Introductions to Csc

Introduction to the trigonometric functions

General

The six trigonometric functions sine $\sin(z)$, cosine $\cos(z)$, tangent $\tan(z)$, cotangent $\cot(z)$, cosecant $\csc(z)$, and secant $\sec(z)$ are well known and among the most frequently used elementary functions. The most popular functions $\sin(z)$, $\cos(z)$, $\tan(z)$, and $\cot(z)$ are taught worldwide in high school programs because of their natural appearance in problems involving angle measurement and their wide applications in the quantitative sciences.

The trigonometric functions share many common properties.

Definitions of trigonometric functions

All trigonometric functions can be defined as simple rational functions of the exponential function of $i\,z$:

$$\sin(z) = \frac{e^{i z} - e^{-i z}}{2 i}$$

$$\cos(z) = \frac{e^{i z} + e^{-i z}}{2}$$

$$\tan(z) = -\frac{i \left(e^{i z} - e^{-i z}\right)}{e^{i z} + e^{-i z}}$$

$$\cot(z) = \frac{i \left(e^{i z} + e^{-i z}\right)}{e^{i z} - e^{-i z}}$$

$$\csc(z) = \frac{2 i}{e^{i z} - e^{-i z}}$$

$$\sec(z) = \frac{2}{e^{i z} + e^{-i z}}.$$

The functions $\tan(z)$, $\cot(z)$, $\csc(z)$, and $\sec(z)$ can also be defined through the functions $\sin(z)$ and $\cos(z)$ using the following formulas:

$$\tan(z) = \frac{\sin(z)}{\cos(z)}$$

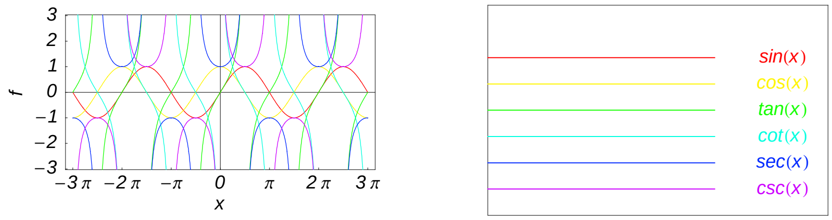
$$\cot(z) = \frac{\cos(z)}{\sin(z)}$$

$$\csc(z) = \frac{1}{\sin(z)}$$

$$\sec(z) = \frac{1}{\cos(z)}.$$

A quick look at the trigonometric functions

Here is a quick look at the graphics for the six trigonometric functions along the real axis.



Connections within the group of trigonometric functions and with other function groups

Representations through more general functions

The trigonometric functions are particular cases of more general functions. Among these more general functions, four different classes of special functions are particularly relevant: Bessel, Jacobi, Mathieu, and hypergeometric functions.

For example, $\sin(z)$ and $\cos(z)$ have the following representations through Bessel, Mathieu, and hypergeometric functions:

$$\sin(z) = \sqrt{\frac{\pi z}{2}}\, J_{1/2}(z) \quad \sin(z) = -i\sqrt{\frac{\pi i z}{2}}\, I_{1/2}(i z) \quad \sin(z) = \sqrt{\frac{\pi z}{2}}\, Y_{-1/2}(z) \quad \sin(z) = \frac{i}{\sqrt{2\pi}}\left(\sqrt{i z}\, K_{1/2}(i z) - \sqrt{-i z}\, K_{1/2}(-i z)\right)$$

$$\cos(z) = \sqrt{\frac{\pi z}{2}}\, J_{-1/2}(z) \quad \cos(z) = \sqrt{\frac{\pi i z}{2}}\, I_{-1/2}(i z) \quad \cos(z) = -\sqrt{\frac{\pi z}{2}}\, Y_{1/2}(z) \quad \cos(z) = \sqrt{\frac{i z}{2\pi}}\, K_{1/2}(i z) + \sqrt{-\frac{i z}{2\pi}}\, K_{1/2}(-i z)$$

$$\sin(z) = \mathrm{Se}(1, 0, z) \quad \cos(z) = \mathrm{Ce}(1, 0, z)$$

$$\sin(z) = z\, {}_0F_1\left(;\, \tfrac{3}{2};\, -\tfrac{z^2}{4}\right) \quad \cos(z) = {}_0F_1\left(;\, \tfrac{1}{2};\, -\tfrac{z^2}{4}\right).$$

On the other hand, all trigonometric functions can be represented as degenerate cases of the corresponding doubly periodic Jacobi elliptic functions when their second parameter is equal to 0 or 1:

$$\begin{array}{ll}
\sin(z) = \mathrm{sd}(z \mid 0) = \mathrm{sn}(z \mid 0) & \sin(z) = -i\, \mathrm{sc}(i z \mid 1) = -i\, \mathrm{sd}(i z \mid 1) \\
\cos(z) = \mathrm{cd}(z \mid 0) = \mathrm{cn}(z \mid 0) & \cos(z) = \mathrm{nc}(i z \mid 1) = \mathrm{nd}(i z \mid 1) \\
\tan(z) = \mathrm{sc}(z \mid 0) & \tan(z) = -i\, \mathrm{sn}(i z \mid 1) \\
\cot(z) = \mathrm{cs}(z \mid 0) & \cot(z) = i\, \mathrm{ns}(i z \mid 1) \\
\csc(z) = \mathrm{ds}(z \mid 0) = \mathrm{ns}(z \mid 0) & \csc(z) = i\, \mathrm{cs}(i z \mid 1) = i\, \mathrm{ds}(i z \mid 1) \\
\sec(z) = \mathrm{dc}(z \mid 0) = \mathrm{nc}(z \mid 0) & \sec(z) = \mathrm{cn}(i z \mid 1) = \mathrm{dn}(i z \mid 1).
\end{array}$$

Representations through related equivalent functions

Each of the six trigonometric functions can be represented through the corresponding hyperbolic function:

$$\begin{array}{ll}
\sin(z) = -i \sinh(i z) & \sin(i z) = i \sinh(z) \\
\cos(z) = \cosh(i z) & \cos(i z) = \cosh(z) \\
\tan(z) = -i \tanh(i z) & \tan(i z) = i \tanh(z) \\
\cot(z) = i \coth(i z) & \cot(i z) = -i \coth(z) \\
\csc(z) = i\, \mathrm{csch}(i z) & \csc(i z) = -i\, \mathrm{csch}(z) \\
\sec(z) = \mathrm{sech}(i z) & \sec(i z) = \mathrm{sech}(z).
\end{array}$$

Relations to inverse functions

Each of the six trigonometric functions is connected with its corresponding inverse trigonometric function by two formulas. One is a simple formula, and the other is much more complicated because of the multivalued nature of the inverse function:

$$\sin(\sin^{-1}(z)) = z \quad \sin^{-1}(\sin(z)) = z \;/;\; -\frac{\pi}{2} < \mathrm{Re}(z) < \frac{\pi}{2} \bigvee \mathrm{Re}(z) = -\frac{\pi}{2} \bigwedge \mathrm{Im}(z) \geq 0 \bigvee \mathrm{Re}(z) = \frac{\pi}{2} \bigwedge \mathrm{Im}(z) \leq 0$$

$$\cos(\cos^{-1}(z)) = z \quad \cos^{-1}(\cos(z)) = z \;/;\; 0 < \mathrm{Re}(z) < \pi \vee \mathrm{Re}(z) = 0 \wedge \mathrm{Im}(z) \geq 0 \vee \mathrm{Re}(z) = \pi \wedge \mathrm{Im}(z) \leq 0$$

$$\tan(\tan^{-1}(z)) = z \quad \tan^{-1}(\tan(z)) = z \;/;\; |\mathrm{Re}(z)| < \frac{\pi}{2} \bigvee \mathrm{Re}(z) = -\frac{\pi}{2} \bigwedge \mathrm{Im}(z) < 0 \bigvee \mathrm{Re}(z) = \frac{\pi}{2} \bigwedge \mathrm{Im}(z) > 0$$

$$\cot(\cot^{-1}(z)) = z \quad \cot^{-1}(\cot(z)) = z \;/;\; |\mathrm{Re}(z)| < \frac{\pi}{2} \bigvee \mathrm{Re}(z) = -\frac{\pi}{2} \bigwedge \mathrm{Im}(z) < 0 \bigvee \mathrm{Re}(z) = \frac{\pi}{2} \bigwedge \mathrm{Im}(z) \geq 0$$

$$\csc(\csc^{-1}(z)) = z \quad \csc^{-1}(\csc(z)) = z \;/;\; |\mathrm{Re}(z)| < \frac{\pi}{2} \bigvee \mathrm{Re}(z) = -\frac{\pi}{2} \bigwedge \mathrm{Im}(z) \leq 0 \bigvee \mathrm{Re}(z) = \frac{\pi}{2} \bigwedge \mathrm{Im}(z) \geq 0$$

$$\sec(\sec^{-1}(z)) = z \quad \sec^{-1}(\sec(z)) = z \;/;\; 0 < \mathrm{Re}(z) < \pi \vee \mathrm{Re}(z) = 0 \wedge \mathrm{Im}(z) \geq 0 \vee \mathrm{Re}(z) = \pi \wedge \mathrm{Im}(z) \leq 0.$$

Representations through other trigonometric functions

Each of the six trigonometric functions can be represented by any other trigonometric function as a rational function of that function with linear arguments. For example, the sine function can be representative as a group-defining function because the other five functions can be expressed as follows:

$$\cos(z) = \sin\left(\frac{\pi}{2} - z\right) \qquad \cos^2(z) = 1 - \sin^2(z)$$

$$\tan(z) = \frac{\sin(z)}{\cos(z)} = \frac{\sin(z)}{\sin\left(\frac{\pi}{2} - z\right)} \qquad \tan^2(z) = \frac{\sin^2(z)}{1 - \sin^2(z)}$$

$$\cot(z) = \frac{\cos(z)}{\sin(z)} = \frac{\sin\left(\frac{\pi}{2} - z\right)}{\sin(z)} \qquad \cot^2(z) = \frac{1 - \sin^2(z)}{\sin^2(z)}$$

$$\csc(z) = \frac{1}{\sin(z)} \qquad \csc^2(z) = \frac{1}{\sin^2(z)}$$

$$\sec(z) = \frac{1}{\cos(z)} = \frac{1}{\sin\left(\frac{\pi}{2} - z\right)} \qquad \sec^2(z) = \frac{1}{1 - \sin^2(z)}.$$

All six trigonometric functions can be transformed into any other trigonometric function of this group if the argument z is replaced by $p\,\pi/2 + q\,z$ with $q^2 = 1 \bigwedge p \in \mathbb{Z}$:

$$\sin(-z - 2\pi) = -\sin(z) \qquad \sin(z - 2\pi) = \sin(z)$$

$$\sin\left(-z - \frac{3\pi}{2}\right) = \cos(z) \qquad \sin\left(z - \frac{3\pi}{2}\right) = \cos(z)$$

$$\sin(-z - \pi) = \sin(z) \qquad \sin(z - \pi) = -\sin(z)$$

$$\sin\left(-z - \frac{\pi}{2}\right) = -\cos(z) \qquad \sin\left(z - \frac{\pi}{2}\right) = -\cos(z)$$

$$\sin\left(z + \frac{\pi}{2}\right) = \cos(z) \qquad \sin\left(\frac{\pi}{2} - z\right) = \cos(z)$$

$$\sin(z + \pi) = -\sin(z) \qquad \sin(\pi - z) = \sin(z)$$

$$\sin\left(z + \frac{3\pi}{2}\right) = -\cos(z) \qquad \sin\left(\frac{3\pi}{2} - z\right) = -\cos(z)$$

$$\sin(z + 2\pi) = \sin(z) \qquad \sin(2\pi - z) = -\sin(z)$$

$\cos(-z - 2\pi) == \cos(z) \quad \cos(z - 2\pi) == \cos(z)$
$\cos\left(-z - \frac{3\pi}{2}\right) == \sin(z) \quad \cos\left(z - \frac{3\pi}{2}\right) == -\sin(z)$
$\cos(-z - \pi) == -\cos(z) \quad \cos(z - \pi) == -\cos(z)$
$\cos\left(-z - \frac{\pi}{2}\right) == -\sin(z) \quad \cos\left(z - \frac{\pi}{2}\right) == \sin(z)$
$\cos\left(z + \frac{\pi}{2}\right) == -\sin(z) \quad \cos\left(\frac{\pi}{2} - z\right) == \sin(z)$
$\cos(z + \pi) == -\cos(z) \quad \cos(\pi - z) == -\cos(z)$
$\cos\left(z + \frac{3\pi}{2}\right) == \sin(z) \quad \cos\left(\frac{3\pi}{2} - z\right) == -\sin(z)$
$\cos(z + 2\pi) == \cos(z) \quad \cos(2\pi - z) == \cos(z)$

$\tan(-z - \pi) == -\tan(z) \quad \tan(z - \pi) == \tan(z)$
$\tan\left(-z - \frac{\pi}{2}\right) == \cot(z) \quad \tan\left(z - \frac{\pi}{2}\right) == -\cot(z)$
$\tan\left(z + \frac{\pi}{2}\right) == -\cot(z) \quad \tan\left(\frac{\pi}{2} - z\right) == \cot(z)$
$\tan(z + \pi) == \tan(z) \quad \tan(\pi - z) == -\tan(z)$

$\cot(-z - \pi) == -\cot(z) \quad \cot(z - \pi) == \cot(z)$
$\cot\left(-z - \frac{\pi}{2}\right) == \tan(z) \quad \cot\left(z - \frac{\pi}{2}\right) == -\tan(z)$
$\cot\left(z + \frac{\pi}{2}\right) == -\tan(z) \quad \cot\left(\frac{\pi}{2} - z\right) == \tan(z)$
$\cot(z + \pi) == \cot(z) \quad \cot(\pi - z) == -\cot(z)$

$\csc(-z - 2\pi) == -\csc(z) \quad \csc(z - 2\pi) == \csc(z)$
$\csc\left(-z - \frac{3\pi}{2}\right) == \sec(z) \quad \csc\left(z - \frac{3\pi}{2}\right) == \sec(z)$
$\csc(-z - \pi) == \csc(z) \quad \csc(z - \pi) == -\csc(z)$
$\csc\left(-z - \frac{\pi}{2}\right) == -\sec(z) \quad \csc\left(z - \frac{\pi}{2}\right) == -\sec(z)$
$\csc\left(z + \frac{\pi}{2}\right) == \sec(z) \quad \csc\left(\frac{\pi}{2} - z\right) == \sec(z)$
$\csc(z + \pi) == -\csc(z) \quad \csc(\pi - z) == \csc(z)$
$\csc\left(z + \frac{3\pi}{2}\right) == -\sec(z) \quad \csc\left(\frac{3\pi}{2} - z\right) == -\sec(z)$
$\csc(z + 2\pi) == \csc(z) \quad \csc(2\pi - z) == -\csc(z)$

$\sec(-z - 2\pi) == \sec(z) \quad \sec(z - 2\pi) == \sec(z)$
$\sec\left(-z - \frac{3\pi}{2}\right) == \csc(z) \quad \sec\left(z - \frac{3\pi}{2}\right) == -\csc(z)$
$\sec(-z - \pi) == -\sec(z) \quad \sec(z - \pi) == -\sec(z)$
$\sec\left(-z - \frac{\pi}{2}\right) == -\csc(z) \quad \sec\left(z - \frac{\pi}{2}\right) == \csc(z)$
$\sec\left(z + \frac{\pi}{2}\right) == -\csc(z) \quad \sec\left(\frac{\pi}{2} - z\right) == \csc(z)$
$\sec(z + \pi) == -\sec(z) \quad \sec(\pi - z) == -\sec(z)$
$\sec\left(z + \frac{3\pi}{2}\right) == \csc(z) \quad \sec\left(\frac{3\pi}{2} - z\right) == -\csc(z)$
$\sec(z + 2\pi) == \sec(z) \quad \sec(2\pi - z) == \sec(z).$

The best-known properties and formulas for trigonometric functions

Real values for real arguments

For real values of argument z, the values of all the trigonometric functions are real (or infinity).

In the points $z = 2\pi n/m /; n \in \mathbb{Z} \bigwedge m \in \mathbb{Z}$, the values of trigonometric functions are algebraic. In several cases they can even be rational numbers or integers (like $\sin(\pi/2) = 1$ or $\sin(\pi/6) = 1/2$). The values of trigonometric functions can be expressed using only square roots if $n \in \mathbb{Z}$ and m is a product of a power of 2 and distinct Fermat primes $\{3, 5, 17, 257, \ldots\}$.

Simple values at zero

All trigonometric functions have rather simple values for arguments $z == 0$ and $z == \pi/2$:

$$\sin(0) == 0 \quad \sin\left(\frac{\pi}{2}\right) == 1$$
$$\cos(0) == 1 \quad \cos\left(\frac{\pi}{2}\right) == 0$$
$$\tan(0) == 0 \quad \tan\left(\frac{\pi}{2}\right) == \tilde{\infty}$$
$$\cot(0) == \tilde{\infty} \quad \cot\left(\frac{\pi}{2}\right) == 0$$
$$\csc(0) == \tilde{\infty} \quad \csc\left(\frac{\pi}{2}\right) == 1$$
$$\sec(0) == 1 \quad \sec\left(\frac{\pi}{2}\right) == \tilde{\infty}.$$

Analyticity

All trigonometric functions are defined for all complex values of z, and they are analytical functions of z over the whole complex z-plane and do not have branch cuts or branch points. The two functions $\sin(z)$ and $\cos(z)$ are entire functions with an essential singular point at $z = \tilde{\infty}$. All other trigonometric functions are meromorphic functions with simple poles at points $z == \pi k /; k \in \mathbb{Z}$ for $\csc(z)$ and $\cot(z)$, and at points $z == \pi/2 + \pi k /; k \in \mathbb{Z}$ for $\sec(z)$ and $\tan(z)$.

Periodicity

All trigonometric functions are periodic functions with a real period (2π or π):

$$\sin(z) == \sin(z + 2\pi) \quad \sin(z + 2\pi k) == \sin(z) /; k \in \mathbb{Z}$$
$$\cos(z) == \cos(z + 2\pi) \quad \cos(z + 2\pi k) == \cos(z) /; k \in \mathbb{Z}$$
$$\tan(z) == \tan(z + \pi) \quad \tan(z + \pi k) == \tan(z) /; k \in \mathbb{Z}$$
$$\cot(z) == \cot(z + \pi) \quad \cot(z + \pi k) == \cot(z) /; k \in \mathbb{Z}$$
$$\csc(z) == \csc(z + 2\pi) \quad \csc(z + 2\pi k) == \csc(z) /; k \in \mathbb{Z}$$
$$\sec(z) == \sec(z + 2\pi) \quad \sec(z + 2\pi k) == \sec(z) /; k \in \mathbb{Z}.$$

Parity and symmetry

All trigonometric functions have parity (either odd or even) and mirror symmetry:

$$\sin(-z) == -\sin(z) \quad \sin(\bar{z}) == \overline{\sin(z)}$$
$$\cos(-z) == \cos(z) \quad \cos(\bar{z}) == \overline{\cos(z)}$$
$$\tan(-z) == -\tan(z) \quad \tan(\bar{z}) == \overline{\tan(z)}$$
$$\cot(-z) == -\cot(z) \quad \cot(\bar{z}) == \overline{\cot(z)}$$
$$\csc(-z) == -\csc(z) \quad \csc(\bar{z}) == \overline{\csc(z)}$$
$$\sec(-z) == \sec(z) \quad \sec(\bar{z}) == \overline{\sec(z)}.$$

Simple representations of derivatives

The derivatives of all trigonometric functions have simple representations that can be expressed through other trigonometric functions:

$$\frac{\partial \sin(z)}{\partial z} == \cos(z) \qquad \frac{\partial \cos(z)}{\partial z} == -\sin(z) \qquad \frac{\partial \tan(z)}{\partial z} == \sec^2(z)$$
$$\frac{\partial \cot(z)}{\partial z} == -\csc^2(z) \qquad \frac{\partial \csc(z)}{\partial z} == -\cot(z)\csc(z) \qquad \frac{\partial \sec(z)}{\partial z} == \sec(z)\tan(z).$$

Simple differential equations

The solutions of the simplest second-order linear ordinary differential equation with constant coefficients can be represented through $\sin(z)$ and $\cos(z)$:

$$w''(z) + w(z) == 0 \text{ /; } w(z) == \cos(z) \land w(0) == 1 \land w'(0) == 0$$
$$w''(z) + w(z) == 0 \text{ /; } w(z) == \sin(z) \land w(0) == 0 \land w'(0) == 1$$
$$w''(z) + w(z) == 0 \text{ /; } w(z) == c_1 \cos(z) + c_2 \sin(z).$$

All six trigonometric functions satisfy first-order nonlinear differential equations:

$$w'(z) - \sqrt{1-(w(z))^2} == 0 \text{ /; } w(z) == \sin(z) \bigwedge w(0) == 0 \bigwedge |\mathrm{Re}(z)| < \frac{\pi}{2}$$
$$w'(z) - \sqrt{1-(w(z))^2} == 0 \text{ /; } w(z) == \cos(z) \bigwedge w(0) == 1 \bigwedge |\mathrm{Re}(z)| < \frac{\pi}{2}$$
$$w'(z) - w(z)^2 - 1 == 0 \text{ /; } w(z) == \tan(z) \land w(0) == 0$$
$$w'(z) + w(z)^2 + 1 == 0 \text{ /; } w(z) == \cot(z) \bigwedge w\left(\frac{\pi}{2}\right) == 0$$
$$w'(z)^2 - w(z)^4 + w(z)^2 == 0 \text{ /; } w(z) == \csc(z)$$
$$w'(z)^2 - w(z)^4 + w(z)^2 == 0 \text{ /; } w(z) == \sec(z).$$

Applications of trigonometric functions

Triangle theorems

The prime application of the trigonometric functions are triangle theorems. In a triangle, a, b, and c represent the lengths of the sides opposite to the angles, Δ the area , R the circumradius, and r the inradius. Then the following identities hold:

$$\alpha + \beta + \gamma == \pi$$
$$\frac{\sin(\alpha)}{a} == \frac{\sin(\beta)}{b} == \frac{\sin(\gamma)}{c}$$

$$\sin(\alpha)\sin(\beta)\sin(\gamma) == \frac{\Delta}{2R^2} \qquad \sin(\alpha) == \frac{2\Delta}{bc}$$
$$\cos(\alpha) == \frac{b^2+c^2-a^2}{2bc} \qquad \cot(\alpha) == \frac{b^2+c^2-a^2}{4\Delta}$$
$$\sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right)\sin\left(\frac{\gamma}{2}\right) == \frac{r}{4R} \qquad \cos(\alpha) + \cos(\beta) + \cos(\gamma) == 1 + \frac{r}{R}$$

$$\cot(\alpha) + \cot(\beta) + \cot(\gamma) == \frac{a^2 + b^2 + c^2}{4\,\Delta}$$

$$\tan(\alpha) + \tan(\beta) + \tan(\gamma) == \tan(\alpha)\tan(\beta)\tan(\gamma)$$

$$\cot(\alpha)\cot(\beta) + \cot(\alpha)\cot(\gamma) + \cot(\beta)\cot(\gamma) == 1$$

$$\cos^2(\alpha) + \cos^2(\beta) + \cos^2(\gamma) == 1 - 2\cos(\alpha)\cos(\beta)\cos(\gamma)$$

$$\frac{\tan\left(\frac{\alpha}{2}\right)\tan\left(\frac{\beta}{2}\right)}{\tan\left(\frac{\alpha}{2}\right) + \tan\left(\frac{\beta}{2}\right)} == \frac{r}{c}.$$

For a right-angle triangle the following relations hold:

$$\sin(\alpha) == \frac{a}{c} \;/;\; \gamma == \frac{\pi}{2} \quad \cos(\alpha) == \frac{b}{c} \;/;\; \gamma == \frac{\pi}{2}$$

$$\tan(\alpha) == \frac{a}{b} \;/;\; \gamma == \frac{\pi}{2} \quad \cot(\alpha) == \frac{b}{a} \;/;\; \gamma == \frac{\pi}{2}$$

$$\csc(\alpha) == \frac{c}{a} \;/;\; \gamma == \frac{\pi}{2} \quad \sec(\alpha) == \frac{c}{b} \;/;\; \gamma == \frac{\pi}{2}.$$

Other applications

Because the trigonometric functions appear virtually everywhere in quantitative sciences, it is impossible to list their numerous applications in teaching, science, engineering, and art.

Introduction to the Cosecant Function

Defining the cosecant function

The cosecant function is an old mathematical function. It was mentioned in the works of G. J. von Lauchen Rheticus (1596) and E. Gunter (around 1620). It was widely used by L. Euler (1748) and T. Olivier, Wait, and Jones (1881).

