi k /; k \in \mathbb{Z}$, where sin($z$) has zeros, the denominator of the last formula equals zero and cot(z) has singularities (poles of the first order).

Here are two graphics showing the real and imaginary parts of the cotangent function over the complex plane.



The best-known properties and formulas for the cotangent function

Values in points

Students usually learn the following basic table of values of the cotangent function for special points of the circle:

$$\cot(0) = \tilde{\infty} \quad \cot\left(\frac{\pi}{6}\right) = \sqrt{3} \quad \cot\left(\frac{\pi}{4}\right) = 1 \quad \cot\left(\frac{\pi}{3}\right) = \frac{1}{\sqrt{3}}$$

$$\cot\left(\frac{\pi}{2}\right) = 0 \quad \cot\left(\frac{2\pi}{3}\right) = -\frac{1}{\sqrt{3}} \quad \cot\left(\frac{3\pi}{4}\right) = -1 \quad \cot\left(\frac{5\pi}{6}\right) = -\sqrt{3}$$

$$\cot(\pi) = \tilde{\infty}$$

$$\cot(\pi m) = \tilde{\infty} /; m \in \mathbb{Z} \quad \cot\left(\pi\left(\frac{1}{2} + m\right)\right) = 0 /; m \in \mathbb{Z}.$$

General characteristics

For real values of argument z, the values of $\cot(z)$ are real.

In the points $z = \pi n / m /; n \in \mathbb{Z} \wedge m \in \mathbb{Z}$, the values of $\cot(z)$ are algebraic. In several cases they can be integers -1, 0, or 1:

$$\cot\left(-\frac{\pi}{4}\right) = -1 \quad \cot\left(\frac{\pi}{2}\right) = 0 \quad \cot\left(\frac{\pi}{4}\right) = 1.$$

The values of $\cot\left(\frac{n\pi}{m}\right)$ can be expressed using only square roots if $n \in \mathbb{Z}$ and m is a product of a power of 2 and distinct Fermat primes $\{3, 5, 17, 257, \ldots\}$.

The function $\cot(z)$ is an analytical function of z that is defined over the whole complex z-plane and does not have branch cuts and branch points. It has an infinite set of singular points:

(a) $z = \pi k /; k \in \mathbb{Z}$ are the simple poles with residues -1.
(b) $z = \tilde{\infty}$ is an essential singular point.

It is a periodic function with the real period π:

$$\cot(z + \pi) = \cot(z)$$

$$\cot(z) = \cot(z + \pi k) /; k \in \mathbb{Z}.$$

The function $\cot(z)$ is an odd function with mirror symmetry:

$$\cot(-z) = -\cot(z) \quad \cot(\bar{z}) = \overline{\cot(z)}.$$

Differentiation

The first derivative of $\cot(z)$ has simple representations using either the $\sin(z)$ function or the $\csc(z)$ function:

$$\frac{\partial \cot(z)}{\partial z} = -\frac{1}{\sin^2(z)} = -\csc^2(z).$$

The n^{th} derivative of $\cot(z)$ has much more complicated representations than symbolic n^{th} derivatives for $\sin(z)$ and $\cos(z)$:

$$\frac{\partial^n \cot(z)}{\partial z^n} = \cot(z)\,\delta_n - \delta_{n-1}\csc^2(z) - n\sum_{k=0}^{n-1}\sum_{j=0}^{k-1}\frac{(-1)^j \sin^{-2k-2}(z)\, 2^{n-2k}\,(k-j)^{n-1}}{k+1}\binom{n-1}{k}\binom{2k}{j}\sin\left(\frac{\pi n}{2}+2(k-j)z\right) /;\, n \in \mathbb{N},$$

where δ_n is the Kronecker delta symbol: $\delta_0 = 1$ and $\delta_n = 0 /; n \neq 0$.

Ordinary differential equation

The function $\cot(z)$ satisfies the following first-order nonlinear differential equation:

$$w'(z) + w(z)^2 + 1 = 0 /; w(z) = \cot(z) \bigwedge w\left(\frac{\pi}{2}\right) = 0.$$

Series representation

The function $\cot(z)$ has a simple Laurent series expansion at the origin that converges for all finite values z with $0 < |z| < \pi$:

$$\cot(z) = \frac{1}{z} - \frac{z}{3} - \frac{z^3}{45} - \ldots = \frac{1}{z} + \sum_{k=1}^{\infty}\frac{(-1)^k 2^{2k} B_{2k} z^{2k-1}}{(2k)!},$$

where B_{2k} are the Bernoulli numbers.

Integral representation

The function $\cot(z)$ has a well-known integral representation through the following definite integral along the positive part of the real axis:

$$\cot(z) = \frac{2}{\pi}\int_0^{\infty}\frac{t^{1-\frac{2z}{\pi}} - 1}{t^2 - 1}\,dt /;\, 0 < \operatorname{Re}(z) < \frac{\pi}{2}.$$

Continued fraction representations

The function $\cot(z)$ has the following simple continued fraction representation:

$$\cot(z) = \frac{1}{z} - \cfrac{4\pi^{-2} z}{1 + \cfrac{1 - 4\pi^{-2} z^2}{3 + \cfrac{4\left(4 - 4\pi^{-2} z^2\right)}{5 + \cfrac{9\left(9 - 4\pi^{-2} z^2\right)}{7 + \cfrac{16\left(16 - 4\pi^{-2} z^2\right)}{9 + \cfrac{25\left(25 - 4\pi^{-2} z^2\right)}{11 + \cfrac{36\left(36 - 4\pi^{-2} z^2\right)}{13 + \ldots}}}}}}}.$$

Indefinite integration

Indefinite integrals of expressions that contain the cotangent function can sometimes be expressed using elementary functions. However, special functions are frequently needed to express the results even when the integrands have a simple form (if they can be evaluated in closed form). Here are some examples:

$$\int \cot(z)\, dz = \log(\sin(z))$$

$$\int \sqrt{\cot(z)}\, dz = \frac{1}{2\sqrt{2}} \left(-2\tan^{-1}\left(\sqrt{2}\cot^{\frac{1}{2}}(z)+1\right)+2\tan^{-1}\left(1-\sqrt{2}\cot^{\frac{1}{2}}(z)\right)-\log\left(-\cot(z)+\sqrt{2}\cot^{\frac{1}{2}}(z)-1\right)+\log\left(\cot(z)+\sqrt{2}\cot^{\frac{1}{2}}(z)+1\right)\right)$$

$$\int \cot^{\nu}(a z)\, dz = -\frac{\cot^{\nu+1}(a z)}{\nu a + a}\, {}_2F_1\left(\frac{\nu+1}{2}, 1; \frac{\nu+3}{2}; -\cot^2(a z)\right).$$

Definite integration

Definite integrals that contain the cotangent function are sometimes simple. For example, the famous Catalan constant C can be defined as the value of the following integral:

$$\int_0^{\frac{\pi}{4}} \log(\cot(t))\, dt = C.$$

This constant also appears in the following integral:

$$\int_0^{\frac{\pi}{4}} t\cot(t)\, dt = \frac{1}{16}\left(8C + i\pi^2 + 4\pi\log(1-i)\right).$$

Some special functions can be used to evaluate more complicated definite integrals. For example, to express the following integral, the Gauss hypergeometric function is needed:

$$\int_0^{\frac{\pi}{4}} \sin^{\alpha-1}(t)\cot^{\beta}(t)\, dt = \frac{2^{\frac{1-\alpha}{2}}}{\alpha-\beta}\, {}_2F_1\left(1, \frac{1-\beta}{2}; \frac{1}{2}(\alpha-\beta+2); -1\right) /; \operatorname{Re}(\alpha-\beta) > 0.$$

Finite summation

The following finite sum that contains a cotangent function can be expressed in terms of a cotangent function:

$$\sum_{k=0}^{n-1} \cot^2\left(\frac{\pi k}{n} + z\right) = \cot^2(n z)\, n^2 + n^2 - n /; n \in \mathbb{N}^+.$$

Other finite sums that contain a cotangent function can be expressed in terms of a polynomial function:

$$\sum_{k=1}^{n-1} \cot^2\left(\frac{\pi k}{n}\right) = \frac{(n-1)(n-2)}{3} /; n \in \mathbb{N}^+$$

$$\sum_{k=1}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \cot^2\left(\frac{k\pi}{n}\right) = \frac{1}{6}(n-1)(n-2) /; n \in \mathbb{N}^+$$

$$\sum_{k=0}^{n-1} (-1)^k \cot\left(\frac{(2k+1)\pi}{4n}\right) = n /; n \in \mathbb{N}^+$$

$$\sum_{k=0}^{2n-1} (-1)^k \cot\left(\frac{(2k+1)\pi}{8n}\right) == 2n \,/;\, n \in \mathbb{N}^+$$

$$\sum_{k=1}^{n} \cot^4\left(\frac{k\pi}{2n+1}\right) == \frac{1}{45} n (2n-1) \left(4n^2 + 10n - 9\right) /;\, n \in \mathbb{N}^+.$$

Infinite summation

The following infinite sum that contains the cotangent function has a very simple value:

$$\sum_{k=1}^{\infty} \frac{1}{k^3} \cot\left(\frac{1}{2} k \pi \left(1 + \sqrt{5}\right)\right) == -\frac{\pi^3}{45\sqrt{5}}.$$

Finite products

The following finite product from the cotangent has a very simple value:

$$\prod_{k=1}^{n-1} \cot\left(\frac{k\pi}{n}\right) == -\frac{(-1)^n}{n} \sin\left(\frac{n\pi}{2}\right) /;\, n \in \mathbb{N}^+.$$

Addition formulas

The cotangent of a sum can be represented by the rule: "the cotangent of a sum is equal to the product of the cotangents minus one divided by a sum of the cotangents." A similar rule is valid for the cotangent of the difference:

$$\cot(a+b) == \frac{\cot(a)\cot(b) - 1}{\cot(a) + \cot(b)}$$

$$\cot(a-b) == \frac{\cot(a)\cot(b) + 1}{\cot(b) - \cot(a)}.$$

Multiple arguments

In the case of multiple arguments z, $2z$, $3z$, …, the function $\cot(z)$ can be represented as the ratio of the finite sums that contains powers of cotangents:

$$\cot(2z) == \frac{\cot^2(z) - 1}{2\cot(z)} == \frac{1}{2} (\cot(z) - \tan(z))$$

$$\cot(3z) == \frac{\cot^3(z) - 3\cot(z)}{3\cot^2(z) - 1}$$

$$\cot(nz) == \frac{(-1)^{n-1}}{\cot^{(-1)^n}(z) \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} (-1)^k \binom{n}{2\left\lfloor \frac{n-1}{2} \right\rfloor - 2k + 1} \cot^{2k}(z)} \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} (-1)^k \binom{n}{2\left\lfloor \frac{n}{2} \right\rfloor - 2k} \cot^{2k}(z) /;\, n \in \mathbb{N}^+.$$

Half-angle formulas

The cotangent of a half-angle can be represented using two trigonometric functions by the following simple formulas:

$$\cot\left(\frac{z}{2}\right) == \cot(z) + \csc(z)$$

$$\cot\left(\frac{z}{2}\right) == \frac{\sin(z)}{1 - \cos(z)}.$$

The sine function in the last formula can be replaced by the cosine function. But it leads to a more complicated representation that is valid in some vertical strip:

$$\cot\left(\frac{z}{2}\right) == \sqrt{\frac{1+\cos(z)}{1-\cos(z)}} \;/;\; 0 < \mathrm{Re}(z) < \pi \lor \mathrm{Re}(z) == 0 \land \mathrm{Im}(z) < 0 \lor \mathrm{Re}(z) == \pi \land \mathrm{Im}(z) \leq 0.$$

To make this formula correct for all complex z, a complicated prefactor is needed:

$$\cot\left(\frac{z}{2}\right) == c(z) \sqrt{\frac{1+\cos(z)}{1-\cos(z)}} \;/;\; c(z) == (-1)^{\left\lfloor \frac{2\,\mathrm{Re}(z)-\pi}{2\pi} \right\rceil} \left(1 - \left(1 + (-1)^{\left\lfloor \frac{\mathrm{Re}(z)}{\pi} \right\rfloor + \left\lfloor -\frac{\mathrm{Re}(z)}{\pi} \right\rfloor}\right) \theta(\mathrm{Im}(z))\right),$$

where $c(z)$ contains the unit step, real part, imaginary part, the floor, and the round functions.

Sums of two direct functions

The sum of two cotangent functions can be described by the rule: "the sum of cotangents is equal to the sine of the sum multiplied by the cosecants." A similar rule is valid for the difference of two cotangents:

$$\cot(a) + \cot(b) == \csc(a)\csc(b)\sin(a+b)$$
$$\cot(a) - \cot(b) == -\csc(a)\csc(b)\sin(a-b).$$

Products involving the direct function

The product of two cotangents and the product of the cotangent and tangent have the following representations:

$$\cot(a)\cot(b) == \frac{\cos(a-b) + \cos(a+b)}{\cos(a-b) - \cos(a+b)}$$

$$\cot(a)\tan(b) == \frac{\sin(a+b) - \sin(a-b)}{\sin(a-b) + \sin(a+b)}.$$

Inequalities

The most famous inequality for the cotangent function is the following:

$$\cot(x) < \frac{1}{x} \;/;\; 0 < x < \pi \land x \in \mathbb{R}.$$

Relations with its inverse function

There are simple relations between the function $\cot(z)$ and its inverse function $\cot^{-1}(z)$:

$$\cot\left(\cot^{-1}(z)\right) == z \quad \cot^{-1}(\cot(z)) == z \;/;\; |\mathrm{Re}(z)| < \frac{\pi}{2} \bigvee \mathrm{Re}(z) == -\frac{\pi}{2} \bigwedge \mathrm{Im}(z) < 0 \bigvee \mathrm{Re}(z) == \frac{\pi}{2} \bigwedge \mathrm{Im}(z) \geq 0.$$

The second formula is valid at least in the vertical strip $-\frac{\pi}{2} < \mathrm{Re}(z) < \frac{\pi}{2}$. Outside of this strip a much more complicated relation (that contains the unit step, real part, and the floor functions) holds:

$$\cot^{-1}(\cot(z)) == z + \pi \left\lfloor \frac{1}{2} - \frac{\mathrm{Re}(z)}{\pi} \right\rfloor - \frac{1}{2}\pi \left(1 + (-1)^{\left\lfloor \frac{\mathrm{Re}(z)}{\pi} + \frac{1}{2} \right\rfloor + \left\lfloor -\frac{\mathrm{Re}(z)}{\pi} - \frac{1}{2} \right\rfloor}\right) \theta(-\mathrm{Im}(z)) \;/;\; \frac{z}{\pi} - \frac{1}{2} \notin \mathbb{Z}.$$

Representations through other trigonometric functions

Cotangent and tangent functions are connected by a very simple formula that contains the linear function in the following argument:

$$\cot(z) == \tan\left(\frac{\pi}{2} - z\right).$$

The cotangent function can also be represented using other trigonometric functions by the following formulas:

$$\cot(z) == \frac{\sin\left(\frac{\pi}{2} - z\right)}{\sin(z)} \quad \cot(z) == \frac{\cos(z)}{\cos\left(\frac{\pi}{2} - z\right)}$$

$$\cot(z) == \frac{\csc(z)}{\csc\left(\frac{\pi}{2} - z\right)} \quad \cot(z) == \frac{\sec\left(\frac{\pi}{2} - z\right)}{\sec(z)}.$$

Representations through hyperbolic functions

The cotangent function has representations using the hyperbolic functions:

$$\cot(z) == \frac{\sinh\left(\frac{i\pi}{2} - i z\right)}{\sinh(i z)} \quad \cot(z) == \frac{\cosh(i z)}{\cosh\left(\frac{i\pi}{2} - i z\right)} \quad \cot(z) == -i \tanh\left(\frac{\pi i}{2} - i z\right) \quad \cot(z) == i \coth(i z)$$

$$\cot(i z) == -i \coth(z) \quad \cot(z) == \frac{\operatorname{csch}(i z)}{\operatorname{csch}\left(\frac{\pi i}{2} - i z\right)} \quad \cot(z) == \frac{\operatorname{sech}\left(\frac{\pi i}{2} - i z\right)}{\operatorname{sech}(i z)}.$$

Applications

The cotangent function is used throughout mathematics, the exact sciences, and engineering.

Introduction to the Trigonometric Functions in *Mathematica*

Overview

The following shows how the six trigonometric functions are realized in *Mathematica*. Examples of evaluating *Mathematica* functions applied to various numeric and exact expressions that involve the trigonometric functions or return them are shown. These involve numeric and symbolic calculations and plots.

Notations

Mathematica forms of notations

All six trigonometric functions are represented as built-in functions in *Mathematica*. Following *Mathematica*'s general naming convention, the `StandardForm` function names are simply capitalized versions of the traditional mathematics names. Here is a list `trigFunctions` of the six trigonometric functions in `StandardForm`.

```
trigFunctions = {Sin[z], Cos[z], Tan[z], Cot[z], Sec[z], Cos[z]}
```

```
{Sin[z], Cos[z], Tan[z], Cot[z], Sec[z], Cos[z]}
```

Here is a list `trigFunctions` of the six trigonometric functions in `TraditionalForm`.

```
trigFunctions // TraditionalForm
```

{sin(*z*), cos(*z*), tan(*z*), cot(*z*), sec(*z*), cos(*z*)}

Additional forms of notations

Mathematica also knows the most popular forms of notations for the trigonometric functions that are used in other programming languages. Here are three examples: `CForm`, `TeXForm`, and `FortranForm`.

```
trigFunctions /. {z → 2 π z} // (CForm /@ #) &
```

```
{Sin(2*Pi*z), Cos(2*Pi*z), Tan(2*Pi*z),
 Cot(2*Pi*z), Sec(2*Pi*z), Cos(2*Pi*z)}
```

```
trigFunctions /. {z → 2 π z} // (TeXForm /@ #) &
```

```
{\sin(2\, \pi \, z), \cos(2\, \pi \, z), \tan(2\, \pi \, z), \cot
  (2\, \pi \, z), \sec(2\, \pi \, z), \cos(2\, \pi \, z)}
```

```
trigFunctions /. {z → 2 π z} // (FortranForm /@ #) &
```

```
{Sin(2*Pi*z), Cos(2*Pi*z), Tan(2*Pi*z),
 Cot(2*Pi*z), Sec(2*Pi*z), Cos(2*Pi*z)}
```

Automatic evaluations and transformations

Evaluation for exact, machine-number, and high-precision arguments

For a simple exact argument, *Mathematica* returns exact results. For instance, for the argument $\pi/6$, the `Sin` function evaluates to $1/2$.

$$\text{Sin}\left[\frac{\pi}{6}\right]$$

$$\frac{1}{2}$$

$$\{\text{Sin}[z], \text{Cos}[z], \text{Tan}[z], \text{Cot}[z], \text{Csc}[z], \text{Sec}[z]\} \text{ /. } z \to \frac{\pi}{6}$$

$$\left\{\frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{1}{\sqrt{3}}, \sqrt{3}, 2, \frac{2}{\sqrt{3}}\right\}$$

For a generic machine-number argument (a numerical argument with a decimal point and not too many digits), a machine number is returned.

```
Cos[3.]
```

```
-0.989992
```

```
{Sin[z], Cos[z], Tan[z], Cot[z], Csc[z], Sec[z]} /. z → 2.
```

```
{0.909297, -0.416147, -2.18504, -0.457658, 1.09975, -2.403}
```

The next inputs calculate 100-digit approximations of the six trigonometric functions at $z = 1$.

```
N[Tan[1], 40]
```

```
1.557407724654902230506974807458360173087
```

```
Cot[1] // N[#, 50] &
```

```
0.64209261593433070300641998659426562023027811391817
```

```
N[{Sin[z], Cos[z], Tan[z], Cot[z], Csc[z], Sec[z]} /. z → 1, 100]
```

```
{0.841470984807896506652502321630298999622563060798371065672751709991910404391239668\
  9486397435430526959,
 0.540302305868139717400936607442976603732310420617922227670097255381100394774471764\
  5179518560871830893,
 1.557407724654902230506974807458360173087250772381520038383946605698861397151727289\
  555099965202242984,
 0.642092615934330703006419986594265620230278113918171379101162280426276856839164672\
  1984829197601968047,
 1.188395105778121216261599452374551003527829834097962625265253666359184367357190487\
  913663568030853023,
 1.850815717680925617911753241398650193470396655094009298835158277858815411261596705\
  921841413287306671}
```

Within a second, it is possible to calculate thousands of digits for the trigonometric functions. The next input calculates 10000 digits for sin(1), cos(1), tan(1), cot(1), sec(1), and csc(1) and analyzes the frequency of the occurrence of the digit k in the resulting decimal number.

```
Map[Function[w, {First[#], Length[#]} & /@ Split[Sort[First[RealDigits[w]]]]],
 N[{Sin[z], Cos[z], Tan[z], Cot[z], Csc[z], Sec[z]} /. z → 1, 10000]]
```

```
{{{0, 983}, {1, 1069}, {2, 1019}, {3, 983}, {4, 972}, {5, 994},
  {6, 994}, {7, 988}, {8, 988}, {9, 1010}}, {{0, 998}, {1, 1034}, {2, 982},
  {3, 1015}, {4, 1013}, {5, 963}, {6, 1034}, {7, 966}, {8, 991}, {9, 1004}},
 {{0, 1024}, {1, 1025}, {2, 1000}, {3, 969}, {4, 1026}, {5, 944}, {6, 999},
  {7, 1001}, {8, 1008}, {9, 1004}}, {{0, 1006}, {1, 1030}, {2, 986},
  {3, 954}, {4, 1003}, {5, 1034}, {6, 999}, {7, 998}, {8, 1009}, {9, 981}},
 {{0, 1031}, {1, 976}, {2, 1045}, {3, 917}, {4, 1001}, {5, 996}, {6, 964},
  {7, 1012}, {8, 982}, {9, 1076}}, {{0, 978}, {1, 1034}, {2, 1016},
  {3, 974}, {4, 987}, {5, 1067}, {6, 943}, {7, 1006}, {8, 1027}, {9, 968}}}
```