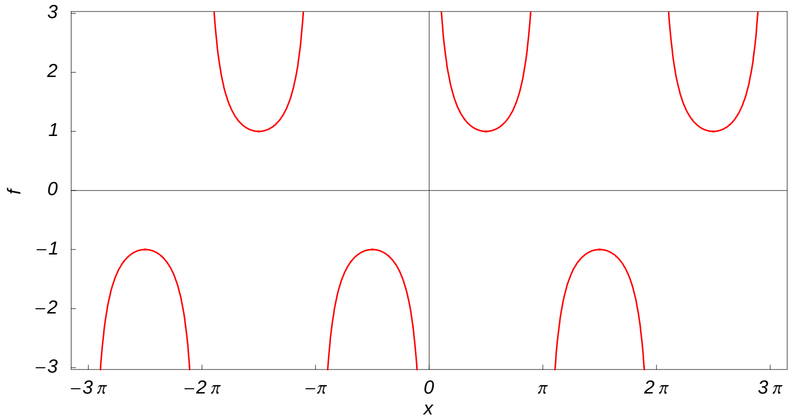
The classical definition of the cosecant function for real arguments is: "the cosecant of an angle α in a right-angle triangle is the ratio of the length of the hypotenuse to the length of the opposite leg." This description of $\csc(\alpha)$ is valid for $0 < \alpha < \pi/2$ when this triangle is nondegenerate. This approach to the cosecant can be expanded to arbitrary real values of α if the arbitrary point $\{x, y\}$ in the x,y-Cartesian plane is considered and $\csc(\alpha)$ is defined as the ratio $\left(x^2 + y^2\right)^{1/2} / y$ assuming that α is the value of the angle between the positive direction of the x-axis and the direction from the origin to the point $\{x, y\}$.

Comparing the classical definition with the definition of the sine function shows that the following formula can also be used as a definition of the cosecant function:

$$\csc(z) == \frac{1}{\sin(z)}.$$

A quick look at the cosecant function

Here is a graphic of the cosecant function $f(x) = \csc(x)$ for real values of its argument x.



Representation through more general functions

The cosecant function $\csc(z)$ can be represented using more general mathematical functions. As the ratio of one divided by the sine function that is a particular case of the generalized hypergeometric, Bessel, Struve, and Mathieu functions, the cosecant function can also be represented as ratios of one and those special functions. Here are some examples:

$$\csc(z) = \frac{1}{z\,{}_0F_1\left(;\frac{3}{2};-\frac{z^2}{4}\right)}$$

$$\csc(z) = \sqrt{\frac{2}{\pi}}\,\frac{1}{\sqrt{z}\,J_{\frac{1}{2}}(z)}$$

$$\csc(z) = \sqrt{\frac{2}{\pi}}\,\frac{i}{\sqrt{i\,z}\,I_{\frac{1}{2}}(i\,z)}$$

$$\csc(z) = \sqrt{\frac{2}{\pi}}\,\frac{1}{\sqrt{z}\,Y_{-\frac{1}{2}}(z)}$$

$$\csc(z) = \frac{\sqrt{\frac{2}{\pi}}}{\sqrt{z}\,\boldsymbol{H}_{-\frac{1}{2}}(z)}$$

$$\csc(z) = \frac{1}{\mathrm{Se}(1,0,z)}.$$

But these representations are not very useful because they include complicated special functions in the denominators.

It is more useful to write the cosecant function as particular cases of one special function. That can be done using doubly periodic Jacobi elliptic functions that degenerate into the cosecant function when their second parameter is equal to 0 or 1:

$$\csc(z) = \text{ds}(z \mid 0) = \text{ns}(z \mid 0) = \text{dc}\left(\frac{\pi}{2} - z \,\middle|\, 0\right) = \text{nc}\left(\frac{\pi}{2} - z \,\middle|\, 0\right) = i\,\text{cs}(i\,z \mid 1) = i\,\text{ds}(i\,z \mid 1) = \text{cn}\left(\frac{\pi\,i}{2} - i\,z \,\middle|\, 1\right) = \text{dn}\left(\frac{\pi\,i}{2} - i\,z \,\middle|\, 1\right).$$

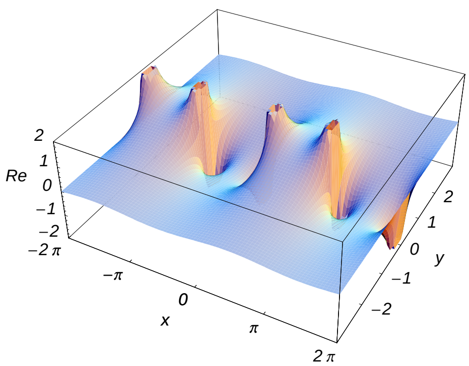
Definition of the cosecant function for a complex argument

In the complex z-plane, the function $\csc(z)$ is defined using $\sin(z)$ or the exponential function e^w in the points $i\,z$ and $-i\,z$ through the formula:

$$\csc(z) = \frac{1}{\sin(z)} = \frac{2\,i}{e^{i\,z} - e^{-i\,z}}.$$

In the points $z = \pi\,k \,/;\, k \in \mathbb{Z}$, where $\sin(z)$ is zero, the denominator of the last formula equals zero and $\csc(z)$ has singularities (poles of the first order).

Here are two graphics showing the real and imaginary parts of the cosecant function over the complex plane.



The best-known properties and formulas for the cosecant function

Values in points

Using the connection between the sine and cosecant functions, the following table of cosecant function values for angles between 0 and $2\,\pi$ can be derived:

$\csc(0) = \tilde\infty$	$\csc\left(\frac{\pi}{6}\right) = 2$	$\csc\left(\frac{\pi}{4}\right) = \sqrt{2}$	$\csc\left(\frac{\pi}{3}\right) = \frac{2}{\sqrt{3}}$
$\csc\left(\frac{\pi}{2}\right) = 1$	$\csc\left(\frac{2\pi}{3}\right) = \frac{2}{\sqrt{3}}$	$\csc\left(\frac{3\pi}{4}\right) = \sqrt{2}$	$\csc\left(\frac{5\pi}{6}\right) = 2$
$\csc(\pi) = \tilde\infty$	$\csc\left(\frac{7\pi}{6}\right) = -2$	$\csc\left(\frac{5\pi}{4}\right) = -\sqrt{2}$	$\csc\left(\frac{4\pi}{3}\right) = -\frac{2}{\sqrt{3}}$
$\csc\left(\frac{3\pi}{2}\right) = -1$	$\csc\left(\frac{5\pi}{3}\right) = -\frac{2}{\sqrt{3}}$	$\csc\left(\frac{7\pi}{4}\right) = -\sqrt{2}$	$\csc\left(\frac{11\pi}{6}\right) = -2$

$$\csc(2\,\pi) = \tilde\infty \quad \csc(\pi\,m) = \tilde\infty \,/;\, m \in \mathbb{Z} \quad \csc\left(\pi\left(\frac{1}{2} + m\right)\right) = (-1)^m \,/;\, m \in \mathbb{Z}\,.$$

General characteristics

For real values of argument z, the values of $\csc(z)$ are real.

In the points $z = 2\pi n / m /; n \in \mathbb{Z} \bigwedge m \in \mathbb{Z}$, the values of $\csc(z)$ are algebraic. In several cases they can be integers -2, -1, 1, or 2:

$$\csc\left(-\frac{\pi}{2}\right) = -1 \quad \csc\left(-\frac{\pi}{6}\right) = -2 \quad \csc\left(\frac{\pi}{2}\right) = 1 \quad \csc\left(\frac{\pi}{6}\right) = 2.$$

The values of $\csc\left(\frac{n\pi}{m}\right)$ can be expressed using only square roots if $n \in \mathbb{Z}$ and m is a product of a power of 2 and distinct Fermat primes {3, 5, 17, 257, …}.

The function $\csc(z)$ is an analytical function of z that is defined over the whole complex z-plane and does not have branch cuts and branch points. It has an infinite set of singular points:

(a) $z = \pi k /; k \in \mathbb{Z}$ are the simple poles with residues $(-1)^k$.
(b) $z = \tilde{\infty}$ is an essential singular point.

It is a periodic function with the real period 2π:

$$\csc(z + 2\pi) = \csc(z)$$

$$\csc(z) = \csc(z + 2\pi k) /; k \in \mathbb{Z} \quad \csc(z) = (-1)^k \csc(z + \pi k) /; k \in \mathbb{Z}.$$

The function $\csc(z)$ is an odd function with mirror symmetry:

$$\csc(-z) = -\csc(z) \quad \csc(\bar{z}) = \overline{\csc(z)}.$$

Differentiation

The first derivative of $\csc(z)$ has simple representations using either the $\cot(z)$ function or the $\csc(z)$ function:

$$\frac{\partial \csc(z)}{\partial z} = -\cot(z)\csc(z).$$

The n^{th} derivative of $\csc(z)$ has much more complicated representations than symbolic n^{th} derivatives for $\sin(z)$ and $\cos(z)$:

$$\frac{\partial^n \csc(z)}{\partial z^n} = \csc(z)\left(\delta_n + (n+1)!\sum_{k=0}^{n}\sum_{j=0}^{\left\lfloor\frac{k-1}{2}\right\rfloor}\frac{(-1)^{j+n}\, 2^{1-k}\,(k-2j)^n \csc^k(z)}{(k+1)\, j!\,(k-j)!\,(n-k)!}\cos\left(\frac{\pi(k-n)}{2} + (k-2j)z\right)\right) /; n \in \mathbb{N},$$

where δ_n is the Kronecker delta symbol: $\delta_0 = 1$ and $\delta_n = 0 /; n \neq 0$.

Ordinary differential equation

The function $\csc(z)$ satisfies the following first-order nonlinear differential equation:

$$w'(z)^2 - w(z)^4 + w(z)^2 = 0 /; w(z) = \csc(z).$$

Series representation

The function $\csc(z)$ has the following Laurent series expansion at the origin that converges for all finite values z with $0 < |z| < \pi$:

$$\csc(z) = \frac{1}{z} + \frac{z}{6} + \frac{7z^3}{360} + \ldots = \frac{1}{z} + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}\, 2\, \left(2^{2k-1} - 1\right) B_{2k}\, z^{2k-1}}{(2k)!},$$

where B_{2k} are the Bernoulli numbers.

The cosecant function csc(z) can also be represented using other kinds of series by the following formulas:

$$\csc(z) = \frac{1}{z} + 2z \sum_{k=1}^{\infty} \frac{(-1)^k}{z^2 - \pi^2 k^2} \;/;\; \frac{z}{\pi} \notin \mathbb{Z}$$

$$\csc^2(z) = \sum_{k=-\infty}^{\infty} \frac{1}{(z - \pi k)^2} \;/;\; \frac{z}{\pi} \notin \mathbb{Z}.$$

Integral representation

The function csc(z) has well-known integral representation through the following definite integral along the positive part of the real axis:

$$\csc(z) = \frac{1}{\pi} \int_0^{\infty} \frac{1}{t^2 + t}\, t^{z/\pi}\, dt \;/;\; 0 < \mathrm{Re}(z) < \pi.$$

Product representation

The famous infinite product representation for sin(z) can be easily rewritten as the following product representation for the cosecant function:

$$\csc(z) = \frac{1}{z} \prod_{k=1}^{\infty} \frac{\pi^2 k^2}{\pi^2 k^2 - z^2}.$$

Limit representation

The cosecant function has the following limit representation:

$$\csc(z) = \lim_{n \to \infty} \sum_{k=-n}^{n} \frac{(-1)^k}{z - \pi k} \;/;\; \frac{z}{\pi} \notin \mathbb{Z}.$$

Indefinite integration

Indefinite integrals of expressions that contain the cosecant function can sometimes be expressed using elementary functions. However, special functions are frequently needed to express the results even when the integrands have a simple form (if they can be evaluated in closed form). Here are some examples:

$$\int \csc(z)\, dz = \log\left(\tan\left(\frac{z}{2}\right)\right)$$

$$\int \sqrt{\csc(z)}\, dz = -2 \csc^{\frac{1}{2}}(z) \sin^{\frac{1}{2}}(z)\, F\left(\frac{1}{4}(\pi - 2z) \,\middle|\, 2\right)$$

$$\int \csc^{\nu}(a z)\, dz = -\frac{\cos(a z) \csc^{\nu-1}(a z) \sin^2(a z)^{\frac{\nu-1}{2}}}{a}\, {}_2F_1\left(\frac{1}{2}, \frac{\nu+1}{2}; \frac{3}{2}; \cos^2(a z)\right).$$

Definite integration

Definite integrals that contain the cosecant function are sometimes simple. For example, the famous Catalan constant C can be defined as the value of the following integral:

$$\frac{1}{2}\int_0^{\pi/2} t\csc(t)\,dt = C.$$

This constant also appears in the following integral:

$$\int_0^{\frac{\pi}{2}} t^2\csc(t)\,dt = 2\,C\,\pi - \frac{7\,\zeta(3)}{2}.$$

Some special functions can be used to evaluate more complicated definite integrals. For example, polylogarithmical, zeta, and gamma functions are needed to express the following integrals:

$$\int_0^{\frac{\pi}{3}} t^3\csc^2(t)\,dt = \frac{1}{162}\left(6\,i\,\pi^3 - 2\sqrt{3}\,\pi^3 + 54\,\pi^2\log\left(-\sqrt[3]{-1}\left(-1+(-1)^{2/3}\right)\right) + 162\,i\,\pi\,\mathrm{Li}_2\left(-\sqrt[3]{-1}\right) + 243\,\mathrm{Li}_3\left(-\sqrt[3]{-1}\right) - 243\,\zeta(3)\right)$$

$$\int_0^{\frac{\pi}{2}} \csc^a(t)\,dt = \frac{\sqrt{\pi}}{2\,\Gamma\left(1-\frac{a}{2}\right)}\,\Gamma\left(\frac{1-a}{2}\right) /; \mathrm{Re}(a) < 1.$$

Finite summation

The following finite sums that contain the cosecant function have simple values:

$$\sum_{k=0}^{n-1}\csc^2\left(\frac{\pi k}{n}+z\right) = n^2\csc^2(n z) /; n\in\mathbb{N}^+$$

$$\sum_{k=1}^{n-1}\csc^2\left(\frac{k\pi}{n}\right) = \frac{1}{3}\left(n^2-1\right) /; n\in\mathbb{N}^+$$

$$\sum_{k=1}^{n}\csc^2\left(\frac{k\pi}{2n+1}\right) = \frac{2}{3}\,n\,(n+1) /; n\in\mathbb{N}^+$$

$$\sum_{k=1}^{\left\lfloor\frac{n-1}{2}\right\rfloor}\csc^2\left(\frac{k\pi}{n}\right) = \frac{1}{12}\left(2\,n^2 - 3\,(-1)^n - 5\right) /; n\in\mathbb{N}^+$$

$$\sum_{k=0}^{\left\lfloor\frac{n-2}{2}\right\rfloor}\csc^2\left(\frac{(2k+1)\pi}{2n}\right) = \frac{1}{4}\left(2\,n^2 + (-1)^n - 1\right) /; n\in\mathbb{N}^+$$

$$\sum_{k=0}^{n}\csc\left(\frac{z}{2^k}\right) = \cot\left(\frac{z}{2^{n+1}}\right) - \cot(z) /; n\in\mathbb{N}^+.$$

Infinite summation

The following infinite sum that contains the cosecant has a simple value:

$$\sum_{k=1}^{\infty} \frac{\csc\left(k \pi \sqrt{2}\right)}{k^3} = -\frac{13 \pi^3}{360 \sqrt{2}}.$$

Finite products

The following finite product from the cosecant can also be represented using the cosecant function:

$$\prod_{k=0}^{n-1} \csc\left(z + \frac{\pi k}{n}\right) = 2^{n-1} \csc(n z) /; n \in \mathbb{N}^+.$$

Addition formulas

The cosecant of a sum and the cosecant of a difference can be represented by the formulas that follow from corresponding formulas for the sine of a sum and the sine of a difference:

$$\csc(a+b) = \frac{1}{\cos(b)\sin(a) + \cos(a)\sin(b)}$$

$$\csc(a-b) = \frac{1}{\cos(b)\sin(a) - \cos(a)\sin(b)}.$$

Multiple arguments

In the case of multiple arguments z, $2z$, $3z$, …, the function $\csc(z)$ can be represented as a rational function that contains powers of cosecants and secants. Here are two examples:

$$\csc(2z) = \frac{1}{2} \csc(z) \sec(z)$$

$$\csc(3z) = \frac{\csc^3(z)}{3\csc^2(z) - 4}.$$

Half-angle formulas

The cosecant of a half-angle can be represented by the following simple formula that is valid in a vertical strip:

$$\csc\left(\frac{z}{2}\right) = \frac{\sqrt{2}}{\sqrt{1-\cos(z)}} /; 0 < \mathrm{Re}(z) < 2\pi \lor \mathrm{Re}(z) = 0 \land \mathrm{Im}(z) > 0 \lor \mathrm{Re}(z) = 2\pi \land \mathrm{Im}(z) < 0.$$

To make this formula correct for all complex z, a complicated prefactor is needed:

$$\csc\left(\frac{z}{2}\right) = \frac{c(z)}{\sqrt{1-\cos(z)}} /; c(z) = \sqrt{2}\, (-1)^{\left\lfloor \frac{\mathrm{Re}(z)}{2\pi} \right\rfloor} \left(1 - \left(1 + (-1)^{\left\lfloor \frac{\mathrm{Re}(z)}{2\pi} \right\rfloor + \left\lfloor -\frac{\mathrm{Re}(z)}{2\pi} \right\rfloor}\right) \theta(-\mathrm{Im}(z))\right),$$

where $c(z)$ contains the unit step, real part, imaginary part, and the floor functions.

Sums of two direct functions

The sum and difference of two cosecant functions can be described by the following formulas:

$$\csc(a)+\csc(b) = 2\sin\left(\frac{a}{2}+\frac{b}{2}\right)\cos\left(\frac{a}{2}-\frac{b}{2}\right)\csc(a)\csc(b)$$

$$\csc(a)-\csc(b) = -2\sin\left(\frac{a}{2}-\frac{b}{2}\right)\cos\left(\frac{a}{2}+\frac{b}{2}\right)\csc(a)\csc(b).$$

Products involving the direct function

The product of two cosecants and the product of the cosecant and secant have the following representations:

$$\csc(a)\csc(b) = \frac{2}{\cos(a-b)-\cos(a+b)}$$

$$\csc(a)\sec(b) = \frac{2}{\sin(a-b)+\sin(a+b)}.$$

Inequalities

Some inequalities for the cosecant function can be easily derived from the corresponding inequalities for the sine function:

$$\csc(x) > \frac{1}{x} \;/;\; x > 0$$

$$|\csc(x)| \geq 1 \;/;\; x \in \mathbb{R}$$

$$x\csc(x) < \frac{\pi}{2} \;/;\; -\frac{\pi}{2} < x < \frac{\pi}{2}.$$

Relations with its inverse function

There are simple relations between the function $\csc(z)$ and its inverse function $\csc^{-1}(z)$:

$$\csc\left(\csc^{-1}(z)\right) = z \quad \csc^{-1}(\csc(z)) = z \;/;\; |\mathrm{Re}(z)| < \tfrac{\pi}{2} \bigvee \mathrm{Re}(z) = -\tfrac{\pi}{2} \bigwedge \mathrm{Im}(z) \leq 0 \bigvee \mathrm{Re}(z) = \tfrac{\pi}{2} \bigwedge \mathrm{Im}(z) \geq 0.$$

The second formula is valid at least in the vertical strip $-\frac{\pi}{2} < \mathrm{Re}(z) < \frac{\pi}{2}$. Outside of this strip a much more complicated relation (that contains the unit step, real part, and the floor functions) holds:

$$\csc^{-1}(\csc(z)) = (-1)^{\left\lfloor \frac{\mathrm{Re}(z)}{\pi}-\frac{1}{2}\right\rfloor}\left(\left(1+(-1)^{\left\lfloor \frac{\mathrm{Re}(z)}{\pi}+\frac{1}{2}\right\rfloor+\left\lfloor -\frac{\mathrm{Re}(z)}{\pi}-\frac{1}{2}\right\rfloor}\right)\theta(-\mathrm{Im}(z))-1\right)\left(z-\pi\left\lfloor\frac{\mathrm{Re}(z)}{\pi}+\frac{1}{2}\right\rfloor+\frac{\pi}{2}\left(1+(-1)^{\left\lfloor \frac{\mathrm{Re}(z)}{\pi}+\frac{1}{2}\right\rfloor+\left\lfloor -\frac{\mathrm{Re}(z)}{\pi}-\frac{1}{2}\right\rfloor}\right)\theta(-\mathrm{Im}(z))\right).$$

Representations through other trigonometric functions

Cosecant and secant functions are connected by a very simple formula that contains the linear function in the argument:

$$\csc(z) = \sec\left(\frac{\pi}{2}-z\right).$$

The cosecant function can also be represented using other trigonometric functions by the following formulas:

$$\csc(z) = \frac{1}{\cos\left(\frac{\pi}{2}-z\right)} \quad \csc(z) = \frac{\tan^2\left(\frac{z}{2}\right)+1}{2\tan\left(\frac{z}{2}\right)} \quad \csc(z) = \frac{\cot^2\left(\frac{z}{2}\right)+1}{2\cot\left(\frac{z}{2}\right)}.$$

Representations through hyperbolic functions

The cosecant function has representations using the hyperbolic functions:

$$\csc(z) = \frac{i}{\sinh(i z)} \quad \csc(z) = \frac{1}{\cosh\left(\frac{\pi i}{2} - i z\right)} \quad \csc(z) = \frac{i\left(1 - \tanh^2\left(\frac{z i}{2}\right)\right)}{2 \tanh\left(\frac{z i}{2}\right)} \quad \csc(z) = \frac{i\left(\coth^2\left(\frac{i z}{2}\right) - 1\right)}{2 \coth\left(\frac{i z}{2}\right)}$$

$$\csc(z) = i \operatorname{csch}(i z) \quad \csc(i z) = -i \operatorname{csch}(z) \quad \csc(z) = \operatorname{sech}\left(\frac{\pi i}{2} - i z\right).$$

Applications

The cosecant function is used throughout mathematics, the exact sciences, and engineering.

Introduction to the Trigonometric Functions in *Mathematica*

Overview

The following shows how the six trigonometric functions are realized in *Mathematica*. Examples of evaluating *Mathematica* functions applied to various numeric and exact expressions that involve the trigonometric functions or return them are shown. These involve numeric and symbolic calculations and plots.

Notations

Mathematica forms of notations

All six trigonometric functions are represented as built-in functions in *Mathematica*. Following *Mathematica*'s general naming convention, the `StandardForm` function names are simply capitalized versions of the traditional mathematics names. Here is a list `trigFunctions` of the six trigonometric functions in `StandardForm`.

```
trigFunctions = {Sin[z], Cos[z], Tan[z], Cot[z], Sec[z], Cos[z]}
```

```
{Sin[z], Cos[z], Tan[z], Cot[z], Sec[z], Cos[z]}
```

Here is a list `trigFunctions` of the six trigonometric functions in `TraditionalForm`.

```
trigFunctions // TraditionalForm
```

$\{\sin(z), \cos(z), \tan(z), \cot(z), \sec(z), \cos(z)\}$

Additional forms of notations

Mathematica also knows the most popular forms of notations for the trigonometric functions that are used in other programming languages. Here are three examples: `CForm`, `TeXForm`, and `FortranForm`.

```
trigFunctions /. {z → 2 π z} // (CForm /@ #) &
```

```
{Sin (2 * Pi * z), Cos (2 * Pi * z), Tan (2 * Pi * z),
 Cot (2 * Pi * z), Sec (2 * Pi * z), Cos (2 * Pi * z)}
```

```
trigFunctions /. {z → 2 π z} // (TeXForm /@ #) &
```

```
{\sin (2\, \pi \, z), \cos (2\, \pi \, z), \tan (2\, \pi \, z), \cot
  (2\, \pi \, z), \sec (2\, \pi \, z), \cos (2\, \pi \, z)}
```

```
trigFunctions /. {z → 2 π z} // (FortranForm /@ #) &
```

```
{Sin (2 * Pi * z), Cos (2 * Pi * z), Tan (2 * Pi * z),
 Cot (2 * Pi * z), Sec (2 * Pi * z), Cos (2 * Pi * z)}
```

Automatic evaluations and transformations

Evaluation for exact, machine-number, and high-precision arguments

For a simple exact argument, *Mathematica* returns exact results. For instance, for the argument $\pi/6$, the `Sin` function evaluates to $1/2$.

```
Sin[π/6]
```

$$\frac{1}{2}$$

```
{Sin[z], Cos[z], Tan[z], Cot[z], Csc[z], Sec[z]} /. z → π/6
```

$$\left\{\frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{1}{\sqrt{3}}, \sqrt{3}, 2, \frac{2}{\sqrt{3}}\right\}$$

For a generic machine-number argument (a numerical argument with a decimal point and not too many digits), a machine number is returned.

```
Cos[3.]
```

```
-0.989992
```

```
{Sin[z], Cos[z], Tan[z], Cot[z], Csc[z], Sec[z]} /. z → 2.
```

```
{0.909297, -0.416147, -2.18504, -0.457658, 1.09975, -2.403}
```

The next inputs calculate 100-digit approximations of the six trigonometric functions at $z = 1$.

```
N[Tan[1], 40]
```

```
1.557407724654902230506974807458360173087
```

```
Cot[1] // N[#, 50] &
```

```
0.64209261593433070300641998659426562023027811391817
```

```
N[{Sin[z], Cos[z], Tan[z], Cot[z], Csc[z], Sec[z]} /. z → 1, 100]
```

```
{0.84147098480789650665250232163029899962256306079837106567275170999191040439123966‌8\
   9486397435430526959,
 0.540302305868139717400936607442976603732310420617922227670097255381100394774471764\
   5179518560871830893,
 1.557407724654902230506974807458360173087250772381520038383946605698861397151727289\
   555099965202242984,
 0.642092615934330703006419986594265620230278113918171379101162280426276856839164672\
   1984829197601968047,
 1.188395105778121216261599452374551003527829834097962625265253666359184367357190487\
   913663568030853023,
 1.850815717680925617911753241398650193470396655094009298835158277858815411261596705\
   921841413287306671}
```

Within a second, it is possible to calculate thousands of digits for the trigonometric functions. The next input calculates 10000 digits for sin(1), cos(1), tan(1), cot(1), sec(1), and csc(1) and analyzes the frequency of the occurrence of the digit *k* in the resulting decimal number.

```
Map[Function[w, {First[#], Length[#]} & /@ Split[Sort[First[RealDigits[w]]]]],
 N[{Sin[z], Cos[z], Tan[z], Cot[z], Csc[z], Sec[z]} /. z → 1, 10 000]]
```

```
{{{0, 983}, {1, 1069}, {2, 1019}, {3, 983}, {4, 972}, {5, 994},
   {6, 994}, {7, 988}, {8, 988}, {9, 1010}}, {{0, 998}, {1, 1034}, {2, 982},
   {3, 1015}, {4, 1013}, {5, 963}, {6, 1034}, {7, 966}, {8, 991}, {9, 1004}},
 {{0, 1024}, {1, 1025}, {2, 1000}, {3, 969}, {4, 1026}, {5, 944}, {6, 999},
   {7, 1001}, {8, 1008}, {9, 1004}}, {{0, 1006}, {1, 1030}, {2, 986},
   {3, 954}, {4, 1003}, {5, 1034}, {6, 999}, {7, 998}, {8, 1009}, {9, 981}},
 {{0, 1031}, {1, 976}, {2, 1045}, {3, 917}, {4, 1001}, {5, 996}, {6, 964},
   {7, 1012}, {8, 982}, {9, 1076}}, {{0, 978}, {1, 1034}, {2, 1016},
   {3, 974}, {4, 987}, {5, 1067}, {6, 943}, {7, 1006}, {8, 1027}, {9, 968}}}
```

Here are 50-digit approximations to the six trigonometric functions at the complex argument $z = 3 + 5\,i$.

```
N[Csc[3 + 5 ⅈ], 100]
```

```
0.001901970423701089996670017296320805840459252512171274310801719695392870034046820‌2\
   96847410109982878354 +
 0.013341591397996678721837322466473194390132347157253190972075437462485814431570118\
    67262664488519840339 ⅈ
```

```
N[{Sin[z], Cos[z], Tan[z], Cot[z], Csc[z], Sec[z]} /. z → 3 + 5 ⅈ, 50]
```

```
{10.4725085339403922766733225368535032711264199503 88 -
  73.4606216956736763667911925050817504072139228144 75 ⅈ,
 -73.467292212645262467746454594833950830814859165299 -
  10.471557674805574377394464224329537808548330651734 ⅈ,
 -0.000025368676207676032417806136707426288195560702602478 +
  0.99991282015135380828209263013972954140566020462086 ⅈ,
 -0.000025373100044545977383763346789469656754050037355986 -
  1.0000871868058967743285316881045218577131612831891 ⅈ,
 0.0019019704237010899966700172963208058404592525121713 +
  0.013341591397996678721837322466473194390132347157253 ⅈ,
 -0.013340476530549737487361100811100839468470481725038 +
  0.0019014661516951513089519270013254277867588978133499 ⅈ}
```

Mathematica always evaluates mathematical functions with machine precision, if the arguments are machine numbers. In this case, only six digits after the decimal point are shown in the results. The remaining digits are suppressed, but can be displayed using the function `InputForm`.

```
{Sin[2.], N[Sin[2]], N[Sin[2], 16], N[Sin[2], 5], N[Sin[2], 20]}
```

```
{0.909297, 0.909297, 0.909297, 0.909297, 0.90929742682568169540}
```

```
% // InputForm
```

```
{0.9092974268256817, 0.9092974268256817, 0.9092974268256817, 0.9092974268256817,
 0.909297426825681695396019865911745`20}
```

```
Precision[%%]
```

```
16
```

Simplification of the argument

Mathematica uses symmetries and periodicities of all the trigonometric functions to simplify expressions. Here are some examples.

```
Sin[-z]
```

```
-Sin[z]
```

```
Sin[z + π]
```

```
-Sin[z]
```

```
Sin[z + 2 π]
```

```
Sin[z]
```

```
Sin[z + 34 π]
```

```
Sin[z]
```

```
{Sin[-z], Cos[-z], Tan[-z], Cot[-z], Csc[-z], Sec[-z]}
```

```
{-Sin[z], Cos[z], -Tan[z], -Cot[z], -Csc[z], Sec[z]}
```

```
{Sin[z + π], Cos[z + π], Tan[z + π], Cot[z + π], Csc[z + π], Sec[z + π]}
```

$\{-\text{Sin}[z], -\text{Cos}[z], \text{Tan}[z], \text{Cot}[z], -\text{Csc}[z], -\text{Sec}[z]\}$

```
{Sin[z + 2 π], Cos[z + 2 π], Tan[z + 2 π], Cot[z + 2 π], Csc[z + 2 π], Sec[z + 2 π]}
```

$\{\text{Sin}[z], \text{Cos}[z], \text{Tan}[z], \text{Cot}[z], \text{Csc}[z], \text{Sec}[z]\}$

```
{Sin[z + 342 π], Cos[z + 342 π], Tan[z + 342 π], Cot[z + 342 π], Csc[z + 342 π], Sec[z + 342 π]}
```

$\{\text{Sin}[z], \text{Cos}[z], \text{Tan}[z], \text{Cot}[z], \text{Csc}[z], \text{Sec}[z]\}$

Mathematica automatically simplifies the composition of the direct and the inverse trigonometric functions into the argument.

```
{Sin[ArcSin[z]], Cos[ArcCos[z]], Tan[ArcTan[z]],
 Cot[ArcCot[z]], Csc[ArcCsc[z]], Sec[ArcSec[z]]}
```

$\{z, z, z, z, z, z\}$

Mathematica also automatically simplifies the composition of the direct and any of the inverse trigonometric functions into algebraic functions of the argument.

```
{Sin[ArcSin[z]], Sin[ArcCos[z]], Sin[ArcTan[z]],
 Sin[ArcCot[z]], Sin[ArcCsc[z]], Sin[ArcSec[z]]}
```

$$\left\{z, \sqrt{1-z^2}, \frac{z}{\sqrt{1+z^2}}, \frac{1}{\sqrt{1+\frac{1}{z^2}}\, z}, \frac{1}{z}, \sqrt{1-\frac{1}{z^2}}\right\}$$

```
{Cos[ArcSin[z]], Cos[ArcCos[z]], Cos[ArcTan[z]],
 Cos[ArcCot[z]], Cos[ArcCsc[z]], Cos[ArcSec[z]]}
```

$$\left\{\sqrt{1-z^2}, z, \frac{1}{\sqrt{1+z^2}}, \frac{1}{\sqrt{1+\frac{1}{z^2}}}, \sqrt{1-\frac{1}{z^2}}, \frac{1}{z}\right\}$$

```
{Tan[ArcSin[z]], Tan[ArcCos[z]], Tan[ArcTan[z]],
 Tan[ArcCot[z]], Tan[ArcCsc[z]], Tan[ArcSec[z]]}
```

$$\left\{\frac{z}{\sqrt{1-z^2}}, \frac{\sqrt{1-z^2}}{z}, z, \frac{1}{z}, \frac{1}{\sqrt{1-\frac{1}{z^2}}\, z}, \sqrt{1-\frac{1}{z^2}}\, z\right\}$$

```
{Cot[ArcSin[z]], Cot[ArcCos[z]], Cot[ArcTan[z]],
 Cot[ArcCot[z]], Cot[ArcCsc[z]], Cot[ArcSec[z]]}
```

$$\left\{\frac{\sqrt{1-z^2}}{z}, \frac{z}{\sqrt{1-z^2}}, \frac{1}{z}, z, \sqrt{1-\frac{1}{z^2}}\, z, \frac{1}{\sqrt{1-\frac{1}{z^2}}\, z}\right\}$$

```
{Csc[ArcSin[z]], Csc[ArcCos[z]], Csc[ArcTan[z]],
 Csc[ArcCot[z]], Csc[ArcCsc[z]], Csc[ArcSec[z]]}
```

$$\left\{\frac{1}{z}, \frac{1}{\sqrt{1-z^2}}, \frac{\sqrt{1+z^2}}{z}, \sqrt{1+\frac{1}{z^2}}\, z, z, \frac{1}{\sqrt{1-\frac{1}{z^2}}}\right\}$$

```
{Sec[ArcSin[z]], Sec[ArcCos[z]], Sec[ArcTan[z]],
 Sec[ArcCot[z]], Sec[ArcCsc[z]], Sec[ArcSec[z]]}
```

$$\left\{\frac{1}{\sqrt{1-z^2}}, \frac{1}{z}, \sqrt{1+z^2}, \sqrt{1+\frac{1}{z^2}}, \frac{1}{\sqrt{1-\frac{1}{z^2}}}, z\right\}$$

In cases where the argument has the structure $\pi k/2+z$ or $\pi k/2-z$, and $\pi k/2+i z$ or $\pi k/2-i z$ with integer k, trigonometric functions can be automatically transformed into other trigonometric or hyperbolic functions. Here are some examples.

$$\text{Tan}\left[\frac{\pi}{2}-z\right]$$

```
Cot[z]
```

```
Csc[ⅈ z]
```

```
-ⅈ Csch[z]
```

$$\left\{\text{Sin}\left[\frac{\pi}{2}-z\right], \text{Cos}\left[\frac{\pi}{2}-z\right], \text{Tan}\left[\frac{\pi}{2}-z\right], \text{Cot}\left[\frac{\pi}{2}-z\right], \text{Csc}\left[\frac{\pi}{2}-z\right], \text{Sec}\left[\frac{\pi}{2}-z\right]\right\}$$

```
{Cos[z], Sin[z], Cot[z], Tan[z], Sec[z], Csc[z]}
```

```
{Sin[ⅈ z], Cos[ⅈ z], Tan[ⅈ z], Cot[ⅈ z], Csc[ⅈ z], Sec[ⅈ z]}
```

```
{ⅈ Sinh[z], Cosh[z], ⅈ Tanh[z], -ⅈ Coth[z], -ⅈ Csch[z], Sech[z]}
```

Simplification of simple expressions containing trigonometric functions

Sometimes simple arithmetic operations containing trigonometric functions can automatically produce other trigonometric functions.

```
1 / Sec[z]
```

```
Cos[z]
```

```
{1/Sin[z], 1/Cos[z], 1/Tan[z], 1/Cot[z], 1/Csc[z], 1/Sec[z],
 Sin[z]/Cos[z], Cos[z]/Sin[z], Sin[z]/Sin[π/2 - z], Cos[z]/Sin[z]^2}
```

```
{Csc[z], Sec[z], Cot[z], Tan[z], Sin[z], Cos[z], Tan[z], Cot[z], Tan[z], Cot[z] Csc[z]}
```

Trigonometric functions arising as special cases from more general functions

All trigonometric functions can be treated as particular cases of some more advanced special functions. For example, sin(z) and cos(z) are sometimes the results of auto-simplifications from Bessel, Mathieu, Jacobi, hypergeometric, and Meijer functions (for appropriate values of their parameters).

```
BesselJ[1/2, z]
```

$$\frac{\sqrt{\frac{2}{\pi}}\,\text{Sin}[z]}{\sqrt{z}}$$

```
MathieuC[1, 0, z]
Cos[z]

JacobiSC[z, 0]
Tan[z]
```

```
In[14]:= {BesselJ[1/2, z], MathieuS[1, 0, z], JacobiSN[z, 0],
          HypergeometricPFQ[{}, {3/2}, -z^2/4], MeijerG[{{}, {}}, {{1/2}, {0}}, z^2/4]}
```

$$\text{Out[14]= } \left\{\frac{\sqrt{\frac{2}{\pi}}\,\text{Sin}[z]}{\sqrt{z}},\ \text{Sin}[z],\ \text{Sin}[z],\ \frac{\text{Sin}\left[\sqrt{z^2}\right]}{\sqrt{z^2}},\ \frac{\sqrt{z^2}\,\text{Sin}[z]}{\sqrt{\pi}\ z}\right\}$$

```
In[15]:= {BesselJ[-1/2, z], MathieuC[1, 0, z], JacobiCD[z, 0],
          Hypergeometric0F1[1/2, -z^2/4], MeijerG[{{}, {}}, {{0}, {1/2}}, z^2/4]}
```

$$\text{Out[15]= } \left\{\frac{\sqrt{\frac{2}{\pi}}\,\text{Cos}[z]}{\sqrt{z}},\ \text{Cos}[z],\ \text{Cos}[z],\ \text{Cos}\left[\sqrt{z^2}\right],\ \frac{\text{Cos}[z]}{\sqrt{\pi}}\right\}$$

```
In[16]:= {JacobiSC[z, 0], JacobiCS[z, 0], JacobiDS[z, 0], JacobiDC[z, 0]}
Out[16]= {Tan[z], Cot[z], Csc[z], Sec[z]}
```

Equivalence transformations carried out by specialized *Mathematica* functions

General remarks

Almost everybody prefers using $\sin(z)/2$ instead of $\cos(\pi/2-z)\sin(\pi/6)$. *Mathematica* automatically transforms the second expression into the first one. The automatic application of transformation rules to mathematical expressions can give overly complicated results. Compact expressions like $\sin(2z)\sin(\pi/16)$ should not be automatically expanded into the more complicated expression $\sin(z)\cos(z)\left(2-\left(2+2^{1/2}\right)^{1/2}\right)^{1/2}$. *Mathematica* has special commands that produce these types of expansions. Some of them are demonstrated in the next section.

TrigExpand

The function `TrigExpand` expands out trigonometric and hyperbolic functions. In more detail, it splits up sums and integer multiples that appear in the arguments of trigonometric and hyperbolic functions, and then expands out the products of the trigonometric and hyperbolic functions into sums of powers, using the trigonometric and hyperbolic identities where possible. Here are some examples.

```
TrigExpand[Sin[x - y]]
```

```
Cos[y] Sin[x] - Cos[x] Sin[y]
```

```
Cos[4 z] // TrigExpand
```

```
Cos[z]^4 - 6 Cos[z]^2 Sin[z]^2 + Sin[z]^4
```

```
TrigExpand[{{Sin[x + y], Sin[3 z]},
            {Cos[x + y], Cos[3 z]},
            {Tan[x + y], Tan[3 z]},
            {Cot[x + y], Cot[3 z]},
            {Csc[x + y], Csc[3 z]},
            {Sec[x + y], Sec[3 z]}}]
```

$$\Big\{\{\text{Cos}[y]\,\text{Sin}[x]+\text{Cos}[x]\,\text{Sin}[y],\ 3\,\text{Cos}[z]^2\,\text{Sin}[z]-\text{Sin}[z]^3\},$$
$$\{\text{Cos}[x]\,\text{Cos}[y]-\text{Sin}[x]\,\text{Sin}[y],\ \text{Cos}[z]^3-3\,\text{Cos}[z]\,\text{Sin}[z]^2\},$$
$$\Big\{\frac{\text{Cos}[y]\,\text{Sin}[x]}{\text{Cos}[x]\,\text{Cos}[y]-\text{Sin}[x]\,\text{Sin}[y]}+\frac{\text{Cos}[x]\,\text{Sin}[y]}{\text{Cos}[x]\,\text{Cos}[y]-\text{Sin}[x]\,\text{Sin}[y]},$$
$$\frac{3\,\text{Cos}[z]^2\,\text{Sin}[z]}{\text{Cos}[z]^3-3\,\text{Cos}[z]\,\text{Sin}[z]^2}-\frac{\text{Sin}[z]^3}{\text{Cos}[z]^3-3\,\text{Cos}[z]\,\text{Sin}[z]^2}\Big\},$$
$$\Big\{\frac{\text{Cos}[x]\,\text{Cos}[y]}{\text{Cos}[y]\,\text{Sin}[x]+\text{Cos}[x]\,\text{Sin}[y]}-\frac{\text{Sin}[x]\,\text{Sin}[y]}{\text{Cos}[y]\,\text{Sin}[x]+\text{Cos}[x]\,\text{Sin}[y]},$$
$$\frac{\text{Cos}[z]^3}{3\,\text{Cos}[z]^2\,\text{Sin}[z]-\text{Sin}[z]^3}-\frac{3\,\text{Cos}[z]\,\text{Sin}[z]^2}{3\,\text{Cos}[z]^2\,\text{Sin}[z]-\text{Sin}[z]^3}\Big\},$$
$$\Big\{\frac{1}{\text{Cos}[y]\,\text{Sin}[x]+\text{Cos}[x]\,\text{Sin}[y]},\ \frac{1}{3\,\text{Cos}[z]^2\,\text{Sin}[z]-\text{Sin}[z]^3}\Big\},$$
$$\Big\{\frac{1}{\text{Cos}[x]\,\text{Cos}[y]-\text{Sin}[x]\,\text{Sin}[y]},\ \frac{1}{\text{Cos}[z]^3-3\,\text{Cos}[z]\,\text{Sin}[z]^2}\Big\}\Big\}$$

```
TableForm[(# == TrigExpand[#]) & /@
  Flatten[{{Sin[x + y], Sin[3 z]}, {Cos[x + y], Cos[3 z]}, {Tan[x + y], Tan[3 z]},
    {Cot[x + y], Cot[3 z]}, {Csc[x + y], Csc[3 z]}, {Sec[x + y], Sec[3 z]}}]]
```

$$\mathrm{Sin}[x+y] == \mathrm{Cos}[y]\,\mathrm{Sin}[x] + \mathrm{Cos}[x]\,\mathrm{Sin}[y]$$
$$\mathrm{Sin}[3z] == 3\,\mathrm{Cos}[z]^2\,\mathrm{Sin}[z] - \mathrm{Sin}[z]^3$$
$$\mathrm{Cos}[x+y] == \mathrm{Cos}[x]\,\mathrm{Cos}[y] - \mathrm{Sin}[x]\,\mathrm{Sin}[y]$$
$$\mathrm{Cos}[3z] == \mathrm{Cos}[z]^3 - 3\,\mathrm{Cos}[z]\,\mathrm{Sin}[z]^2$$
$$\mathrm{Tan}[x+y] == \frac{\mathrm{Cos}[y]\,\mathrm{Sin}[x]}{\mathrm{Cos}[x]\,\mathrm{Cos}[y]-\mathrm{Sin}[x]\,\mathrm{Sin}[y]} + \frac{\mathrm{Cos}[x]\,\mathrm{Sin}[y]}{\mathrm{Cos}[x]\,\mathrm{Cos}[y]-\mathrm{Sin}[x]\,\mathrm{Sin}[y]}$$
$$\mathrm{Tan}[3z] == \frac{3\,\mathrm{Cos}[z]^2\,\mathrm{Sin}[z]}{\mathrm{Cos}[z]^3-3\,\mathrm{Cos}[z]\,\mathrm{Sin}[z]^2} - \frac{\mathrm{Sin}[z]^3}{\mathrm{Cos}[z]^3-3\,\mathrm{Cos}[z]\,\mathrm{Sin}[z]^2}$$
$$\mathrm{Cot}[x+y] == \frac{\mathrm{Cos}[x]\,\mathrm{Cos}[y]}{\mathrm{Cos}[y]\,\mathrm{Sin}[x]+\mathrm{Cos}[x]\,\mathrm{Sin}[y]} - \frac{\mathrm{Sin}[x]\,\mathrm{Sin}[y]}{\mathrm{Cos}[y]\,\mathrm{Sin}[x]+\mathrm{Cos}[x]\,\mathrm{Sin}[y]}$$
$$\mathrm{Cot}[3z] == \frac{\mathrm{Cos}[z]^3}{3\,\mathrm{Cos}[z]^2\,\mathrm{Sin}[z]-\mathrm{Sin}[z]^3} - \frac{3\,\mathrm{Cos}[z]\,\mathrm{Sin}[z]^2}{3\,\mathrm{Cos}[z]^2\,\mathrm{Sin}[z]-\mathrm{Sin}[z]^3}$$
$$\mathrm{Csc}[x+y] == \frac{1}{\mathrm{Cos}[y]\,\mathrm{Sin}[x]+\mathrm{Cos}[x]\,\mathrm{Sin}[y]}$$
$$\mathrm{Csc}[3z] == \frac{1}{3\,\mathrm{Cos}[z]^2\,\mathrm{Sin}[z]-\mathrm{Sin}[z]^3}$$
$$\mathrm{Sec}[x+y] == \frac{1}{\mathrm{Cos}[x]\,\mathrm{Cos}[y]-\mathrm{Sin}[x]\,\mathrm{Sin}[y]}$$
$$\mathrm{Sec}[3z] == \frac{1}{\mathrm{Cos}[z]^3-3\,\mathrm{Cos}[z]\,\mathrm{Sin}[z]^2}$$

TrigFactor

The function `TrigFactor` factors trigonometric and hyperbolic functions. In more detail, it splits up sums and integer multiples that appear in the arguments of trigonometric and hyperbolic functions, and then factors the resulting polynomials in the trigonometric and hyperbolic functions, using the corresponding identities where possible. Here are some examples.

```
TrigFactor[Sin[x] + Cos[y]]
```

$$\left(\mathrm{Cos}\left[\frac{x}{2}-\frac{y}{2}\right]+\mathrm{Sin}\left[\frac{x}{2}-\frac{y}{2}\right]\right)\left(\mathrm{Cos}\left[\frac{x}{2}+\frac{y}{2}\right]+\mathrm{Sin}\left[\frac{x}{2}+\frac{y}{2}\right]\right)$$

```
Tan[x] - Cot[y] // TrigFactor
```

$$-\mathrm{Cos}[x+y]\,\mathrm{Csc}[y]\,\mathrm{Sec}[x]$$

```
TrigFactor[{Sin[x] + Sin[y],
            Cos[x] + Cos[y],
            Tan[x] + Tan[y],
            Cot[x] + Cot[y],
            Csc[x] + Csc[y],
            Sec[x] + Sec[y]}]
```

$$\left\{2\,\mathrm{Cos}\left[\frac{x}{2}-\frac{y}{2}\right]\mathrm{Sin}\left[\frac{x}{2}+\frac{y}{2}\right],\ 2\,\mathrm{Cos}\left[\frac{x}{2}-\frac{y}{2}\right]\mathrm{Cos}\left[\frac{x}{2}+\frac{y}{2}\right],\ \mathrm{Sec}[x]\,\mathrm{Sec}[y]\,\mathrm{Sin}[x+y],\right.$$
$$\mathrm{Csc}[x]\,\mathrm{Csc}[y]\,\mathrm{Sin}[x+y],\ \frac{1}{2}\mathrm{Cos}\left[\frac{x}{2}-\frac{y}{2}\right]\mathrm{Csc}\left[\frac{x}{2}\right]\mathrm{Csc}\left[\frac{y}{2}\right]\mathrm{Sec}\left[\frac{x}{2}\right]\mathrm{Sec}\left[\frac{y}{2}\right]\mathrm{Sin}\left[\frac{x}{2}+\frac{y}{2}\right],$$
$$\left.\frac{2\,\mathrm{Cos}\left[\frac{x}{2}-\frac{y}{2}\right]\mathrm{Cos}\left[\frac{x}{2}+\frac{y}{2}\right]}{\left(\mathrm{Cos}\left[\frac{x}{2}\right]-\mathrm{Sin}\left[\frac{x}{2}\right]\right)\left(\mathrm{Cos}\left[\frac{x}{2}\right]+\mathrm{Sin}\left[\frac{x}{2}\right]\right)\left(\mathrm{Cos}\left[\frac{y}{2}\right]-\mathrm{Sin}\left[\frac{y}{2}\right]\right)\left(\mathrm{Cos}\left[\frac{y}{2}\right]+\mathrm{Sin}\left[\frac{y}{2}\right]\right)}\right\}$$