Here are 50-digit approximations to the six trigonometric functions at the complex argument $z = 3 + 5\,i$.

```
N[Csc[3 + 5 i], 100]
```

```
0.0019019704237010899966700172963208058404592525121712743108017196953928700340468202\
  96847410109982878354 +
 0.0133415913979966787218373224664731943901323471572531909720754374624858144315701 18\
   67262664488519840339 ⅈ
```

```
N[{Sin[z], Cos[z], Tan[z], Cot[z], Csc[z], Sec[z]} /. z → 3 + 5 ⅈ, 50]
```

```
{10.472508533940392276673322536853503271126419950388 -
  73.460621695673676366791192505081750407213922814475 ⅈ,
 -73.467292212645262467746454594833950830814859165299 -
  10.471557674805574377394464224329537808548330651734 ⅈ,
 -0.000025368676207676032417806136707426288195560702602478 +
  0.99991282015135380828209263013972954140566020462086 ⅈ,
 -0.000025373100044545977383763346789469656754050037355986 -
  1.0000871868058967743285316881045218577131612831891 ⅈ,
 0.0019019704237010899966700172963208058404592525121713 +
  0.013341591397996678721837322466473194390132347157253 ⅈ,
 -0.013340476530549737487361100811100839468470481725038 +
  0.0019014661516951513089519270013254277867588978133499 ⅈ}
```

Mathematica always evaluates mathematical functions with machine precision, if the arguments are machine numbers. In this case, only six digits after the decimal point are shown in the results. The remaining digits are suppressed, but can be displayed using the function `InputForm`.

```
{Sin[2.], N[Sin[2]], N[Sin[2], 16], N[Sin[2], 5], N[Sin[2], 20]}
```

```
{0.909297, 0.909297, 0.909297, 0.909297, 0.90929742682568169540}
```

```
% // InputForm
```

```
{0.9092974268256817, 0.9092974268256817, 0.9092974268256817, 0.9092974268256817,
 0.909297426825681695396019865911745`20}
```

```
Precision[%%]
```

```
16
```

Simplification of the argument

Mathematica uses symmetries and periodicities of all the trigonometric functions to simplify expressions. Here are some examples.

```
Sin[-z]
```

```
-Sin[z]
```

```
Sin[z + π]
```

```
-Sin[z]
```

```
Sin[z + 2 π]
```

```
Sin[z]
```

```
Sin[z + 34 π]
```

```
Sin[z]
```

```
{Sin[-z], Cos[-z], Tan[-z], Cot[-z], Csc[-z], Sec[-z]}
```

```
{-Sin[z], Cos[z], -Tan[z], -Cot[z], -Csc[z], Sec[z]}
```

```
{Sin[z + π], Cos[z + π], Tan[z + π], Cot[z + π], Csc[z + π], Sec[z + π]}
```

```
{-Sin[z], -Cos[z], Tan[z], Cot[z], -Csc[z], -Sec[z]}
```

```
{Sin[z + 2 π], Cos[z + 2 π], Tan[z + 2 π], Cot[z + 2 π], Csc[z + 2 π], Sec[z + 2 π]}
```

```
{Sin[z], Cos[z], Tan[z], Cot[z], Csc[z], Sec[z]}
```

```
{Sin[z + 342 π], Cos[z + 342 π], Tan[z + 342 π], Cot[z + 342 π], Csc[z + 342 π], Sec[z + 342 π]}
```

```
{Sin[z], Cos[z], Tan[z], Cot[z], Csc[z], Sec[z]}
```

Mathematica automatically simplifies the composition of the direct and the inverse trigonometric functions into the argument.

```
{Sin[ArcSin[z]], Cos[ArcCos[z]], Tan[ArcTan[z]],
 Cot[ArcCot[z]], Csc[ArcCsc[z]], Sec[ArcSec[z]]}
```

```
{z, z, z, z, z, z}
```

Mathematica also automatically simplifies the composition of the direct and any of the inverse trigonometric functions into algebraic functions of the argument.

```
{Sin[ArcSin[z]], Sin[ArcCos[z]], Sin[ArcTan[z]],
 Sin[ArcCot[z]], Sin[ArcCsc[z]], Sin[ArcSec[z]]}
```

$$\left\{z, \sqrt{1-z^2}, \frac{z}{\sqrt{1+z^2}}, \frac{1}{\sqrt{1+\frac{1}{z^2}}\, z}, \frac{1}{z}, \sqrt{1-\frac{1}{z^2}}\right\}$$

```
{Cos[ArcSin[z]], Cos[ArcCos[z]], Cos[ArcTan[z]],
 Cos[ArcCot[z]], Cos[ArcCsc[z]], Cos[ArcSec[z]]}
```

$$\left\{\sqrt{1-z^2}, z, \frac{1}{\sqrt{1+z^2}}, \frac{1}{\sqrt{1+\frac{1}{z^2}}}, \sqrt{1-\frac{1}{z^2}}, \frac{1}{z}\right\}$$

```
{Tan[ArcSin[z]], Tan[ArcCos[z]], Tan[ArcTan[z]],
 Tan[ArcCot[z]], Tan[ArcCsc[z]], Tan[ArcSec[z]]}
```

$$\left\{\frac{z}{\sqrt{1-z^2}}, \frac{\sqrt{1-z^2}}{z}, z, \frac{1}{z}, \frac{1}{\sqrt{1-\frac{1}{z^2}}\, z}, \sqrt{1-\frac{1}{z^2}}\, z\right\}$$

```
{Cot[ArcSin[z]], Cot[ArcCos[z]], Cot[ArcTan[z]],
 Cot[ArcCot[z]], Cot[ArcCsc[z]], Cot[ArcSec[z]]}
```

$$\left\{\frac{\sqrt{1-z^2}}{z},\ \frac{z}{\sqrt{1-z^2}},\ \frac{1}{z},\ z,\ \sqrt{1-\frac{1}{z^2}}\ z,\ \frac{1}{\sqrt{1-\frac{1}{z^2}}\ z}\right\}$$

```
{Csc[ArcSin[z]], Csc[ArcCos[z]], Csc[ArcTan[z]],
 Csc[ArcCot[z]], Csc[ArcCsc[z]], Csc[ArcSec[z]]}
```

$$\left\{\frac{1}{z},\ \frac{1}{\sqrt{1-z^2}},\ \frac{\sqrt{1+z^2}}{z},\ \sqrt{1+\frac{1}{z^2}}\ z,\ z,\ \frac{1}{\sqrt{1-\frac{1}{z^2}}}\right\}$$

```
{Sec[ArcSin[z]], Sec[ArcCos[z]], Sec[ArcTan[z]],
 Sec[ArcCot[z]], Sec[ArcCsc[z]], Sec[ArcSec[z]]}
```

$$\left\{\frac{1}{\sqrt{1-z^2}},\ \frac{1}{z},\ \sqrt{1+z^2},\ \sqrt{1+\frac{1}{z^2}},\ \frac{1}{\sqrt{1-\frac{1}{z^2}}},\ z\right\}$$

In cases where the argument has the structure $\pi k/2+z$ or $\pi k/2-z$, and $\pi k/2+i z$ or $\pi k/2-i z$ with integer k, trigonometric functions can be automatically transformed into other trigonometric or hyperbolic functions. Here are some examples.

$$\mathrm{Tan}\left[\frac{\pi}{2}-z\right]$$

```
Cot[z]
```

```
Csc[ⅈ z]
```

```
-ⅈ Csch[z]
```

$$\left\{\mathrm{Sin}\left[\frac{\pi}{2}-z\right],\ \mathrm{Cos}\left[\frac{\pi}{2}-z\right],\ \mathrm{Tan}\left[\frac{\pi}{2}-z\right],\ \mathrm{Cot}\left[\frac{\pi}{2}-z\right],\ \mathrm{Csc}\left[\frac{\pi}{2}-z\right],\ \mathrm{Sec}\left[\frac{\pi}{2}-z\right]\right\}$$

```
{Cos[z], Sin[z], Cot[z], Tan[z], Sec[z], Csc[z]}
```

```
{Sin[ⅈ z], Cos[ⅈ z], Tan[ⅈ z], Cot[ⅈ z], Csc[ⅈ z], Sec[ⅈ z]}
```

```
{ⅈ Sinh[z], Cosh[z], ⅈ Tanh[z], -ⅈ Coth[z], -ⅈ Csch[z], Sech[z]}
```

Simplification of simple expressions containing trigonometric functions

Sometimes simple arithmetic operations containing trigonometric functions can automatically produce other trigonometric functions.

```
1 / Sec[z]
```

```
Cos[z]
```

```
{1/Sin[z], 1/Cos[z], 1/Tan[z], 1/Cot[z], 1/Csc[z], 1/Sec[z],
 Sin[z]/Cos[z], Cos[z]/Sin[z], Sin[z]/Sin[π/2 - z], Cos[z]/Sin[z]^2}
```

```
{Csc[z], Sec[z], Cot[z], Tan[z], Sin[z], Cos[z], Tan[z], Cot[z], Tan[z], Cot[z] Csc[z]}
```

Trigonometric functions arising as special cases from more general functions

All trigonometric functions can be treated as particular cases of some more advanced special functions. For example, sin(z) and cos(z) are sometimes the results of auto-simplifications from Bessel, Mathieu, Jacobi, hypergeometric, and Meijer functions (for appropriate values of their parameters).

```
BesselJ[1/2, z]
```

$$\frac{\sqrt{\frac{2}{\pi}}\,\mathrm{Sin}[z]}{\sqrt{z}}$$

```
MathieuC[1, 0, z]
```

```
Cos[z]
```

```
JacobiSC[z, 0]
```

```
Tan[z]
```

In[14]:=
```
{BesselJ[1/2, z], MathieuS[1, 0, z], JacobiSN[z, 0],
 HypergeometricPFQ[{}, {3/2}, -z^2/4], MeijerG[{{}, {}}, {{1/2}, {0}}, z^2/4]}
```

Out[14]=
$$\left\{\frac{\sqrt{\frac{2}{\pi}}\,\mathrm{Sin}[z]}{\sqrt{z}},\ \mathrm{Sin}[z],\ \mathrm{Sin}[z],\ \frac{\mathrm{Sin}\left[\sqrt{z^2}\right]}{\sqrt{z^2}},\ \frac{\sqrt{z^2}\,\mathrm{Sin}[z]}{\sqrt{\pi}\,z}\right\}$$

In[15]:=
```
{BesselJ[-1/2, z], MathieuC[1, 0, z], JacobiCD[z, 0],
 Hypergeometric0F1[1/2, -z^2/4], MeijerG[{{}, {}}, {{0}, {1/2}}, z^2/4]}
```

Out[15]=
$$\left\{\frac{\sqrt{\frac{2}{\pi}}\,\mathrm{Cos}[z]}{\sqrt{z}},\ \mathrm{Cos}[z],\ \mathrm{Cos}[z],\ \mathrm{Cos}\left[\sqrt{z^2}\right],\ \frac{\mathrm{Cos}[z]}{\sqrt{\pi}}\right\}$$

In[16]:=
```
{JacobiSC[z, 0], JacobiCS[z, 0], JacobiDS[z, 0], JacobiDC[z, 0]}
```

Out[16]=
```
{Tan[z], Cot[z], Csc[z], Sec[z]}
```

Equivalence transformations carried out by specialized *Mathematica* functions

General remarks

Almost everybody prefers using $\sin(z)/2$ instead of $\cos(\pi/2 - z)\sin(\pi/6)$. *Mathematica* automatically transforms the second expression into the first one. The automatic application of transformation rules to mathematical expressions can give overly complicated results. Compact expressions like $\sin(2z)\sin(\pi/16)$ should not be automatically expanded into the more complicated expression $\sin(z)\cos(z)\left(2-\left(2+2^{1/2}\right)^{1/2}\right)^{1/2}$. *Mathematica* has special commands that produce these types of expansions. Some of them are demonstrated in the next section.

TrigExpand

The function `TrigExpand` expands out trigonometric and hyperbolic functions. In more detail, it splits up sums and integer multiples that appear in the arguments of trigonometric and hyperbolic functions, and then expands out the products of the trigonometric and hyperbolic functions into sums of powers, using the trigonometric and hyperbolic identities where possible. Here are some examples.

```
TrigExpand[Sin[x - y]]
```

```
Cos[y] Sin[x] - Cos[x] Sin[y]
```

```
Cos[4 z] // TrigExpand
```

$$\mathrm{Cos}[z]^4 - 6\,\mathrm{Cos}[z]^2\,\mathrm{Sin}[z]^2 + \mathrm{Sin}[z]^4$$

```
TrigExpand[{{Sin[x + y], Sin[3 z]},
            {Cos[x + y], Cos[3 z]},
            {Tan[x + y], Tan[3 z]},
            {Cot[x + y], Cot[3 z]},
            {Csc[x + y], Csc[3 z]},
            {Sec[x + y], Sec[3 z]}}]
```

$$\Big\{\{\mathrm{Cos}[y]\,\mathrm{Sin}[x] + \mathrm{Cos}[x]\,\mathrm{Sin}[y],\ 3\,\mathrm{Cos}[z]^2\,\mathrm{Sin}[z] - \mathrm{Sin}[z]^3\},$$
$$\{\mathrm{Cos}[x]\,\mathrm{Cos}[y] - \mathrm{Sin}[x]\,\mathrm{Sin}[y],\ \mathrm{Cos}[z]^3 - 3\,\mathrm{Cos}[z]\,\mathrm{Sin}[z]^2\},$$
$$\Big\{\frac{\mathrm{Cos}[y]\,\mathrm{Sin}[x]}{\mathrm{Cos}[x]\,\mathrm{Cos}[y] - \mathrm{Sin}[x]\,\mathrm{Sin}[y]} + \frac{\mathrm{Cos}[x]\,\mathrm{Sin}[y]}{\mathrm{Cos}[x]\,\mathrm{Cos}[y] - \mathrm{Sin}[x]\,\mathrm{Sin}[y]},$$
$$\frac{3\,\mathrm{Cos}[z]^2\,\mathrm{Sin}[z]}{\mathrm{Cos}[z]^3 - 3\,\mathrm{Cos}[z]\,\mathrm{Sin}[z]^2} - \frac{\mathrm{Sin}[z]^3}{\mathrm{Cos}[z]^3 - 3\,\mathrm{Cos}[z]\,\mathrm{Sin}[z]^2}\Big\},$$
$$\Big\{\frac{\mathrm{Cos}[x]\,\mathrm{Cos}[y]}{\mathrm{Cos}[y]\,\mathrm{Sin}[x] + \mathrm{Cos}[x]\,\mathrm{Sin}[y]} - \frac{\mathrm{Sin}[x]\,\mathrm{Sin}[y]}{\mathrm{Cos}[y]\,\mathrm{Sin}[x] + \mathrm{Cos}[x]\,\mathrm{Sin}[y]},$$
$$\frac{\mathrm{Cos}[z]^3}{3\,\mathrm{Cos}[z]^2\,\mathrm{Sin}[z] - \mathrm{Sin}[z]^3} - \frac{3\,\mathrm{Cos}[z]\,\mathrm{Sin}[z]^2}{3\,\mathrm{Cos}[z]^2\,\mathrm{Sin}[z] - \mathrm{Sin}[z]^3}\Big\},$$
$$\Big\{\frac{1}{\mathrm{Cos}[y]\,\mathrm{Sin}[x] + \mathrm{Cos}[x]\,\mathrm{Sin}[y]},\ \frac{1}{3\,\mathrm{Cos}[z]^2\,\mathrm{Sin}[z] - \mathrm{Sin}[z]^3}\Big\},$$
$$\Big\{\frac{1}{\mathrm{Cos}[x]\,\mathrm{Cos}[y] - \mathrm{Sin}[x]\,\mathrm{Sin}[y]},\ \frac{1}{\mathrm{Cos}[z]^3 - 3\,\mathrm{Cos}[z]\,\mathrm{Sin}[z]^2}\Big\}\Big\}$$

```
TableForm[(# == TrigExpand[#]) & /@
  Flatten[{{Sin[x + y], Sin[3 z]}, {Cos[x + y], Cos[3 z]}, {Tan[x + y], Tan[3 z]},
    {Cot[x + y], Cot[3 z]}, {Csc[x + y], Csc[3 z]}, {Sec[x + y], Sec[3 z]}}]]
```

```
Sin[x + y] == Cos[y] Sin[x] + Cos[x] Sin[y]
Sin[3 z] == 3 Cos[z]^2 Sin[z] - Sin[z]^3
Cos[x + y] == Cos[x] Cos[y] - Sin[x] Sin[y]
Cos[3 z] == Cos[z]^3 - 3 Cos[z] Sin[z]^2
Tan[x + y] == Cos[y] Sin[x]/(Cos[x] Cos[y] - Sin[x] Sin[y]) + Cos[x] Sin[y]/(Cos[x] Cos[y] - Sin[x] Sin[y])
Tan[3 z] == 3 Cos[z]^2 Sin[z]/(Cos[z]^3 - 3 Cos[z] Sin[z]^2) - Sin[z]^3/(Cos[z]^3 - 3 Cos[z] Sin[z]^2)
Cot[x + y] == Cos[x] Cos[y]/(Cos[y] Sin[x] + Cos[x] Sin[y]) - Sin[x] Sin[y]/(Cos[y] Sin[x] + Cos[x] Sin[y])
Cot[3 z] == Cos[z]^3/(3 Cos[z]^2 Sin[z] - Sin[z]^3) - 3 Cos[z] Sin[z]^2/(3 Cos[z]^2 Sin[z] - Sin[z]^3)
Csc[x + y] == 1/(Cos[y] Sin[x] + Cos[x] Sin[y])
Csc[3 z] == 1/(3 Cos[z]^2 Sin[z] - Sin[z]^3)
Sec[x + y] == 1/(Cos[x] Cos[y] - Sin[x] Sin[y])
Sec[3 z] == 1/(Cos[z]^3 - 3 Cos[z] Sin[z]^2)
```

TrigFactor

The function `TrigFactor` factors trigonometric and hyperbolic functions. In more detail, it splits up sums and integer multiples that appear in the arguments of trigonometric and hyperbolic functions, and then factors the resulting polynomials in the trigonometric and hyperbolic functions, using the corresponding identities where possible. Here are some examples.

```
TrigFactor[Sin[x] + Cos[y]]
```

```
(Cos[x/2 - y/2] + Sin[x/2 - y/2]) (Cos[x/2 + y/2] + Sin[x/2 + y/2])
```

```
Tan[x] - Cot[y] // TrigFactor
```

```
-Cos[x + y] Csc[y] Sec[x]
```

```
TrigFactor[{Sin[x] + Sin[y],
            Cos[x] + Cos[y],
            Tan[x] + Tan[y],
            Cot[x] + Cot[y],
            Csc[x] + Csc[y],
            Sec[x] + Sec[y]}]
```

```
{2 Cos[x/2 - y/2] Sin[x/2 + y/2], 2 Cos[x/2 - y/2] Cos[x/2 + y/2], Sec[x] Sec[y] Sin[x + y],
 Csc[x] Csc[y] Sin[x + y], 1/2 Cos[x/2 - y/2] Csc[x/2] Csc[y/2] Sec[x/2] Sec[y/2] Sin[x/2 + y/2],
 2 Cos[x/2 - y/2] Cos[x/2 + y/2]/((Cos[x/2] - Sin[x/2]) (Cos[x/2] + Sin[x/2]) (Cos[y/2] - Sin[y/2]) (Cos[y/2] + Sin[y/2]))}
```

TrigReduce

The function `TrigReduce` rewrites products and powers of trigonometric and hyperbolic functions in terms of those functions with combined arguments. In more detail, it typically yields a linear expression involving trigonometric and hyperbolic functions with more complicated arguments. `TrigReduce` is approximately inverse to `TrigExpand` and `TrigFactor`. Here are some examples.

```
TrigReduce[Sin[z]^3]
```

$$\frac{1}{4}\left(3\,\mathrm{Sin}[z]-\mathrm{Sin}[3\,z]\right)$$

```
Sin[x] Cos[y] // TrigReduce
```

$$\frac{1}{2}\left(\mathrm{Sin}[x-y]+\mathrm{Sin}[x+y]\right)$$

```
TrigReduce[{Sin[z]^2, Cos[z]^2, Tan[z]^2, Cot[z]^2, Csc[z]^2, Sec[z]^2}]
```

$$\left\{\frac{1}{2}\left(1-\mathrm{Cos}[2\,z]\right),\ \frac{1}{2}\left(1+\mathrm{Cos}[2\,z]\right),\ \frac{1-\mathrm{Cos}[2\,z]}{1+\mathrm{Cos}[2\,z]},\ \frac{-1-\mathrm{Cos}[2\,z]}{-1+\mathrm{Cos}[2\,z]},\ -\frac{2}{-1+\mathrm{Cos}[2\,z]},\ \frac{2}{1+\mathrm{Cos}[2\,z]}\right\}$$

```
TrigReduce[TrigExpand[{{Sin[x + y], Sin[3 z], Sin[x] Sin[y]},
                        {Cos[x + y], Cos[3 z], Cos[x] Cos[y]},
                        {Tan[x + y], Tan[3 z], Tan[x] Tan[y]},
                        {Cot[x + y], Cot[3 z], Cot[x] Cot[y]},
                        {Csc[x + y], Csc[3 z], Csc[x] Csc[y]},
                        {Sec[x + y], Sec[3 z], Sec[x] Sec[y]}}]]
```

$$\Big\{\left\{\mathrm{Sin}[x+y],\ \mathrm{Sin}[3\,z],\ \frac{1}{2}\left(\mathrm{Cos}[x-y]-\mathrm{Cos}[x+y]\right)\right\},$$
$$\left\{\mathrm{Cos}[x+y],\ \mathrm{Cos}[3\,z],\ \frac{1}{2}\left(\mathrm{Cos}[x-y]+\mathrm{Cos}[x+y]\right)\right\},$$
$$\left\{\mathrm{Tan}[x+y],\ \mathrm{Tan}[3\,z],\ \frac{\mathrm{Cos}[x-y]-\mathrm{Cos}[x+y]}{\mathrm{Cos}[x-y]+\mathrm{Cos}[x+y]}\right\},$$
$$\left\{\mathrm{Cot}[x+y],\ \mathrm{Cot}[3\,z],\ \frac{\mathrm{Cos}[x-y]+\mathrm{Cos}[x+y]}{\mathrm{Cos}[x-y]-\mathrm{Cos}[x+y]}\right\},$$
$$\left\{\mathrm{Csc}[x+y],\ \mathrm{Csc}[3\,z],\ \frac{2}{\mathrm{Cos}[x-y]-\mathrm{Cos}[x+y]}\right\},$$
$$\left\{\mathrm{Sec}[x+y],\ \mathrm{Sec}[3\,z],\ \frac{2}{\mathrm{Cos}[x-y]+\mathrm{Cos}[x+y]}\right\}\Big\}$$

```
TrigReduce[TrigFactor[{Sin[x] + Sin[y], Cos[x] + Cos[y],
   Tan[x] + Tan[y], Cot[x] + Cot[y], Csc[x] + Csc[y], Sec[x] + Sec[y]}]]
```

$$\Big\{\mathrm{Sin}[x]+\mathrm{Sin}[y],\ \mathrm{Cos}[x]+\mathrm{Cos}[y],\ \frac{2\,\mathrm{Sin}[x+y]}{\mathrm{Cos}[x-y]+\mathrm{Cos}[x+y]},$$
$$\frac{2\,\mathrm{Sin}[x+y]}{\mathrm{Cos}[x-y]-\mathrm{Cos}[x+y]},\ \frac{2\left(\mathrm{Sin}[x]+\mathrm{Sin}[y]\right)}{\mathrm{Cos}[x-y]-\mathrm{Cos}[x+y]},\ \frac{2\left(\mathrm{Cos}[x]+\mathrm{Cos}[y]\right)}{\mathrm{Cos}[x-y]+\mathrm{Cos}[x+y]}\Big\}$$