TrigReduce

The function `TrigReduce` rewrites products and powers of trigonometric and hyperbolic functions in terms of those functions with combined arguments. In more detail, it typically yields a linear expression involving trigonometric and hyperbolic functions with more complicated arguments. `TrigReduce` is approximately inverse to `TrigExpand` and `TrigFactor`. Here are some examples.

```
TrigReduce[Sin[z]^3]
```

$$\frac{1}{4}\,(3\,\mathrm{Sin}[z]-\mathrm{Sin}[3\,z])$$

```
Sin[x] Cos[y] // TrigReduce
```

$$\frac{1}{2}\,(\mathrm{Sin}[x-y]+\mathrm{Sin}[x+y])$$

```
TrigReduce[{Sin[z]^2, Cos[z]^2, Tan[z]^2, Cot[z]^2, Csc[z]^2, Sec[z]^2}]
```

$$\left\{\frac{1}{2}\,(1-\mathrm{Cos}[2\,z]),\ \frac{1}{2}\,(1+\mathrm{Cos}[2\,z]),\ \frac{1-\mathrm{Cos}[2\,z]}{1+\mathrm{Cos}[2\,z]},\ \frac{-1-\mathrm{Cos}[2\,z]}{-1+\mathrm{Cos}[2\,z]},\ -\frac{2}{-1+\mathrm{Cos}[2\,z]},\ \frac{2}{1+\mathrm{Cos}[2\,z]}\right\}$$

```
TrigReduce[TrigExpand[{{Sin[x + y], Sin[3 z], Sin[x] Sin[y]},
                       {Cos[x + y], Cos[3 z], Cos[x] Cos[y]},
                       {Tan[x + y], Tan[3 z], Tan[x] Tan[y]},
                       {Cot[x + y], Cot[3 z], Cot[x] Cot[y]},
                       {Csc[x + y], Csc[3 z], Csc[x] Csc[y]},
                       {Sec[x + y], Sec[3 z], Sec[x] Sec[y]}}]]
```

$$\left\{\left\{\mathrm{Sin}[x+y],\ \mathrm{Sin}[3\,z],\ \frac{1}{2}\,(\mathrm{Cos}[x-y]-\mathrm{Cos}[x+y])\right\},\right.$$
$$\left\{\mathrm{Cos}[x+y],\ \mathrm{Cos}[3\,z],\ \frac{1}{2}\,(\mathrm{Cos}[x-y]+\mathrm{Cos}[x+y])\right\},$$
$$\left\{\mathrm{Tan}[x+y],\ \mathrm{Tan}[3\,z],\ \frac{\mathrm{Cos}[x-y]-\mathrm{Cos}[x+y]}{\mathrm{Cos}[x-y]+\mathrm{Cos}[x+y]}\right\},$$
$$\left\{\mathrm{Cot}[x+y],\ \mathrm{Cot}[3\,z],\ \frac{\mathrm{Cos}[x-y]+\mathrm{Cos}[x+y]}{\mathrm{Cos}[x-y]-\mathrm{Cos}[x+y]}\right\},$$
$$\left\{\mathrm{Csc}[x+y],\ \mathrm{Csc}[3\,z],\ \frac{2}{\mathrm{Cos}[x-y]-\mathrm{Cos}[x+y]}\right\},$$
$$\left.\left\{\mathrm{Sec}[x+y],\ \mathrm{Sec}[3\,z],\ \frac{2}{\mathrm{Cos}[x-y]+\mathrm{Cos}[x+y]}\right\}\right\}$$

```
TrigReduce[TrigFactor[{Sin[x] + Sin[y], Cos[x] + Cos[y],
   Tan[x] + Tan[y], Cot[x] + Cot[y], Csc[x] + Csc[y], Sec[x] + Sec[y]}]]
```

$$\left\{\mathrm{Sin}[x]+\mathrm{Sin}[y],\ \mathrm{Cos}[x]+\mathrm{Cos}[y],\ \frac{2\,\mathrm{Sin}[x+y]}{\mathrm{Cos}[x-y]+\mathrm{Cos}[x+y]},\right.$$
$$\left.\frac{2\,\mathrm{Sin}[x+y]}{\mathrm{Cos}[x-y]-\mathrm{Cos}[x+y]},\ \frac{2\,(\mathrm{Sin}[x]+\mathrm{Sin}[y])}{\mathrm{Cos}[x-y]-\mathrm{Cos}[x+y]},\ \frac{2\,(\mathrm{Cos}[x]+\mathrm{Cos}[y])}{\mathrm{Cos}[x-y]+\mathrm{Cos}[x+y]}\right\}$$

TrigToExp

The function TrigToExp converts direct and inverse trigonometric and hyperbolic functions to exponential or logarithmic functions. It tries, where possible, to give results that do not involve explicit complex numbers. Here are some examples.

```
TrigToExp[Sin[2 z]]
```

$$\frac{1}{2} \mathbb{i}\, e^{-2 \mathbb{i} z} - \frac{1}{2} \mathbb{i}\, e^{2 \mathbb{i} z}$$

```
Sin[z] Tan[2 z] // TrigToExp
```

$$-\frac{\left(e^{-\mathbb{i} z} - e^{\mathbb{i} z}\right)\left(e^{-2 \mathbb{i} z} - e^{2 \mathbb{i} z}\right)}{2\left(e^{-2 \mathbb{i} z} + e^{2 \mathbb{i} z}\right)}$$

```
TrigToExp[{Sin[z], Cos[z], Tan[z], Cot[z], Csc[z], Sec[z]}]
```

$$\left\{\frac{1}{2} \mathbb{i}\, e^{-\mathbb{i} z} - \frac{1}{2} \mathbb{i}\, e^{\mathbb{i} z}, \frac{e^{-\mathbb{i} z}}{2} + \frac{e^{\mathbb{i} z}}{2}, \frac{\mathbb{i}\left(e^{-\mathbb{i} z} - e^{\mathbb{i} z}\right)}{e^{-\mathbb{i} z} + e^{\mathbb{i} z}}, -\frac{\mathbb{i}\left(e^{-\mathbb{i} z} + e^{\mathbb{i} z}\right)}{e^{-\mathbb{i} z} - e^{\mathbb{i} z}}, -\frac{2 \mathbb{i}}{e^{-\mathbb{i} z} - e^{\mathbb{i} z}}, \frac{2}{e^{-\mathbb{i} z} + e^{\mathbb{i} z}}\right\}$$

ExpToTrig

The function ExpToTrig converts exponentials to trigonometric or hyperbolic functions. It tries, where possible, to give results that do not involve explicit complex numbers. It is approximately inverse to TrigToExp. Here are some examples.

```
ExpToTrig[ⅇ^(ⅈ x β)]
```

Cos[x β] + ⅈ Sin[x β]

$$\frac{e^{\mathbb{i} x \alpha} - e^{\mathbb{i} x \beta}}{e^{\mathbb{i} x \gamma} + e^{\mathbb{i} x \delta}} \text{ // ExpToTrig}$$

$$\frac{\text{Cos}[x\,\alpha] - \text{Cos}[x\,\beta] + \mathbb{i}\,\text{Sin}[x\,\alpha] - \mathbb{i}\,\text{Sin}[x\,\beta]}{\text{Cos}[x\,\gamma] + \text{Cos}[x\,\delta] + \mathbb{i}\,\text{Sin}[x\,\gamma] + \mathbb{i}\,\text{Sin}[x\,\delta]}$$

```
ExpToTrig[TrigToExp[{Sin[z], Cos[z], Tan[z], Cot[z], Csc[z], Sec[z]}]]
```

{Sin[z], Cos[z], Tan[z], Cot[z], Csc[z], Sec[z]}

```
ExpToTrig[{α ⅇ^(-ⅈ x β) + α ⅇ^(ⅈ x β), α ⅇ^(-ⅈ x β) + γ ⅇ^(ⅈ x β)}]
```

{2 α Cos[x β], α Cos[x β] + γ Cos[x β] − ⅈ α Sin[x β] + ⅈ γ Sin[x β]}

ComplexExpand

The function ComplexExpand expands expressions assuming that all the occurring variables are real. The value option TargetFunctions is a list of functions from the set {Re, Im, Abs, Arg, Conjugate, Sign}. ComplexExpand tries to give results in terms of the specified functions. Here are some examples

```
ComplexExpand[Sin[x + ⅈ y] Cos[x − ⅈ y]]
```

```
Cos[x] Cosh[y]^2 Sin[x] - Cos[x] Sin[x] Sinh[y]^2 +
 ⅈ (Cos[x]^2 Cosh[y] Sinh[y] + Cosh[y] Sin[x]^2 Sinh[y])
```

```
Csc[x + ⅈ y] Sec[x - ⅈ y] // ComplexExpand
```

```
-(4 Cos[x] Cosh[y]^2 Sin[x])/((Cos[2 x] - Cosh[2 y]) (Cos[2 x] + Cosh[2 y])) + (4 Cos[x] Sin[x] Sinh[y]^2)/((Cos[2 x] - Cosh[2 y]) (Cos[2 x] + Cosh[2 y])) +
 ⅈ ((4 Cos[x]^2 Cosh[y] Sinh[y])/((Cos[2 x] - Cosh[2 y]) (Cos[2 x] + Cosh[2 y])) +
   (4 Cosh[y] Sin[x]^2 Sinh[y])/((Cos[2 x] - Cosh[2 y]) (Cos[2 x] + Cosh[2 y])))
```

```
In[17]:= li1 = {Sin[x + ⅈ y], Cos[x + ⅈ y], Tan[x + ⅈ y], Cot[x + ⅈ y], Csc[x + ⅈ y], Sec[x + ⅈ y]}

Out[17]= {Sin[x + ⅈ y], Cos[x + ⅈ y], Tan[x + ⅈ y], Cot[x + ⅈ y], Csc[x + ⅈ y], Sec[x + ⅈ y]}

In[18]:= ComplexExpand[li1]

Out[18]= {Cosh[y] Sin[x] + ⅈ Cos[x] Sinh[y], Cos[x] Cosh[y] - ⅈ Sin[x] Sinh[y],
  Sin[2 x]/(Cos[2 x] + Cosh[2 y]) + (ⅈ Sinh[2 y])/(Cos[2 x] + Cosh[2 y]), -Sin[2 x]/(Cos[2 x] - Cosh[2 y]) + (ⅈ Sinh[2 y])/(Cos[2 x] - Cosh[2 y]),
  -(2 Cosh[y] Sin[x])/(Cos[2 x] - Cosh[2 y]) + (2 ⅈ Cos[x] Sinh[y])/(Cos[2 x] - Cosh[2 y]), (2 Cos[x] Cosh[y])/(Cos[2 x] + Cosh[2 y]) + (2 ⅈ Sin[x] Sinh[y])/(Cos[2 x] + Cosh[2 y])}

In[19]:= ComplexExpand[Re[#] & /@ li1, TargetFunctions → {Re, Im}]

Out[19]= {Cosh[y] Sin[x], Cos[x] Cosh[y], Sin[2 x]/(Cos[2 x] + Cosh[2 y]),
  -Sin[2 x]/(Cos[2 x] - Cosh[2 y]), -(2 Cosh[y] Sin[x])/(Cos[2 x] - Cosh[2 y]), (2 Cos[x] Cosh[y])/(Cos[2 x] + Cosh[2 y])}

In[20]:= ComplexExpand[Im[#] & /@ li1, TargetFunctions → {Re, Im}]

Out[20]= {Cos[x] Sinh[y], -Sin[x] Sinh[y], Sinh[2 y]/(Cos[2 x] + Cosh[2 y]),
  Sinh[2 y]/(Cos[2 x] - Cosh[2 y]), (2 Cos[x] Sinh[y])/(Cos[2 x] - Cosh[2 y]), (2 Sin[x] Sinh[y])/(Cos[2 x] + Cosh[2 y])}

In[21]:= ComplexExpand[Abs[#] & /@ li1, TargetFunctions → {Re, Im}]
```

Out[21]=

$$\left\{\sqrt{\text{Cosh}[y]^2\,\text{Sin}[x]^2+\text{Cos}[x]^2\,\text{Sinh}[y]^2},\ \sqrt{\text{Cos}[x]^2\,\text{Cosh}[y]^2+\text{Sin}[x]^2\,\text{Sinh}[y]^2},\right.$$
$$\sqrt{\frac{\text{Sin}[2\,x]^2}{(\text{Cos}[2\,x]+\text{Cosh}[2\,y])^2}+\frac{\text{Sinh}[2\,y]^2}{(\text{Cos}[2\,x]+\text{Cosh}[2\,y])^2}},$$
$$\sqrt{\frac{\text{Sin}[2\,x]^2}{(\text{Cos}[2\,x]-\text{Cosh}[2\,y])^2}+\frac{\text{Sinh}[2\,y]^2}{(\text{Cos}[2\,x]-\text{Cosh}[2\,y])^2}},$$
$$\sqrt{\frac{4\,\text{Cosh}[y]^2\,\text{Sin}[x]^2}{(\text{Cos}[2\,x]-\text{Cosh}[2\,y])^2}+\frac{4\,\text{Cos}[x]^2\,\text{Sinh}[y]^2}{(\text{Cos}[2\,x]-\text{Cosh}[2\,y])^2}},$$
$$\left.\sqrt{\frac{4\,\text{Cos}[x]^2\,\text{Cosh}[y]^2}{(\text{Cos}[2\,x]+\text{Cosh}[2\,y])^2}+\frac{4\,\text{Sin}[x]^2\,\text{Sinh}[y]^2}{(\text{Cos}[2\,x]+\text{Cosh}[2\,y])^2}}\right\}$$

In[22]:= `% // Simplify[#, {x, y} ∈ Reals] &`

Out[22]=

$$\left\{\frac{\sqrt{-\text{Cos}[2\,x]+\text{Cosh}[2\,y]}}{\sqrt{2}},\ \frac{\sqrt{\text{Cos}[2\,x]+\text{Cosh}[2\,y]}}{\sqrt{2}},\ \frac{\sqrt{\text{Sin}[2\,x]^2+\text{Sinh}[2\,y]^2}}{\text{Cos}[2\,x]+\text{Cosh}[2\,y]},\right.$$
$$\left.\sqrt{-\frac{\text{Cos}[2\,x]+\text{Cosh}[2\,y]}{\text{Cos}[2\,x]-\text{Cosh}[2\,y]}},\ \frac{\sqrt{2}}{\sqrt{-\text{Cos}[2\,x]+\text{Cosh}[2\,y]}},\ \frac{\sqrt{2}}{\sqrt{\text{Cos}[2\,x]+\text{Cosh}[2\,y]}}\right\}$$

In[23]:= `ComplexExpand[Arg[#] & /@ li1, TargetFunctions → {Re, Im}]`

Out[23]=

$$\{\text{ArcTan}[\text{Cosh}[y]\,\text{Sin}[x],\ \text{Cos}[x]\,\text{Sinh}[y]],\ \text{ArcTan}[\text{Cos}[x]\,\text{Cosh}[y],\ -\text{Sin}[x]\,\text{Sinh}[y]],$$
$$\text{ArcTan}\left[\frac{\text{Sin}[2\,x]}{\text{Cos}[2\,x]+\text{Cosh}[2\,y]},\ \frac{\text{Sinh}[2\,y]}{\text{Cos}[2\,x]+\text{Cosh}[2\,y]}\right],$$
$$\text{ArcTan}\left[-\frac{\text{Sin}[2\,x]}{\text{Cos}[2\,x]-\text{Cosh}[2\,y]},\ \frac{\text{Sinh}[2\,y]}{\text{Cos}[2\,x]-\text{Cosh}[2\,y]}\right],$$
$$\text{ArcTan}\left[-\frac{2\,\text{Cosh}[y]\,\text{Sin}[x]}{\text{Cos}[2\,x]-\text{Cosh}[2\,y]},\ \frac{2\,\text{Cos}[x]\,\text{Sinh}[y]}{\text{Cos}[2\,x]-\text{Cosh}[2\,y]}\right],$$
$$\text{ArcTan}\left[\frac{2\,\text{Cos}[x]\,\text{Cosh}[y]}{\text{Cos}[2\,x]+\text{Cosh}[2\,y]},\ \frac{2\,\text{Sin}[x]\,\text{Sinh}[y]}{\text{Cos}[2\,x]+\text{Cosh}[2\,y]}\right]\}$$

In[24]:= `ComplexExpand[Conjugate[#] & /@ li1, TargetFunctions → {Re, Im}] // Simplify`

Out[24]=

$$\{\text{Cosh}[y]\,\text{Sin}[x]-i\,\text{Cos}[x]\,\text{Sinh}[y],\ \text{Cos}[x]\,\text{Cosh}[y]+i\,\text{Sin}[x]\,\text{Sinh}[y],$$
$$\frac{\text{Sin}[2\,x]-i\,\text{Sinh}[2\,y]}{\text{Cos}[2\,x]+\text{Cosh}[2\,y]},\ -\frac{\text{Sin}[2\,x]+i\,\text{Sinh}[2\,y]}{\text{Cos}[2\,x]-\text{Cosh}[2\,y]},$$
$$\frac{1}{\text{Cosh}[y]\,\text{Sin}[x]-i\,\text{Cos}[x]\,\text{Sinh}[y]},\ \frac{1}{\text{Cos}[x]\,\text{Cosh}[y]+i\,\text{Sin}[x]\,\text{Sinh}[y]}\}$$

Simplify

The function `Simplify` performs a sequence of algebraic transformations on its argument, and returns the simplest form it finds. Here are two examples.

```
Simplify[Sin[2 z] / Sin[z]]
```

```
2 Cos[z]
```

```
Sin[2 z] / Cos[z] // Simplify
```

```
2 Sin[z]
```

Here is a large collection of trigonometric identities. All are written as one large logical conjunction.

```
Simplify[#] & /@ (Cos[z]^2 + Sin[z]^2 == 1 ∧
Sin[z]^2 == (1 - Cos[2 z])/2 ∧ Cos[z]^2 == (1 + Cos[2 z])/2 ∧
Tan[z]^2 == (1 - Cos[2 z])/(1 + Cos[2 z]) ∧ Cot[z]^2 == (1 + Cos[2 z])/(1 - Cos[2 z]) ∧
Sin[2 z] == 2 Sin[z] Cos[z] ∧ Cos[2 z] == Cos[z]^2 - Sin[z]^2 == 2 Cos[z]^2 - 1 ∧
Sin[a + b] == Sin[a] Cos[b] + Cos[a] Sin[b] ∧ Sin[a - b] == Sin[a] Cos[b] - Cos[a] Sin[b] ∧
Cos[a + b] == Cos[a] Cos[b] - Sin[a] Sin[b] ∧ Cos[a - b] == Cos[a] Cos[b] + Sin[a] Sin[b] ∧
Sin[a] + Sin[b] == 2 Sin[(a + b)/2] Cos[(a - b)/2] ∧ Sin[a] - Sin[b] == 2 Cos[(a + b)/2] Sin[(a - b)/2] ∧
Cos[a] + Cos[b] == 2 Cos[(a + b)/2] Cos[(a - b)/2] ∧ Cos[a] - Cos[b] == 2 Sin[(a + b)/2] Sin[(b - a)/2] ∧
Tan[a] + Tan[b] == Sin[a + b]/(Cos[a] Cos[b]) ∧ Tan[a] - Tan[b] == Sin[a - b]/(Cos[a] Cos[b]) ∧
A Sin[z] + B Cos[z] == A Sqrt[1 + B^2/A^2] Sin[z + ArcTan[B/A]] ∧
Sin[a] Sin[b] == (Cos[a - b] - Cos[a + b])/2 ∧
  Cos[a] Cos[b] == (Cos[a - b] + Cos[a + b])/2 ∧ Sin[a] Cos[b] == (Sin[a + b] + Sin[a - b])/2 ∧
Sin[z/2]^2 == (1 - Cos[z])/2 ∧ Cos[z/2]^2 == (1 + Cos[z])/2 ∧
Tan[z/2] == (1 - Cos[z])/Sin[z] == Sin[z]/(1 + Cos[z]) ∧ Cot[z/2] == Sin[z]/(1 - Cos[z]) == (1 + Cos[z])/Sin[z])
```

```
True
```

The function `Simplify` has the `Assumption` option. For example, *Mathematica* knows that $-1 \leq \sin(x) \leq 1$ for all real x, and uses the periodicity of trigonometric functions for the symbolic integer coefficient k of $k\,\pi$.

```
Simplify[Abs[Sin[x]] ≤ 1, x ∈ Reals]
```

```
True
```

```
Abs[Sin[x]] ≤ 1 // Simplify[#, x ∈ Reals] &
```

```
True
```

```
Simplify[{Sin[z + 2 k π], Cos[z + 2 k π], Tan[z + k π],
  Cot[z + k π], Csc[z + 2 k π], Sec[z + 2 k π]}, k ∈ Integers]
```

```
{Sin[z], Cos[z], Tan[z], Cot[z], Csc[z], Sec[z]}
```

```
Simplify[{Sin[z + k π] / Sin[z], Cos[z + k π] / Cos[z], Tan[z + k π] / Tan[z],
  Cot[z + k π] / Cot[z], Csc[z + k π] / Csc[z], Sec[z + k π] / Sec[z]}, k ∈ Integers]
```

$\{(-1)^k, (-1)^k, 1, 1, (-1)^k, (-1)^k\}$

Mathematica also knows that the composition of inverse and direct trigonometric functions produces the value of the inner argument under the appropriate restriction. Here are some examples.

```
Simplify[{ArcSin[Sin[z]], ArcTan[Tan[z]], ArcCot[Cot[z]], ArcCsc[Csc[z]]},
 -π / 2 < Re[z] < π / 2]
```

```
{z, z, z, z}
```

```
Simplify[{ArcCos[Cos[z]], ArcSec[Sec[z]]}, 0 < Re[z] < π]
```

```
{z, z}
```

FunctionExpand (and Together)

While the trigonometric functions auto-evaluate for simple fractions of π, for more complicated cases they stay as trigonometric functions to avoid the build up of large expressions. Using the function `FunctionExpand`, such expressions can be transformed into explicit radicals.

$$\mathrm{Cos}\left[\frac{\pi}{32}\right]$$

$$\mathrm{Cos}\left[\frac{\pi}{32}\right]$$

$$\mathrm{FunctionExpand}\left[\mathrm{Cos}\left[\frac{\pi}{32}\right]\right]$$

$$\frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}$$

$$\mathrm{Cot}\left[\frac{\pi}{24}\right] \,//\, \mathrm{FunctionExpand}$$

$$\frac{\frac{\sqrt{2-\sqrt{2}}}{4}+\frac{1}{4}\sqrt{3\left(2+\sqrt{2}\right)}}{-\frac{1}{4}\sqrt{3\left(2-\sqrt{2}\right)}+\frac{\sqrt{2+\sqrt{2}}}{4}}$$

{Sin[π/16], Cos[π/16], Tan[π/16], Cot[π/16], Csc[π/16], Sec[π/16]}

$$\left\{\text{Sin}\left[\frac{\pi}{16}\right],\ \text{Cos}\left[\frac{\pi}{16}\right],\ \text{Tan}\left[\frac{\pi}{16}\right],\ \text{Cot}\left[\frac{\pi}{16}\right],\ \text{Csc}\left[\frac{\pi}{16}\right],\ \text{Sec}\left[\frac{\pi}{16}\right]\right\}$$

FunctionExpand[%]

$$\left\{\frac{1}{2}\sqrt{2-\sqrt{2+\sqrt{2}}},\ \frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2}}},\ \sqrt{\frac{2-\sqrt{2+\sqrt{2}}}{2+\sqrt{2+\sqrt{2}}}},\ \sqrt{\frac{2+\sqrt{2+\sqrt{2}}}{2-\sqrt{2+\sqrt{2}}}},\ \frac{2}{\sqrt{2-\sqrt{2+\sqrt{2}}}},\ \frac{2}{\sqrt{2+\sqrt{2+\sqrt{2}}}}\right\}$$

{Sin[π/60], Cos[π/60], Tan[π/60], Cot[π/60], Csc[π/60], Sec[π/60]}

$$\left\{\text{Sin}\left[\frac{\pi}{60}\right],\ \text{Cos}\left[\frac{\pi}{60}\right],\ \text{Tan}\left[\frac{\pi}{60}\right],\ \text{Cot}\left[\frac{\pi}{60}\right],\ \text{Csc}\left[\frac{\pi}{60}\right],\ \text{Sec}\left[\frac{\pi}{60}\right]\right\}$$

Together[FunctionExpand[%]]

$$\Big\{\frac{1}{16}\left(-\sqrt{2}-\sqrt{6}+\sqrt{10}+\sqrt{30}+2\sqrt{5+\sqrt{5}}-2\sqrt{3\left(5+\sqrt{5}\right)}\right),$$

$$\frac{1}{16}\left(\sqrt{2}-\sqrt{6}-\sqrt{10}+\sqrt{30}+2\sqrt{5+\sqrt{5}}+2\sqrt{3\left(5+\sqrt{5}\right)}\right),$$

$$\frac{-1-\sqrt{3}+\sqrt{5}+\sqrt{15}+\sqrt{2\left(5+\sqrt{5}\right)}-\sqrt{6\left(5+\sqrt{5}\right)}}{1-\sqrt{3}-\sqrt{5}+\sqrt{15}+\sqrt{2\left(5+\sqrt{5}\right)}+\sqrt{6\left(5+\sqrt{5}\right)}},$$

$$\frac{-1+\sqrt{3}+\sqrt{5}-\sqrt{15}-\sqrt{2\left(5+\sqrt{5}\right)}-\sqrt{6\left(5+\sqrt{5}\right)}}{1+\sqrt{3}-\sqrt{5}-\sqrt{15}-\sqrt{2\left(5+\sqrt{5}\right)}+\sqrt{6\left(5+\sqrt{5}\right)}},$$

$$\frac{16}{-\sqrt{2}-\sqrt{6}+\sqrt{10}+\sqrt{30}+2\sqrt{5+\sqrt{5}}-2\sqrt{3\left(5+\sqrt{5}\right)}},$$

$$\frac{16}{\sqrt{2}-\sqrt{6}-\sqrt{10}+\sqrt{30}+2\sqrt{5+\sqrt{5}}+2\sqrt{3\left(5+\sqrt{5}\right)}}\Big\}$$

If the denominator contains squares of integers other than 2, the results always contain complex numbers (meaning that the imaginary number $i=\sqrt{-1}$ appears unavoidably).

```
{Sin[π/9], Cos[π/9], Tan[π/9], Cot[π/9], Csc[π/9], Sec[π/9]}
```

$$\left\{\mathrm{Sin}\left[\frac{\pi}{9}\right],\ \mathrm{Cos}\left[\frac{\pi}{9}\right],\ \mathrm{Tan}\left[\frac{\pi}{9}\right],\ \mathrm{Cot}\left[\frac{\pi}{9}\right],\ \mathrm{Csc}\left[\frac{\pi}{9}\right],\ \mathrm{Sec}\left[\frac{\pi}{9}\right]\right\}$$

```
FunctionExpand[%] // Together
```

$$\Big\{\frac{1}{8}\left(-i\,2^{2/3}\left(-1-i\sqrt{3}\right)^{1/3}+2^{2/3}\sqrt{3}\left(-1-i\sqrt{3}\right)^{1/3}+i\,2^{2/3}\left(-1+i\sqrt{3}\right)^{1/3}+2^{2/3}\sqrt{3}\left(-1+i\sqrt{3}\right)^{1/3}\right),$$

$$\frac{1}{8}\left(2^{2/3}\left(-1-i\sqrt{3}\right)^{1/3}+i\,2^{2/3}\sqrt{3}\left(-1-i\sqrt{3}\right)^{1/3}+2^{2/3}\left(-1+i\sqrt{3}\right)^{1/3}-i\,2^{2/3}\sqrt{3}\left(-1+i\sqrt{3}\right)^{1/3}\right),$$

$$\frac{-\left(-1-i\sqrt{3}\right)^{1/3}-i\sqrt{3}\left(-1-i\sqrt{3}\right)^{1/3}+\left(-1+i\sqrt{3}\right)^{1/3}-i\sqrt{3}\left(-1+i\sqrt{3}\right)^{1/3}}{-i\left(-1-i\sqrt{3}\right)^{1/3}+\sqrt{3}\left(-1-i\sqrt{3}\right)^{1/3}-i\left(-1+i\sqrt{3}\right)^{1/3}-\sqrt{3}\left(-1+i\sqrt{3}\right)^{1/3}},$$

$$\frac{\left(-1-i\sqrt{3}\right)^{1/3}+i\sqrt{3}\left(-1-i\sqrt{3}\right)^{1/3}+\left(-1+i\sqrt{3}\right)^{1/3}-i\sqrt{3}\left(-1+i\sqrt{3}\right)^{1/3}}{-i\left(-1-i\sqrt{3}\right)^{1/3}+\sqrt{3}\left(-1-i\sqrt{3}\right)^{1/3}+i\left(-1+i\sqrt{3}\right)^{1/3}+\sqrt{3}\left(-1+i\sqrt{3}\right)^{1/3}},$$

$$8\Big/\left(-i\,2^{2/3}\left(-1-i\sqrt{3}\right)^{1/3}+2^{2/3}\sqrt{3}\left(-1-i\sqrt{3}\right)^{1/3}+i\,2^{2/3}\left(-1+i\sqrt{3}\right)^{1/3}+2^{2/3}\sqrt{3}\left(-1+i\sqrt{3}\right)^{1/3}\right),$$

$$-(8\,i)\Big/\left(-i\,2^{2/3}\left(-1-i\sqrt{3}\right)^{1/3}+2^{2/3}\sqrt{3}\left(-1-i\sqrt{3}\right)^{1/3}-i\,2^{2/3}\left(-1+i\sqrt{3}\right)^{1/3}-2^{2/3}\sqrt{3}\left(-1+i\sqrt{3}\right)^{1/3}\right)\Big\}$$

Here the function `RootReduce` is used to express the previous algebraic numbers as numbered roots of polynomial equations.

```
RootReduce[Simplify[%]]
```

```
{Root[-3 + 36 #1^2 - 96 #1^4 + 64 #1^6 &, 4], Root[-1 - 6 #1 + 8 #1^3 &, 3],
 Root[-3 + 27 #1^2 - 33 #1^4 + #1^6 &, 4], Root[-1 + 33 #1^2 - 27 #1^4 + 3 #1^6 &, 6],
 Root[-64 + 96 #1^2 - 36 #1^4 + 3 #1^6 &, 6], Root[-8 + 6 #1^2 + #1^3 &, 3]}
```

The function `FunctionExpand` also reduces trigonometric expressions with compound arguments or compositions, including hyperbolic functions, to simpler ones. Here are some examples.

```
FunctionExpand[Cot[Sqrt[-z^2]]]
```

$$-\frac{\sqrt{-z}\ \mathrm{Coth}[z]}{\sqrt{z}}$$

```
Tan[Sqrt[i z^2]] // FunctionExpand
```

$$-\frac{(-1)^{3/4}\sqrt{-(-1)^{3/4}\,z}\ \sqrt{(-1)^{3/4}\,z}\ \mathrm{Tan}\left[(-1)^{1/4}\,z\right]}{z}$$

$$\left\{\mathrm{Sin}\left[\sqrt{z^2}\right],\ \mathrm{Cos}\left[\sqrt{z^2}\right],\ \mathrm{Tan}\left[\sqrt{z^2}\right],\ \mathrm{Cot}\left[\sqrt{z^2}\right],\ \mathrm{Csc}\left[\sqrt{z^2}\right],\ \mathrm{Sec}\left[\sqrt{z^2}\right]\right\} \text{ // FunctionExpand}$$

$$\left\{\frac{\sqrt{-i z}\ \sqrt{i z}\ \mathrm{Sin}[z]}{z},\ \mathrm{Cos}[z],\ \frac{\sqrt{-i z}\ \sqrt{i z}\ \mathrm{Tan}[z]}{z},\ \frac{\sqrt{-i z}\ \sqrt{i z}\ \mathrm{Cot}[z]}{z},\ \frac{\sqrt{-i z}\ \sqrt{i z}\ \mathrm{Csc}[z]}{z},\ \mathrm{Sec}[z]\right\}$$

Applying `Simplify` to the last expression gives a more compact result.

```
Simplify[%]
```

$$\left\{\frac{\sqrt{z^2}\ \mathrm{Sin}[z]}{z},\ \mathrm{Cos}[z],\ \frac{\sqrt{z^2}\ \mathrm{Tan}[z]}{z},\ \frac{\sqrt{z^2}\ \mathrm{Cot}[z]}{z},\ \frac{\sqrt{z^2}\ \mathrm{Csc}[z]}{z},\ \mathrm{Sec}[z]\right\}$$

Here are some similar examples.

```
Sin[2 ArcTan[z]] // FunctionExpand
```

$$\frac{2 z}{1 + z^2}$$

$$\mathrm{Cos}\left[\frac{\mathrm{ArcCot}[z]}{2}\right] \text{ // FunctionExpand}$$

$$\frac{\sqrt{1 + \frac{\sqrt{-z}\ \sqrt{z}}{\sqrt{-1-z^2}}}}{\sqrt{2}}$$

```
{Sin[2 ArcSin[z]], Cos[2 ArcCos[z]], Tan[2 ArcTan[z]],
  Cot[2 ArcCot[z]], Csc[2 ArcCsc[z]], Sec[2 ArcSec[z]]} // FunctionExpand
```

$$\left\{2 \sqrt{1 - z}\ z \sqrt{1 + z},\ -1 + 2 z^2,\ -\frac{2 z}{(-1 + z)\ (1 + z)},\ \frac{1}{2} \left(1 + \frac{1}{z^2}\right) z \left(\frac{1}{-1 - z^2} - \frac{z^2}{-1 - z^2}\right),\ \frac{\sqrt{-i z}\ \sqrt{i z}\ z}{2 \sqrt{(-1 + z)\ (1 + z)}},\ \frac{z^2}{2 - z^2}\right\}$$

$$\left\{\mathrm{Sin}\left[\frac{\mathrm{ArcSin}[z]}{2}\right],\ \mathrm{Cos}\left[\frac{\mathrm{ArcCos}[z]}{2}\right],\ \mathrm{Tan}\left[\frac{\mathrm{ArcTan}[z]}{2}\right],\ \mathrm{Cot}\left[\frac{\mathrm{ArcCot}[z]}{2}\right],\ \mathrm{Csc}\left[\frac{\mathrm{ArcCsc}[z]}{2}\right],\ \mathrm{Sec}\left[\frac{\mathrm{ArcSec}[z]}{2}\right]\right\} \text{ // FunctionExpand}$$

$$\left\{\frac{z\sqrt{1-\sqrt{1-z}\,\sqrt{1+z}}}{\sqrt{2}\,\sqrt{-i\,z}\,\sqrt{i\,z}},\ \frac{\sqrt{1+z}}{\sqrt{2}},\ \frac{z}{1+\sqrt{i\,(-i+z)}\,\sqrt{-i\,(i+z)}},\right.$$

$$\left. z\left(1+\frac{\sqrt{-1-z^2}}{\sqrt{-z}\,\sqrt{z}}\right),\ \frac{\sqrt{2}\,\sqrt{-\frac{i}{z}}\,\sqrt{\frac{i}{z}}\,z}{\sqrt{1-\frac{\sqrt{(-1+z)\,(1+z)}}{\sqrt{-i\,z}\,\sqrt{i\,z}}}},\ \frac{\sqrt{2}\,\sqrt{-z}}{\sqrt{-1-z}}\right\}$$

Simplify[%]

$$\left\{\frac{z\sqrt{1-\sqrt{1-z^2}}}{\sqrt{2}\,\sqrt{z^2}},\ \frac{\sqrt{1+z}}{\sqrt{2}},\ \frac{z}{1+\sqrt{1+z^2}},\ z+\frac{\sqrt{z}\,\sqrt{-1-z^2}}{\sqrt{-z}},\ \frac{\sqrt{2}\,\sqrt{\frac{1}{z^2}}\,z}{\sqrt{1-\frac{\sqrt{z^2}\,\sqrt{-1+z^2}}{z^2}}},\ \frac{\sqrt{2}}{\sqrt{1+\frac{1}{z}}}\right\}$$

FullSimplify

The function `FullSimplify` tries a wider range of transformations than `Simplify` and returns the simplest form it finds. Here are some examples that contrast the results of applying these functions to the same expressions.

$$\mathbf{Cos}\left[\frac{1}{2}\,i\,\mathbf{Log}[1-i\,z]-\frac{1}{2}\,i\,\mathbf{Log}[1+i\,z]\right]\ //\ \mathbf{Simplify}$$

$$\mathrm{Cosh}\left[\frac{1}{2}\,(\mathrm{Log}[1-i\,z]-\mathrm{Log}[1+i\,z])\right]$$

$$\mathbf{Cos}\left[\frac{1}{2}\,i\,\mathbf{Log}[1-i\,z]-\frac{1}{2}\,i\,\mathbf{Log}[1+i\,z]\right]\ //\ \mathbf{FullSimplify}$$

$$\frac{1}{\sqrt{1+z^2}}$$

$$\left\{\mathbf{Sin}\left[-i\,\mathbf{Log}\left[i\,z+\sqrt{1-z^2}\right]\right],\ \mathbf{Cos}\left[-i\,\mathbf{Log}\left[i\,z+\sqrt{1-z^2}\right]\right],\right.$$
$$\mathbf{Tan}\left[-i\,\mathbf{Log}\left[i\,z+\sqrt{1-z^2}\right]\right],\ \mathbf{Cot}\left[-i\,\mathbf{Log}\left[i\,z+\sqrt{1-z^2}\right]\right],$$
$$\left.\mathbf{Csc}\left[-i\,\mathbf{Log}\left[i\,z+\sqrt{1-z^2}\right]\right],\ \mathbf{Sec}\left[-i\,\mathbf{Log}\left[i\,z+\sqrt{1-z^2}\right]\right]\right\}\ //\ \mathbf{Simplify}$$

$$\left\{z,\ \frac{1-z^2+i\,z\sqrt{1-z^2}}{i\,z+\sqrt{1-z^2}},\ \frac{z\left(z-i\sqrt{1-z^2}\right)}{-i+i\,z^2+z\sqrt{1-z^2}},\ \frac{1-z^2+i\,z\sqrt{1-z^2}}{i\,z^2+z\sqrt{1-z^2}},\ \frac{1}{z},\ \frac{2\left(i\,z+\sqrt{1-z^2}\right)}{1+\left(i\,z+\sqrt{1-z^2}\right)^2}\right\}$$

$$\left\{\mathbf{Sin}\left[-i\,\mathbf{Log}\left[i\,z+\sqrt{1-z^2}\right]\right],\ \mathbf{Cos}\left[-i\,\mathbf{Log}\left[i\,z+\sqrt{1-z^2}\right]\right],\right.$$
$$\mathbf{Tan}\left[-i\,\mathbf{Log}\left[i\,z+\sqrt{1-z^2}\right]\right],\ \mathbf{Cot}\left[-i\,\mathbf{Log}\left[i\,z+\sqrt{1-z^2}\right]\right],$$
$$\left.\mathbf{Csc}\left[-i\,\mathbf{Log}\left[i\,z+\sqrt{1-z^2}\right]\right],\ \mathbf{Sec}\left[-i\,\mathbf{Log}\left[i\,z+\sqrt{1-z^2}\right]\right]\right\}\ //\ \mathbf{FullSimplify}$$

$$\left\{z,\ \sqrt{1-z^2},\ \frac{z}{\sqrt{1-z^2}},\ \frac{\sqrt{1-z^2}}{z},\ \frac{1}{z},\ \frac{1}{\sqrt{1-z^2}}\right\}$$

Operations carried out by specialized *Mathematica* functions

Series expansions

Calculating the series expansion of trigonometric functions to hundreds of terms can be done in seconds. Here are some examples.

```
Series[Sin[z], {z, 0, 5}]
```

$$z-\frac{z^3}{6}+\frac{z^5}{120}+O[z]^6$$

```
Normal[%]
```

$$z-\frac{z^3}{6}+\frac{z^5}{120}$$

```
Series[{Sin[z], Cos[z], Tan[z], Cot[z], Csc[z], Sec[z]}, {z, 0, 3}]
```

$$\left\{z-\frac{z^3}{6}+O[z]^4,\ 1-\frac{z^2}{2}+O[z]^4,\ z+\frac{z^3}{3}+O[z]^4,\ \frac{1}{z}-\frac{z}{3}-\frac{z^3}{45}+O[z]^4,\ \frac{1}{z}+\frac{z}{6}+\frac{7z^3}{360}+O[z]^4,\ 1+\frac{z^2}{2}+O[z]^4\right\}$$

```
Series[Cot[z], {z, 0, 100}] // Timing
```

$$\left\{1.442\ \text{Second},\ \frac{1}{z}-\frac{z}{3}-\frac{z^3}{45}-\frac{2z^5}{945}-\frac{z^7}{4725}-\frac{2z^9}{93\,555}-\frac{1382z^{11}}{638\,512\,875}-\right.$$
$$\frac{4z^{13}}{18\,243\,225}-\frac{3617z^{15}}{162\,820\,783\,125}-\frac{87\,734z^{17}}{38\,979\,295\,480\,125}-\frac{349\,222z^{19}}{1\,531\,329\,465\,290\,625}-$$
$$\frac{310\,732z^{21}}{13\,447\,856\,940\,643\,125}-\frac{472\,728\,182z^{23}}{201\,919\,571\,963\,756\,521\,875}-\frac{2\,631\,724z^{25}}{11\,094\,481\,976\,030\,578\,125}-$$
$$\frac{13\,571\,120\,588z^{27}}{564\,653\,660\,170\,076\,273\,671\,875}-\frac{13\,785\,346\,041\,608z^{29}}{5\,660\,878\,804\,669\,082\,674\,070\,015\,625}-$$
$$\frac{7\,709\,321\,041\,217z^{31}}{31\,245\,110\,285\,511\,170\,603\,633\,203\,125}-\frac{303\,257\,395\,102z^{33}}{12\,130\,454\,581\,433\,748\,587\,292\,890\,625}-$$
$$\frac{52\,630\,543\,106\,106\,954\,746z^{35}}{20\,777\,977\,561\,866\,588\,586\,487\,628\,662\,044\,921\,875}-\frac{616\,840\,823\,966\,644z^{37}}{2\,403\,467\,618\,492\,375\,776\,343\,276\,883\,984\,375}-$$
$$\frac{522\,165\,436\,992\,898\,244\,102z^{39}}{20\,080\,431\,172\,289\,638\,826\,798\,401\,128\,390\,556\,640\,625}-$$
$$\frac{6\,080\,390\,575\,672\,283\,210\,764z^{41}}{2\,307\,789\,189\,818\,960\,127\,712\,594\,427\,864\,667\,427\,734\,375}-$$
$$\frac{10\,121\,188\,937\,927\,645\,176\,372z^{43}}{37\,913\,679\,547\,025\,773\,526\,706\,908\,457\,776\,679\,169\,921\,875}-$$

$$\frac{207\,461\,256\,206\,578\,143\,748\,856\,z^{45}}{7\,670\,102\,214\,448\,301\,053\,033\,358\,480\,610\,212\,529\,462\,890\,625} -$$

$$\frac{11\,218\,806\,737\,995\,635\,372\,498\,255\,094\,z^{47}}{4\,093\,648\,603\,384\,274\,996\,519\,698\,921\,478\,879\,580\,162\,286\,669\,921\,875} -$$

$$\frac{79\,209\,152\,838\,572\,743\,713\,996\,404\,z^{49}}{285\,258\,771\,457\,546\,764\,463\,363\,635\,252\,374\,414\,183\,254\,365\,234\,375} -$$

$$\frac{246\,512\,528\,657\,073\,833\,030\,130\,766\,724\,z^{51}}{8\,761\,982\,491\,474\,419\,367\,550\,817\,114\,626\,909\,562\,924\,278\,968\,505\,859\,375} -$$

$$\frac{233\,199\,709\,079\,078\,899\,371\,344\,990\,501\,528\,z^{53}}{81\,807\,125\,729\,900\,063\,867\,074\,959\,072\,425\,603\,825\,198\,823\,017\,351\,806\,640\,625} -$$

$$\frac{1\,416\,795\,959\,607\,558\,144\,963\,094\,708\,378\,988\,z^{55}}{4\,905\,352\,087\,939\,496\,310\,826\,487\,207\,538\,302\,184\,255\,342\,959\,123\,162\,841\,796\,875} -$$

$$\frac{23\,305\,824\,372\,104\,839\,134\,357\,731\,308\,699\,592\,z^{57}}{796\,392\,368\,980\,577\,121\,745\,974\,726\,570\,063\,253\,238\,310\,542\,073\,919\,837\,646\,484\,375} -$$

$$\frac{9\,721\,865\,123\,870\,044\,576\,322\,439\,952\,638\,561\,968\,331\,928\,z^{59}}{3\,278\,777\,586\,273\,629\,598\,615\,520\,165\,380\,455\,583\,231\,003\,564\,645\,636\,125\,000\,418\,914\,794\,921\,875} -$$

$$\frac{6\,348\,689\,256\,302\,894\,731\,330\,601\,216\,724\,328\,336\,z^{61}}{21\,132\,271\,510\,899\,613\,925\,529\,439\,369\,536\,628\,424\,678\,570\,233\,931\,462\,891\,949\,462\,890\,625} -$$

$$\frac{106\,783\,830\,147\,866\,529\,886\,385\,444\,979\,142\,647\,942\,017\,z^{63}}{3\,508\,062\,732\,166\,890\,409\,707\,514\,582\,539\,928\,001\,638\,766\,051\,683\,792\,497\,378\,070\,587\,158\,203\,125} -$$

$$\left(267\,745\,458\,568\,424\,664\,373\,021\,714\,282\,169\,516\,771\,254\,382\,z^{65}\right) / 86\,812\,790\,293\,146\,213\,360\,651\,966\,604\,262\,937\,105\,495\,141\,563\,588\,806\,888\,204\,273\,501\,373\,291\,015\,625 -$$

$$\left(250\,471\,004\,320\,250\,327\,955\,196\,022\,920\,428\,000\,776\,938\,z^{67}\right) / 801\,528\,196\,428\,242\,695\,121\,010\,267\,455\,843\,804\,062\,822\,357\,897\,831\,858\,125\,102\,407\,684\,326\,171\,875 -$$

$$\left(172\,043\,582\,552\,384\,800\,434\,637\,321\,986\,040\,823\,829\,878\,646\,884\,z^{69}\right) / 5\,433\,748\,964\,547\,053\,581\,149\,916\,185\,708\,338\,218\,048\,392\,402\,830\,337\,634\,114\,958\,370\,880\,742\,156\,982\,421\,875 -$$

$$\left(11\,655\,909\,923\,339\,888\,220\,876\,554\,489\,282\,134\,730\,564\,976\,603\,688\,520\,858\,z^{71}\right) / 3\,633\,348\,205\,269\,879\,230\,856\,840\,004\,304\,821\,536\,968\,049\,780\,112\,803\,650\,817\,771\,432\,558\,560\,793\,458\,452\,606\,201\,171\,875 -$$

$$\left(3\,692\,153\,220\,456\,342\,488\,035\,683\,646\,645\,690\,290\,452\,790\,030\,604\,z^{73}\right) / 11\,359\,005\,221\,796\,317\,918\,049\,302\,062\,760\,294\,302\,183\,889\,391\,189\,419\,445\,133\,951\,612\,582\,060\,536\,346\,435\,546\,875 -$$

$$\left(5\,190\,545\,015\,986\,394\,254\,249\,936\,008\,544\,252\,611\,445\,319\,542\,919\,116\,z^{75}\right) / 157\,606\,197\,452\,423\,911\,112\,934\,066\,120\,799\,083\,442\,801\,465\,302\,753\,194\,801\,233\,578\,624\,576\,089\,941\,806\,793\,212\,890\,625 -$$

$$\left(255\,290\,071\,123\,323\,586\,643\,187\,098\,799\,718\,199\,072\,122\,692\,536\,861\,835\,992\,z^{77}\right) / 76\,505\,736\,228\,426\,953\,173\,738\,238\,352\,183\,101\,801\,688\,392\,812\,244\,485\,181\,277\,127\,930\,109\,049\,138\,257\,655\,704\,498\,291\,015\,625 -$$

$$\left(9\,207\,568\,598\,958\,915\,293\,871\,149\,938\,038\,093\,699\,588\,515\,745\,502\,577\,839\,313\,734\,z^{79}\right) / 27\,233\,582\,984\,369\,795\,892\,070\,228\,410\,001\,578\,355\,986\,013\,571\,390\,071\,723\,225\,259\,349\,721\,067\,988\,068\,852\,863\,296\,604\,156\,494\,140\,625 -$$

$$\left(163\,611\,136\,505\,867\,886\,519\,332\,147\,296\,221\,453\,678\,803\,514\,884\,902\,772\,183\,572\,z^{81}\right) / 4\,776\,089\,171\,877\,348\,057\,451\,105\,924\,101\,750\,653\,118\,402\,745\,283\,825\,543\,113\,171\,217\,116\,857\,704\,024\,700\,607\,798\,175\,811\,767\,578\,125 -$$

$$\left(8\,098\,304\,783\,741\,161\,440\,924\,524\,640\,446\,924\,039\,959\,669\,564\,792\,363\,509\,124\,335\,729\,908\,z^{83}\right) /$$

```
  2 333 207 846 470 426 678 843 707 227 616 712 214 909 162 634 745 895 349 325 948 586 531 533 393
   530 725 143 500 144 033 328 342 437 744 140 625 -
(122 923 650 124 219 284 385 832 157 660 699 813 260 991 755 656 444 452 420 836 648 z^85) /
  349 538 086 043 843 717 584 559 187 055 386 621 548 470 304 913 596 772 372 737 435 524 697 231
   069 047 713 981 709 496 784 210 205 078 125 -
(476 882 359 517 824 548 362 004 154 188 840 670 307 545 554 753 464 961 562 516 323 845 108 z^87) /
  13 383 510 964 174 348 021 497 060 628 653 950 829 663 288 548 327 870 152 944 013 988 358 928 114
   528 962 242 087 062 453 152 690 410 614 013 671 875 -
(1 886 491 646 433 732 479 814 597 361 998 744 134 040 407 919 471 435 385 970 472 345 164 676 056
   z^89) /
  522 532 651 330 971 490 226 753 590 247 329 744 050 384 290 675 644 135 735 656 667 608 610 471
   400 391 047 234 539 824 350 830 981 313 610 076 904 296 875 -
(450 638 590 680 882 618 431 105 331 665 591 912 924 988 342 163 281 788 877 675 244 114 763 912
   z^91) /
  1 231 931 818 039 911 948 327 467 370 123 161 265 684 460 571 086 659 079 080 437 659 781 065 743
   269 173 212 919 832 661 978 537 311 246 395 111 083 984 375 -
(415 596 189 473 955 564 121 634 614 268 323 814 113 534 779 643 471 190 276 158 333 713 923 216
   z^93) /
  11 213 200 675 690 943 223 287 032 785 929 540 201 272 600 687 465 377 745 332 153 847 964 679 254
   692 602 138 023 498 144 562 090 675 557 613 372 802 734 375 -
(423 200 899 194 533 026 195 195 456 219 648 467 346 087 908 778 120 468 301 277 466 840 101 336
    699 974 518 z^95) /
  112 694 926 530 960 148 011 367 752 417 874 063 473 378 698 369 880 587 800 838 274 234 349 237
   591 647 453 413 782 021 538 312 594 164 677 406 144 702 434 539 794 921 875 -
(5 543 531 483 502 489 438 698 050 411 951 314 743 456 505 773 755 468 368 087 670 306 121 873 229
    244 z^97) /
  14 569 479 835 935 377 894 165 191 004 250 040 526 616 509 162 234 077 285 176 247 476 968 227 225
   810 918 346 966 001 491 701 692 846 112 140 419 483 184 814 453 125 -
(378 392 151 276 488 501 180 909 732 277 974 887 490 811 366 132 267 744 533 542 784 817 245 581
    660 788 990 844 z^99) /
  9 815 205 420 757 514 710 108 178 059 369 553 458 327 392 260 750 404 049 930 407 987 933 582 359
   080 767 225 644 716 670 683 512 153 512 547 802 166 033 089 160 919 189 453 125 + O[z]^101}
```