TrigToExp

The function `TrigToExp` converts direct and inverse trigonometric and hyperbolic functions to exponential or logarithmic functions. It tries, where possible, to give results that do not involve explicit complex numbers. Here are some examples.

```
TrigToExp[Sin[2 z]]
```

$$\frac{1}{2} i e^{-2 i z} - \frac{1}{2} i e^{2 i z}$$

```
Sin[z] Tan[2 z] // TrigToExp
```

$$-\frac{\left(e^{-i z} - e^{i z}\right)\left(e^{-2 i z} - e^{2 i z}\right)}{2\left(e^{-2 i z} + e^{2 i z}\right)}$$

```
TrigToExp[{Sin[z], Cos[z], Tan[z], Cot[z], Csc[z], Sec[z]}]
```

$$\left\{\frac{1}{2} i e^{-i z} - \frac{1}{2} i e^{i z}, \frac{e^{-i z}}{2} + \frac{e^{i z}}{2}, \frac{i\left(e^{-i z} - e^{i z}\right)}{e^{-i z} + e^{i z}}, -\frac{i\left(e^{-i z} + e^{i z}\right)}{e^{-i z} - e^{i z}}, -\frac{2 i}{e^{-i z} - e^{i z}}, \frac{2}{e^{-i z} + e^{i z}}\right\}$$

ExpToTrig

The function `ExpToTrig` converts exponentials to trigonometric or hyperbolic functions. It tries, where possible, to give results that do not involve explicit complex numbers. It is approximately inverse to `TrigToExp`. Here are some examples.

$$\text{ExpToTrig}\left[e^{i x \beta}\right]$$

$$\text{Cos}[x \beta] + i \, \text{Sin}[x \beta]$$

$$\frac{e^{i x \alpha} - e^{i x \beta}}{e^{i x \gamma} + e^{i x \delta}} \; // \; \text{ExpToTrig}$$

$$\frac{\text{Cos}[x \alpha] - \text{Cos}[x \beta] + i \, \text{Sin}[x \alpha] - i \, \text{Sin}[x \beta]}{\text{Cos}[x \gamma] + \text{Cos}[x \delta] + i \, \text{Sin}[x \gamma] + i \, \text{Sin}[x \delta]}$$

```
ExpToTrig[TrigToExp[{Sin[z], Cos[z], Tan[z], Cot[z], Csc[z], Sec[z]}]]
```

```
{Sin[z], Cos[z], Tan[z], Cot[z], Csc[z], Sec[z]}
```

$$\text{ExpToTrig}\left[\left\{\alpha e^{-i x \beta} + \alpha e^{i x \beta}, \alpha e^{-i x \beta} + \gamma e^{i x \beta}\right\}\right]$$

$$\{2 \alpha \, \text{Cos}[x \beta], \alpha \, \text{Cos}[x \beta] + \gamma \, \text{Cos}[x \beta] - i \alpha \, \text{Sin}[x \beta] + i \gamma \, \text{Sin}[x \beta]\}$$

ComplexExpand

The function `ComplexExpand` expands expressions assuming that all the occurring variables are real. The value option `TargetFunctions` is a list of functions from the set `{Re, Im, Abs, Arg, Conjugate, Sign}`. `ComplexExpand` tries to give results in terms of the specified functions. Here are some examples

$$\text{ComplexExpand}[\text{Sin}[x + i y] \, \text{Cos}[x - i y]]$$

$$\mathrm{Cos}[x]\,\mathrm{Cosh}[y]^2\,\mathrm{Sin}[x] - \mathrm{Cos}[x]\,\mathrm{Sin}[x]\,\mathrm{Sinh}[y]^2 + i\left(\mathrm{Cos}[x]^2\,\mathrm{Cosh}[y]\,\mathrm{Sinh}[y] + \mathrm{Cosh}[y]\,\mathrm{Sin}[x]^2\,\mathrm{Sinh}[y]\right)$$

```
Csc[x + ⅈ y] Sec[x - ⅈ y] // ComplexExpand
```

$$-\frac{4\,\mathrm{Cos}[x]\,\mathrm{Cosh}[y]^2\,\mathrm{Sin}[x]}{(\mathrm{Cos}[2x]-\mathrm{Cosh}[2y])\,(\mathrm{Cos}[2x]+\mathrm{Cosh}[2y])} + \frac{4\,\mathrm{Cos}[x]\,\mathrm{Sin}[x]\,\mathrm{Sinh}[y]^2}{(\mathrm{Cos}[2x]-\mathrm{Cosh}[2y])\,(\mathrm{Cos}[2x]+\mathrm{Cosh}[2y])} + i\left(\frac{4\,\mathrm{Cos}[x]^2\,\mathrm{Cosh}[y]\,\mathrm{Sinh}[y]}{(\mathrm{Cos}[2x]-\mathrm{Cosh}[2y])\,(\mathrm{Cos}[2x]+\mathrm{Cosh}[2y])} + \frac{4\,\mathrm{Cosh}[y]\,\mathrm{Sin}[x]^2\,\mathrm{Sinh}[y]}{(\mathrm{Cos}[2x]-\mathrm{Cosh}[2y])\,(\mathrm{Cos}[2x]+\mathrm{Cosh}[2y])}\right)$$

In[17]:=
```
li1 = {Sin[x + ⅈ y], Cos[x + ⅈ y], Tan[x + ⅈ y], Cot[x + ⅈ y], Csc[x + ⅈ y], Sec[x + ⅈ y]}
```

Out[17]=
```
{Sin[x + ⅈ y], Cos[x + ⅈ y], Tan[x + ⅈ y], Cot[x + ⅈ y], Csc[x + ⅈ y], Sec[x + ⅈ y]}
```

In[18]:=
```
ComplexExpand[li1]
```

Out[18]=

$$\left\{\mathrm{Cosh}[y]\,\mathrm{Sin}[x] + i\,\mathrm{Cos}[x]\,\mathrm{Sinh}[y],\ \mathrm{Cos}[x]\,\mathrm{Cosh}[y] - i\,\mathrm{Sin}[x]\,\mathrm{Sinh}[y],\ \frac{\mathrm{Sin}[2x]}{\mathrm{Cos}[2x]+\mathrm{Cosh}[2y]} + \frac{i\,\mathrm{Sinh}[2y]}{\mathrm{Cos}[2x]+\mathrm{Cosh}[2y]},\ -\frac{\mathrm{Sin}[2x]}{\mathrm{Cos}[2x]-\mathrm{Cosh}[2y]} + \frac{i\,\mathrm{Sinh}[2y]}{\mathrm{Cos}[2x]-\mathrm{Cosh}[2y]},\ -\frac{2\,\mathrm{Cosh}[y]\,\mathrm{Sin}[x]}{\mathrm{Cos}[2x]-\mathrm{Cosh}[2y]} + \frac{2\,i\,\mathrm{Cos}[x]\,\mathrm{Sinh}[y]}{\mathrm{Cos}[2x]-\mathrm{Cosh}[2y]},\ \frac{2\,\mathrm{Cos}[x]\,\mathrm{Cosh}[y]}{\mathrm{Cos}[2x]+\mathrm{Cosh}[2y]} + \frac{2\,i\,\mathrm{Sin}[x]\,\mathrm{Sinh}[y]}{\mathrm{Cos}[2x]+\mathrm{Cosh}[2y]}\right\}$$

In[19]:=
```
ComplexExpand[Re[#] & /@ li1, TargetFunctions → {Re, Im}]
```

Out[19]=

$$\left\{\mathrm{Cosh}[y]\,\mathrm{Sin}[x],\ \mathrm{Cos}[x]\,\mathrm{Cosh}[y],\ \frac{\mathrm{Sin}[2x]}{\mathrm{Cos}[2x]+\mathrm{Cosh}[2y]},\ -\frac{\mathrm{Sin}[2x]}{\mathrm{Cos}[2x]-\mathrm{Cosh}[2y]},\ -\frac{2\,\mathrm{Cosh}[y]\,\mathrm{Sin}[x]}{\mathrm{Cos}[2x]-\mathrm{Cosh}[2y]},\ \frac{2\,\mathrm{Cos}[x]\,\mathrm{Cosh}[y]}{\mathrm{Cos}[2x]+\mathrm{Cosh}[2y]}\right\}$$

In[20]:=
```
ComplexExpand[Im[#] & /@ li1, TargetFunctions → {Re, Im}]
```

Out[20]=

$$\left\{\mathrm{Cos}[x]\,\mathrm{Sinh}[y],\ -\mathrm{Sin}[x]\,\mathrm{Sinh}[y],\ \frac{\mathrm{Sinh}[2y]}{\mathrm{Cos}[2x]+\mathrm{Cosh}[2y]},\ \frac{\mathrm{Sinh}[2y]}{\mathrm{Cos}[2x]-\mathrm{Cosh}[2y]},\ \frac{2\,\mathrm{Cos}[x]\,\mathrm{Sinh}[y]}{\mathrm{Cos}[2x]-\mathrm{Cosh}[2y]},\ \frac{2\,\mathrm{Sin}[x]\,\mathrm{Sinh}[y]}{\mathrm{Cos}[2x]+\mathrm{Cosh}[2y]}\right\}$$

In[21]:=
```
ComplexExpand[Abs[#] & /@ li1, TargetFunctions → {Re, Im}]
```

Out[21]= $\left\{\sqrt{\text{Cosh}[y]^2 \text{Sin}[x]^2+\text{Cos}[x]^2 \text{Sinh}[y]^2}, \sqrt{\text{Cos}[x]^2 \text{Cosh}[y]^2+\text{Sin}[x]^2 \text{Sinh}[y]^2},\right.$
$\sqrt{\frac{\text{Sin}[2 x]^2}{(\text{Cos}[2 x]+\text{Cosh}[2 y])^2}+\frac{\text{Sinh}[2 y]^2}{(\text{Cos}[2 x]+\text{Cosh}[2 y])^2}},$
$\sqrt{\frac{\text{Sin}[2 x]^2}{(\text{Cos}[2 x]-\text{Cosh}[2 y])^2}+\frac{\text{Sinh}[2 y]^2}{(\text{Cos}[2 x]-\text{Cosh}[2 y])^2}},$
$\sqrt{\frac{4 \text{Cosh}[y]^2 \text{Sin}[x]^2}{(\text{Cos}[2 x]-\text{Cosh}[2 y])^2}+\frac{4 \text{Cos}[x]^2 \text{Sinh}[y]^2}{(\text{Cos}[2 x]-\text{Cosh}[2 y])^2}},$
$\left.\sqrt{\frac{4 \text{Cos}[x]^2 \text{Cosh}[y]^2}{(\text{Cos}[2 x]+\text{Cosh}[2 y])^2}+\frac{4 \text{Sin}[x]^2 \text{Sinh}[y]^2}{(\text{Cos}[2 x]+\text{Cosh}[2 y])^2}}\right\}$

In[22]:= **% // Simplify[#, {x, y} ∈ Reals] &**

Out[22]= $\left\{\frac{\sqrt{-\text{Cos}[2 x]+\text{Cosh}[2 y]}}{\sqrt{2}}, \frac{\sqrt{\text{Cos}[2 x]+\text{Cosh}[2 y]}}{\sqrt{2}}, \frac{\sqrt{\text{Sin}[2 x]^2+\text{Sinh}[2 y]^2}}{\text{Cos}[2 x]+\text{Cosh}[2 y]},\right.$
$\left.\sqrt{-\frac{\text{Cos}[2 x]+\text{Cosh}[2 y]}{\text{Cos}[2 x]-\text{Cosh}[2 y]}}, \frac{\sqrt{2}}{\sqrt{-\text{Cos}[2 x]+\text{Cosh}[2 y]}}, \frac{\sqrt{2}}{\sqrt{\text{Cos}[2 x]+\text{Cosh}[2 y]}}\right\}$

In[23]:= **ComplexExpand[Arg[#] & /@ li1, TargetFunctions → {Re, Im}]**

Out[23]= $\{\text{ArcTan}[\text{Cosh}[y] \text{Sin}[x], \text{Cos}[x] \text{Sinh}[y]], \text{ArcTan}[\text{Cos}[x] \text{Cosh}[y], -\text{Sin}[x] \text{Sinh}[y]],$
$\text{ArcTan}\left[\frac{\text{Sin}[2 x]}{\text{Cos}[2 x]+\text{Cosh}[2 y]}, \frac{\text{Sinh}[2 y]}{\text{Cos}[2 x]+\text{Cosh}[2 y]}\right],$
$\text{ArcTan}\left[-\frac{\text{Sin}[2 x]}{\text{Cos}[2 x]-\text{Cosh}[2 y]}, \frac{\text{Sinh}[2 y]}{\text{Cos}[2 x]-\text{Cosh}[2 y]}\right],$
$\text{ArcTan}\left[-\frac{2 \text{Cosh}[y] \text{Sin}[x]}{\text{Cos}[2 x]-\text{Cosh}[2 y]}, \frac{2 \text{Cos}[x] \text{Sinh}[y]}{\text{Cos}[2 x]-\text{Cosh}[2 y]}\right],$
$\left.\text{ArcTan}\left[\frac{2 \text{Cos}[x] \text{Cosh}[y]}{\text{Cos}[2 x]+\text{Cosh}[2 y]}, \frac{2 \text{Sin}[x] \text{Sinh}[y]}{\text{Cos}[2 x]+\text{Cosh}[2 y]}\right]\right\}$

In[24]:= **ComplexExpand[Conjugate[#] & /@ li1, TargetFunctions → {Re, Im}] // Simplify**

Out[24]= $\{\text{Cosh}[y] \text{Sin}[x]-i \text{Cos}[x] \text{Sinh}[y], \text{Cos}[x] \text{Cosh}[y]+i \text{Sin}[x] \text{Sinh}[y],$
$\frac{\text{Sin}[2 x]-i \text{Sinh}[2 y]}{\text{Cos}[2 x]+\text{Cosh}[2 y]}, -\frac{\text{Sin}[2 x]+i \text{Sinh}[2 y]}{\text{Cos}[2 x]-\text{Cosh}[2 y]},$
$\left.\frac{1}{\text{Cosh}[y] \text{Sin}[x]-i \text{Cos}[x] \text{Sinh}[y]}, \frac{1}{\text{Cos}[x] \text{Cosh}[y]+i \text{Sin}[x] \text{Sinh}[y]}\right\}$

Simplify

The function `Simplify` performs a sequence of algebraic transformations on its argument, and returns the simplest form it finds. Here are two examples.

```
Simplify[Sin[2 z] / Sin[z]]
```

```
2 Cos[z]
```

```
Sin[2 z] / Cos[z] // Simplify
```

```
2 Sin[z]
```

Here is a large collection of trigonometric identities. All are written as one large logical conjunction.

```
Simplify[#] & /@ (Cos[z]^2 + Sin[z]^2 == 1 ∧
Sin[z]^2 == (1 - Cos[2 z])/2 ∧ Cos[z]^2 == (1 + Cos[2 z])/2 ∧
Tan[z]^2 == (1 - Cos[2 z])/(1 + Cos[2 z]) ∧ Cot[z]^2 == (1 + Cos[2 z])/(1 - Cos[2 z]) ∧
Sin[2 z] == 2 Sin[z] Cos[z] ∧ Cos[2 z] == Cos[z]^2 - Sin[z]^2 == 2 Cos[z]^2 - 1 ∧
Sin[a + b] == Sin[a] Cos[b] + Cos[a] Sin[b] ∧ Sin[a - b] == Sin[a] Cos[b] - Cos[a] Sin[b] ∧
Cos[a + b] == Cos[a] Cos[b] - Sin[a] Sin[b] ∧ Cos[a - b] == Cos[a] Cos[b] + Sin[a] Sin[b] ∧
Sin[a] + Sin[b] == 2 Sin[(a + b)/2] Cos[(a - b)/2] ∧ Sin[a] - Sin[b] == 2 Cos[(a + b)/2] Sin[(a - b)/2] ∧
Cos[a] + Cos[b] == 2 Cos[(a + b)/2] Cos[(a - b)/2] ∧ Cos[a] - Cos[b] == 2 Sin[(a + b)/2] Sin[(b - a)/2] ∧
Tan[a] + Tan[b] == Sin[a + b]/(Cos[a] Cos[b]) ∧ Tan[a] - Tan[b] == Sin[a - b]/(Cos[a] Cos[b]) ∧
A Sin[z] + B Cos[z] == A Sqrt[1 + B^2/A^2] Sin[z + ArcTan[B/A]] ∧
Sin[a] Sin[b] == (Cos[a - b] - Cos[a + b])/2 ∧
  Cos[a] Cos[b] == (Cos[a - b] + Cos[a + b])/2 ∧ Sin[a] Cos[b] == (Sin[a + b] + Sin[a - b])/2 ∧
Sin[z/2]^2 == (1 - Cos[z])/2 ∧ Cos[z/2]^2 == (1 + Cos[z])/2 ∧
Tan[z/2] == (1 - Cos[z])/Sin[z] == Sin[z]/(1 + Cos[z]) ∧ Cot[z/2] == Sin[z]/(1 - Cos[z]) == (1 + Cos[z])/Sin[z])
```

```
True
```

The function `Simplify` has the `Assumption` option. For example, *Mathematica* knows that $-1 \leq \sin(x) \leq 1$ for all real x, and uses the periodicity of trigonometric functions for the symbolic integer coefficient k of $k\,\pi$.

```
Simplify[Abs[Sin[x]] ≤ 1, x ∈ Reals]
```

```
True
```

```
Abs[Sin[x]] ≤ 1 // Simplify[#, x ∈ Reals] &
```

```
True
```

```
Simplify[{Sin[z + 2 k π], Cos[z + 2 k π], Tan[z + k π],
  Cot[z + k π], Csc[z + 2 k π], Sec[z + 2 k π]}, k ∈ Integers]
```

```
{Sin[z], Cos[z], Tan[z], Cot[z], Csc[z], Sec[z]}
```

```
Simplify[{Sin[z + k π] / Sin[z], Cos[z + k π] / Cos[z], Tan[z + k π] / Tan[z],
  Cot[z + k π] / Cot[z], Csc[z + k π] / Csc[z], Sec[z + k π] / Sec[z]}, k ∈ Integers]
```

$\left\{(-1)^k, (-1)^k, 1, 1, (-1)^k, (-1)^k\right\}$

Mathematica also knows that the composition of inverse and direct trigonometric functions produces the value of the inner argument under the appropriate restriction. Here are some examples.

```
Simplify[{ArcSin[Sin[z]], ArcTan[Tan[z]], ArcCot[Cot[z]], ArcCsc[Csc[z]]},
 -π / 2 < Re[z] < π / 2]
```

```
{z, z, z, z}
```

```
Simplify[{ArcCos[Cos[z]], ArcSec[Sec[z]]}, 0 < Re[z] < π]
```

```
{z, z}
```

FunctionExpand (and Together)

While the trigonometric functions auto-evaluate for simple fractions of π, for more complicated cases they stay as trigonometric functions to avoid the build up of large expressions. Using the function `FunctionExpand`, such expressions can be transformed into explicit radicals.