Mathematica comes with the add-on package `DiscreteMath`RSolve`` that allows finding the general terms of series for many functions. After loading this package, and using the package function `SeriesTerm`, the following n^{th} term for odd trigonometric functions can be evaluated.

```
<< DiscreteMath`RSolve`
```

```
SeriesTerm[{Sin[z], Tan[z], Cot[z], Csc[z], Cos[z], Sec[z]}, {z, 0, n}]
```

$$\left\{\frac{i^{-1+n}\,\text{KroneckerDelta}[\text{Mod}[-1+n,2]]\,\text{UnitStep}[-1+n]}{\text{Gamma}[1+n]},\right.$$
$$\text{If}\left[\text{Odd}[n],\ \frac{i^{-1+n}\,2^{1+n}\left(-1+2^{1+n}\right)\text{BernoulliB}[1+n]}{(1+n)!},\ 0\right],\ \frac{i\,i^n\,2^{1+n}\,\text{BernoulliB}[1+n]}{(1+n)!},$$
$$\left.\frac{i\,i^n\,2^{1+n}\,\text{BernoulliB}\left[1+n,\frac{1}{2}\right]}{(1+n)!},\ \frac{i^n\,\text{KroneckerDelta}[\text{Mod}[n,2]]}{\text{Gamma}[1+n]},\ \frac{i^n\,\text{EulerE}[n]}{n!}\right\}$$

Differentiation

Mathematica can evaluate derivatives of trigonometric functions of an arbitrary positive integer order.

```
D[Sin[z], z]
```

```
Cos[z]
```

```
Sin[z] // D[#, z] &
```

```
Cos[z]
```

```
∂_z {Sin[z], Cos[z], Tan[z], Cot[z], Csc[z], Sec[z]}
```

$$\{\text{Cos}[z],\ -\text{Sin}[z],\ \text{Sec}[z]^2,\ -\text{Csc}[z]^2,\ -\text{Cot}[z]\,\text{Csc}[z],\ \text{Sec}[z]\,\text{Tan}[z]\}$$

```
∂_{z,2} {Sin[z], Cos[z], Tan[z], Cot[z], Csc[z], Sec[z]}
```

$$\{-\text{Sin}[z],\ -\text{Cos}[z],\ 2\,\text{Sec}[z]^2\,\text{Tan}[z],\ 2\,\text{Cot}[z]\,\text{Csc}[z]^2,\ \text{Cot}[z]^2\,\text{Csc}[z]+\text{Csc}[z]^3,\ \text{Sec}[z]^3+\text{Sec}[z]\,\text{Tan}[z]^2\}$$

```
Table[D[{Sin[z], Cos[z], Tan[z], Cot[z], Csc[z], Sec[z]}, {z, n}], {n, 4}]
```

$$\{\{\text{Cos}[z],\ -\text{Sin}[z],\ \text{Sec}[z]^2,\ -\text{Csc}[z]^2,\ -\text{Cot}[z]\,\text{Csc}[z],\ \text{Sec}[z]\,\text{Tan}[z]\},\ \{-\text{Sin}[z],\ -\text{Cos}[z],$$
$$2\,\text{Sec}[z]^2\,\text{Tan}[z],\ 2\,\text{Cot}[z]\,\text{Csc}[z]^2,\ \text{Cot}[z]^2\,\text{Csc}[z]+\text{Csc}[z]^3,\ \text{Sec}[z]^3+\text{Sec}[z]\,\text{Tan}[z]^2\},$$
$$\{-\text{Cos}[z],\ \text{Sin}[z],\ 2\,\text{Sec}[z]^4+4\,\text{Sec}[z]^2\,\text{Tan}[z]^2,\ -4\,\text{Cot}[z]^2\,\text{Csc}[z]^2-2\,\text{Csc}[z]^4,$$
$$-\text{Cot}[z]^3\,\text{Csc}[z]-5\,\text{Cot}[z]\,\text{Csc}[z]^3,\ 5\,\text{Sec}[z]^3\,\text{Tan}[z]+\text{Sec}[z]\,\text{Tan}[z]^3\},$$
$$\{\text{Sin}[z],\ \text{Cos}[z],\ 16\,\text{Sec}[z]^4\,\text{Tan}[z]+8\,\text{Sec}[z]^2\,\text{Tan}[z]^3,$$
$$8\,\text{Cot}[z]^3\,\text{Csc}[z]^2+16\,\text{Cot}[z]\,\text{Csc}[z]^4,\ \text{Cot}[z]^4\,\text{Csc}[z]+18\,\text{Cot}[z]^2\,\text{Csc}[z]^3+5\,\text{Csc}[z]^5,$$
$$5\,\text{Sec}[z]^5+18\,\text{Sec}[z]^3\,\text{Tan}[z]^2+\text{Sec}[z]\,\text{Tan}[z]^4\}\}$$

Finite summation

Mathematica can calculate finite sums that contain trigonometric functions. Here are two examples.

```
Sum[Sin[a k], {k, 0, n}]
```

$$\frac{1}{2}\left(\text{Cos}\left[\frac{a}{2}\right]-\text{Cos}\left[\frac{a}{2}+a\,n\right]\right)\text{Csc}\left[\frac{a}{2}\right]$$

$$\sum_{k=0}^{n}(-1)^k\,\text{Sin}[a\,k]$$

$$\frac{1}{2}\,\text{Sec}\left[\frac{a}{2}\right]\left(-\text{Sin}\left[\frac{a}{2}\right]+\text{Sin}\left[\frac{a}{2}+a\,n+n\,\pi\right]\right)$$

Infinite summation

Mathematica can calculate infinite sums that contain trigonometric functions. Here are some examples.

$$\sum_{k=1}^{\infty} z^k\,\text{Sin}[k\,x]$$

$$\frac{i\left(-1+e^{2\,i\,x}\right)z}{2\left(e^{i\,x}-z\right)\left(-1+e^{i\,x}\,z\right)}$$

$$\sum_{k=1}^{\infty}\frac{\text{Sin}[k\,x]}{k\,!}$$

$$\frac{1}{2}\,i\left(e^{e^{-i\,x}}-e^{e^{i\,x}}\right)$$

$$\sum_{k=1}^{\infty}\frac{\text{Cos}[k\,x]}{k}$$

$$\frac{1}{2}\left(-\text{Log}\left[1-e^{-i\,x}\right]-\text{Log}\left[1-e^{i\,x}\right]\right)$$

Finite products

Mathematica can calculate some finite symbolic products that contain the trigonometric functions. Here are two examples.

$$\text{Product}\left[\text{Sin}\left[\frac{\pi\,k}{n}\right],\,\{k,\,1,\,n-1\}\right]$$

$$2^{1-n}\,n$$

$$\prod_{k=1}^{n-1}\text{Cos}\left[z+\frac{\pi\,k}{n}\right]$$

$$-(-1)^n\,2^{1-n}\,\text{Sec}[z]\,\text{Sin}\left[\frac{1}{2}\,n\,(\pi-2\,z)\right]$$

Infinite products

Mathematica can calculate infinite products that contain trigonometric functions. Here are some examples.

In[2]:= $$\prod_{k=1}^{\infty}\text{Exp}\left[z^k\,\text{Sin}[k\,x]\right]$$

Out[2]= $$e^{\frac{i\left(-1+e^{2\,i\,x}\right)z}{2\left(z+e^{2\,i\,x}\,z-e^{i\,x}\left(1+z^2\right)\right)}}$$

In[3]:= $\prod_{k=1}^{\infty} \text{Exp}\left[\frac{\text{Cos}[k\,x]}{k\,!}\right]$

Out[3]= $e^{\frac{1}{2}\left(-2+e^{e^{-i x}}+e^{e^{i x}}\right)}$

Indefinite integration

Mathematica can calculate a huge number of doable indefinite integrals that contain trigonometric functions. Here are some examples.

$$\int \text{Sin}[7\,z]\,dz$$

$$-\frac{1}{7}\,\text{Cos}[7\,z]$$

$$\int \{\{\text{Sin}[z],\ \text{Sin}[z]^a\},\ \{\text{Cos}[z],\ \text{Cos}[z]^a\},\ \{\text{Tan}[z],\ \text{Tan}[z]^a\},\ \{\text{Cot}[z],\ \text{Cot}[z]^a\},\ \{\text{Csc}[z],\ \text{Csc}[z]^a\},\ \{\text{Sec}[z],\ \text{Sec}[z]^a\}\}\,dz$$

$$\Big\{\Big\{-\text{Cos}[z],\ -\text{Cos}[z]\ \text{Hypergeometric2F1}\left[\frac{1}{2},\ \frac{1-a}{2},\ \frac{3}{2},\ \text{Cos}[z]^2\right]\ \text{Sin}[z]^{1+a}\ \left(\text{Sin}[z]^2\right)^{\frac{1}{2}(-1-a)}\Big\},$$

$$\Big\{\text{Sin}[z],\ -\frac{\text{Cos}[z]^{1+a}\ \text{Hypergeometric2F1}\left[\frac{1+a}{2},\ \frac{1}{2},\ \frac{3+a}{2},\ \text{Cos}[z]^2\right]\ \text{Sin}[z]}{(1+a)\ \sqrt{\text{Sin}[z]^2}}\Big\},$$

$$\Big\{-\text{Log}[\text{Cos}[z]],\ \frac{\text{Hypergeometric2F1}\left[\frac{1+a}{2},\ 1,\ 1+\frac{1+a}{2},\ -\text{Tan}[z]^2\right]\ \text{Tan}[z]^{1+a}}{1+a}\Big\},$$

$$\Big\{\text{Log}[\text{Sin}[z]],\ -\frac{\text{Cot}[z]^{1+a}\ \text{Hypergeometric2F1}\left[\frac{1+a}{2},\ 1,\ 1+\frac{1+a}{2},\ -\text{Cot}[z]^2\right]}{1+a}\Big\},$$

$$\Big\{-\text{Log}\left[\text{Cos}\left[\frac{z}{2}\right]\right]+\text{Log}\left[\text{Sin}\left[\frac{z}{2}\right]\right],$$

$$-\text{Cos}[z]\ \text{Csc}[z]^{-1+a}\ \text{Hypergeometric2F1}\left[\frac{1}{2},\ \frac{1+a}{2},\ \frac{3}{2},\ \text{Cos}[z]^2\right]\ \left(\text{Sin}[z]^2\right)^{\frac{1}{2}(-1+a)}\Big\},$$

$$\Big\{-\text{Log}\left[\text{Cos}\left[\frac{z}{2}\right]-\text{Sin}\left[\frac{z}{2}\right]\right]+\text{Log}\left[\text{Cos}\left[\frac{z}{2}\right]+\text{Sin}\left[\frac{z}{2}\right]\right],$$

$$-\frac{\text{Hypergeometric2F1}\left[\frac{1-a}{2},\ \frac{1}{2},\ \frac{3-a}{2},\ \text{Cos}[z]^2\right]\ \text{Sec}[z]^{-1+a}\ \text{Sin}[z]}{(1-a)\ \sqrt{\text{Sin}[z]^2}}\Big\}\Big\}$$

Definite integration

Mathematica can calculate wide classes of definite integrals that contain trigonometric functions. Here are some examples.

$$\int_0^{\pi/2} \sqrt[3]{\text{Sin}[z]}\,dz$$

$$\frac{\sqrt{\pi}\ \text{Gamma}\left[\frac{2}{3}\right]}{2\ \text{Gamma}\left[\frac{7}{6}\right]}$$

$$\int_0^{\pi/2}\left\{\sqrt{\text{Sin}[z]},\ \sqrt{\text{Cos}[z]},\ \sqrt{\text{Tan}[z]},\ \sqrt{\text{Cot}[z]},\ \sqrt{\text{Csc}[z]},\ \sqrt{\text{Sec}[z]}\right\} dz$$

$$\left\{2\ \text{EllipticE}\left[\frac{\pi}{4}, 2\right], 2\ \text{EllipticE}\left[\frac{\pi}{4}, 2\right], \frac{\pi}{\sqrt{2}}, \frac{\pi}{\sqrt{2}}, \frac{2\sqrt{\pi}\ \text{Gamma}\left[\frac{5}{4}\right]}{\text{Gamma}\left[\frac{3}{4}\right]}, \frac{2\sqrt{\pi}\ \text{Gamma}\left[\frac{5}{4}\right]}{\text{Gamma}\left[\frac{3}{4}\right]}\right\}$$

$$\int_0^{\frac{\pi}{2}}\left\{\{\text{Sin}[z], \text{Sin}[z]^a\}, \{\text{Cos}[z], \text{Cos}[z]^a\}, \{\text{Tan}[z], \text{Tan}[z]^a\}, \{\text{Cot}[z], \text{Cot}[z]^a\}, \{\text{Csc}[z], \text{Csc}[z]^a\}, \{\text{Sec}[z], \text{Sec}[z]^a\}\right\} dz$$

$$\left\{\left\{1, \frac{\sqrt{\pi}\ \text{Gamma}\left[\frac{1+a}{2}\right]}{a\ \text{Gamma}\left[\frac{a}{2}\right]}\right\}, \left\{1, \frac{\sqrt{\pi}\ \text{Gamma}\left[\frac{1+a}{2}\right]}{a\ \text{Gamma}\left[\frac{a}{2}\right]}\right\},\right.$$
$$\left\{\int_0^{\frac{\pi}{2}}\text{Tan}[z]\,dz, \text{If}\left[\text{Re}[a] < 1, \frac{1}{2}\pi\,\text{Sec}\left[\frac{a\pi}{2}\right], \int_0^{\frac{\pi}{2}}\text{Tan}[z]^a\,dz\right]\right\},$$
$$\left\{\int_0^{\frac{\pi}{2}}\text{Cot}[z]\,dz, \text{If}\left[\text{Re}[a] < 1, \frac{1}{2}\pi\,\text{Sec}\left[\frac{a\pi}{2}\right], \int_0^{\frac{\pi}{2}}\text{Cot}[z]^a\,dz\right]\right\},$$
$$\left.\left\{\int_0^{\frac{\pi}{2}}\text{Csc}[z]\,dz, \frac{\sqrt{\pi}\ \text{Gamma}\left[\frac{1}{2}-\frac{a}{2}\right]}{2\ \text{Gamma}\left[1-\frac{a}{2}\right]}\right\}, \left\{\int_0^{\frac{\pi}{2}}\text{Sec}[z]\,dz, \frac{\sqrt{\pi}\ \text{Gamma}\left[\frac{1}{2}-\frac{a}{2}\right]}{2\ \text{Gamma}\left[1-\frac{a}{2}\right]}\right\}\right\}$$

Limit operation

Mathematica can calculate limits that contain trigonometric functions.

$$\text{Limit}\left[\frac{\text{Sin}[z]}{z} + \text{Cos}[z]^3, z \to 0\right]$$

2

$$\text{Limit}\left[\left(\frac{\text{Tan}[x]}{x}\right)^{\frac{1}{x^2}}, x \to 0\right]$$

$e^{1/3}$

Solving equations

The next input solves equations that contain trigonometric functions. The message indicates that the multivalued functions are used to express the result and that some solutions might be absent.

$$\text{Solve}\left[\text{Tan}[z]^2 + 3\,\text{Sin}\left[z + \text{Pi}/6\right] == 4, z\right]$$

```
Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found.
```

```
{{z → ArcCos[Root[4 - 40 #1^2 + 12 #1^3 + 73 #1^4 - 60 #1^5 + 36 #1^6 &, 1]]},
 {z → -ArcCos[Root[4 - 40 #1^2 + 12 #1^3 + 73 #1^4 - 60 #1^5 + 36 #1^6 &, 2]]},
 {z → -ArcCos[Root[4 - 40 #1^2 + 12 #1^3 + 73 #1^4 - 60 #1^5 + 36 #1^6 &, 3]]},
 {z → ArcCos[Root[4 - 40 #1^2 + 12 #1^3 + 73 #1^4 - 60 #1^5 + 36 #1^6 &, 4]]},
 {z → ArcCos[Root[4 - 40 #1^2 + 12 #1^3 + 73 #1^4 - 60 #1^5 + 36 #1^6 &, 5]]},
 {z → ArcCos[Root[4 - 40 #1^2 + 12 #1^3 + 73 #1^4 - 60 #1^5 + 36 #1^6 &, 6]]}}
```

Complete solutions can be obtained by using the function `Reduce`.

```
Reduce[Sin[x] == a, x] // TraditionalForm
```

```
// InputForm =
C[1] ∈ Integers && (x == Pi - ArcSin[a] + 2 * Pi * C[1] || x == ArcSin[a] + 2 * Pi * C[1])
```

```
Reduce[Cos[x] == a, x] // TraditionalForm
```

```
// InputForm = C[1] ∈ Integers && (x == -ArcCos[a] + 2 * Pi * C[1] || x == ArcCos[a] + 2 * Pi * C[1])
```

```
Reduce[Tan[x] == a, x] // TraditionalForm
```

```
// InputForm = C[1] ∈ Integers && 1 + a^2 ≠ 0 && x == ArcTan[a] + Pi * C[1]
```

```
Reduce[Cot[x] == a, x] // TraditionalForm
```

```
// InputForm = C[1] ∈ Integers && 1 + a^2 ≠ 0 && x == ArcCot[a] + Pi * C[1]
```

```
Reduce[Csc[x] == a, x] // TraditionalForm
```

$$c_1 \in \mathbb{Z} \bigwedge a \neq 0 \bigwedge \left(x == -\sin^{-1}\left(\frac{1}{a}\right) + 2\pi c_1 + \pi \bigvee x == \sin^{-1}\left(\frac{1}{a}\right) + 2\pi c_1 \right)$$

```
Reduce[Sec[x] == a, x] // TraditionalForm
```

```
// InputForm = C[1] ∈ Integers && a ≠ 0 &&
  (x == -ArcCos[a^(-1)] + 2 * Pi * C[1] || x == ArcCos[a^(-1)] + 2 * Pi * C[1])
```

Solving differential equations

Here are differential equations whose linear-independent solutions are trigonometric functions. The solutions of the simplest second-order linear ordinary differential equation with constant coefficients can be represented through sin(z) and cos(z).

```
DSolve[w''[z] + w[z] == 0, w[z], z]
```

```
{{w[z] → C[1] Cos[z] + C[2] Sin[z]}}
```

```
dsol1 = DSolve[2 w[z] + 3 w''[z] + w^(4)[z] == 0, w[z], z]
```

```
{{w[z] → C[3] Cos[z] + C[1] Cos[√2 z] + C[4] Sin[z] + C[2] Sin[√2 z]}}
```

In the last input, the differential equation was solved for $w(z)$. If the argument is suppressed, the result is returned as a pure function (in the sense of the λ-calculus).

```
dsol2 = DSolve[2 w[z] + 3 w''[z] + w^(4)[z] == 0, w, z]
```

$\{\{w \to \text{Function}[\{z\}, C[3]\ \text{Cos}[z] + C[1]\ \text{Cos}[\sqrt{2}\ z] + C[4]\ \text{Sin}[z] + C[2]\ \text{Sin}[\sqrt{2}\ z]]\}\}$

The advantage of such a pure function is that it can be used for different arguments, derivatives, and more.

```
w'[ζ] /. dsol1
```

{w′[ζ]}

```
w'[ζ] /. dsol2
```

$\{C[4]\ \text{Cos}[\zeta] + \sqrt{2}\ C[2]\ \text{Cos}[\sqrt{2}\ \zeta] - C[3]\ \text{Sin}[\zeta] - \sqrt{2}\ C[1]\ \text{Sin}[\sqrt{2}\ \zeta]\}$

All trigonometric functions satisfy first-order nonlinear differential equations. In carrying out the algorithm to solve the nonlinear differential equation, *Mathematica* has to solve a transcendental equation. In doing so, the generically multivariate inverse of a function is encountered, and a message is issued that a solution branch is potentially missed.

$\text{DSolve}\left[\left\{w'[z] == \sqrt{1 - w[z]^2},\ w[0] == 0\right\},\ w[z],\ z\right]$

```
Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found.
```

{{w[z] → Sin[z]}}

$\text{DSolve}\left[\left\{w'[z] == \sqrt{1 - w[z]^2},\ w[0] == 1\right\},\ w[z],\ z\right]$

```
Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found.
```

{{w[z] → Cos[z]}}

$\text{DSolve}\left[\left\{w'[z] - w[z]^2 - 1 == 0,\ w[0] == 0\right\},\ w[z],\ z\right]$

```
Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found.
```

{{w[z] → Tan[z]}}

$\text{DSolve}\left[\left\{w'[z] + w[z]^2 + 1 == 0,\ w\left[\frac{\pi}{2}\right] == 0\right\},\ w[z],\ z\right]$

```
Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found.
```

{{w[z] → Cot[z]}}

$\text{DSolve}\left[\left\{w'[z] == \sqrt{w[z]^4 - w[z]^2},\ 1/w[0] == 0\right\},\ w[z],\ z\right]\ //\ \text{Simplify}[\#,\ 0 < z < \text{Pi}/2]\ \&$

```
Solve::verif : Potential solution {C[1] → Indeterminate} (possibly
   discarded by verifier) should be checked by hand. May require use of limits.

Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found.

Solve::verif : Potential solution {C[1] → Indeterminate} (possibly
   discarded by verifier) should be checked by hand. May require use of limits.

Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found.
```

```
{{w[z] → -Csc[z]}, {w[z] → Csc[z]}}
```

```
DSolve[{w'[z] == Sqrt[w[z]^4 - w[z]^2], 1/w[π/2] == 0}, w[z], z] // Simplify[#, 0 < z < Pi/2] &
```

```
Solve::verif : Potential solution {C[1] → Indeterminate} (possibly
   discarded by verifier) should be checked by hand. May require use of limits.

Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found.

Solve::verif : Potential solution {C[1] → Indeterminate} (possibly
   discarded by verifier) should be checked by hand. May require use of limits.

Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found.
```

```
{{w[z] → -Sec[z]}, {w[z] → Sec[z]}}
```

Integral transforms

Mathematica supports the main integral transforms like direct and inverse Fourier, Laplace, and Z transforms that can give results that contain classical or generalized functions. Here are some transforms of trigonometric functions.