$\mathbf{Cos}\left[\frac{\pi}{32}\right]$

$\mathrm{Cos}\left[\frac{\pi}{32}\right]$

$\mathbf{FunctionExpand}\left[\mathbf{Cos}\left[\frac{\pi}{32}\right]\right]$

$$\frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}$$

$\mathbf{Cot}\left[\frac{\pi}{24}\right]$ `// FunctionExpand`

$$\frac{\frac{\sqrt{2-\sqrt{2}}}{4}+\frac{1}{4}\sqrt{3\left(2+\sqrt{2}\right)}}{-\frac{1}{4}\sqrt{3\left(2-\sqrt{2}\right)}+\frac{\sqrt{2+\sqrt{2}}}{4}}$$

{Sin[π/16], Cos[π/16], Tan[π/16], Cot[π/16], Csc[π/16], Sec[π/16]}

$$\left\{\mathrm{Sin}\left[\frac{\pi}{16}\right],\ \mathrm{Cos}\left[\frac{\pi}{16}\right],\ \mathrm{Tan}\left[\frac{\pi}{16}\right],\ \mathrm{Cot}\left[\frac{\pi}{16}\right],\ \mathrm{Csc}\left[\frac{\pi}{16}\right],\ \mathrm{Sec}\left[\frac{\pi}{16}\right]\right\}$$

FunctionExpand[%]

$$\left\{\frac{1}{2}\sqrt{2-\sqrt{2+\sqrt{2}}},\ \frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2}}},\ \sqrt{\frac{2-\sqrt{2+\sqrt{2}}}{2+\sqrt{2+\sqrt{2}}}},\ \sqrt{\frac{2+\sqrt{2+\sqrt{2}}}{2-\sqrt{2+\sqrt{2}}}},\ \frac{2}{\sqrt{2-\sqrt{2+\sqrt{2}}}},\ \frac{2}{\sqrt{2+\sqrt{2+\sqrt{2}}}}\right\}$$

{Sin[π/60], Cos[π/60], Tan[π/60], Cot[π/60], Csc[π/60], Sec[π/60]}

$$\left\{\mathrm{Sin}\left[\frac{\pi}{60}\right],\ \mathrm{Cos}\left[\frac{\pi}{60}\right],\ \mathrm{Tan}\left[\frac{\pi}{60}\right],\ \mathrm{Cot}\left[\frac{\pi}{60}\right],\ \mathrm{Csc}\left[\frac{\pi}{60}\right],\ \mathrm{Sec}\left[\frac{\pi}{60}\right]\right\}$$

Together[FunctionExpand[%]]

$$\left\{\frac{1}{16}\left(-\sqrt{2}-\sqrt{6}+\sqrt{10}+\sqrt{30}+2\sqrt{5+\sqrt{5}}-2\sqrt{3\left(5+\sqrt{5}\right)}\right),\right.$$

$$\frac{1}{16}\left(\sqrt{2}-\sqrt{6}-\sqrt{10}+\sqrt{30}+2\sqrt{5+\sqrt{5}}+2\sqrt{3\left(5+\sqrt{5}\right)}\right),$$

$$\frac{-1-\sqrt{3}+\sqrt{5}+\sqrt{15}+\sqrt{2\left(5+\sqrt{5}\right)}-\sqrt{6\left(5+\sqrt{5}\right)}}{1-\sqrt{3}-\sqrt{5}+\sqrt{15}+\sqrt{2\left(5+\sqrt{5}\right)}+\sqrt{6\left(5+\sqrt{5}\right)}},$$

$$\frac{-1+\sqrt{3}+\sqrt{5}-\sqrt{15}-\sqrt{2\left(5+\sqrt{5}\right)}-\sqrt{6\left(5+\sqrt{5}\right)}}{1+\sqrt{3}-\sqrt{5}-\sqrt{15}-\sqrt{2\left(5+\sqrt{5}\right)}+\sqrt{6\left(5+\sqrt{5}\right)}},$$

$$\frac{16}{-\sqrt{2}-\sqrt{6}+\sqrt{10}+\sqrt{30}+2\sqrt{5+\sqrt{5}}-2\sqrt{3\left(5+\sqrt{5}\right)}},$$

$$\left.\frac{16}{\sqrt{2}-\sqrt{6}-\sqrt{10}+\sqrt{30}+2\sqrt{5+\sqrt{5}}+2\sqrt{3\left(5+\sqrt{5}\right)}}\right\}$$

If the denominator contains squares of integers other than 2, the results always contain complex numbers (meaning that the imaginary number $i=\sqrt{-1}$ appears unavoidably).

$$\left\{\mathbf{Sin}\left[\frac{\pi}{9}\right],\ \mathbf{Cos}\left[\frac{\pi}{9}\right],\ \mathbf{Tan}\left[\frac{\pi}{9}\right],\ \mathbf{Cot}\left[\frac{\pi}{9}\right],\ \mathbf{Csc}\left[\frac{\pi}{9}\right],\ \mathbf{Sec}\left[\frac{\pi}{9}\right]\right\}$$

$$\left\{\mathrm{Sin}\left[\frac{\pi}{9}\right],\ \mathrm{Cos}\left[\frac{\pi}{9}\right],\ \mathrm{Tan}\left[\frac{\pi}{9}\right],\ \mathrm{Cot}\left[\frac{\pi}{9}\right],\ \mathrm{Csc}\left[\frac{\pi}{9}\right],\ \mathrm{Sec}\left[\frac{\pi}{9}\right]\right\}$$

```
FunctionExpand[%] // Together
```

$$\Big\{\frac{1}{8}\Big(-i\,2^{2/3}\left(-1-i\sqrt{3}\right)^{1/3}+2^{2/3}\sqrt{3}\left(-1-i\sqrt{3}\right)^{1/3}+i\,2^{2/3}\left(-1+i\sqrt{3}\right)^{1/3}+2^{2/3}\sqrt{3}\left(-1+i\sqrt{3}\right)^{1/3}\Big),$$

$$\frac{1}{8}\Big(2^{2/3}\left(-1-i\sqrt{3}\right)^{1/3}+i\,2^{2/3}\sqrt{3}\left(-1-i\sqrt{3}\right)^{1/3}+2^{2/3}\left(-1+i\sqrt{3}\right)^{1/3}-i\,2^{2/3}\sqrt{3}\left(-1+i\sqrt{3}\right)^{1/3}\Big),$$

$$\frac{-\left(-1-i\sqrt{3}\right)^{1/3}-i\sqrt{3}\left(-1-i\sqrt{3}\right)^{1/3}+\left(-1+i\sqrt{3}\right)^{1/3}-i\sqrt{3}\left(-1+i\sqrt{3}\right)^{1/3}}{-i\left(-1-i\sqrt{3}\right)^{1/3}+\sqrt{3}\left(-1-i\sqrt{3}\right)^{1/3}-i\left(-1+i\sqrt{3}\right)^{1/3}-\sqrt{3}\left(-1+i\sqrt{3}\right)^{1/3}},$$

$$\frac{\left(-1-i\sqrt{3}\right)^{1/3}+i\sqrt{3}\left(-1-i\sqrt{3}\right)^{1/3}+\left(-1+i\sqrt{3}\right)^{1/3}-i\sqrt{3}\left(-1+i\sqrt{3}\right)^{1/3}}{-i\left(-1-i\sqrt{3}\right)^{1/3}+\sqrt{3}\left(-1-i\sqrt{3}\right)^{1/3}+i\left(-1+i\sqrt{3}\right)^{1/3}+\sqrt{3}\left(-1+i\sqrt{3}\right)^{1/3}},$$

$$8\Big/\Big(-i\,2^{2/3}\left(-1-i\sqrt{3}\right)^{1/3}+2^{2/3}\sqrt{3}\left(-1-i\sqrt{3}\right)^{1/3}+i\,2^{2/3}\left(-1+i\sqrt{3}\right)^{1/3}+2^{2/3}\sqrt{3}\left(-1+i\sqrt{3}\right)^{1/3}\Big),$$

$$-(8\,i)\Big/\Big(-i\,2^{2/3}\left(-1-i\sqrt{3}\right)^{1/3}+2^{2/3}\sqrt{3}\left(-1-i\sqrt{3}\right)^{1/3}-i\,2^{2/3}\left(-1+i\sqrt{3}\right)^{1/3}-2^{2/3}\sqrt{3}\left(-1+i\sqrt{3}\right)^{1/3}\Big)\Big\}$$

Here the function `RootReduce` is used to express the previous algebraic numbers as numbered roots of polynomial equations.

```
RootReduce[Simplify[%]]
```

```
{Root[-3 + 36 #1^2 - 96 #1^4 + 64 #1^6 &, 4], Root[-1 - 6 #1 + 8 #1^3 &, 3],
 Root[-3 + 27 #1^2 - 33 #1^4 + #1^6 &, 4], Root[-1 + 33 #1^2 - 27 #1^4 + 3 #1^6 &, 6],
 Root[-64 + 96 #1^2 - 36 #1^4 + 3 #1^6 &, 6], Root[-8 + 6 #1^2 + #1^3 &, 3]}
```

The function `FunctionExpand` also reduces trigonometric expressions with compound arguments or compositions, including hyperbolic functions, to simpler ones. Here are some examples.

```
FunctionExpand[Cot[Sqrt[-z^2]]]
```

$$-\frac{\sqrt{-z}\,\mathrm{Coth}[z]}{\sqrt{z}}$$

```
Tan[Sqrt[i z^2]] // FunctionExpand
```

$$-\frac{(-1)^{3/4}\sqrt{-(-1)^{3/4}\,z}\,\sqrt{(-1)^{3/4}\,z}\,\mathrm{Tan}\left[(-1)^{1/4}\,z\right]}{z}$$

$$\left\{\text{Sin}\left[\sqrt{z^2}\right],\ \text{Cos}\left[\sqrt{z^2}\right],\ \text{Tan}\left[\sqrt{z^2}\right],\ \text{Cot}\left[\sqrt{z^2}\right],\ \text{Csc}\left[\sqrt{z^2}\right],\ \text{Sec}\left[\sqrt{z^2}\right]\right\} \text{ // FunctionExpand}$$

$$\left\{\frac{\sqrt{-i z}\,\sqrt{i z}\,\text{Sin}[z]}{z},\ \text{Cos}[z],\ \frac{\sqrt{-i z}\,\sqrt{i z}\,\text{Tan}[z]}{z},\ \frac{\sqrt{-i z}\,\sqrt{i z}\,\text{Cot}[z]}{z},\ \frac{\sqrt{-i z}\,\sqrt{i z}\,\text{Csc}[z]}{z},\ \text{Sec}[z]\right\}$$

Applying `Simplify` to the last expression gives a more compact result.

```
Simplify[%]
```

$$\left\{\frac{\sqrt{z^2}\,\text{Sin}[z]}{z},\ \text{Cos}[z],\ \frac{\sqrt{z^2}\,\text{Tan}[z]}{z},\ \frac{\sqrt{z^2}\,\text{Cot}[z]}{z},\ \frac{\sqrt{z^2}\,\text{Csc}[z]}{z},\ \text{Sec}[z]\right\}$$

Here are some similar examples.

```
Sin[2 ArcTan[z]] // FunctionExpand
```

$$\frac{2 z}{1 + z^2}$$

$$\text{Cos}\left[\frac{\text{ArcCot}[z]}{2}\right] \text{ // FunctionExpand}$$

$$\frac{\sqrt{1 + \frac{\sqrt{-z}\,\sqrt{z}}{\sqrt{-1-z^2}}}}{\sqrt{2}}$$

```
{Sin[2 ArcSin[z]], Cos[2 ArcCos[z]], Tan[2 ArcTan[z]],
  Cot[2 ArcCot[z]], Csc[2 ArcCsc[z]], Sec[2 ArcSec[z]]} // FunctionExpand
```

$$\left\{2\sqrt{1-z}\,z\sqrt{1+z},\ -1+2 z^2,\ -\frac{2 z}{(-1+z)\,(1+z)},\ \frac{1}{2}\left(1+\frac{1}{z^2}\right) z\left(\frac{1}{-1-z^2}-\frac{z^2}{-1-z^2}\right),\ \frac{\sqrt{-i z}\,\sqrt{i z}\,z}{2\sqrt{(-1+z)\,(1+z)}},\ \frac{z^2}{2-z^2}\right\}$$

$$\left\{\text{Sin}\left[\frac{\text{ArcSin}[z]}{2}\right],\ \text{Cos}\left[\frac{\text{ArcCos}[z]}{2}\right],\ \text{Tan}\left[\frac{\text{ArcTan}[z]}{2}\right],\ \text{Cot}\left[\frac{\text{ArcCot}[z]}{2}\right],\ \text{Csc}\left[\frac{\text{ArcCsc}[z]}{2}\right],\ \text{Sec}\left[\frac{\text{ArcSec}[z]}{2}\right]\right\} \text{ // FunctionExpand}$$

$$\left\{\frac{z\sqrt{1-\sqrt{1-z}\ \sqrt{1+z}}}{\sqrt{2}\ \sqrt{-i z}\ \sqrt{i z}},\ \frac{\sqrt{1+z}}{\sqrt{2}},\ \frac{z}{1+\sqrt{i\,(-i+z)}\ \sqrt{-i\,(i+z)}},\ z\left(1+\frac{\sqrt{-1-z^2}}{\sqrt{-z}\ \sqrt{z}}\right),\ \frac{\sqrt{2}\ \sqrt{-\frac{i}{z}}\ \sqrt{\frac{i}{z}}\ z}{\sqrt{1-\frac{\sqrt{(-1+z)\,(1+z)}}{\sqrt{-i z}\ \sqrt{i z}}}},\ \frac{\sqrt{2}\ \sqrt{-z}}{\sqrt{-1-z}}\right\}$$

Simplify[%]

$$\left\{\frac{z\sqrt{1-\sqrt{1-z^2}}}{\sqrt{2}\ \sqrt{z^2}},\ \frac{\sqrt{1+z}}{\sqrt{2}},\ \frac{z}{1+\sqrt{1+z^2}},\ z+\frac{\sqrt{z}\ \sqrt{-1-z^2}}{\sqrt{-z}},\ \frac{\sqrt{2}\ \sqrt{\frac{1}{z^2}}\ z}{\sqrt{1-\frac{\sqrt{z^2}\ \sqrt{-1+z^2}}{z^2}}},\ \frac{\sqrt{2}}{\sqrt{1+\frac{1}{z}}}\right\}$$

FullSimplify

The function `FullSimplify` tries a wider range of transformations than `Simplify` and returns the simplest form it finds. Here are some examples that contrast the results of applying these functions to the same expressions.

$$\mathbf{Cos}\left[\frac{1}{2}\, i\, \mathbf{Log}[1-i z]-\frac{1}{2}\, i\, \mathbf{Log}[1+i z]\right] \mathbin{//} \mathbf{Simplify}$$

$$\mathrm{Cosh}\left[\frac{1}{2}\,(\mathrm{Log}[1-i z]-\mathrm{Log}[1+i z])\right]$$

$$\mathbf{Cos}\left[\frac{1}{2}\, i\, \mathbf{Log}[1-i z]-\frac{1}{2}\, i\, \mathbf{Log}[1+i z]\right] \mathbin{//} \mathbf{FullSimplify}$$

$$\frac{1}{\sqrt{1+z^2}}$$

$$\left\{\mathbf{Sin}\left[-i\,\mathbf{Log}\left[i z+\sqrt{1-z^2}\right]\right],\ \mathbf{Cos}\left[-i\,\mathbf{Log}\left[i z+\sqrt{1-z^2}\right]\right],\ \mathbf{Tan}\left[-i\,\mathbf{Log}\left[i z+\sqrt{1-z^2}\right]\right],\ \mathbf{Cot}\left[-i\,\mathbf{Log}\left[i z+\sqrt{1-z^2}\right]\right],\ \mathbf{Csc}\left[-i\,\mathbf{Log}\left[i z+\sqrt{1-z^2}\right]\right],\ \mathbf{Sec}\left[-i\,\mathbf{Log}\left[i z+\sqrt{1-z^2}\right]\right]\right\} \mathbin{//} \mathbf{Simplify}$$

$$\left\{z,\ \frac{1-z^2+i z\sqrt{1-z^2}}{i z+\sqrt{1-z^2}},\ \frac{z\left(z-i\sqrt{1-z^2}\right)}{-i+i z^2+z\sqrt{1-z^2}},\ \frac{1-z^2+i z\sqrt{1-z^2}}{i z^2+z\sqrt{1-z^2}},\ \frac{1}{z},\ \frac{2\left(i z+\sqrt{1-z^2}\right)}{1+\left(i z+\sqrt{1-z^2}\right)^2}\right\}$$

$$\left\{\mathbf{Sin}\left[-i\,\mathbf{Log}\left[i z+\sqrt{1-z^2}\right]\right],\ \mathbf{Cos}\left[-i\,\mathbf{Log}\left[i z+\sqrt{1-z^2}\right]\right],\ \mathbf{Tan}\left[-i\,\mathbf{Log}\left[i z+\sqrt{1-z^2}\right]\right],\ \mathbf{Cot}\left[-i\,\mathbf{Log}\left[i z+\sqrt{1-z^2}\right]\right],\ \mathbf{Csc}\left[-i\,\mathbf{Log}\left[i z+\sqrt{1-z^2}\right]\right],\ \mathbf{Sec}\left[-i\,\mathbf{Log}\left[i z+\sqrt{1-z^2}\right]\right]\right\} \mathbin{//} \mathbf{FullSimplify}$$

$$\left\{z,\ \sqrt{1-z^2},\ \frac{z}{\sqrt{1-z^2}},\ \frac{\sqrt{1-z^2}}{z},\ \frac{1}{z},\ \frac{1}{\sqrt{1-z^2}}\right\}$$

Operations carried out by specialized *Mathematica* functions

Series expansions

Calculating the series expansion of trigonometric functions to hundreds of terms can be done in seconds. Here are some examples.

```
Series[Sin[z], {z, 0, 5}]
```

$$z-\frac{z^3}{6}+\frac{z^5}{120}+O[z]^6$$

```
Normal[%]
```

$$z-\frac{z^3}{6}+\frac{z^5}{120}$$

```
Series[{Sin[z], Cos[z], Tan[z], Cot[z], Csc[z], Sec[z]}, {z, 0, 3}]
```

$$\left\{z-\frac{z^3}{6}+O[z]^4,\ 1-\frac{z^2}{2}+O[z]^4,\ z+\frac{z^3}{3}+O[z]^4,\ \frac{1}{z}-\frac{z}{3}-\frac{z^3}{45}+O[z]^4,\ \frac{1}{z}+\frac{z}{6}+\frac{7z^3}{360}+O[z]^4,\ 1+\frac{z^2}{2}+O[z]^4\right\}$$

```
Series[Cot[z], {z, 0, 100}] // Timing
```

$$\left\{1.442\ \text{Second},\ \frac{1}{z}-\frac{z}{3}-\frac{z^3}{45}-\frac{2z^5}{945}-\frac{z^7}{4725}-\frac{2z^9}{93\,555}-\frac{1382z^{11}}{638\,512\,875}-\right.$$
$$\frac{4z^{13}}{18\,243\,225}-\frac{3617z^{15}}{162\,820\,783\,125}-\frac{87\,734z^{17}}{38\,979\,295\,480\,125}-\frac{349\,222z^{19}}{1\,531\,329\,465\,290\,625}-$$
$$\frac{310\,732z^{21}}{13\,447\,856\,940\,643\,125}-\frac{472\,728\,182z^{23}}{201\,919\,571\,963\,756\,521\,875}-\frac{2\,631\,724z^{25}}{11\,094\,481\,976\,030\,578\,125}-$$
$$\frac{13\,571\,120\,588z^{27}}{564\,653\,660\,170\,076\,273\,671\,875}-\frac{13\,785\,346\,041\,608z^{29}}{5\,660\,878\,804\,669\,082\,674\,070\,015\,625}-$$
$$\frac{7\,709\,321\,041\,217z^{31}}{31\,245\,110\,285\,511\,170\,603\,633\,203\,125}-\frac{303\,257\,395\,102z^{33}}{12\,130\,454\,581\,433\,748\,587\,292\,890\,625}-$$
$$\frac{52\,630\,543\,106\,106\,954\,746z^{35}}{20\,777\,977\,561\,866\,588\,586\,487\,628\,662\,044\,921\,875}-\frac{616\,840\,823\,966\,644z^{37}}{2\,403\,467\,618\,492\,375\,776\,343\,276\,883\,984\,375}-$$
$$\frac{522\,165\,436\,992\,898\,244\,102z^{39}}{20\,080\,431\,172\,289\,638\,826\,798\,401\,128\,390\,556\,640\,625}-$$
$$\frac{6\,080\,390\,575\,672\,283\,210\,764z^{41}}{2\,307\,789\,189\,818\,960\,127\,712\,594\,427\,864\,667\,427\,734\,375}-$$
$$\frac{10\,121\,188\,937\,927\,645\,176\,372z^{43}}{37\,913\,679\,547\,025\,773\,526\,706\,908\,457\,776\,679\,169\,921\,875}-$$

$$\frac{207\,461\,256\,206\,578\,143\,748\,856\, z^{45}}{7\,670\,102\,214\,448\,301\,053\,033\,358\,480\,610\,212\,529\,462\,890\,625} - \frac{11\,218\,806\,737\,995\,635\,372\,498\,255\,094\, z^{47}}{4\,093\,648\,603\,384\,274\,996\,519\,698\,921\,478\,879\,580\,162\,286\,669\,921\,875} - \frac{79\,209\,152\,838\,572\,743\,713\,996\,404\, z^{49}}{285\,258\,771\,457\,546\,764\,463\,363\,635\,252\,374\,414\,183\,254\,365\,234\,375} - \frac{246\,512\,528\,657\,073\,833\,030\,130\,766\,724\, z^{51}}{8\,761\,982\,491\,474\,419\,367\,550\,817\,114\,626\,909\,562\,924\,278\,968\,505\,859\,375} - \frac{233\,199\,709\,079\,078\,899\,371\,344\,990\,501\,528\, z^{53}}{81\,807\,125\,729\,900\,063\,867\,074\,959\,072\,425\,603\,825\,198\,823\,017\,351\,806\,640\,625} - \frac{1\,416\,795\,959\,607\,558\,144\,963\,094\,708\,378\,988\, z^{55}}{4\,905\,352\,087\,939\,496\,310\,826\,487\,207\,538\,302\,184\,255\,342\,959\,123\,162\,841\,796\,875} - \frac{23\,305\,824\,372\,104\,839\,134\,357\,731\,308\,699\,592\, z^{57}}{796\,392\,368\,980\,577\,121\,745\,974\,726\,570\,063\,253\,238\,310\,542\,073\,919\,837\,646\,484\,375} - \frac{9\,721\,865\,123\,870\,044\,576\,322\,439\,952\,638\,561\,968\,331\,928\, z^{59}}{3\,278\,777\,586\,273\,629\,598\,615\,520\,165\,380\,455\,583\,231\,003\,564\,645\,636\,125\,000\,418\,914\,794\,921\,875} - \frac{6\,348\,689\,256\,302\,894\,731\,330\,601\,216\,724\,328\,336\, z^{61}}{21\,132\,271\,510\,899\,613\,925\,529\,439\,369\,536\,628\,424\,678\,570\,233\,931\,462\,891\,949\,462\,890\,625} - \frac{106\,783\,830\,147\,866\,529\,886\,385\,444\,979\,142\,647\,942\,017\, z^{63}}{3\,508\,062\,732\,166\,890\,409\,707\,514\,582\,539\,928\,001\,638\,766\,051\,683\,792\,497\,378\,070\,587\,158\,203\,125} -$$

$$(267\,745\,458\,568\,424\,664\,373\,021\,714\,282\,169\,516\,771\,254\,382\, z^{65}) / 86\,812\,790\,293\,146\,213\,360\,651\,966\,604\,262\,937\,105\,495\,141\,563\,588\,806\,888\,204\,273\,501\,373\,291\,015\,625 - (250\,471\,004\,320\,250\,327\,955\,196\,022\,920\,428\,000\,776\,938\, z^{67}) / 801\,528\,196\,428\,242\,695\,121\,010\,267\,455\,843\,804\,062\,822\,357\,897\,831\,858\,125\,102\,407\,684\,326\,171\,875 - (172\,043\,582\,552\,384\,800\,434\,637\,321\,986\,040\,823\,829\,878\,646\,884\, z^{69}) / 5\,433\,748\,964\,547\,053\,581\,149\,916\,185\,708\,338\,218\,048\,392\,402\,830\,337\,634\,114\,958\,370\,880\,742\,156\,982\,421\,875 - (11\,655\,909\,923\,339\,888\,220\,876\,554\,489\,282\,134\,730\,564\,976\,603\,688\,520\,858\, z^{71}) / 3\,633\,348\,205\,269\,879\,230\,856\,840\,004\,304\,821\,536\,968\,049\,780\,112\,803\,650\,817\,771\,432\,558\,560\,793\,458\,452\,606\,201\,171\,875 - (3\,692\,153\,220\,456\,342\,488\,035\,683\,646\,645\,690\,290\,452\,790\,030\,604\, z^{73}) / 11\,359\,005\,221\,796\,317\,918\,049\,302\,062\,760\,294\,302\,183\,889\,391\,189\,419\,445\,133\,951\,612\,582\,060\,536\,346\,435\,546\,875 - (5\,190\,545\,015\,986\,394\,254\,249\,936\,008\,544\,252\,611\,445\,319\,542\,919\,116\, z^{75}) / 157\,606\,197\,452\,423\,911\,112\,934\,066\,120\,799\,083\,442\,801\,465\,302\,753\,194\,801\,233\,578\,624\,576\,089\,941\,806\,793\,212\,890\,625 - (255\,290\,071\,123\,323\,586\,643\,187\,098\,799\,718\,199\,072\,122\,692\,536\,861\,835\,992\, z^{77}) / 76\,505\,736\,228\,426\,953\,173\,738\,238\,352\,183\,101\,801\,688\,392\,812\,244\,485\,181\,277\,127\,930\,109\,049\,138\,257\,655\,704\,498\,291\,015\,625 - (9\,207\,568\,598\,958\,915\,293\,871\,149\,938\,038\,093\,699\,588\,515\,745\,502\,577\,839\,313\,734\, z^{79}) / 27\,233\,582\,984\,369\,795\,892\,070\,228\,410\,001\,578\,355\,986\,013\,571\,390\,071\,723\,225\,259\,349\,721\,067\,988\,068\,852\,863\,296\,604\,156\,494\,140\,625 - (163\,611\,136\,505\,867\,886\,519\,332\,147\,296\,221\,453\,678\,803\,514\,884\,902\,772\,183\,572\, z^{81}) / 4\,776\,089\,171\,877\,348\,057\,451\,105\,924\,101\,750\,653\,118\,402\,745\,283\,825\,543\,113\,171\,217\,116\,857\,704\,024\,700\,607\,798\,175\,811\,767\,578\,125 - (8\,098\,304\,783\,741\,161\,440\,924\,524\,640\,446\,924\,039\,959\,669\,564\,792\,363\,509\,124\,335\,729\,908\, z^{83}) /$$

```
2 333 207 846 470 426 678 843 707 227 616 712 214 909 162 634 745 895 349 325 948 586 531 533 393 
  530 725 143 500 144 033 328 342 437 744 140 625 -
(122 923 650 124 219 284 385 832 157 660 699 813 260 991 755 656 444 452 420 836 648 z^85) /
 349 538 086 043 843 717 584 559 187 055 386 621 548 470 304 913 596 772 372 737 435 524 697 231 
  069 047 713 981 709 496 784 210 205 078 125 -
(476 882 359 517 824 548 362 004 154 188 840 670 307 545 554 753 464 961 562 516 323 845 108 z^87) /
 13 383 510 964 174 348 021 497 060 628 653 950 829 663 288 548 327 870 152 944 013 988 358 928 114 
  528 962 242 087 062 453 152 690 410 614 013 671 875 -
(1 886 491 646 433 732 479 814 597 361 998 744 134 040 407 919 471 435 385 970 472 345 164 676 056
   z^89) /
 522 532 651 330 971 490 226 753 590 247 329 744 050 384 290 675 644 135 735 656 667 608 610 471 
  400 391 047 234 539 824 350 830 981 313 610 076 904 296 875 -
(450 638 590 680 882 618 431 105 331 665 591 912 924 988 342 163 281 788 877 675 244 114 763 912
   z^91) /
 1 231 931 818 039 911 948 327 467 370 123 161 265 684 460 571 086 659 079 080 437 659 781 065 743 
  269 173 212 919 832 661 978 537 311 246 395 111 083 984 375 -
(415 596 189 473 955 564 121 634 614 268 323 814 113 534 779 643 471 190 276 158 333 713 923 216
   z^93) /
 11 213 200 675 690 943 223 287 032 785 929 540 201 272 600 687 465 377 745 332 153 847 964 679 254 
  692 602 138 023 498 144 562 090 675 557 613 372 802 734 375 -
(423 200 899 194 533 026 195 195 456 219 648 467 346 087 908 778 120 468 301 277 466 840 101 336 
    699 974 518 z^95) /
 112 694 926 530 960 148 011 367 752 417 874 063 473 378 698 369 880 587 800 838 274 234 349 237 
  591 647 453 413 782 021 538 312 594 164 677 406 144 702 434 539 794 921 875 -
(5 543 531 483 502 489 438 698 050 411 951 314 743 456 505 773 755 468 368 087 670 306 121 873 229 
    244 z^97) /
 14 569 479 835 935 377 894 165 191 004 250 040 526 616 509 162 234 077 285 176 247 476 968 227 225 
  810 918 346 966 001 491 701 692 846 112 140 419 483 184 814 453 125 -
(378 392 151 276 488 501 180 909 732 277 974 887 490 811 366 132 267 744 533 542 784 817 245 581 
    660 788 990 844 z^99) /
 9 815 205 420 757 514 710 108 178 059 369 553 458 327 392 260 750 404 049 930 407 987 933 582 359 
  080 767 225 644 716 670 683 512 153 512 547 802 166 033 089 160 919 189 453 125 + O[z]^101}
```

Mathematica comes with the add-on package `DiscreteMath`RSolve`` that allows finding the general terms of series for many functions. After loading this package, and using the package function `SeriesTerm`, the following n^{th} term for odd trigonometric functions can be evaluated.

```
<< DiscreteMath`RSolve`
```

```
SeriesTerm[{Sin[z], Tan[z], Cot[z], Csc[z], Cos[z], Sec[z]}, {z, 0, n}]
```

$$\left\{\frac{i^{-1+n}\,\text{KroneckerDelta}[\text{Mod}[-1+n,2]]\,\text{UnitStep}[-1+n]}{\text{Gamma}[1+n]},\ \text{If}\left[\text{Odd}[n],\ \frac{i^{-1+n}\,2^{1+n}\,(-1+2^{1+n})\,\text{BernoulliB}[1+n]}{(1+n)!},\ 0\right],\ \frac{i\,i^{n}\,2^{1+n}\,\text{BernoulliB}[1+n]}{(1+n)!},\ \frac{i\,i^{n}\,2^{1+n}\,\text{BernoulliB}\left[1+n,\frac{1}{2}\right]}{(1+n)!},\ \frac{i^{n}\,\text{KroneckerDelta}[\text{Mod}[n,2]]}{\text{Gamma}[1+n]},\ \frac{i^{n}\,\text{EulerE}[n]}{n!}\right\}$$

Differentiation

Mathematica can evaluate derivatives of trigonometric functions of an arbitrary positive integer order.

```
D[Sin[z], z]
```

```
Cos[z]
```

```
Sin[z] // D[#, z] &
```

```
Cos[z]
```

```
∂z {Sin[z], Cos[z], Tan[z], Cot[z], Csc[z], Sec[z]}
```

```
{Cos[z], -Sin[z], Sec[z]^2, -Csc[z]^2, -Cot[z] Csc[z], Sec[z] Tan[z]}
```

```
∂{z,2} {Sin[z], Cos[z], Tan[z], Cot[z], Csc[z], Sec[z]}
```

```
{-Sin[z], -Cos[z], 2 Sec[z]^2 Tan[z], 2 Cot[z] Csc[z]^2,
 Cot[z]^2 Csc[z] + Csc[z]^3, Sec[z]^3 + Sec[z] Tan[z]^2}
```

```
Table[D[{Sin[z], Cos[z], Tan[z], Cot[z], Csc[z], Sec[z]}, {z, n}], {n, 4}]
```

```
{{Cos[z], -Sin[z], Sec[z]^2, -Csc[z]^2, -Cot[z] Csc[z], Sec[z] Tan[z]}, {-Sin[z], -Cos[z],
  2 Sec[z]^2 Tan[z], 2 Cot[z] Csc[z]^2, Cot[z]^2 Csc[z] + Csc[z]^3, Sec[z]^3 + Sec[z] Tan[z]^2},
 {-Cos[z], Sin[z], 2 Sec[z]^4 + 4 Sec[z]^2 Tan[z]^2, -4 Cot[z]^2 Csc[z]^2 - 2 Csc[z]^4,
  -Cot[z]^3 Csc[z] - 5 Cot[z] Csc[z]^3, 5 Sec[z]^3 Tan[z] + Sec[z] Tan[z]^3},
 {Sin[z], Cos[z], 16 Sec[z]^4 Tan[z] + 8 Sec[z]^2 Tan[z]^3,
  8 Cot[z]^3 Csc[z]^2 + 16 Cot[z] Csc[z]^4, Cot[z]^4 Csc[z] + 18 Cot[z]^2 Csc[z]^3 + 5 Csc[z]^5,
  5 Sec[z]^5 + 18 Sec[z]^3 Tan[z]^2 + Sec[z] Tan[z]^4}}
```

Finite summation

Mathematica can calculate finite sums that contain trigonometric functions. Here are two examples.

```
Sum[Sin[a k], {k, 0, n}]
```

$$\frac{1}{2}\left(\text{Cos}\left[\frac{a}{2}\right]-\text{Cos}\left[\frac{a}{2}+a\,n\right]\right)\text{Csc}\left[\frac{a}{2}\right]$$

$$\sum_{k=0}^{n}(-1)^{k}\,\text{Sin}[a\,k]$$

$$\frac{1}{2} \operatorname{Sec}\left[\frac{a}{2}\right] \left(-\operatorname{Sin}\left[\frac{a}{2}\right] + \operatorname{Sin}\left[\frac{a}{2} + a\, n + n\, \pi\right]\right)$$

Infinite summation

Mathematica can calculate infinite sums that contain trigonometric functions. Here are some examples.

$$\sum_{k=1}^{\infty} z^k \operatorname{Sin}[k\, x]$$

$$\frac{i \left(-1 + e^{2 i x}\right) z}{2 \left(e^{i x} - z\right) \left(-1 + e^{i x} z\right)}$$

$$\sum_{k=1}^{\infty} \frac{\operatorname{Sin}[k\, x]}{k\,!}$$

$$\frac{1}{2}\, i \left(e^{e^{-i x}} - e^{e^{i x}}\right)$$

$$\sum_{k=1}^{\infty} \frac{\operatorname{Cos}[k\, x]}{k}$$

$$\frac{1}{2} \left(-\operatorname{Log}\left[1 - e^{-i x}\right] - \operatorname{Log}\left[1 - e^{i x}\right]\right)$$

Finite products

Mathematica can calculate some finite symbolic products that contain the trigonometric functions. Here are two examples.

$$\operatorname{Product}\left[\operatorname{Sin}\left[\frac{\pi\, k}{n}\right], \{k, 1, n-1\}\right]$$

$$2^{1-n}\, n$$

$$\prod_{k=1}^{n-1} \operatorname{Cos}\left[z + \frac{\pi\, k}{n}\right]$$

$$-(-1)^n\, 2^{1-n} \operatorname{Sec}[z] \operatorname{Sin}\left[\frac{1}{2}\, n\, (\pi - 2\, z)\right]$$

Infinite products

Mathematica can calculate infinite products that contain trigonometric functions. Here are some examples.

In[2]:= $$\prod_{k=1}^{\infty} \operatorname{Exp}\left[z^k \operatorname{Sin}[k\, x]\right]$$

Out[2]= $$e^{\frac{i \left(-1+e^{2 i x}\right) z}{2 \left(z+e^{2 i x} z-e^{i x} \left(1+z^2\right)\right)}}$$

In[3]:= $$\prod_{k=1}^{\infty} \text{Exp}\left[\frac{\text{Cos}[k\,x]}{k\,!}\right]$$

Out[3]= $$e^{\frac{1}{2}\left(-2+e^{e^{-i x}}+e^{e^{i x}}\right)}$$

Indefinite integration

Mathematica can calculate a huge number of doable indefinite integrals that contain trigonometric functions. Here are some examples.

$$\int \text{Sin}[7\,z]\,dz$$

$$-\frac{1}{7}\,\text{Cos}[7\,z]$$

$$\int \{\{\text{Sin}[z],\ \text{Sin}[z]^a\},\ \{\text{Cos}[z],\ \text{Cos}[z]^a\},\ \{\text{Tan}[z],\ \text{Tan}[z]^a\},\ \{\text{Cot}[z],\ \text{Cot}[z]^a\},\ \{\text{Csc}[z],\ \text{Csc}[z]^a\},\ \{\text{Sec}[z],\ \text{Sec}[z]^a\}\}\,dz$$

$$\Big\{\Big\{-\text{Cos}[z],\ -\text{Cos}[z]\ \text{Hypergeometric2F1}\Big[\frac{1}{2},\ \frac{1-a}{2},\ \frac{3}{2},\ \text{Cos}[z]^2\Big]\ \text{Sin}[z]^{1+a}\ \left(\text{Sin}[z]^2\right)^{\frac{1}{2}(-1-a)}\Big\},$$

$$\Big\{\text{Sin}[z],\ -\frac{\text{Cos}[z]^{1+a}\ \text{Hypergeometric2F1}\left[\frac{1+a}{2},\ \frac{1}{2},\ \frac{3+a}{2},\ \text{Cos}[z]^2\right]\ \text{Sin}[z]}{(1+a)\ \sqrt{\text{Sin}[z]^2}}\Big\},$$

$$\Big\{-\text{Log}[\text{Cos}[z]],\ \frac{\text{Hypergeometric2F1}\left[\frac{1+a}{2},\ 1,\ 1+\frac{1+a}{2},\ -\text{Tan}[z]^2\right]\ \text{Tan}[z]^{1+a}}{1+a}\Big\},$$

$$\Big\{\text{Log}[\text{Sin}[z]],\ -\frac{\text{Cot}[z]^{1+a}\ \text{Hypergeometric2F1}\left[\frac{1+a}{2},\ 1,\ 1+\frac{1+a}{2},\ -\text{Cot}[z]^2\right]}{1+a}\Big\},$$

$$\Big\{-\text{Log}\Big[\text{Cos}\Big[\frac{z}{2}\Big]\Big]+\text{Log}\Big[\text{Sin}\Big[\frac{z}{2}\Big]\Big],$$

$$-\text{Cos}[z]\ \text{Csc}[z]^{-1+a}\ \text{Hypergeometric2F1}\Big[\frac{1}{2},\ \frac{1+a}{2},\ \frac{3}{2},\ \text{Cos}[z]^2\Big]\ \left(\text{Sin}[z]^2\right)^{\frac{1}{2}(-1+a)}\Big\},$$

$$\Big\{-\text{Log}\Big[\text{Cos}\Big[\frac{z}{2}\Big]-\text{Sin}\Big[\frac{z}{2}\Big]\Big]+\text{Log}\Big[\text{Cos}\Big[\frac{z}{2}\Big]+\text{Sin}\Big[\frac{z}{2}\Big]\Big],$$

$$-\frac{\text{Hypergeometric2F1}\left[\frac{1-a}{2},\ \frac{1}{2},\ \frac{3-a}{2},\ \text{Cos}[z]^2\right]\ \text{Sec}[z]^{-1+a}\ \text{Sin}[z]}{(1-a)\ \sqrt{\text{Sin}[z]^2}}\Big\}\Big\}$$

Definite integration

Mathematica can calculate wide classes of definite integrals that contain trigonometric functions. Here are some examples.

$$\int_0^{\pi/2} \sqrt[3]{\text{Sin}[z]}\ dz$$

$$\frac{\sqrt{\pi}\ \text{Gamma}\left[\frac{2}{3}\right]}{2\ \text{Gamma}\left[\frac{7}{6}\right]}$$

$$\int_0^{\pi/2}\left\{\sqrt{\text{Sin}[z]},\ \sqrt{\text{Cos}[z]},\ \sqrt{\text{Tan}[z]},\ \sqrt{\text{Cot}[z]},\ \sqrt{\text{Csc}[z]},\ \sqrt{\text{Sec}[z]}\right\} dz$$

$$\left\{2\ \text{EllipticE}\left[\frac{\pi}{4}, 2\right], 2\ \text{EllipticE}\left[\frac{\pi}{4}, 2\right], \frac{\pi}{\sqrt{2}}, \frac{\pi}{\sqrt{2}}, \frac{2\sqrt{\pi}\ \text{Gamma}\left[\frac{5}{4}\right]}{\text{Gamma}\left[\frac{3}{4}\right]}, \frac{2\sqrt{\pi}\ \text{Gamma}\left[\frac{5}{4}\right]}{\text{Gamma}\left[\frac{3}{4}\right]}\right\}$$

$$\int_0^{\frac{\pi}{2}}\left\{\{\text{Sin}[z], \text{Sin}[z]^a\}, \{\text{Cos}[z], \text{Cos}[z]^a\}, \{\text{Tan}[z], \text{Tan}[z]^a\}, \{\text{Cot}[z], \text{Cot}[z]^a\}, \{\text{Csc}[z], \text{Csc}[z]^a\}, \{\text{Sec}[z], \text{Sec}[z]^a\}\right\} dz$$

$$\left\{\left\{1, \frac{\sqrt{\pi}\ \text{Gamma}\left[\frac{1+a}{2}\right]}{a\ \text{Gamma}\left[\frac{a}{2}\right]}\right\}, \left\{1, \frac{\sqrt{\pi}\ \text{Gamma}\left[\frac{1+a}{2}\right]}{a\ \text{Gamma}\left[\frac{a}{2}\right]}\right\},\right.$$

$$\left\{\int_0^{\frac{\pi}{2}}\text{Tan}[z]\,dz, \text{If}\left[\text{Re}[a] < 1, \frac{1}{2}\pi\ \text{Sec}\left[\frac{a\pi}{2}\right], \int_0^{\frac{\pi}{2}}\text{Tan}[z]^a\,dz\right]\right\},$$

$$\left\{\int_0^{\frac{\pi}{2}}\text{Cot}[z]\,dz, \text{If}\left[\text{Re}[a] < 1, \frac{1}{2}\pi\ \text{Sec}\left[\frac{a\pi}{2}\right], \int_0^{\frac{\pi}{2}}\text{Cot}[z]^a\,dz\right]\right\},$$

$$\left.\left\{\int_0^{\frac{\pi}{2}}\text{Csc}[z]\,dz, \frac{\sqrt{\pi}\ \text{Gamma}\left[\frac{1}{2}-\frac{a}{2}\right]}{2\ \text{Gamma}\left[1-\frac{a}{2}\right]}\right\}, \left\{\int_0^{\frac{\pi}{2}}\text{Sec}[z]\,dz, \frac{\sqrt{\pi}\ \text{Gamma}\left[\frac{1}{2}-\frac{a}{2}\right]}{2\ \text{Gamma}\left[1-\frac{a}{2}\right]}\right\}\right\}$$

Limit operation

Mathematica can calculate limits that contain trigonometric functions.

$$\text{Limit}\left[\frac{\text{Sin}[z]}{z} + \text{Cos}[z]^3, z \to 0\right]$$

2

$$\text{Limit}\left[\left(\frac{\text{Tan}[x]}{x}\right)^{\frac{1}{x^2}}, x \to 0\right]$$

$$e^{1/3}$$

Solving equations

The next input solves equations that contain trigonometric functions. The message indicates that the multivalued functions are used to express the result and that some solutions might be absent.

$$\text{Solve}\left[\text{Tan}[z]^2 + 3\ \text{Sin}[z + \text{Pi}/6] == 4, z\right]$$

Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found.

```
{{z → ArcCos[Root[4 - 40 #1^2 + 12 #1^3 + 73 #1^4 - 60 #1^5 + 36 #1^6 &, 1]]},
 {z → -ArcCos[Root[4 - 40 #1^2 + 12 #1^3 + 73 #1^4 - 60 #1^5 + 36 #1^6 &, 2]]},
 {z → -ArcCos[Root[4 - 40 #1^2 + 12 #1^3 + 73 #1^4 - 60 #1^5 + 36 #1^6 &, 3]]},
 {z → ArcCos[Root[4 - 40 #1^2 + 12 #1^3 + 73 #1^4 - 60 #1^5 + 36 #1^6 &, 4]]},
 {z → ArcCos[Root[4 - 40 #1^2 + 12 #1^3 + 73 #1^4 - 60 #1^5 + 36 #1^6 &, 5]]},
 {z → ArcCos[Root[4 - 40 #1^2 + 12 #1^3 + 73 #1^4 - 60 #1^5 + 36 #1^6 &, 6]]}}
```

Complete solutions can be obtained by using the function `Reduce`.

```
Reduce[Sin[x] == a, x] // TraditionalForm
```

```
// InputForm =
C[1] ∈ Integers && (x == Pi - ArcSin[a] + 2 * Pi * C[1] || x == ArcSin[a] + 2 * Pi * C[1])
```

```
Reduce[Cos[x] == a, x] // TraditionalForm
```

```
// InputForm = C[1] ∈ Integers && (x == -ArcCos[a] + 2 * Pi * C[1] || x == ArcCos[a] + 2 * Pi * C[1])
```

```
Reduce[Tan[x] == a, x] // TraditionalForm
```

```
// InputForm = C[1] ∈ Integers && 1 + a^2 ≠ 0 && x == ArcTan[a] + Pi * C[1]
```

```
Reduce[Cot[x] == a, x] // TraditionalForm
```

```
// InputForm = C[1] ∈ Integers && 1 + a^2 ≠ 0 && x == ArcCot[a] + Pi * C[1]
```

```
Reduce[Csc[x] == a, x] // TraditionalForm
```

$$c_1 \in \mathbb{Z} \bigwedge a \neq 0 \bigwedge \left(x == -\sin^{-1}\left(\frac{1}{a}\right) + 2\pi c_1 + \pi \bigvee x == \sin^{-1}\left(\frac{1}{a}\right) + 2\pi c_1 \right)$$

```
Reduce[Sec[x] == a, x] // TraditionalForm
```

```
// InputForm = C[1] ∈ Integers && a ≠ 0 &&
  (x == -ArcCos[a^(-1)] + 2 * Pi * C[1] || x == ArcCos[a^(-1)] + 2 * Pi * C[1])
```

Solving differential equations

Here are differential equations whose linear-independent solutions are trigonometric functions. The solutions of the simplest second-order linear ordinary differential equation with constant coefficients can be represented through $\sin(z)$ and $\cos(z)$.

```
DSolve[w''[z] + w[z] == 0, w[z], z]
```

```
{{w[z] → C[1] Cos[z] + C[2] Sin[z]}}
```

```
dsol1 = DSolve[2 w[z] + 3 w''[z] + w^(4)[z] == 0, w[z], z]
```

```
{{w[z] → C[3] Cos[z] + C[1] Cos[√2 z] + C[4] Sin[z] + C[2] Sin[√2 z]}}
```

In the last input, the differential equation was solved for $w(z)$. If the argument is suppressed, the result is returned as a pure function (in the sense of the λ-calculus).

```
dsol2 = DSolve[2 w[z] + 3 w''[z] + w^(4)[z] == 0, w, z]
```

```
{{w → Function[{z}, C[3] Cos[z] + C[1] Cos[√2 z] + C[4] Sin[z] + C[2] Sin[√2 z]]}}
```

The advantage of such a pure function is that it can be used for different arguments, derivatives, and more.

```
w'[ζ] /. dsol1
```

```
{w'[ζ]}
```

```
w'[ζ] /. dsol2
```

```
{C[4] Cos[ζ] + √2 C[2] Cos[√2 ζ] - C[3] Sin[ζ] - √2 C[1] Sin[√2 ζ]}
```

All trigonometric functions satisfy first-order nonlinear differential equations. In carrying out the algorithm to solve the nonlinear differential equation, *Mathematica* has to solve a transcendental equation. In doing so, the generically multivariate inverse of a function is encountered, and a message is issued that a solution branch is potentially missed.

```
DSolve[{w'[z] == √(1 - w[z]^2), w[0] == 0}, w[z], z]
```

```
Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found.
```

```
{{w[z] → Sin[z]}}
```

```
DSolve[{w'[z] == √(1 - w[z]^2), w[0] == 1}, w[z], z]
```

```
Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found.
```

```
{{w[z] → Cos[z]}}
```

```
DSolve[{w'[z] - w[z]^2 - 1 == 0, w[0] == 0}, w[z], z]
```

```
Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found.
```

```
{{w[z] → Tan[z]}}
```

```
DSolve[{w'[z] + w[z]^2 + 1 == 0, w[π/2] == 0}, w[z], z]
```

```
Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found.
```

```
{{w[z] → Cot[z]}}
```

```
DSolve[{w'[z] == √(w[z]^4 - w[z]^2), 1 / w[0] == 0}, w[z], z] // Simplify[#, 0 < z < Pi/2] &
```

```
Solve::verif : Potential solution {C[1] → Indeterminate} (possibly
   discarded by verifier) should be checked by hand. May require use of limits.

Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found.

Solve::verif : Potential solution {C[1] → Indeterminate} (possibly
   discarded by verifier) should be checked by hand. May require use of limits.

Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found.
```

$\{\{w[z] \to -\text{Csc}[z]\}, \{w[z] \to \text{Csc}[z]\}\}$

$$\text{DSolve}\left[\left\{w'[z] == \sqrt{w[z]^4 - w[z]^2}, 1 \Big/ w\left[\frac{\pi}{2}\right] == 0\right\}, w[z], z\right] \text{ // Simplify}\left[\#, 0 < z < \text{Pi}\big/2\right] \&$$

```
Solve::verif : Potential solution {C[1] → Indeterminate} (possibly
   discarded by verifier) should be checked by hand. May require use of limits.

Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found.

Solve::verif : Potential solution {C[1] → Indeterminate} (possibly
   discarded by verifier) should be checked by hand. May require use of limits.

Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found.
```

$\{\{w[z] \to -\text{Sec}[z]\}, \{w[z] \to \text{Sec}[z]\}\}$

Integral transforms

Mathematica supports the main integral transforms like direct and inverse Fourier, Laplace, and Z transforms that can give results that contain classical or generalized functions. Here are some transforms of trigonometric functions.