LaplaceTransform[Sin[t], t, s]

$$\frac{1}{1+s^2}$$

FourierTransform[Sin[t], t, s]

$$i\sqrt{\frac{\pi}{2}}\ \text{DiracDelta}[-1+s] - i\sqrt{\frac{\pi}{2}}\ \text{DiracDelta}[1+s]$$

FourierSinTransform[Sin[t], t, s]

$$\sqrt{\frac{\pi}{2}}\ \text{DiracDelta}[-1+s] - \sqrt{\frac{\pi}{2}}\ \text{DiracDelta}[1+s]$$

FourierCosTransform[Sin[t], t, s]

$$-\frac{1}{\sqrt{2\pi}\ (-1+s)} + \frac{1}{\sqrt{2\pi}\ (1+s)}$$

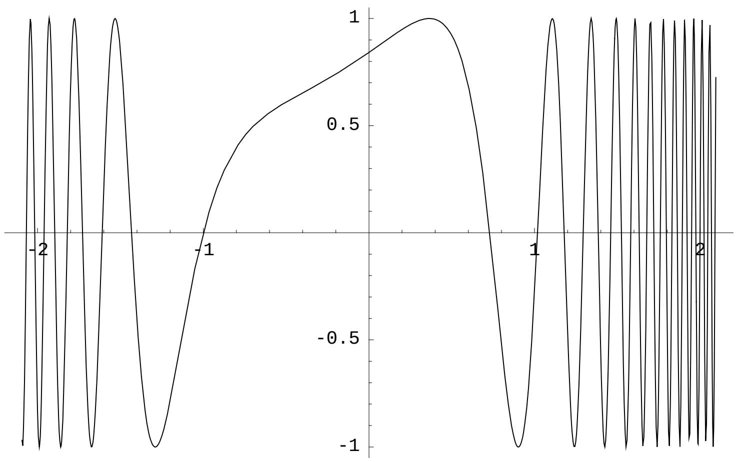
ZTransform[Sin[π t], t, s]

0

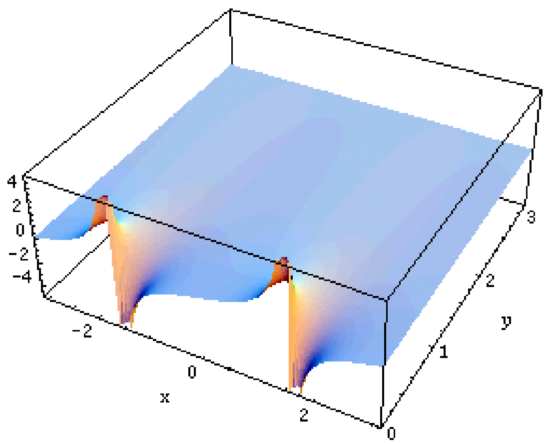
Plotting

Mathematica has built-in functions for 2D and 3D graphics. Here are some examples.

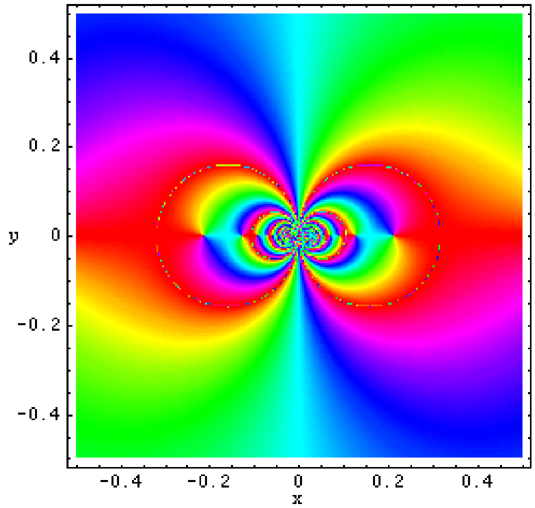
Plot$\left[\text{Sin}\left[\sum_{k=0}^{5} z^k\right], \left\{z, -\frac{2\pi}{3}, \frac{2\pi}{3}\right\}\right]$;




```
Plot3D[Re[Tan[x + ⅈ y]], {x, -π, π}, {y, 0, π},
  PlotPoints → 240, PlotRange → {-5, 5},
  ClipFill → None, Mesh → False, AxesLabel → {"x", "y", None}];
```




```
ContourPlot[Arg[Sec[1/(x + ⅈ y)]], {x, -1/2, 1/2}, {y, -1/2, 1/2},
  PlotPoints → 400, PlotRange → {-π, π}, FrameLabel → {"x", "y", None, None},
  ColorFunction → Hue, ContourLines → False, Contours → 200];
```



Introduction to the Cosecant Function in *Mathematica*

Overview

The following shows how the cosecant function is realized in *Mathematica*. Examples of evaluating *Mathematica* functions applied to various numeric and exact expressions that involve the cosecant function or return it are shown. These involve numeric and symbolic calculations and plots.

Notations

Mathematica forms of notations

Following *Mathematica*'s general naming convention, function names in `StandardForm` are just the capitalized versions of their traditional mathematics names. This shows the cosecant function in `StandardForm`.

```
Csc[z]
```

```
Csc[z]
```

This shows the cosecant function in `TraditionalForm`.

```
% // TraditionalForm
```

csc(z)

Additional forms of notations

Mathematica has other popular forms of notations that are used for print and electronic publications. In this particular instance the task is not difficult. However, it must be made to work in *Mathematica*'s `CForm`, `TeXForm`, and `FortranForm`.

```
{CForm[Csc[2 π z]], FortranForm[Csc[2 π z]], TeXForm[Csc[2 π z]]}
```

```
{Csc(2*Pi*z), Csc(2*Pi*z), \csc(2\, \pi \, z)}
```

Automatic evaluations and transformations

Evaluation for exact and machine-number values of arguments

For the exact argument $z = \pi/4$, *Mathematica* returns an exact result.

```
Csc[π/4]
```

$\sqrt{2}$

```
Csc[z] /. z → π/4
```

$\sqrt{2}$

For a machine-number argument (numerical argument with a decimal point), a machine number is also returned.

```
Csc[4.]
```

```
-1.32135
```

```
Csc[z] /. z → 2.
```

```
1.09975
```

The next inputs calculate 100-digit approximations at $z = 1$ and $z = 2$.

```
N[Csc[z] /. z → 1, 100]
```

```
1.188395105778121216261599452374551003527829834097962625265253666359184367357190487‚
 13663568030853023
```

```
N[Csc[2], 100]
```

```
1.099750170294616466756697397026312896658764443149845708742554443062569126995445980‚
 76791442481219894
```

```
Csc[2] // N[#, 100] &
```

```
1.099750170294616466756697397026312896658764443149845708742554443062569126995445980‚
 76791442481219894
```

Within a second, it is possible to calculate thousands of digits for the cosecant function. The next input calculates 10000 digits for csc(1) and analyzes the frequency of the digit k in the resulting decimal number.

```
Map[Function[w, {First[#], Length[#]} & /@ Split[Sort[First[RealDigits[w]]]]],
 N[{Csc[z]} /. z → 1, 10 000]]
```

```
{{{0, 1031}, {1, 976}, {2, 1045}, {3, 917},
  {4, 1001}, {5, 996}, {6, 964}, {7, 1012}, {8, 982}, {9, 1076}}}
```

Here is a 50-digit approximation to the cosecant function at the complex argument $z = 3 - 2\,i$.

```
N[Csc[3 - 2 ⅈ], 50]
```

```
0.040300578856891521875132479542869867780842475424268 -
 0.272548661462940199512498477932708924053539864495451 ⅈ
```

```
{N[Csc[z] /. z → 3 - 2 ⅈ, 50], Csc[3 - 2 ⅈ] // N[#, 50] &}
```

```
{0.040300578856891521875132479542869867780842475424268 -
  0.272548661462940199512498477932708924053539864495451 ⅈ,
 0.040300578856891521875132479542869867780842475424268 -
  0.272548661462940199512498477932708924053539864495451 ⅈ}
```

Mathematica automatically evaluates mathematical functions with machine precision, if the arguments of the function are numerical values and include machine-number elements. In this case only six digits after the decimal point are shown in the results. The remaining digits are suppressed, but can be displayed using the function `InputForm`.

```
{Csc[3.], N[Csc[3]], N[Csc[3], 16], N[Csc[3], 5], N[Csc[3], 20]}
```

```
{7.08617, 7.08617, 7.08617, 7.08617, 7.0861673957371859182}
```

```
% // InputForm
```

```
{7.086167395737187, 7.086167395737187, 7.086167395737187, 7.086167395737187,
 7.08616739573718591821753227245268`20}
```

Simplification of the argument

Mathematica knows the symmetry and periodicity of the cosecant function. Here are some examples.

```
Csc[-3]
```

```
-Csc[3]
```

```
{Csc[-z], Csc[z + π], Csc[z + 2 π], Csc[-z + 21 π]}
```

```
{-Csc[z], -Csc[z], Csc[z], Csc[z]}
```

Mathematica automatically simplifies the composition of the direct and the inverse cosecant functions into its argument.

```
Csc[ArcCsc[z]]
```

```
z
```

Mathematica also automatically simplifies the composition of the direct and any of the inverse trigonometric functions into algebraic functions of the argument.

```
{Csc[ArcSin[z]], Csc[ArcCos[z]], Csc[ArcTan[z]],
 Csc[ArcCot[z]], Csc[ArcCsc[z]], Csc[ArcSec[z]]}
```

$$\left\{\frac{1}{z}, \frac{1}{\sqrt{1-z^2}}, \frac{\sqrt{1+z^2}}{z}, \sqrt{1+\frac{1}{z^2}}\, z, z, \frac{1}{\sqrt{1-\frac{1}{z^2}}}\right\}$$

In cases where the argument has the structure $\pi k/2+z$ or $\pi k/2-z$, and $\pi k/2+i z$ or $\pi k/2-i z$ with integer k, the cosecant function can be automatically transformed into trigonometric or hyperbolic cosecant or secant functions.

$\text{Csc}\left[\frac{\pi}{2}-4\right]$

Sec[4]

$\left\{\text{Csc}\left[\frac{\pi}{2}-z\right], \text{Csc}\left[\frac{\pi}{2}+z\right], \text{Csc}\left[-\frac{\pi}{2}-z\right], \text{Csc}\left[-\frac{\pi}{2}+z\right], \text{Csc}[\pi-z], \text{Csc}[\pi+z]\right\}$

{Sec[z], Sec[z], -Sec[z], -Sec[z], Csc[z], -Csc[z]}

Csc[ⅈ 5]

-ⅈ Csch[5]

$\left\{\text{Csc}[i z], \text{Csc}\left[\frac{\pi}{2}-i z\right], \text{Csc}\left[\frac{\pi}{2}+i z\right], \text{Csc}[\pi-i z], \text{Csc}[\pi+i z]\right\}$

{-ⅈ Csch[z], Sech[z], Sech[z], -ⅈ Csch[z], ⅈ Csch[z]}

Simplification of combinations of cosecant functions

Sometimes simple arithmetic operations containing the cosecant function can automatically generate other equal trigonometric functions.

1 / Csc[4]

Sin[4]

{1 / Csc[z], 1 / Csc[π / 2 - z], Csc[π / 2 - z] / Csc[z],
 Csc[z] / Csc[π / 2 - z], 1 / Csc[π / 2 - z], Csc[π / 2 - z] / Csc[z] ^2}

{Sin[z], Cos[z], Tan[z], Cot[z], Cos[z], Sin[z] Tan[z]}

The cosecant function arising as special cases from more general functions

The cosecant function can be treated as a particular case of some more general special functions. For example, $\csc(z)$ appears automatically from Bessel, Struve, Mathieu, Jacobi, hypergeometric, and Meijer functions for appropriate values of their parameters.

$$\left\{\sqrt{\frac{2}{\pi}}\Big/\left(\sqrt{z}\,\text{BesselJ}\left[\frac{1}{2}, z\right]\right), \frac{\sqrt{\frac{2}{\pi}}}{\sqrt{z}\,\text{StruveH}\left[-\frac{1}{2}, z\right]}, \frac{1}{\text{MathieuS}[1, 0, z]},\right.$$
$$\text{JacobiDS}[z, 0], \text{JacobiNS}[z, 0], \text{JacobiDC}\left[\frac{\pi}{2}-z, 0\right], \text{JacobiNC}\left[\frac{\pi}{2}-z, 0\right],$$
$$i\,\text{JacobiCS}[i z, 1], i\,\text{JacobiDS}[i z, 1], \text{JacobiCN}\left[\frac{\pi i}{2}-i z, 1\right], \text{JacobiDN}\left[\frac{\pi i}{2}-i z, 1\right],$$
$$\left.1\Big/\text{HypergeometricPFQ}\left[\{\}, \left\{\frac{3}{2}\right\}, -\frac{z^2}{4}\right], 1\Big/\text{MeijerG}\left[\{\{\}, \{\}\}, \left\{\left\{\frac{1}{2}\right\}, \{0\}\right\}, \frac{z^2}{4}\right]\right\}$$

$\{\text{Csc}[z], \text{Csc}[z], \text{Csc}[z], \text{Csc}[z], \text{Csc}[z], \text{Csc}[z], \text{Csc}[z],$
$\text{Csc}[z], \text{Csc}[z], \text{Csc}[z], \text{Csc}[z], \sqrt{z^2}\,\text{Csc}\left[\sqrt{z^2}\right], \frac{\sqrt{\pi}\, z\,\text{Csc}[z]}{\sqrt{z^2}}\}$

Equivalence transformations using specialized *Mathematica* functions

General remarks

Almost everybody prefers using $1+\csc(z)$ instead of $\csc(\pi-z)+\csc(\pi/2)$. *Mathematica* automatically transforms the second expression into the first one. The automatic application of transformation rules to mathematical expressions can result in overly complicated results. Compact expressions like $\csc(\pi/16)$ should not be automatically expanded into the more complicated expression $2\Big/\left(2-\left(2+2^{1/2}\right)\right)$. *Mathematica* has special functions that produce such expansions. Some are demonstrated in the next section.

TrigExpand

The function `TrigExpand` expands out trigonometric and hyperbolic functions. In more detail, it splits up sums and integer multiples that appear in the arguments of trigonometric and hyperbolic functions, and then expands out products of trigonometric and hyperbolic functions into sums of powers, using trigonometric and hyperbolic identities where possible. Here are some examples.

```
TrigExpand[Csc[x - y]]
```

$\frac{1}{\text{Cos}[y]\,\text{Sin}[x]-\text{Cos}[x]\,\text{Sin}[y]}$

```
Csc[4 z] // TrigExpand
```

$\frac{1}{4\,\text{Cos}[z]^3\,\text{Sin}[z]-4\,\text{Cos}[z]\,\text{Sin}[z]^3}$

```
Csc[2 z]^2 // TrigExpand
```

$\frac{1}{4}\,\text{Csc}[z]^2\,\text{Sec}[z]^2$

```
TrigExpand[{Csc[x + y + z], Csc[3 z]}]
```

$\{1/(\text{Cos}[y]\,\text{Cos}[z]\,\text{Sin}[x]+\text{Cos}[x]\,\text{Cos}[z]\,\text{Sin}[y]+$
$\text{Cos}[x]\,\text{Cos}[y]\,\text{Sin}[z]-\text{Sin}[x]\,\text{Sin}[y]\,\text{Sin}[z]), \frac{1}{3\,\text{Cos}[z]^2\,\text{Sin}[z]-\text{Sin}[z]^3}\}$

TrigFactor

The function `TrigFactor` factors trigonometric and hyperbolic functions. In more detail, it splits up sums and integer multiples that appear in the arguments of trigonometric and hyperbolic functions, and then factors the resulting polynomials into trigonometric and hyperbolic functions, using trigonometric and hyperbolic identities where possible. Here are some examples.

```
TrigFactor[Csc[x] + Csc[y]]
```

$$\frac{1}{2}\,\text{Cos}\left[\frac{x}{2}-\frac{y}{2}\right]\text{Csc}\left[\frac{x}{2}\right]\text{Csc}\left[\frac{y}{2}\right]\text{Sec}\left[\frac{x}{2}\right]\text{Sec}\left[\frac{y}{2}\right]\text{Sin}\left[\frac{x}{2}+\frac{y}{2}\right]$$

```
Csc[x] - Sec[y] // TrigFactor
```

$$\frac{\text{Csc}\left[\frac{x}{2}\right]\text{Sec}\left[\frac{x}{2}\right]\left(\text{Cos}\left[\frac{x}{2}-\frac{y}{2}\right]-\text{Sin}\left[\frac{x}{2}-\frac{y}{2}\right]\right)\left(\text{Cos}\left[\frac{x}{2}+\frac{y}{2}\right]-\text{Sin}\left[\frac{x}{2}+\frac{y}{2}\right]\right)}{2\left(\text{Cos}\left[\frac{y}{2}\right]-\text{Sin}\left[\frac{y}{2}\right]\right)\left(\text{Cos}\left[\frac{y}{2}\right]+\text{Sin}\left[\frac{y}{2}\right]\right)}$$

TrigReduce

The function `TrigReduce` rewrites the products and powers of trigonometric and hyperbolic functions in terms of trigonometric and hyperbolic functions with combined arguments. In more detail, it typically yields a linear expression involving trigonometric and hyperbolic functions with more complicated arguments. `TrigReduce` is approximately opposite to `TrigExpand` and `TrigFactor`. Here are some examples.

```
TrigReduce[Csc[x] Csc[y]]
```

$$\frac{2}{\text{Cos}[x-y]-\text{Cos}[x+y]}$$

```
Csc[x] Sec[y] // TrigReduce
```

$$\frac{2}{\text{Sin}[x-y]+\text{Sin}[x+y]}$$

```
Table[TrigReduce[Csc[z]^n], {n, 2, 5}]
```

$$\left\{-\frac{2}{-1+\text{Cos}[2z]},\ \frac{4}{3\,\text{Sin}[z]-\text{Sin}[3z]},\ -\frac{8}{-3+4\,\text{Cos}[2z]-\text{Cos}[4z]},\ \frac{16}{10\,\text{Sin}[z]-5\,\text{Sin}[3z]+\text{Sin}[5z]}\right\}$$

```
TrigReduce[TrigExpand[{Csc[x + y + z], Csc[3 z], Csc[x] Csc[y]}]]
```

$$\left\{\text{Csc}[x+y+z],\ \text{Csc}[3z],\ \frac{2}{\text{Cos}[x-y]-\text{Cos}[x+y]}\right\}$$

```
TrigFactor[Csc[x] + Csc[y]] // TrigReduce
```

$$\frac{2\,(\text{Sin}[x]+\text{Sin}[y])}{\text{Cos}[x-y]-\text{Cos}[x+y]}$$

TrigToExp

The function `TrigToExp` converts trigonometric and hyperbolic functions to exponentials. It tries, where possible, to give results that do not involve explicit complex numbers. Here are some examples.

```
TrigToExp[Csc[z]]
```

$$-\frac{2\,i}{e^{-i z}-e^{i z}}$$

```
Csc[a z] + Csc[b z] // TrigToExp
```

$$-\frac{2\,i}{e^{-i\,a\,z}-e^{i\,a\,z}}-\frac{2\,i}{e^{-i\,b\,z}-e^{i\,b\,z}}$$

ExpToTrig

The function `ExpToTrig` converts exponentials to trigonometric and hyperbolic functions. It is approximately opposite to `TrigToExp`. Here are some examples.

```
ExpToTrig[TrigToExp[Csc[z]]]
```

```
Csc[z]
```

$$\left\{\alpha\,e^{-i\,x\,\beta}+\alpha\,e^{i\,x\,\beta}\Big/\left(\alpha\,e^{-i\,x\,\beta}+\gamma\,e^{i\,x\,\beta}\right)\right\}\;\text{// ExpToTrig}$$

$$\left\{\alpha\,\text{Cos}[x\,\beta]-i\,\alpha\,\text{Sin}[x\,\beta]+\frac{\alpha\,(\text{Cos}[x\,\beta]+i\,\text{Sin}[x\,\beta])}{\alpha\,\text{Cos}[x\,\beta]+\gamma\,\text{Cos}[x\,\beta]-i\,\alpha\,\text{Sin}[x\,\beta]+i\,\gamma\,\text{Sin}[x\,\beta]}\right\}$$

ComplexExpand

The function `ComplexExpand` expands expressions assuming that all the variables are real. The option `TargetFunctions` can be given as a list of functions from the set `{Re, Im, Abs, Arg, Conjugate, Sign}`. `ComplexExpand` will try to give results in terms of the specified functions. Here are some examples.

```
ComplexExpand[Csc[x + ⅈ y]]
```

$$-\frac{2\,\text{Cosh}[y]\,\text{Sin}[x]}{\text{Cos}[2\,x]-\text{Cosh}[2\,y]}+\frac{2\,i\,\text{Cos}[x]\,\text{Sinh}[y]}{\text{Cos}[2\,x]-\text{Cosh}[2\,y]}$$

```
Csc[x + ⅈ y] + Csc[x - ⅈ y] // ComplexExpand
```

$$-\frac{4\,\text{Cosh}[y]\,\text{Sin}[x]}{\text{Cos}[2\,x]-\text{Cosh}[2\,y]}$$

```
ComplexExpand[Re[Csc[x + ⅈ y]], TargetFunctions → {Re, Im}]
```

$$-\frac{2\,\text{Cosh}[y]\,\text{Sin}[x]}{\text{Cos}[2\,x]-\text{Cosh}[2\,y]}$$

```
ComplexExpand[Im[Csc[x + ⅈ y]], TargetFunctions → {Re, Im}]
```

$$\frac{2\,\text{Cos}[x]\,\text{Sinh}[y]}{\text{Cos}[2\,x]-\text{Cosh}[2\,y]}$$

```
ComplexExpand[Abs[Csc[x + ⅈ y]], TargetFunctions → {Re, Im}]
```

$$\sqrt{\frac{4\,\text{Cosh}[y]^2\,\text{Sin}[x]^2}{(\text{Cos}[2\,x]-\text{Cosh}[2\,y])^2}+\frac{4\,\text{Cos}[x]^2\,\text{Sinh}[y]^2}{(\text{Cos}[2\,x]-\text{Cosh}[2\,y])^2}}$$

```
ComplexExpand[Abs[Csc[x + ⅈ y]], TargetFunctions → {Re, Im}] //
 Simplify[#, {x, y} ∈ Reals] &
```

$$\frac{\sqrt{2}}{\sqrt{-\text{Cos}[2\,x]+\text{Cosh}[2\,y]}}$$

$$\text{ComplexExpand}[\text{Re}[\text{Csc}[x + i\, y]] + \text{Im}[\text{Csc}[x + i\, y]], \text{TargetFunctions} \to \{\text{Re}, \text{Im}\}]$$

$$-\frac{2 \cosh[y] \sin[x]}{\cos[2 x] - \cosh[2 y]} + \frac{2 \cos[x] \sinh[y]}{\cos[2 x] - \cosh[2 y]}$$

$$\text{ComplexExpand}[\text{Arg}[\text{Csc}[x + i\, y]], \text{TargetFunctions} \to \{\text{Re}, \text{Im}\}]$$

$$\text{ArcTan}\left[-\frac{2 \cosh[y] \sin[x]}{\cos[2 x] - \cosh[2 y]}, \frac{2 \cos[x] \sinh[y]}{\cos[2 x] - \cosh[2 y]}\right]$$

$$\text{ComplexExpand}[\text{Arg}[\text{Csc}[x + i\, y]], \text{TargetFunctions} \to \{\text{Re}, \text{Im}\}] \mathbin{//} \text{Simplify}[\#, \{x, y\} \in \text{Reals}] \&$$

$$\text{ArcTan}[\cosh[y] \sin[x], -\cos[x] \sinh[y]]$$

$$\text{ComplexExpand}[\text{Conjugate}[\text{Csc}[x + i\, y]], \text{TargetFunctions} \to \{\text{Re}, \text{Im}\}] \mathbin{//} \text{Simplify}$$

$$\frac{1}{\cosh[y] \sin[x] - i \cos[x] \sinh[y]}$$

Simplify

The function `Simplify` performs a sequence of algebraic transformations on the expression, and returns the simplest form it finds. Here are some examples.

$$\left(\text{Csc}[z_1]^2 - \text{Csc}[z_2]^2\right)^2 \text{Csc}[z_1 + z_2]^4 - 2 \,\text{Csc}[z_1]^2 \,\text{Csc}[z_2]^2 \left(\text{Csc}[z_1]^2 + \text{Csc}[z_2]^2 - 2\right) \text{Csc}[z_1 + z_2]^2 \mathbin{//} \text{Simplify}$$

$$-\text{Csc}[z_1]^4 \,\text{Csc}[z_2]^4$$

$$\text{Simplify}\left[4 \,\text{Csc}[2 z]^2 \left(\text{Csc}[z]^2 - 1\right)\right]$$

$$\text{Csc}[z]^4$$

Here is a collection of trigonometric identities. Each is written as a logical conjunction.