LaplaceTransform[Sin[t], t, s]

$$\frac{1}{1+s^2}$$

FourierTransform[Sin[t], t, s]

$$i\sqrt{\frac{\pi}{2}}\ \text{DiracDelta}[-1+s] - i\sqrt{\frac{\pi}{2}}\ \text{DiracDelta}[1+s]$$

FourierSinTransform[Sin[t], t, s]

$$\sqrt{\frac{\pi}{2}}\ \text{DiracDelta}[-1+s] - \sqrt{\frac{\pi}{2}}\ \text{DiracDelta}[1+s]$$

FourierCosTransform[Sin[t], t, s]

$$-\frac{1}{\sqrt{2\pi}\,(-1+s)} + \frac{1}{\sqrt{2\pi}\,(1+s)}$$

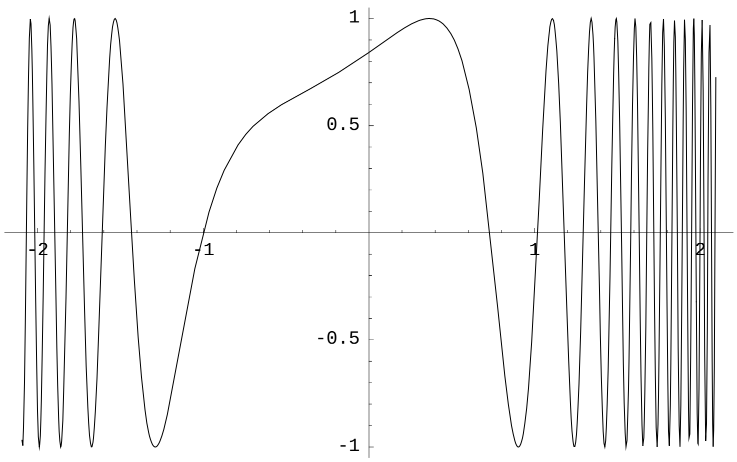
ZTransform[Sin[π t], t, s]

0

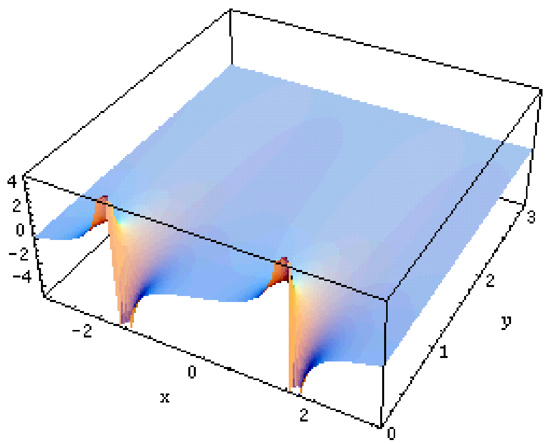
Plotting

Mathematica has built-in functions for 2D and 3D graphics. Here are some examples.

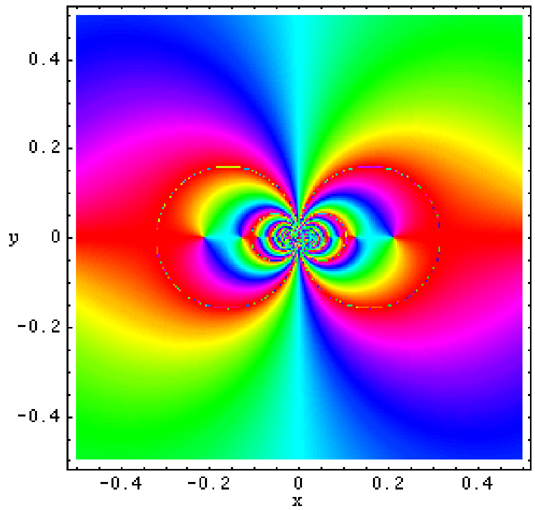
$$\text{Plot}\left[\text{Sin}\left[\sum_{k=0}^{5} z^k\right], \left\{z, -\frac{2\pi}{3}, \frac{2\pi}{3}\right\}\right];$$




```
Plot3D[Re[Tan[x + ⅈ y]], {x, -π, π}, {y, 0, π},
  PlotPoints → 240, PlotRange → {-5, 5},
  ClipFill → None, Mesh → False, AxesLabel → {"x", "y", None}];
```




```
ContourPlot[Arg[Sec[1/(x + ⅈ y)]], {x, -1/2, 1/2}, {y, -1/2, 1/2},
  PlotPoints → 400, PlotRange → {-π, π}, FrameLabel → {"x", "y", None, None},
  ColorFunction → Hue, ContourLines → False, Contours → 200];
```



Introduction to the Cotangent Function in *Mathematica*

Overview

The following shows how the cotangent function is realized in *Mathematica*. Examples of evaluating *Mathematica* functions applied to various numeric and exact expressions that involve the cotangent function or return it are shown. These involve numeric and symbolic calculations and plots.

Notations

Mathematica forms of notations

Following *Mathematica*'s general naming convention, function names in `StandardForm` are just the capitalized versions of their traditional mathematics names. This shows the cotangent function in `StandardForm`.

```
Cot[z]
```

```
Cot[z]
```

This shows the cotangent function in `TraditionalForm`.

```
% // TraditionalForm
```

$\cot(z)$

Additional forms of notations

Mathematica has other popular forms of notations that are used for print and electronic publication. In this particular instance the task is not difficult. However, it must be made to work in *Mathematica*'s `CForm`, `TeXForm`, and `FortranForm`.

```
{CForm[Cot[2 π z]], FortranForm[Cot[2 π z]], TeXForm[Cot[2 π z]]}
```

```
{Cot(2*Pi*z), Cot(2*Pi*z), \cot (2\, \pi \, z)}
```

Automatic evaluations and transformations

Evaluation for exact and machine-number values of arguments

For the exact argument $z = \pi/4$, *Mathematica* returns an exact result.

```
Cot[π/4]
```

```
1
```

```
Cot[z] /. z → π/4
```

```
1
```

For a machine-number argument (numerical argument with a decimal point), a machine number is also returned.

```
Cot[3.]
```

```
-7.01525
```

```
Cot[z] /. z → 2.
```

```
-0.457658
```

The next inputs calculate 100-digit approximations at $z = 1$ and $z = 2$.

```
N[Cot[z] /. z → 1, 100]
```

```
0.642092615934330703006419986594265620230278113918171379101162280426276856839164672\
 1984829197601968047
```

```
N[Cot[2], 100]
```

```
-0.457657554360285763750277410432047276428486329231674329641392162636292270156281308\
  6783085227210230896
```

```
Cot[2] // N[#, 100] &
```

```
-0.457657554360285763750277410432047276428486329231674329641392162636292270156281308\
  6783085227210230896
```

Within a second, it is possible to calculate thousands of digits for the cotangent function. The next input calculates 10000 digits for cot(1) and analyzes the frequency of the digit k in the resulting decimal number.

```
Map[Function[w, {First[#], Length[#]} & /@ Split[Sort[First[RealDigits[w]]]]],
 N[{Cot[z]} /. z → 1, 10000]]
```

```
{{{0, 1006}, {1, 1030}, {2, 986}, {3, 954},
  {4, 1003}, {5, 1034}, {6, 999}, {7, 998}, {8, 1009}, {9, 981}}}
```

Here is a 50-digit approximation to the cotangent function at the complex argument $z = 3 - 2i$.

```
N[Cot[3 - 2 ⅈ], 50]
```

```
-0.0106047834703371017503168962077792923972675909391411 +
 1.0357466377649953961127586568979083202483069599232 ⅈ
```

```
{N[Cot[z] /. z → 3 - 2 ⅈ, 50], Cot[3 - 2 ⅈ] // N[#, 50] &}
```

```
{-0.0106047834703371017503168962077792923972675909391411 +
  1.0357466377649953961127586568979083202483069599232 ⅈ,
 -0.0106047834703371017503168962077792923972675909391411 +
  1.0357466377649953961127586568979083202483069599232 ⅈ}
```

Mathematica automatically evaluates mathematical functions with machine precision, if the arguments of the function are numerical values and include machine-number elements. In this case only six digits after the decimal point are shown in the results. The remaining digits are suppressed, but can be displayed using the function `InputForm`.

```
{Cot[3.], N[Cot[3]], N[Cot[3], 16], N[Cot[3], 5], N[Cot[3], 20]}
```

```
{-7.01525, -7.01525, -7.01525, -7.01525, -7.0152525514345334694}
```

```
% // InputForm
```

```
{-7.015252551434534, -7.015252551434534, -7.015252551434534, -7.015252551434534,
 -7.01525255143453346942855137951906 8`20}
```

Simplification of the argument

Mathematica knows the symmetry and periodicity of the cotangent function. Here are some examples.

```
Cot[-3]
```

```
-Cot[3]
```

```
{Cot[-z], Cot[z + π], Cot[z + 2 π], Cot[-z + 21 π]}
```

```
{-Cot[z], Cot[z], Cot[z], -Cot[z]}
```

Mathematica automatically simplifies the composition of the direct and the inverse cotangent functions into its argument.

```
Cot[ArcCot[z]]
```

```
z
```

Mathematica also automatically simplifies the composition of the direct and any of the inverse trigonometric functions into algebraic functions of the argument.

```
{Cot[ArcSin[z]], Cot[ArcCos[z]], Cot[ArcTan[z]],
 Cot[ArcCot[z]], Cot[ArcCsc[z]], Cot[ArcSec[z]]}
```

$$\left\{\frac{\sqrt{1-z^2}}{z}, \frac{z}{\sqrt{1-z^2}}, \frac{1}{z}, z, \sqrt{1-\frac{1}{z^2}}\, z, \frac{1}{\sqrt{1-\frac{1}{z^2}}\, z}\right\}$$

In the cases where the argument has the structure $\pi k/2+z$ or $\pi k/2-z$, and $\pi k/2+i z$ or $\pi k/2-i z$ with integer k, the cotangent function can be automatically transformed into trigonometric or hyperbolic cotangent or tangent functions.

$$\text{Cot}\left[\frac{\pi}{2}-4\right]$$

Tan[4]

$$\left\{\text{Cot}\left[\frac{\pi}{2}-z\right], \text{Cot}\left[\frac{\pi}{2}+z\right], \text{Cot}\left[-\frac{\pi}{2}-z\right], \text{Cot}\left[-\frac{\pi}{2}+z\right], \text{Cot}[\pi-z], \text{Cot}[\pi+z]\right\}$$

{Tan[z], -Tan[z], Tan[z], -Tan[z], -Cot[z], Cot[z]}

Cot[i 5]

-i Coth[5]

$$\left\{\text{Cot}[i z], \text{Cot}\left[\frac{\pi}{2}-i z\right], \text{Cot}\left[\frac{\pi}{2}+i z\right], \text{Cot}[\pi-i z], \text{Cot}[\pi+i z]\right\}$$

{-i Coth[z], i Tanh[z], -i Tanh[z], i Coth[z], -i Coth[z]}

Simplification of combinations of cotangent functions

Sometimes simple arithmetic operations containing the cotangent function can automatically generate other equal trigonometric functions.

1 / Cot[4]

Tan[4]

{1 / Cot[z], 1 / Cot[π / 2 - z], Cot[π / 2 - z] / Cot[z],
Cot[z] / Cot[π / 2 - z], 1 / Cot[π / 2 - z], Cot[π / 2 - z] / Cot[z] ^2}

$$\left\{\text{Tan}[z], \text{Cot}[z], \text{Tan}[z]^2, \text{Cot}[z]^2, \text{Cot}[z], \text{Tan}[z]^3\right\}$$

The cotangent function arising as special cases from more general functions

The cotangent function can be treated as a particular case of some more general special functions. For example, cot(z) appears automatically from Bessel, Mathieu, Jacobi, hypergeometric, and Meijer functions for appropriate values of their parameters.

$$\left\{\text{BesselJ}\left[-\frac{1}{2}, z\right] \Big/ \text{BesselJ}\left[\frac{1}{2}, z\right], \frac{\text{MathieuC}[1, 0, z]}{\text{MathieuS}[1, 0, z]}, \text{JacobiCS}[z, 0],\right.$$
$$\text{JacobiSC}\left[\frac{\pi}{2}-z, 0\right], -i\,\text{JacobiNS}[i z, 1], i\,\text{JacobiSN}\left[\frac{\pi i}{2}-i z, 1\right],$$
$$\text{HypergeometricPFQ}\left[\{\}, \left\{\frac{1}{2}\right\}, -\frac{z^2}{4}\right] \Big/ \text{HypergeometricPFQ}\left[\{\}, \left\{\frac{3}{2}\right\}, -\frac{z^2}{4}\right],$$
$$\left.\text{MeijerG}\left[\{\{\}, \{\}\}, \left\{\left\{-\frac{1}{2}\right\}, \{0\}\right\}, \frac{z^2}{4}\right] \Big/ \text{MeijerG}\left[\{\{\}, \{\}\}, \left\{\left\{\frac{1}{2}\right\}, \{0\}\right\}, \frac{z^2}{4}\right]\right\}$$

$$\left\{\text{Cot}[z],\ \text{Cot}[z],\ \text{Cot}[z],\ \text{Cot}[z],\ -\text{Cot}[z],\ -\text{Cot}[z],\ \sqrt{z^2}\ \text{Cot}\left[\sqrt{z^2}\right],\ \frac{2\,\text{Cot}[z]}{z}\right\}$$

Equivalence transformations using specialized *Mathematica* functions

General remarks

Almost everybody prefers using $1-\cot(z)$ instead of $\cot(\pi - z) + \cot(\pi/4)$. *Mathematica* automatically transforms the second expression into the first one. The automatic application of transformation rules to mathematical expressions can result in overly complicated results. Compact expressions like $\cot(\pi/16)$ should not be automatically expanded into the more complicated expression $\left(\left(2+\left(2+2^{1/2}\right)^{1/2}\right)\Big/\left(2-\left(2+2^{1/2}\right)\right)\right)^{1/2}$. *Mathematica* has special functions that produce such expansions. Some are demonstrated in the next section.

TrigExpand

The function `TrigExpand` expands out trigonometric and hyperbolic functions. In more detail, it splits up sums and integer multiples that appear in arguments of trigonometric and hyperbolic functions, and then expands out products of trigonometric and hyperbolic functions into sums of powers, using trigonometric and hyperbolic identities where possible. Here are some examples.

```
TrigExpand[Cot[x - y]]
```

$$-\frac{\text{Cos}[x]\ \text{Cos}[y]}{-\text{Cos}[y]\ \text{Sin}[x]+\text{Cos}[x]\ \text{Sin}[y]}-\frac{\text{Sin}[x]\ \text{Sin}[y]}{-\text{Cos}[y]\ \text{Sin}[x]+\text{Cos}[x]\ \text{Sin}[y]}$$

```
Cot[4 z] // TrigExpand
```

$$\frac{\text{Cos}[z]^4}{4\,\text{Cos}[z]^3\,\text{Sin}[z]-4\,\text{Cos}[z]\,\text{Sin}[z]^3}-\frac{6\,\text{Cos}[z]^2\,\text{Sin}[z]^2}{4\,\text{Cos}[z]^3\,\text{Sin}[z]-4\,\text{Cos}[z]\,\text{Sin}[z]^3}+\frac{\text{Sin}[z]^4}{4\,\text{Cos}[z]^3\,\text{Sin}[z]-4\,\text{Cos}[z]\,\text{Sin}[z]^3}$$

```
Cot[2 z]^2 // TrigExpand
```

$$-\frac{3}{4}+\frac{\text{Cot}[z]^2}{8}+\frac{1}{8}\,\text{Csc}[z]^2\,\text{Sec}[z]^2+\frac{\text{Tan}[z]^2}{8}$$

```
TrigExpand[{Cot[x + y + z], Cot[3 z]}]
```

$$\left\{\frac{\text{Cos}[x]\,\text{Cos}[y]\,\text{Cos}[z]}{\text{Cos}[y]\,\text{Cos}[z]\,\text{Sin}[x]+\text{Cos}[x]\,\text{Cos}[z]\,\text{Sin}[y]+\text{Cos}[x]\,\text{Cos}[y]\,\text{Sin}[z]-\text{Sin}[x]\,\text{Sin}[y]\,\text{Sin}[z]}-\frac{\text{Cos}[z]\,\text{Sin}[x]\,\text{Sin}[y]}{\text{Cos}[y]\,\text{Cos}[z]\,\text{Sin}[x]+\text{Cos}[x]\,\text{Cos}[z]\,\text{Sin}[y]+\text{Cos}[x]\,\text{Cos}[y]\,\text{Sin}[z]-\text{Sin}[x]\,\text{Sin}[y]\,\text{Sin}[z]}-\frac{\text{Cos}[y]\,\text{Sin}[x]\,\text{Sin}[z]}{\text{Cos}[y]\,\text{Cos}[z]\,\text{Sin}[x]+\text{Cos}[x]\,\text{Cos}[z]\,\text{Sin}[y]+\text{Cos}[x]\,\text{Cos}[y]\,\text{Sin}[z]-\text{Sin}[x]\,\text{Sin}[y]\,\text{Sin}[z]}-\frac{\text{Cos}[x]\,\text{Sin}[y]\,\text{Sin}[z]}{\text{Cos}[y]\,\text{Cos}[z]\,\text{Sin}[x]+\text{Cos}[x]\,\text{Cos}[z]\,\text{Sin}[y]+\text{Cos}[x]\,\text{Cos}[y]\,\text{Sin}[z]-\text{Sin}[x]\,\text{Sin}[y]\,\text{Sin}[z]},\ \frac{\text{Cos}[z]^3}{3\,\text{Cos}[z]^2\,\text{Sin}[z]-\text{Sin}[z]^3}-\frac{3\,\text{Cos}[z]\,\text{Sin}[z]^2}{3\,\text{Cos}[z]^2\,\text{Sin}[z]-\text{Sin}[z]^3}\right\}$$

TrigFactor

The function `TrigFactor` factors trigonometric and hyperbolic functions. In more detail, it splits up sums and integer multiples that appear in the arguments of trigonometric and hyperbolic functions, and then factors the resulting polynomials into trigonometric and hyperbolic functions, using trigonometric and hyperbolic identities where possible. Here are some examples.

```
TrigFactor[Cot[x] + Cot[y]]
```

```
Csc[x] Csc[y] Sin[x + y]
```

```
Cot[x] - Tan[y] // TrigFactor
```

```
Cos[x + y] Csc[x] Sec[y]
```

TrigReduce

The function `TrigReduce` rewrites the products and powers of trigonometric and hyperbolic functions in terms of trigonometric and hyperbolic functions with combined arguments. In more detail, it typically yields a linear expression involving trigonometric and hyperbolic functions with more complicated arguments. `TrigReduce` is approximately opposite to `TrigExpand` and `TrigFactor`. Here are some examples.

```
TrigReduce[Cot[x] Cot[y]]
```

$$\frac{\text{Cos}[x-y]+\text{Cos}[x+y]}{\text{Cos}[x-y]-\text{Cos}[x+y]}$$

```
Cot[x] Tan[y] // TrigReduce
```

$$\frac{-\text{Sin}[x-y]+\text{Sin}[x+y]}{\text{Sin}[x-y]+\text{Sin}[x+y]}$$

```
Table[TrigReduce[Cot[z]^n], {n, 2, 5}]
```

$$\left\{\frac{-1-\text{Cos}[2\,z]}{-1+\text{Cos}[2\,z]},\ \frac{3\,\text{Cos}[z]+\text{Cos}[3\,z]}{3\,\text{Sin}[z]-\text{Sin}[3\,z]},\ \frac{-3-4\,\text{Cos}[2\,z]-\text{Cos}[4\,z]}{-3+4\,\text{Cos}[2\,z]-\text{Cos}[4\,z]},\ \frac{10\,\text{Cos}[z]+5\,\text{Cos}[3\,z]+\text{Cos}[5\,z]}{10\,\text{Sin}[z]-5\,\text{Sin}[3\,z]+\text{Sin}[5\,z]}\right\}$$

```
TrigReduce[TrigExpand[{Cot[x + y + z], Cot[3 z], Cot[x] Cot[y]}]]
```

$\{\text{Cot}[x+y+z],\ \text{Cot}[3\,z],\ \frac{\text{Cos}[x-y]+\text{Cos}[x+y]}{\text{Cos}[x-y]-\text{Cos}[x+y]}\}$

```
TrigFactor[Cot[x] + Cot[y]] // TrigReduce
```

$$\frac{2\,\text{Sin}[x+y]}{\text{Cos}[x-y]-\text{Cos}[x+y]}$$

TrigToExp

The function `TrigToExp` converts trigonometric and hyperbolic functions to exponentials. It tries, where possible, to give results that do not involve explicit complex numbers. Here are some examples.

```
TrigToExp[Cot[z]]
```

$$-\frac{i\left(e^{-i z}+e^{i z}\right)}{e^{-i z}-e^{i z}}$$

```
Cot[a z] + Cot[b z] // TrigToExp
```

$$-\frac{i\left(e^{-i a z}+e^{i a z}\right)}{e^{-i a z}-e^{i a z}}-\frac{i\left(e^{-i b z}+e^{i b z}\right)}{e^{-i b z}-e^{i b z}}$$

ExpToTrig

The function `ExpToTrig` converts exponentials to trigonometric and hyperbolic functions. It is approximately opposite to `TrigToExp`. Here are some examples.

```
ExpToTrig[TrigToExp[Cot[z]]]
```

Cot[z]

$\{\alpha\, e^{-i x \beta}+\alpha\, e^{i x \beta} / \left(\alpha\, e^{-i x \beta}+\gamma\, e^{i x \beta}\right)\}$ **// ExpToTrig**

$$\left\{\alpha\,\text{Cos}[x\,\beta]-i\,\alpha\,\text{Sin}[x\,\beta]+\frac{\alpha\,(\text{Cos}[x\,\beta]+i\,\text{Sin}[x\,\beta])}{\alpha\,\text{Cos}[x\,\beta]+\gamma\,\text{Cos}[x\,\beta]-i\,\alpha\,\text{Sin}[x\,\beta]+i\,\gamma\,\text{Sin}[x\,\beta]}\right\}$$

ComplexExpand

The function `ComplexExpand` expands expressions assuming that all the variables are real. The option `TargetFunctions` can be given as a list of functions from the set `{Re, Im, Abs, Arg, Conjugate, Sign}`. `ComplexExpand` tries to give results in terms of the functions specified. Here are some examples.

```
ComplexExpand[Cot[x + i y]]
```

$$-\frac{\text{Sin}[2\,x]}{\text{Cos}[2\,x]-\text{Cosh}[2\,y]}+\frac{i\,\text{Sinh}[2\,y]}{\text{Cos}[2\,x]-\text{Cosh}[2\,y]}$$

```
Cot[x + i y] + Cot[x - i y] // ComplexExpand
```

$$-\frac{2\,\text{Sin}[2\,x]}{\text{Cos}[2\,x]-\text{Cosh}[2\,y]}$$

```
ComplexExpand[Re[Cot[x + i y]], TargetFunctions → {Re, Im}]
```

$$-\frac{2\,\text{Cosh}[y]\,\text{Sin}[x]}{\text{Cos}[2\,x]-\text{Cosh}[2\,y]}$$

ComplexExpand[Im[Cot[x + ⅈ y]], TargetFunctions → {Re, Im}]

$$\frac{2\,\text{Cos}[x]\,\text{Sinh}[y]}{\text{Cos}[2\,x]-\text{Cosh}[2\,y]}$$

ComplexExpand[Abs[Cot[x + ⅈ y]], TargetFunctions → {Re, Im}]

$$\sqrt{\frac{4\,\text{Cosh}[y]^2\,\text{Sin}[x]^2}{(\text{Cos}[2\,x]-\text{Cosh}[2\,y])^2}+\frac{4\,\text{Cos}[x]^2\,\text{Sinh}[y]^2}{(\text{Cos}[2\,x]-\text{Cosh}[2\,y])^2}}$$

**ComplexExpand[Abs[Cot[x + ⅈ y]], TargetFunctions → {Re, Im}] //
Simplify[#, {x, y} ∈ Reals] &**

$$\frac{\sqrt{2}}{\sqrt{-\text{Cos}[2\,x]+\text{Cosh}[2\,y]}}$$

ComplexExpand[Re[Cot[x + ⅈ y]] + Im[Cot[x + ⅈ y]], TargetFunctions → {Re, Im}]

$$-\frac{2\,\text{Cosh}[y]\,\text{Sin}[x]}{\text{Cos}[2\,x]-\text{Cosh}[2\,y]}+\frac{2\,\text{Cos}[x]\,\text{Sinh}[y]}{\text{Cos}[2\,x]-\text{Cosh}[2\,y]}$$

ComplexExpand[Arg[Cot[x + ⅈ y]], TargetFunctions → {Re, Im}]

$$\text{ArcTan}\left[-\frac{2\,\text{Cosh}[y]\,\text{Sin}[x]}{\text{Cos}[2\,x]-\text{Cosh}[2\,y]},\ \frac{2\,\text{Cos}[x]\,\text{Sinh}[y]}{\text{Cos}[2\,x]-\text{Cosh}[2\,y]}\right]$$

**ComplexExpand[Arg[Cot[x + ⅈ y]], TargetFunctions → {Re, Im}] //
Simplify[#, {x, y} ∈ Reals] &**

ArcTan[Cosh[y] Sin[x], -Cos[x] Sinh[y]]

ComplexExpand[Conjugate[Cot[x + ⅈ y]], TargetFunctions → {Re, Im}] // Simplify

$$\frac{1}{\text{Cosh}[y]\,\text{Sin}[x]-i\,\text{Cos}[x]\,\text{Sinh}[y]}$$

Simplify

The function Simplify performs a sequence of algebraic transformations on the expression, and returns the simplest form it finds. Here are some examples.

$$\frac{\text{Cot}[z_1]+\text{Cot}[z_2]+\text{Cot}[z_3]-\text{Cot}[z_1]\,\text{Cot}[z_2]\,\text{Cot}[z_3]}{1-\text{Cot}[z_1]\,\text{Cot}[z_2]-\text{Cot}[z_1]\,\text{Cot}[z_3]-\text{Cot}[z_2]\,\text{Cot}[z_3]}\ //\ \text{Simplify}$$

$\text{Cot}[z_1+z_2+z_3]$

$$\text{Simplify}\left[\text{Cot}\left[z-\frac{\pi}{3}\right]\text{Cot}\left[\frac{\pi}{3}+z\right]+\text{Cot}\left[z-\frac{\pi}{3}\right]\text{Cot}[z]+\text{Cot}[z]\,\text{Cot}\left[\frac{\pi}{3}+z\right]\right]$$

-3

Here is a collection of trigonometric identities. Each is written as a logical conjunction.

Simplify[#] & /@

$$\left(\mathrm{Cot}\left[\frac{z}{2}\right] == \mathrm{Cot}[z] + \mathrm{Csc}[z] \bigwedge \mathrm{Cot}[z]^2 == \frac{1+\mathrm{Cos}[2z]}{1-\mathrm{Cos}[2z]} \bigwedge \mathrm{Cot}[z]^2 == \frac{1}{\mathrm{Sech}[i z]^2 - 1} \bigwedge \right.$$
$$\mathrm{Cot}\left[\frac{z}{2}\right] == \frac{\mathrm{Sin}[z]}{1-\mathrm{Cos}[z]} == \frac{1+\mathrm{Cos}[z]}{\mathrm{Sin}[z]} \bigwedge \mathrm{Cot}[z]\,\mathrm{Cot}[2z] == \frac{1}{2}\left(\mathrm{Cot}[z]^2 - 1\right) \bigwedge$$
$$\mathrm{Cot}[a]^2 - \mathrm{Cot}[b]^2 == -\mathrm{Csc}[a]^2\,\mathrm{Csc}[b]^2\,\mathrm{Sin}[a-b]\,\mathrm{Sin}[a+b] \bigwedge$$
$$\left.\mathrm{Cot}[z]^3 == \frac{3\,\mathrm{Cos}[z] + \mathrm{Cos}[3z]}{3\,\mathrm{Sin}[z] - \mathrm{Sin}[3z]} \bigwedge \mathrm{Cot}[3z] == \frac{\mathrm{Cot}[z]^3 - 3\,\mathrm{Cot}[z]}{3\,\mathrm{Cot}[z]^2 - 1}\right)$$

```
True
```

The function `Simplify` has the `Assumption` option. For example, *Mathematica* treats the periodicity of trigonometric functions for the symbolic integer coefficient k of $k\pi$.

```
Simplify[{Cot[z + 2 k π], Cot[z + k π] / Cot[z]}, k ∈ Integers]
```

```
{Cot[z], 1}
```

Mathematica also knows that the composition of the inverse and direct trigonometric functions produces the value of the internal argument under the corresponding restriction.

```
ArcCot[Cot[z]]
```

```
ArcCot[Cot[z]]
```

```
Simplify[ArcCot[Cot[z]], -π / 2 < Re[z] < π / 2]
```

```
z
```

FunctionExpand (and Together)

While the cotangent function auto-evaluates for simple fractions of π, for more complicated cases it stays as a cotangent function to avoid the build up of large expressions. Using the function `FunctionExpand`, the cotangent function can sometimes be transformed into explicit radicals. Here are some examples.

$$\left\{\mathrm{Cot}\left[\frac{\pi}{16}\right], \mathrm{Cot}\left[\frac{\pi}{60}\right]\right\}$$

$$\left\{\mathrm{Cot}\left[\frac{\pi}{16}\right], \mathrm{Cot}\left[\frac{\pi}{60}\right]\right\}$$

```
FunctionExpand[%]
```

$$\left\{\sqrt{\frac{2+\sqrt{2+\sqrt{2}}}{2-\sqrt{2+\sqrt{2}}}}, \frac{-\frac{-\frac{1}{8}\sqrt{3}\left(-1+\sqrt{5}\right)-\frac{1}{4}\sqrt{\frac{1}{2}\left(5+\sqrt{5}\right)}}{\sqrt{2}} - \frac{\frac{1}{8}\left(-1+\sqrt{5}\right)-\frac{1}{4}\sqrt{\frac{3}{2}\left(5+\sqrt{5}\right)}}{\sqrt{2}}}{-\frac{-\frac{1}{8}\sqrt{3}\left(-1+\sqrt{5}\right)-\frac{1}{4}\sqrt{\frac{1}{2}\left(5+\sqrt{5}\right)}}{\sqrt{2}} + \frac{\frac{1}{8}\left(-1+\sqrt{5}\right)-\frac{1}{4}\sqrt{\frac{3}{2}\left(5+\sqrt{5}\right)}}{\sqrt{2}}}\right\}$$

```
Together[%]
```

$$\left\{\sqrt{\frac{2+\sqrt{2+\sqrt{2}}}{2-\sqrt{2+\sqrt{2}}}},\ \frac{-1+\sqrt{3}+\sqrt{5}-\sqrt{15}-\sqrt{2\left(5+\sqrt{5}\right)}-\sqrt{6\left(5+\sqrt{5}\right)}}{1+\sqrt{3}-\sqrt{5}-\sqrt{15}-\sqrt{2\left(5+\sqrt{5}\right)}+\sqrt{6\left(5+\sqrt{5}\right)}}\right\}$$

If the denominator contains squares of integers other than 2, the results always contain complex numbers (meaning that the imaginary number $i=\sqrt{-1}$ appears unavoidably).

$$\left\{\text{Cot}\left[\frac{\pi}{9}\right]\right\}$$

$$\left\{\text{Cot}\left[\frac{\pi}{9}\right]\right\}$$

```
FunctionExpand[%] // Together
```

$$\left\{\frac{\left(-1-i\sqrt{3}\right)^{1/3}+i\sqrt{3}\left(-1-i\sqrt{3}\right)^{1/3}+\left(-1+i\sqrt{3}\right)^{1/3}-i\sqrt{3}\left(-1+i\sqrt{3}\right)^{1/3}}{-i\left(-1-i\sqrt{3}\right)^{1/3}+\sqrt{3}\left(-1-i\sqrt{3}\right)^{1/3}+i\left(-1+i\sqrt{3}\right)^{1/3}+\sqrt{3}\left(-1+i\sqrt{3}\right)^{1/3}}\right\}$$

Here the function `RootReduce` is used to express the previous algebraic numbers as roots of polynomial equations.

```
RootReduce[Simplify[%]]
```

$$\left\{\text{Root}\left[-1+33\,\#1^2-27\,\#1^4+3\,\#1^6\ \&,\ 6\right]\right\}$$

The function `FunctionExpand` also reduces trigonometric expressions with compound arguments or compositions, including inverse trigonometric functions, to simpler ones. Here are some examples.

$$\left\{\text{Cot}\left[\sqrt{z^2}\right],\ \text{Cot}\left[\frac{\text{ArcCot}[z]}{2}\right],\ \text{Cot}[2\,\text{ArcCot}[z]],\ \text{Cot}[3\,\text{ArcSin}[z]]\right\}\ //\ \text{FunctionExpand}$$

$$\left\{\frac{\sqrt{-i\,z}\,\sqrt{i\,z}\,\text{Cot}[z]}{z},\ z\left(1+\frac{\sqrt{-1-z^2}}{\sqrt{-z}\,\sqrt{z}}\right),\right.$$

$$\left.\frac{1}{2}\left(1+\frac{1}{z^2}\right)z\left(\frac{1}{-1-z^2}-\frac{z^2}{-1-z^2}\right),\ \frac{-3\sqrt{1-z}\,z^2\sqrt{1+z}+(1-z)^{3/2}\,(1+z)^{3/2}}{-z^3+3\,z\left(1-z^2\right)}\right\}$$

Applying `Simplify` to the last expression gives a more compact result.

```
Simplify[%]
```

$$\left\{\frac{\sqrt{z^2}\,\text{Cot}[z]}{z},\ z+\frac{\sqrt{z}\,\sqrt{-1-z^2}}{\sqrt{-z}},\ \frac{-1+z^2}{2\,z},\ \frac{\sqrt{1-z^2}\left(-1+4\,z^2\right)}{z\left(-3+4\,z^2\right)}\right\}$$

FullSimplify

The function `FullSimplify` tries a wider range of transformations than `Simplify` and returns the simplest form it finds. Here are some examples that contrast the results of applying these functions to the same expressions.

$$\text{set1} = \left\{\text{Cot}\left[-i\,\text{Log}\left[i\,z + \sqrt{1-z^2}\right]\right],\ \text{Cot}\left[\frac{\pi}{2} + i\,\text{Log}\left[i\,z + \sqrt{1-z^2}\right]\right],\ \text{Cot}\left[\frac{1}{2}\,i\,\text{Log}[1 - i\,z] - \frac{1}{2}\,i\,\text{Log}[1 + i\,z]\right],\ \text{Cot}\left[\frac{1}{2}\,i\,\text{Log}\left[1 - \frac{i}{z}\right] - \frac{1}{2}\,i\,\text{Log}\left[1 + \frac{i}{z}\right]\right],\ \text{Cot}\left[-i\,\text{Log}\left[\sqrt{1-\frac{1}{z^2}} + \frac{i}{z}\right]\right],\ \text{Cot}\left[\frac{\pi}{2} + i\,\text{Log}\left[\sqrt{1-\frac{1}{z^2}} + \frac{i}{z}\right]\right]\right\}$$

$$\left\{\frac{i\left(1 + \left(i\,z + \sqrt{1-z^2}\right)^2\right)}{-1 + \left(i\,z + \sqrt{1-z^2}\right)^2},\ -\frac{i\left(-1 + \left(i\,z + \sqrt{1-z^2}\right)^2\right)}{1 + \left(i\,z + \sqrt{1-z^2}\right)^2},\ -i\,\text{Coth}\left[\frac{1}{2}\,\text{Log}[1 - i\,z] - \frac{1}{2}\,\text{Log}[1 + i\,z]\right],\ -i\,\text{Coth}\left[\frac{1}{2}\,\text{Log}\left[1 - \frac{i}{z}\right] - \frac{1}{2}\,\text{Log}\left[1 + \frac{i}{z}\right]\right],\ \frac{i\left(1 + \left(\sqrt{1-\frac{1}{z^2}} + \frac{i}{z}\right)^2\right)}{-1 + \left(\sqrt{1-\frac{1}{z^2}} + \frac{i}{z}\right)^2},\ -\frac{i\left(-1 + \left(\sqrt{1-\frac{1}{z^2}} + \frac{i}{z}\right)^2\right)}{1 + \left(\sqrt{1-\frac{1}{z^2}} + \frac{i}{z}\right)^2}\right\}$$

```
set1 // Simplify
```

$$\left\{\frac{1 - z^2 + i\,z\,\sqrt{1-z^2}}{i\,z^2 + z\,\sqrt{1-z^2}},\ \frac{z\left(z - i\,\sqrt{1-z^2}\right)}{-i + i\,z^2 + z\,\sqrt{1-z^2}},\ -i\,\text{Coth}\left[\frac{1}{2}\,(\text{Log}[1 - i\,z] - \text{Log}[1 + i\,z])\right],\ -i\,\text{Coth}\left[\frac{1}{2}\left(\text{Log}\left[\frac{-i + z}{z}\right] - \text{Log}\left[\frac{i + z}{z}\right]\right)\right],\ \frac{-1 + i\,\sqrt{1-\frac{1}{z^2}}\,z + z^2}{i + \sqrt{1-\frac{1}{z^2}}\,z},\ \frac{1 - i\,\sqrt{1-\frac{1}{z^2}}\,z}{i + \sqrt{1-\frac{1}{z^2}}\,z - i\,z^2}\right\}$$

```
set1 // FullSimplify
```

$$\left\{\frac{\sqrt{1-z^2}}{z},\ \frac{z}{\sqrt{1-z^2}},\ \frac{1}{z},\ z,\ \sqrt{1-\frac{1}{z^2}}\,z,\ \frac{1}{\sqrt{1-\frac{1}{z^2}}\,z}\right\}$$

Operations under special *Mathematica* functions

Series expansions

Calculating the series expansion of a cotangent function to hundreds of terms can be done in seconds.

```
Series[Cot[z], {z, 0, 3}]
```

$$\frac{1}{z}-\frac{z}{3}-\frac{z^3}{45}+O[z]^4$$

Normal[**%**]

$$\frac{1}{z}-\frac{z}{3}-\frac{z^3}{45}$$

Series[**Cot**[**z**], {**z, 0, 100**}] // **Timing**

$$\Big\{1.442\ \text{Second},\ \frac{1}{z}-\frac{z}{3}-\frac{z^3}{45}-\frac{2z^5}{945}-\frac{z^7}{4725}-\frac{2z^9}{93\,555}-\frac{1382z^{11}}{638\,512\,875}-$$
$$\frac{4z^{13}}{18\,243\,225}-\frac{3617z^{15}}{162\,820\,783\,125}-\frac{87\,734z^{17}}{38\,979\,295\,480\,125}-\frac{349\,222z^{19}}{1\,531\,329\,465\,290\,625}-$$
$$\frac{310\,732z^{21}}{13\,447\,856\,940\,643\,125}-\frac{472\,728\,182z^{23}}{201\,919\,571\,963\,756\,521\,875}-\frac{2\,631\,724z^{25}}{11\,094\,481\,976\,030\,578\,125}-$$
$$\frac{13\,571\,120\,588z^{27}}{564\,653\,660\,170\,076\,273\,671\,875}-\frac{13\,785\,346\,041\,608z^{29}}{5\,660\,878\,804\,669\,082\,674\,070\,015\,625}-$$
$$\frac{7\,709\,321\,041\,217z^{31}}{31\,245\,110\,285\,511\,170\,603\,633\,203\,125}-\frac{303\,257\,395\,102z^{33}}{12\,130\,454\,581\,433\,748\,587\,292\,890\,625}-$$
$$\frac{52\,630\,543\,106\,106\,954\,746z^{35}}{20\,777\,977\,561\,866\,588\,586\,487\,628\,662\,044\,921\,875}-\frac{616\,840\,823\,966\,644z^{37}}{2\,403\,467\,618\,492\,375\,776\,343\,276\,883\,984\,375}-$$
$$\frac{522\,165\,436\,992\,898\,244\,102z^{39}}{20\,080\,431\,172\,289\,638\,826\,798\,401\,128\,390\,556\,640\,625}-$$
$$\frac{6\,080\,390\,575\,672\,283\,210\,764z^{41}}{2\,307\,789\,189\,818\,960\,127\,712\,594\,427\,864\,667\,427\,734\,375}-$$
$$\frac{10\,121\,188\,937\,927\,645\,176\,372z^{43}}{37\,913\,679\,547\,025\,773\,526\,706\,908\,457\,776\,679\,169\,921\,875}-$$
$$\frac{207\,461\,256\,206\,578\,143\,748\,856z^{45}}{7\,670\,102\,214\,448\,301\,053\,033\,358\,480\,610\,212\,529\,462\,890\,625}-$$
$$\frac{11\,218\,806\,737\,995\,635\,372\,498\,255\,094z^{47}}{4\,093\,648\,603\,384\,274\,996\,519\,698\,921\,478\,879\,580\,162\,286\,669\,921\,875}-$$
$$\frac{79\,209\,152\,838\,572\,743\,713\,996\,404z^{49}}{285\,258\,771\,457\,546\,764\,463\,363\,635\,252\,374\,414\,183\,254\,365\,234\,375}-$$
$$\frac{246\,512\,528\,657\,073\,833\,030\,130\,766\,724z^{51}}{8\,761\,982\,491\,474\,419\,367\,550\,817\,114\,626\,909\,562\,924\,278\,968\,505\,859\,375}-$$
$$\frac{233\,199\,709\,079\,078\,899\,371\,344\,990\,501\,528z^{53}}{81\,807\,125\,729\,900\,063\,867\,074\,959\,072\,425\,603\,825\,198\,823\,017\,351\,806\,640\,625}-$$
$$\frac{1\,416\,795\,959\,607\,558\,144\,963\,094\,708\,378\,988z^{55}}{4\,905\,352\,087\,939\,496\,310\,826\,487\,207\,538\,302\,184\,255\,342\,959\,123\,162\,841\,796\,875}-$$
$$\frac{23\,305\,824\,372\,104\,839\,134\,357\,731\,308\,699\,592z^{57}}{796\,392\,368\,980\,577\,121\,745\,974\,726\,570\,063\,253\,238\,310\,542\,073\,919\,837\,646\,484\,375}-$$
$$\frac{9\,721\,865\,123\,870\,044\,576\,322\,439\,952\,638\,561\,968\,331\,928z^{59}}{3\,278\,777\,586\,273\,629\,598\,615\,520\,165\,380\,455\,583\,231\,003\,564\,645\,636\,125\,000\,418\,914\,794\,921\,875}-$$

$$\frac{6\,348\,689\,256\,302\,894\,731\,330\,601\,216\,724\,328\,336\, z^{61}}{21\,132\,271\,510\,899\,613\,925\,529\,439\,369\,536\,628\,424\,678\,570\,233\,931\,462\,891\,949\,462\,890\,625} -$$

$$\frac{106\,783\,830\,147\,866\,529\,886\,385\,444\,979\,142\,647\,942\,017\, z^{63}}{3\,508\,062\,732\,166\,890\,409\,707\,514\,582\,539\,928\,001\,638\,766\,051\,683\,792\,497\,378\,070\,587\,158\,203\,125} -$$

$$\left(267\,745\,458\,568\,424\,664\,373\,021\,714\,282\,169\,516\,771\,254\,382\, z^{65}\right) \Big/ 86\,812\,790\,293\,146\,213\,360\,651\,966\,604\,262\,937\,105\,495\,141\,563\,588\,806\,888\,204\,273\,501\,373\,291\,015\,625 -$$

$$\left(250\,471\,004\,320\,250\,327\,955\,196\,022\,920\,428\,000\,776\,938\, z^{67}\right) \Big/ 801\,528\,196\,428\,242\,695\,121\,010\,267\,455\,843\,804\,062\,822\,357\,897\,831\,858\,125\,102\,407\,684\,326\,171\,875 -$$

$$\left(172\,043\,582\,552\,384\,800\,434\,637\,321\,986\,040\,823\,829\,878\,646\,884\, z^{69}\right) \Big/ 5\,433\,748\,964\,547\,053\,581\,149\,916\,185\,708\,338\,218\,048\,392\,402\,830\,337\,634\,114\,958\,370\,880\,742\,156\,982\,421\,875 -$$

$$\left(11\,655\,909\,923\,339\,888\,220\,876\,554\,489\,282\,134\,730\,564\,976\,603\,688\,520\,858\, z^{71}\right) \Big/ 3\,633\,348\,205\,269\,879\,230\,856\,840\,004\,304\,821\,536\,968\,049\,780\,112\,803\,650\,817\,771\,432\,558\,560\,793\,458\,452\,606\,201\,171\,875 -$$

$$\left(3\,692\,153\,220\,456\,342\,488\,035\,683\,646\,645\,690\,290\,452\,790\,030\,604\, z^{73}\right) \Big/ 11\,359\,005\,221\,796\,317\,918\,049\,302\,062\,760\,294\,302\,183\,889\,391\,189\,419\,445\,133\,951\,612\,582\,060\,536\,346\,435\,546\,875 -$$

$$\left(5\,190\,545\,015\,986\,394\,254\,249\,936\,008\,544\,252\,611\,445\,319\,542\,919\,116\, z^{75}\right) \Big/ 157\,606\,197\,452\,423\,911\,112\,934\,066\,120\,799\,083\,442\,801\,465\,302\,753\,194\,801\,233\,578\,624\,576\,089\,941\,806\,793\,212\,890\,625 -$$

$$\left(255\,290\,071\,123\,323\,586\,643\,187\,098\,799\,718\,199\,072\,122\,692\,536\,861\,835\,992\, z^{77}\right) \Big/ 76\,505\,736\,228\,426\,953\,173\,738\,238\,352\,183\,101\,801\,688\,392\,812\,244\,485\,181\,277\,127\,930\,109\,049\,138\,257\,655\,704\,498\,291\,015\,625 -$$

$$\left(9\,207\,568\,598\,958\,915\,293\,871\,149\,938\,038\,093\,699\,588\,515\,745\,502\,577\,839\,313\,734\, z^{79}\right) \Big/ 27\,233\,582\,984\,369\,795\,892\,070\,228\,410\,001\,578\,355\,986\,013\,571\,390\,071\,723\,225\,259\,349\,721\,067\,988\,068\,852\,863\,296\,604\,156\,494\,140\,625 -$$

$$\left(163\,611\,136\,505\,867\,886\,519\,332\,147\,296\,221\,453\,678\,803\,514\,884\,902\,772\,183\,572\, z^{81}\right) \Big/ 4\,776\,089\,171\,877\,348\,057\,451\,105\,924\,101\,750\,653\,118\,402\,745\,283\,825\,543\,113\,171\,217\,116\,857\,704\,024\,700\,607\,798\,175\,811\,767\,578\,125 -$$

$$\left(8\,098\,304\,783\,741\,161\,440\,924\,524\,640\,446\,924\,039\,959\,669\,564\,792\,363\,509\,124\,335\,729\,908\, z^{83}\right) \Big/ 2\,333\,207\,846\,470\,426\,678\,843\,707\,227\,616\,712\,214\,909\,162\,634\,745\,895\,349\,325\,948\,586\,531\,533\,393\,530\,725\,143\,500\,144\,033\,328\,342\,437\,744\,140\,625 -$$

$$\left(122\,923\,650\,124\,219\,284\,385\,832\,157\,660\,699\,813\,260\,991\,755\,656\,444\,452\,420\,836\,648\, z^{85}\right) \Big/ 349\,538\,086\,043\,843\,717\,584\,559\,187\,055\,386\,621\,548\,470\,304\,913\,596\,772\,372\,737\,435\,524\,697\,231\,069\,047\,713\,981\,709\,496\,784\,210\,205\,078\,125 -$$

$$\left(476\,882\,359\,517\,824\,548\,362\,004\,154\,188\,840\,670\,307\,545\,554\,753\,464\,961\,562\,516\,323\,845\,108\, z^{87}\right) \Big/ 13\,383\,510\,964\,174\,348\,021\,497\,060\,628\,653\,950\,829\,663\,288\,548\,327\,870\,152\,944\,013\,988\,358\,928\,114\,528\,962\,242\,087\,062\,453\,152\,690\,410\,614\,013\,671\,875 -$$

$$\left(1\,886\,491\,646\,433\,732\,479\,814\,597\,361\,998\,744\,134\,040\,407\,919\,471\,435\,385\,970\,472\,345\,164\,676\,056\, z^{89}\right) \Big/ 522\,532\,651\,330\,971\,490\,226\,753\,590\,247\,329\,744\,050\,384\,290\,675\,644\,135\,735\,656\,667\,608\,610\,471\,400\,391\,047\,234\,539\,824\,350\,830\,981\,313\,610\,076\,904\,296\,875 -$$

$$\left(450\,638\,590\,680\,882\,618\,431\,105\,331\,665\,591\,912\,924\,988\,342\,163\,281\,788\,877\,675\,244\,114\,763\,912\, z^{91}\right) \Big/ 1\,231\,931\,818\,039\,911\,948\,327\,467\,370\,123\,161\,265\,684\,460\,571\,086\,659\,079\,080\,437\,659\,781\,065\,743\,269\,173\,212\,919\,832\,661\,978\,537\,311\,246\,395\,111\,083\,984\,375 -$$

$$\left(415\,596\,189\,473\,955\,564\,121\,634\,614\,268\,323\,814\,113\,534\,779\,643\,471\,190\,276\,158\,333\,713\,923\,216\, z^{93}\right) \Big/ 11\,213\,200\,675\,690\,943\,223\,287\,032\,785\,929\,540\,201\,272\,600\,687\,465\,377\,745\,332\,153\,847\,964\,679\,254$$

```
692 602 138 023 498 144 562 090 675 557 613 372 802 734 375 −
(423 200 899 194 533 026 195 195 456 219 648 467 346 087 908 778 120 468 301 277 466 840 101 336 ⋱
    699 974 518 z^95) /
 112 694 926 530 960 148 011 367 752 417 874 063 473 378 698 369 880 587 800 838 274 234 349 237 ⋱
  591 647 453 413 782 021 538 312 594 164 677 406 144 702 434 539 794 921 875 −
(5 543 531 483 502 489 438 698 050 411 951 314 743 456 505 773 755 468 368 087 670 306 121 873 229 ⋱
    244 z^97) /
 14 569 479 835 935 377 894 165 191 004 250 040 526 616 509 162 234 077 285 176 247 476 968 227 225 ⋱
  810 918 346 966 001 491 701 692 846 112 140 419 483 184 814 453 125 −
(378 392 151 276 488 501 180 909 732 277 974 887 490 811 366 132 267 744 533 542 784 817 245 581 ⋱
    660 788 990 844 z^99) /
 9 815 205 420 757 514 710 108 178 059 369 553 458 327 392 260 750 404 049 930 407 987 933 582 359 ⋱
  080 767 225 644 716 670 683 512 153 512 547 802 166 033 089 160 919 189 453 125 + O[z]^101}
```

Mathematica comes with the add-on package `DiscreteMath`RSolve`` that allows finding the general terms of the series for many functions. After loading this package, and using the package function `SeriesTerm`, the following n^{th} term of cot(z) can be evaluated.

```
<< DiscreteMath`RSolve`
```

```
SeriesTerm[Cot[z], {z, 0, n}] z^n
```

$$z^n \left(\frac{i\,(2\,i)^n\,\text{BernoulliB}[1+n]}{(1+n)!} + \frac{i\,(2\,i)^n\,\text{BernoulliB}[1+n,\,1]}{(1+n)!}\right)$$

This result can be verified by the following process.

```
% /. {Odd[n_] :> Element[(n + 1) / 2, Integers], BernoulliB[m_, 1] -> BernoulliB[m]} /.
  {n → 2 k − 1}
```

$$\frac{i\; i^{-1+2\,k}\; 2^{2\,k}\; z^{-1+2\,k}\;\text{BernoulliB}[2\,k]}{(2\,k)!}$$

```
Simplify[%, k ∈ Integers]
```

$$\frac{(2\,i)^{2\,k}\; z^{-1+2\,k}\;\text{BernoulliB}[2\,k]}{(2\,k)!}$$

```
Sum[%, {k, 1, ∞}]
```

$$\frac{\text{Csc}\left[\sqrt{z^2}\right]\left(\sqrt{z^2}\;\text{Cos}\left[\sqrt{z^2}\right] - \text{Sin}\left[\sqrt{z^2}\right]\right)}{z}$$

```
% // FullSimplify
```

$$-\frac{1}{z} + \text{Cot}[z]$$

Differentiation

Mathematica can evaluate derivatives of the cotangent function of an arbitrary positive integer order.

∂_z **Cot[z]**

$-\text{Csc}[z]^2$

$\partial_{\{z,2\}}$ **Cot[z]**

$2\,\text{Cot}[z]\,\text{Csc}[z]^2$

Table[D[Cot[z], {z, n}], {n, 10}]

$$\{-\text{Csc}[z]^2,\ 2\,\text{Cot}[z]\,\text{Csc}[z]^2,\ -4\,\text{Cot}[z]^2\,\text{Csc}[z]^2-2\,\text{Csc}[z]^4,\ 8\,\text{Cot}[z]^3\,\text{Csc}[z]^2+16\,\text{Cot}[z]\,\text{Csc}[z]^4,\ -16\,\text{Cot}[z]^4\,\text{Csc}[z]^2-88\,\text{Cot}[z]^2\,\text{Csc}[z]^4-16\,\text{Csc}[z]^6,\ 32\,\text{Cot}[z]^5\,\text{Csc}[z]^2+416\,\text{Cot}[z]^3\,\text{Csc}[z]^4+272\,\text{Cot}[z]\,\text{Csc}[z]^6,\ -64\,\text{Cot}[z]^6\,\text{Csc}[z]^2-1824\,\text{Cot}[z]^4\,\text{Csc}[z]^4-2880\,\text{Cot}[z]^2\,\text{Csc}[z]^6-272\,\text{Csc}[z]^8,\ 128\,\text{Cot}[z]^7\,\text{Csc}[z]^2+7680\,\text{Cot}[z]^5\,\text{Csc}[z]^4+24\,576\,\text{Cot}[z]^3\,\text{Csc}[z]^6+7936\,\text{Cot}[z]\,\text{Csc}[z]^8,\ -256\,\text{Cot}[z]^8\,\text{Csc}[z]^2-31\,616\,\text{Cot}[z]^6\,\text{Csc}[z]^4-185\,856\,\text{Cot}[z]^4\,\text{Csc}[z]^6-137\,216\,\text{Cot}[z]^2\,\text{Csc}[z]^8-7936\,\text{Csc}[z]^{10},\ 512\,\text{Cot}[z]^9\,\text{Csc}[z]^2+128\,512\,\text{Cot}[z]^7\,\text{Csc}[z]^4+1\,304\,832\,\text{Cot}[z]^5\,\text{Csc}[z]^6+1\,841\,152\,\text{Cot}[z]^3\,\text{Csc}[z]^8+353\,792\,\text{Cot}[z]\,\text{Csc}[z]^{10}\}$$

Finite products

Mathematica can calculate some finite symbolic products that contain the cotangent function. Here is an example.

$$\prod_{k=1}^{n-1} \text{Cot}\left[\frac{k\,\pi}{n}\right]$$

$$-\frac{(-1)^n\,\text{Sin}\left[\frac{n\,\pi}{2}\right]}{n}$$

Indefinite integration

Mathematica can calculate a huge number of doable indefinite integrals that contain the cotangent function. The results can contain special functions. Here are some examples.

$$\int \text{Cot}[z]\, dz$$

$\text{Log}[\text{Sin}[z]]$

$$\int \text{Cot}[z]^a\, dz$$

$$-\frac{\text{Cot}[z]^{1+a}\,\text{Hypergeometric2F1}\left[\frac{1+a}{2},\,1,\,1+\frac{1+a}{2},\,-\text{Cot}[z]^2\right]}{1+a}$$

Definite integration

Mathematica can calculate wide classes of definite integrals that contain the cotangent function. Here are some examples.

$$\int_0^{\pi/2}\sqrt{\text{Cot}[z]}\,dz$$

$$\frac{\pi}{\sqrt{2}}$$

$$\int_0^{\frac{\pi}{2}}\text{Cot}[z]^a\,dz$$

$$\text{If}\left[\text{Re}[a]<1,\ \frac{1}{2}\pi\,\text{Sec}\left[\frac{a\,\pi}{2}\right],\ \int_0^{\frac{\pi}{2}}\text{Cot}[z]^a\,dz\right]$$

Limit operation

Mathematica can calculate limits that contain the cotangent function. Here are some examples.

$$\text{Limit}\left[z^2\,\text{Cot}[3\,z]^2,\ z\to 0\right]$$

$$\frac{1}{9}$$

$$\text{Limit}\left[z\,\text{Cot}\left[2\sqrt{z^2}\right],\ z\to 0,\ \text{Direction}\to 1\right]$$

$$-\frac{1}{2}$$

$$\text{Limit}\left[z\,\text{Cot}\left[2\sqrt{z^2}\right],\ z\to 0,\ \text{Direction}\to -1\right]$$

$$\frac{1}{2}$$

Solving equations

The next inputs solve two equations that contain the cotangent function. Because of the multivalued nature of the inverse cotangent function, a printed message indicates that only some of the possible solutions are returned.

$$\text{Solve}\left[\text{Cot}[z]^2+3\,\text{Cot}[z+\text{Pi}/3]==4,\ z\right]$$

Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found.

$$\Bigg\{\Bigg\{z\to -\text{ArcCos}\Bigg[-\sqrt{\Bigg(\frac{71}{102}+\frac{5\sqrt{3}}{68}+}$$

$$1051\Big/\left(204\left(-93\,680+48\,159\sqrt{3}+3\sqrt{3\left(-5\,909\,472+34\,957\,440\sqrt{3}\right)}\right)^{1/3}\right)-$$

$$\left(104\sqrt{3}\right)\Big/\left(17\left(-93\,680+48\,159\sqrt{3}+3\sqrt{3\left(-5\,909\,472+34\,957\,440\sqrt{3}\right)}\right)^{1/3}\right)+$$

$$\frac{1}{204}\left(-93\,680+48\,159\sqrt{3}+3\sqrt{3\left(-5\,909\,472+34\,957\,440\sqrt{3}\right)}\right)^{1/3}\Bigg]\Bigg]\Bigg\},\ \Big\{z\to\text{ArcCos}\Big[$$

$$\sqrt{\left(\frac{71}{102}+\frac{5\sqrt{3}}{68}+1051\Big/\left(204\left(-93\,680+48\,159\sqrt{3}+3\sqrt{3\left(-5\,909\,472+34\,957\,440\sqrt{3}\right)}\right)^{1/3}\right)-\right.}$$

$$\left(104\sqrt{3}\right)\Big/\left(17\left(-93\,680+48\,159\sqrt{3}+3\sqrt{3\left(-5\,909\,472+34\,957\,440\sqrt{3}\right)}\right)^{1/3}\right)+$$

$$\left.\frac{1}{204}\left(-93\,680+48\,159\sqrt{3}+3\sqrt{3\left(-5\,909\,472+34\,957\,440\sqrt{3}\right)}\right)^{1/3}\right)\Big]\Big\},$$

$$\Big\{z\to\text{ArcCos}\Big[-\sqrt{\left(\frac{71}{102}+\frac{5\sqrt{3}}{68}-\left(\frac{1051}{408}+\frac{156\,i}{17}\right)\Big/\right.}$$

$$\left(-93\,680+48\,159\sqrt{3}+3\sqrt{3\left(-5\,909\,472+34\,957\,440\sqrt{3}\right)}\right)^{1/3}+$$

$$\left(\frac{156}{17}+\frac{1051\,i}{136}\right)\Big/\left(\sqrt{3}\left(-93\,680+48\,159\sqrt{3}+3\sqrt{3\left(-5\,909\,472+34\,957\,440\sqrt{3}\right)}\right)^{1/3}\right)-$$

$$\frac{1}{408}\left(-93\,680+48\,159\sqrt{3}+3\sqrt{3\left(-5\,909\,472+34\,957\,440\sqrt{3}\right)}\right)^{1/3}-$$

$$\left.\frac{1}{136\sqrt{3}}\left(i\left(-93\,680+48\,159\sqrt{3}+3\sqrt{3\left(-5\,909\,472+34\,957\,440\sqrt{3}\right)}\right)^{1/3}\right)\right)\Big]\Big\},$$

$$\Big\{z\to-\text{ArcCos}\Big[\sqrt{\left(\frac{71}{102}+\frac{5\sqrt{3}}{68}-\left(\frac{1051}{408}+\frac{156\,i}{17}\right)\Big/\right.}$$

$$\left(-93\,680+48\,159\sqrt{3}+3\sqrt{3\left(-5\,909\,472+34\,957\,440\sqrt{3}\right)}\right)^{1/3}+$$

$$\left(\frac{156}{17}+\frac{1051\,i}{136}\right)\Big/\left(\sqrt{3}\left(-93\,680+48\,159\sqrt{3}+3\sqrt{3\left(-5\,909\,472+34\,957\,440\sqrt{3}\right)}\right)^{1/3}\right)-$$

$$\frac{1}{408}\left(-93\,680+48\,159\sqrt{3}+3\sqrt{3\left(-5\,909\,472+34\,957\,440\sqrt{3}\right)}\right)^{1/3}-$$

$$\left.\frac{1}{136\sqrt{3}}\left(i\left(-93\,680+48\,159\sqrt{3}+3\sqrt{3\left(-5\,909\,472+34\,957\,440\sqrt{3}\right)}\right)^{1/3}\right)\right)\Big]\Big\},$$

$$\Bigg\{z \to \text{ArcCos}\Bigg[-\sqrt{\left(\frac{71}{102}+\frac{5\sqrt{3}}{68}-\left(\frac{1051}{408}-\frac{156\,i}{17}\right)\Big/\left(-93\,680+48\,159\sqrt{3}+3\sqrt{3\left(-5\,909\,472+34\,957\,440\sqrt{3}\right)}\right)^{1/3}+\left(\frac{156}{17}-\frac{1051\,i}{136}\right)\Big/\left(\sqrt{3}\left(-93\,680+48\,159\sqrt{3}+3\sqrt{3\left(-5\,909\,472+34\,957\,440\sqrt{3}\right)}\right)^{1/3}\right)-\frac{1}{408}\left(-93\,680+48\,159\sqrt{3}+3\sqrt{3\left(-5\,909\,472+34\,957\,440\sqrt{3}\right)}\right)^{1/3}+\frac{1}{136\sqrt{3}}\left(i\left(-93\,680+48\,159\sqrt{3}+3\sqrt{3\left(-5\,909\,472+34\,957\,440\sqrt{3}\right)}\right)^{1/3}\right)\right)}\Bigg]\Bigg\},$$

$$\Bigg\{z \to -\text{ArcCos}\Bigg[\sqrt{\left(\frac{71}{102}+\frac{5\sqrt{3}}{68}-\left(\frac{1051}{408}-\frac{156\,i}{17}\right)\Big/\left(-93\,680+48\,159\sqrt{3}+3\sqrt{3\left(-5\,909\,472+34\,957\,440\sqrt{3}\right)}\right)^{1/3}+\left(\frac{156}{17}-\frac{1051\,i}{136}\right)\Big/\left(\sqrt{3}\left(-93\,680+48\,159\sqrt{3}+3\sqrt{3\left(-5\,909\,472+34\,957\,440\sqrt{3}\right)}\right)^{1/3}\right)-\frac{1}{408}\left(-93\,680+48\,159\sqrt{3}+3\sqrt{3\left(-5\,909\,472+34\,957\,440\sqrt{3}\right)}\right)^{1/3}+\frac{1}{136\sqrt{3}}\left(i\left(-93\,680+48\,159\sqrt{3}+3\sqrt{3\left(-5\,909\,472+34\,957\,440\sqrt{3}\right)}\right)^{1/3}\right)\right)}\Bigg]\Bigg\}\Bigg\}$$

```
Solve[Cot[x] == a, x]
```

```
Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found.
```

```
{{x → ArcCot[a]}}
```

A complete solution of the previous equation can be obtained using the function `Reduce`.

```
Reduce[Cot[x] == a, x] // InputForm
```

```
// InputForm = C[1] ∈ Integers && 1 + a^2 ≠ 0 && x == ArcCot[a] + Pi * C[1]
```

Solving differential equations

Here is a nonlinear differential equation whose independent solutions include the cotangent function. In carrying out the algorithm to solve the following nonlinear differential equation, *Mathematica* has to solve a transcendental equation. In doing so, the generically multivariate inverse of a function is encountered, and a message is issued that a solution branch is potentially missed.

```
In[15]:= DSolve[{w′[z] + w[z]^2 + 1 == 0, w[π/2] == 0}, w, z]

Solve::ifun : Inverse functions are being used by Solve, so some
   solutions may not be found; use Reduce for complete solution information. More…

Out[15]= {{w → Function[{z}, Cot[z]]}}
```

Plotting

Mathematica has built-in functions for 2D and 3D graphics. Here are some examples.

```
Plot[Cot[Sum[z^k, {k, 0, 5}]], {z, -2π/3, 2π/3}];

Plot3D[Re[Cot[x + ⅈ y]], {x, -π, π}, {y, 0, π},
  PlotPoints → 240, PlotRange → {-5, 5},
  ClipFill → None, Mesh → False, AxesLabel → {"x", "y", None}];

ContourPlot[Arg[Cot[1/(x + ⅈ y)]], {x, -1/2, 1/2}, {y, -1/2, 1/2},
  PlotPoints → 400, PlotRange → {-π, π}, FrameLabel → {"x", "y", None, None},
  ColorFunction → Hue, ContourLines → False, Contours → 200];
```