$$\text{Simplify}[\#] \,\&\, /@ \left(\text{Csc}[3 z] == \frac{\text{Csc}[z]^3}{3 \,\text{Csc}[z]^2 - 4} \bigwedge \text{Csc}\left[\frac{z}{2}\right]^2 == \frac{2}{1 - \cos[z]} \bigwedge \text{Csc}[z]^2 == \frac{2 \sec[2 z]}{\sec[2 z] - 1} \bigwedge \text{Csc}[a + i\, b] == \frac{2 \cosh[b] \sin[a] - 2 i \cos[a] \sinh[b]}{\cosh[2 b] - \cos[2 a]} \bigwedge \text{Csc}[a] + \text{Csc}[b] == 2 \sin\left[\frac{a}{2} + \frac{b}{2}\right] \cos\left[\frac{a}{2} - \frac{b}{2}\right] \text{Csc}[a] \,\text{Csc}[b] \bigwedge \text{Csc}[a] \,\text{Csc}[b] == \frac{2}{\cos[a - b] - \cos[a + b]} \bigwedge \text{Csc}[a]^2 - \sec[b]^2 == \cos[a - b] \cos[a + b] \,\text{Csc}[a]^2 \sec[b]^2\right)$$

True

The function `Simplify` has the `Assumption` option. For example, *Mathematica* treats the periodicity of trigonometric functions for the symbolic integer coefficient k of $k\,\pi$.

```
Simplify[{Csc[z + 2 k π], Csc[z + k π] / Csc[z]}, k ∈ Integers]
```

$$\left\{\text{Csc}[z],\ (-1)^k\right\}$$

Mathematica also knows that the composition of the inverse and direct trigonometric functions produces the value of the internal argument under the corresponding restriction.

```
ArcCsc[Csc[z]]
```

```
ArcCsc[Csc[z]]
```

```
Simplify[ArcCsc[Csc[z]], -π / 2 < Re[z] < π / 2]
```

```
z
```

FunctionExpand (and Together)

While the cosecant function auto-evaluates for simple fractions of π, for more complicated cases it stays as a cosecant function to avoid the build up of large expressions. Using the function `FunctionExpand`, the cosecant function can sometimes be transformed into explicit radicals. Here are some examples.

$$\left\{\text{Csc}\left[\frac{\pi}{16}\right],\ \text{Csc}\left[\frac{\pi}{60}\right]\right\}$$

$$\left\{\text{Csc}\left[\frac{\pi}{16}\right],\ \text{Csc}\left[\frac{\pi}{60}\right]\right\}$$

```
FunctionExpand[%]
```

$$\left\{\frac{2}{\sqrt{2-\sqrt{2+\sqrt{2}}}},\ \frac{1}{-\frac{-\frac{1}{8}\sqrt{3}\left(-1+\sqrt{5}\right)-\frac{1}{4}\sqrt{\frac{1}{2}\left(5+\sqrt{5}\right)}}{\sqrt{2}}+\frac{\frac{1}{8}\left(-1+\sqrt{5}\right)-\frac{1}{4}\sqrt{\frac{3}{2}\left(5+\sqrt{5}\right)}}{\sqrt{2}}}\right\}$$

```
Together[%]
```

$$\left\{\frac{2}{\sqrt{2-\sqrt{2+\sqrt{2}}}},\ \frac{16}{-\sqrt{2}-\sqrt{6}+\sqrt{10}+\sqrt{30}+2\sqrt{5+\sqrt{5}}-2\sqrt{3\left(5+\sqrt{5}\right)}}\right\}$$

If the denominator contains squares of integers other than 2, the results always contain complex numbers (meaning that the imaginary number $i = \sqrt{-1}$ appears unavoidably).

$$\left\{\text{Csc}\left[\frac{\pi}{9}\right]\right\}$$

$$\left\{\text{Csc}\left[\frac{\pi}{9}\right]\right\}$$

```
FunctionExpand[%] // Together
```

$$\left\{8 \Big/ \left(-i\, 2^{2/3} \left(-1 - i\sqrt{3}\right)^{1/3} + 2^{2/3}\sqrt{3}\left(-1 - i\sqrt{3}\right)^{1/3} + i\, 2^{2/3}\left(-1 + i\sqrt{3}\right)^{1/3} + 2^{2/3}\sqrt{3}\left(-1 + i\sqrt{3}\right)^{1/3}\right)\right\}$$

Here the function `RootReduce` is used to express the previous algebraic numbers as roots of polynomial equations.

```
RootReduce[Simplify[%]]
```

$$\left\{\text{Root}\left[-64 + 96\, \#1^2 - 36\, \#1^4 + 3\, \#1^6\, \&,\, 6\right]\right\}$$

The function `FunctionExpand` also reduces trigonometric expressions with compound arguments or compositions, including trigonometric functions, to simpler ones. Here are some examples.

$$\left\{\text{Csc}\left[\sqrt{z^2}\right],\ \text{Csc}\left[\frac{\text{ArcCsc}[z]}{2}\right],\ \text{Csc}[2\,\text{ArcCsc}[z]],\ \text{Csc}[3\,\text{ArcSin}[z]]\right\}\ //\ \text{FunctionExpand}$$

$$\left\{\frac{\sqrt{-i z}\,\sqrt{i z}\,\text{Csc}[z]}{z},\ \frac{\sqrt{2}\,\sqrt{-\frac{i}{z}}\,\sqrt{\frac{i}{z}}\,z}{\sqrt{1-\frac{\sqrt{(-1+z)\,(1+z)}}{\sqrt{-i z}\,\sqrt{i z}}}},\ \frac{\sqrt{-i z}\,\sqrt{i z}\,z}{2\sqrt{(-1+z)\,(1+z)}},\ \frac{1}{-z^3+3z\left(1-z^2\right)}\right\}$$

Applying `Simplify` to the last expression gives a more compact result.

```
Simplify[%]
```

$$\left\{\frac{\sqrt{z^2}\,\text{Csc}[z]}{z},\ \frac{\sqrt{2}\,\sqrt{\frac{1}{z^2}}\,z}{\sqrt{1-\frac{\sqrt{z^2}\,\sqrt{-1+z^2}}{z^2}}},\ \frac{z}{2\sqrt{1-\frac{1}{z^2}}},\ \frac{1}{3z-4z^3}\right\}$$

FullSimplify

The function `FullSimplify` tries a wider range of transformations than `Simplify` and returns the simplest form it finds. Here are some examples that contrast the results of applying these functions to the same expressions.

$$\text{set1} = \left\{\text{Csc}\left[-i\,\text{Log}\left[i z + \sqrt{1-z^2}\right]\right],\ \text{Csc}\left[\frac{\pi}{2} + i\,\text{Log}\left[i z + \sqrt{1-z^2}\right]\right],\right.$$
$$\text{Csc}\left[\frac{1}{2} i\,\text{Log}[1 - i z] - \frac{1}{2} i\,\text{Log}[1 + i z]\right],\ \text{Csc}\left[\frac{1}{2} i\,\text{Log}\left[1 - \frac{i}{z}\right] - \frac{1}{2} i\,\text{Log}\left[1 + \frac{i}{z}\right]\right],$$
$$\left.\text{Csc}\left[-i\,\text{Log}\left[\sqrt{1-\frac{1}{z^2}} + \frac{i}{z}\right]\right],\ \text{Csc}\left[\frac{\pi}{2} + i\,\text{Log}\left[\sqrt{1-\frac{1}{z^2}} + \frac{i}{z}\right]\right]\right\}$$

$$\left\{\frac{2\,i\left(i\,z+\sqrt{1-z^2}\right)}{-1+\left(i\,z+\sqrt{1-z^2}\right)^2},\ \frac{2\left(i\,z+\sqrt{1-z^2}\right)}{1+\left(i\,z+\sqrt{1-z^2}\right)^2},\ -i\,\mathrm{Csch}\left[\frac{1}{2}\,\mathrm{Log}[1-i\,z]-\frac{1}{2}\,\mathrm{Log}[1+i\,z]\right],\right.$$

$$\left.-i\,\mathrm{Csch}\left[\frac{1}{2}\,\mathrm{Log}\left[1-\frac{i}{z}\right]-\frac{1}{2}\,\mathrm{Log}\left[1+\frac{i}{z}\right]\right],\ \frac{2\,i\left(\sqrt{1-\frac{1}{z^2}}+\frac{i}{z}\right)}{-1+\left(\sqrt{1-\frac{1}{z^2}}+\frac{i}{z}\right)^2},\ \frac{2\left(\sqrt{1-\frac{1}{z^2}}+\frac{i}{z}\right)}{1+\left(\sqrt{1-\frac{1}{z^2}}+\frac{i}{z}\right)^2}\right\}$$

```
set1 // Simplify
```

$$\left\{\frac{1}{z},\ \frac{2\left(i\,z+\sqrt{1-z^2}\right)}{1+\left(i\,z+\sqrt{1-z^2}\right)^2},\ -i\,\mathrm{Csch}\left[\frac{1}{2}\,(\mathrm{Log}[1-i\,z]-\mathrm{Log}[1+i\,z])\right],\right.$$

$$\left.-i\,\mathrm{Csch}\left[\frac{1}{2}\left(\mathrm{Log}\left[\frac{-i+z}{z}\right]-\mathrm{Log}\left[\frac{i+z}{z}\right]\right)\right],\ z,\ \frac{z\left(i+\sqrt{1-\frac{1}{z^2}}\,z\right)}{-1+i\sqrt{1-\frac{1}{z^2}}\,z+z^2}\right\}$$

```
set1 // FullSimplify
```

$$\left\{\frac{1}{z},\ \frac{1}{\sqrt{1-z^2}},\ \frac{\sqrt{1+z^2}}{z},\ \sqrt{1+\frac{1}{z^2}}\,z,\ z,\ \frac{1}{\sqrt{1-\frac{1}{z^2}}}\right\}$$

Operations under special *Mathematica* functions

Series expansions

Calculating the series expansion of a cosecant function to hundreds of terms can be done in seconds.

```
Series[Csc[z], {z, 0, 3}]
```

$$\frac{1}{z}+\frac{z}{6}+\frac{7\,z^3}{360}+O[z]^4$$

```
Normal[%]
```

$$\frac{1}{z}+\frac{z}{6}+\frac{7\,z^3}{360}$$

```
Series[Csc[z], {z, 0, 100}] // Timing
```

$$\Big\{0.731\ \text{Second},\ \frac{1}{z}+\frac{z}{6}+\frac{7 z^3}{360}+\frac{31 z^5}{15120}+\frac{127 z^7}{604800}+$$
$$\frac{73 z^9}{3421440}+\frac{1414477 z^{11}}{653837184000}+\frac{8191 z^{13}}{37362124800}+\frac{16931177 z^{15}}{762187345920000}+$$
$$\frac{5749691557 z^{17}}{2554547108585472000}+\frac{91546277357 z^{19}}{401428831349145600000}+\frac{3324754717 z^{21}}{143888775912161280000}+$$
$$\frac{1982765468311237 z^{23}}{846912068365871834726400000}+\frac{22076500342261 z^{25}}{93067260259985915904000000}+$$
$$\frac{65053034220152267 z^{27}}{2706661834818276108533760000000}+\frac{925118910976041358111 z^{29}}{379895145823020034178921005056000000}+$$
$$\frac{16555640865486520478399 z^{31}}{67098363418091850192640593100800000000}+$$
$$\frac{8089941578146657681 z^{33}}{323601898837710055733597306880000000}+$$
$$\frac{29167285342563717499865628061 z^{35}}{11514933432934040605684030014461313024000000000}+$$
$$\frac{21194489326041221593005331 z^{37}}{82582518547358070277895877489485414400000000}+$$
$$\frac{20504534628406556573233443388891 z^{39}}{788523841596004024692580034116817717821440000000000}+$$
$$\frac{3342730069684120811652882591487741 z^{41}}{1268720524330850548384457773438793947652554752000000000}+$$
$$\frac{22256729848336009246732182756923251 z^{43}}{83373063027455893179550082254549316560025026560000000000}+$$
$$\frac{4204720064739945338027515648494601 z^{45}}{155453761677720805836283924155647353217309736960000000000}+$$
$$\left(789453341324662409540561918158225679892869 z^{47}\right)/$$
$$288064911324799566327359417344299167768891090860297420800000000000+$$
$$\left(11147697225254007513111810575214137741411 z^{49}\right)/$$
$$40146603026261659017568642779205494874578486871027875840000000000+$$
$$\left(19824888075038498895325555597751708575618601 z^{51}\right)/$$
$$704651090779125093016156113196054887311595129950573690880000000000000+$$
$$\left(262559530727861921881866927518259047914603716781 z^{53}\right)/$$
$$92106635238357310808365004330528004852228147264151368511782912000000000000+$$
$$\left(17898125542071873735730356715155014721251666173 z^{55}\right)/$$
$$61968420295555119526449580574377950938765101345082804954726400000000000000+$$
$$\left(59977201118727758119227899690327133403726311223297 z^{57}\right)/$$
$$2049504214960933533072990439217577433683293745409173879913615196160000000000000+$$
$$\left(700534210387317657846086373720757772905116340448339145899117 z^{59}\right)/$$
$$236260824252362440973840816079792861390752198085416178182664985620516962304000000000000000+$$
$$\left(914942546207257218347013581951612877187089501496318471 z^{61}\right)/$$

3 045 481 283 263 348 792 805 140 873 059 751 363 356 027 173 928 216 018 596 435 636 885 913 600
000 000 000 000 +
(20 100 142 713 756 778 191 911 006 382 652 359 986 295 781 679 578 907 558 831 z^{63}) /
660 329 953 212 254 323 395 418 658 785 024 032 749 606 257 643 041 306 340 703 736 034 280 603
648 000 000 000 000 000 +
(159 323 611 326 120 450 488 389 245 384 865 575 324 738 828 109 795 992 797 082 391 z^{65}) /
51 658 494 350 396 290 305 644 228 573 784 279 525 870 873 480 017 361 814 539 256 790 702 128 613
031 936 000 000 000 000 000 +
(18 481 498 058 322 628 947 716 344 686 182 741 586 483 134 684 345 384 453 338 763 z^{67}) /
59 142 342 029 495 165 332 297 068 805 306 158 530 542 111 320 080 310 151 948 586 757 191 290 438
287 360 000 000 000 000 000 +
(77 170 673 236 910 986 782 979 489 874 724 949 419 948 359 134 226 987 131 130 744 239 z^{69}) /
2 437 324 656 772 857 705 444 944 182 363 661 167 131 853 414 501 630 550 087 035 622 390 223 747
501 225 148 416 000 000 000 000 000 +
(13 760 869 587 331 995 975 993 746 457 467 711 498 463 585 881 697 886 664 960 973 906 542 546 557
363 z^{71}) /
4 289 500 446 290 264 330 097 545 893 459 108 419 974 147 567 987 974 238 654 910 161 238 195 198
952 348 775 092 835 686 809 600 000 000 000 000 000 +
(8 717 850 308 951 125 943 106 241 478 955 633 907 151 863 144 161 504 932 373 403 782 468 541 z^{73}) /
26 820 692 769 076 106 045 347 542 828 750 306 660 460 804 201 697 430 094 914 480 554 222 532 624
403 099 902 101 422 080 000 000 000 000 000 +
(225 913 878 445 347 788 239 012 263 656 524 774 222 058 591 589 057 113 117 183 235 138 438 229
z^{75}) /
6 859 670 270 431 907 306 528 979 847 906 184 422 375 919 968 636 900 323 814 533 044 973 965 717
301 253 661 597 368 320 000 000 000 000 000 000 +
(209 664 917 440 381 075 210 488 947 910 949 681 099 079 009 142 171 617 465 640 196 874 495 298
645 923 z^{77}) /
62 832 717 306 502 502 014 675 677 599 787 507 336 006 597 165 120 434 954 820 164 975 868 220 082
152 304 585 955 177 350 037 504 000 000 000 000 000 000 +
(2 782 816 853 788 582 377 005 851 633 530 926 719 138 614 127 020 726 458 440 267 606 052 697 114
593 813 198 429 z^{79}) /
8 230 845 407 605 568 582 914 800 435 835 067 131 703 063 517 658 773 375 130 471 728 921 400 649
576 494 517 651 142 004 461 630 023 270 400 000 000 000 000 000 000 +
(193 535 936 691 230 245 420 277 570 327 999 527 314 162 100 147 724 925 070 210 964 685 489 914
214 919 813 z^{81}) /
5 649 645 319 632 464 299 350 504 202 203 323 550 871 534 066 294 226 472 515 539 715 770 572 034
222 227 483 396 438 156 666 092 912 640 000 000 000 000 000 000 +
(19 580 499 496 346 711 729 461 614 442 098 338 002 401 418 332 067 606 586 224 352 059 717 880 243
995 978 807 707 091 139 z^{83}) /
5 641 350 416 251 088 890 189 342 564 754 248 622 526 774 122 458 335 744 746 779 368 250 986 015
623 852 318 863 005 632 779 816 064 968 554 027 417 600 000 000 000 000 000 000 +
(19 174 912 835 670 156 353 478 151 878 992 306 221 151 224 536 350 971 084 299 575 297 208 876 737
808 561 882 181 z^{85}) /
54 524 595 762 204 198 908 699 602 108 269 901 834 849 671 160 756 298 013 890 692 790 818 069 579
315 341 958 265 861 894 001 469 164 827 443 200 000 000 000 000 000 000 +
(2 635 498 959 267 859 649 322 127 517 302 966 806 019 641 757 161 948 876 000 210 570 808 277 950
696 291 425 285 834 088 997 z^{87}) /

```
73 964 214 681 992 455 493 764 118 773 331 151 453 177 725 345 341 648 570 716 423 950 681 090 998
  189 426 754 454 153 726 256 132 839 500 425 092 661 248 000 000 000 000 000 000 000 +
(145 960 221 431 107 281 351 963 087 682 715 030 380 082 072 086 568 444 045 058 765 037 532 667
    787 093 417 143 826 510 295 814 277 z^89) /
 40 429 005 682 284 797 512 487 270 722 743 185 814 194 552 672 486 693 788 278 216 851 767 321 519
  797 101 819 394 228 680 438 222 854 264 607 060 950 150 388 121 600 000 000 000 000 000 000 +
(139 465 888 662 750 057 525 922 213 557 499 999 477 386 634 608 852 751 750 285 481 117 999 879
    995 472 867 750 432 983 815 705 583 z^91) /
 381 264 430 805 309 635 439 569 723 901 776 300 856 194 963 464 876 831 372 711 691 563 973 351
  575 799 185 239 232 500 752 396 267 386 258 294 709 685 783 101 440 000 000 000 000 000 000 000 +
(36 748 797 366 017 349 666 579 236 513 579 509 683 801 362 413 034 457 447 931 359 205 704 321 783
    477 255 551 013 260 133 381 713 z^93) /
 991 519 291 784 264 206 936 461 567 691 063 231 041 217 053 922 640 173 016 693 383 897 068 691
  818 400 555 898 745 676 728 724 153 117 576 282 533 760 352 649 216 000 000 000 000 000 000 000 +
(270 398 625 948 123 082 314 611 963 393 016 865 616 785 654 770 755 643 282 191 138 740 659 737
    107 488 544 884 649 347 789 848 812 668 206 363 z^95) /
 72 004 935 110 709 599 716 015 681 424 504 052 536 127 038 774 013 190 701 574 657 941 778 695 707
  665 868 417 735 175 040 953 046 157 178 003 213 597 320 383 233 877 711 257 600 000 000 000 000
  000 000 000 +
(219 601 906 638 938 053 472 410 975 845 541 383 947 744 568 855 462 550 031 915 350 118 687 904
    096 064 182 124 956 938 408 313 871 105 681 z^97) /
 577 156 558 094 892 719 119 970 555 697 382 192 274 062 883 515 037 856 948 626 863 916 002 473
  674 891 252 992 835 275 053 534 435 907 567 705 138 078 069 005 882 406 993 920 000 000 000 000
  000 000 000 +
(57 320 894 125 919 037 242 220 236 514 238 821 884 006 539 223 225 037 716 059 395 657 008 088 741
    358 897 843 919 575 897 472 694 823 084 321 z^99) /
 1 486 860 519 832 210 475 011 895 423 821 682 213 269 747 634 241 049 353 666 433 474 134 551 641
  022 255 416 138 561 320 860 605 876 616 171 511 410 030 318 753 190 437 191 680 000 000 000 000
  000 000 000 000 + O[z]^101}
```

Mathematica comes with the add-on package `DiscreteMath`RSolve`` that allows finding the general terms of the series for many functions. After loading this package, and using the package function `SeriesTerm`, the following n^{th} term of csc(z) can be evaluated.

```
<< DiscreteMath`RSolve`
```

```
SeriesTerm[Csc[z], {z, 0, n}] z^n
```

$$\frac{i\, i^n\, 2^{1+n} z^n \,\text{BernoulliB}\left[1+n, \frac{1}{2}\right]}{(1+n)!}$$

This result can be verified by the following process.

```
% /. {Odd[n_] :→ Element[(n + 1) / 2, Integers]} /.
  {BernoulliB[m_, 1/2] :> -(1 - 2^(1-m)) BernoulliB[m] } /. {n → 2 k - 1}
```

$$\frac{\mathbb{i}\,\mathbb{i}^{-1+2k}\,2^{2k}\left(-1+2^{1-2k}\right)z^{-1+2k}\,\text{BernoulliB}[2k]}{(2k)!}$$

Sum[%, {k, 1, ∞}]

$$-\frac{\text{Csc}\left[\frac{\sqrt{z^2}}{2}\right]\text{Sec}\left[\frac{\sqrt{z^2}}{2}\right]\left(-\sqrt{z^2}+\text{Sin}\left[\sqrt{z^2}\right]\right)}{2z}$$

FullSimplify[%]

$$-\frac{1}{z}+\text{Csc}[z]$$

Differentiation

Mathematica can evaluate derivatives of the cosecant function of an arbitrary positive integer order.

∂_z **Csc[z]**

$$-\text{Cot}[z]\,\text{Csc}[z]$$

$\partial_{\{z,2\}}$ **Csc[z]**

$$\text{Cot}[z]^2\,\text{Csc}[z]+\text{Csc}[z]^3$$

Table[D[Csc[z], {z, n}], {n, 10}]

$$\{-\text{Cot}[z]\,\text{Csc}[z],\ \text{Cot}[z]^2\,\text{Csc}[z]+\text{Csc}[z]^3,$$
$$-\text{Cot}[z]^3\,\text{Csc}[z]-5\,\text{Cot}[z]\,\text{Csc}[z]^3,\ \text{Cot}[z]^4\,\text{Csc}[z]+18\,\text{Cot}[z]^2\,\text{Csc}[z]^3+5\,\text{Csc}[z]^5,$$
$$-\text{Cot}[z]^5\,\text{Csc}[z]-58\,\text{Cot}[z]^3\,\text{Csc}[z]^3-61\,\text{Cot}[z]\,\text{Csc}[z]^5,$$
$$\text{Cot}[z]^6\,\text{Csc}[z]+179\,\text{Cot}[z]^4\,\text{Csc}[z]^3+479\,\text{Cot}[z]^2\,\text{Csc}[z]^5+61\,\text{Csc}[z]^7,$$
$$-\text{Cot}[z]^7\,\text{Csc}[z]-543\,\text{Cot}[z]^5\,\text{Csc}[z]^3-3111\,\text{Cot}[z]^3\,\text{Csc}[z]^5-1385\,\text{Cot}[z]\,\text{Csc}[z]^7,$$
$$\text{Cot}[z]^8\,\text{Csc}[z]+1636\,\text{Cot}[z]^6\,\text{Csc}[z]^3+18\,270\,\text{Cot}[z]^4\,\text{Csc}[z]^5+$$
$$19\,028\,\text{Cot}[z]^2\,\text{Csc}[z]^7+1385\,\text{Csc}[z]^9,\ -\text{Cot}[z]^9\,\text{Csc}[z]-4916\,\text{Cot}[z]^7\,\text{Csc}[z]^3-$$
$$101\,166\,\text{Cot}[z]^5\,\text{Csc}[z]^5-206\,276\,\text{Cot}[z]^3\,\text{Csc}[z]^7-50\,521\,\text{Cot}[z]\,\text{Csc}[z]^9,$$
$$\text{Cot}[z]^{10}\,\text{Csc}[z]+14\,757\,\text{Cot}[z]^8\,\text{Csc}[z]^3+540\,242\,\text{Cot}[z]^6\,\text{Csc}[z]^5+$$
$$1\,949\,762\,\text{Cot}[z]^4\,\text{Csc}[z]^7+1\,073\,517\,\text{Cot}[z]^2\,\text{Csc}[z]^9+50\,521\,\text{Csc}[z]^{11}\}$$

Finite products

Mathematica can calculate some finite symbolic and nonsymbolic products that contain the cosecant function. Here are two examples.

$$\prod_{k=0}^{n-1}\text{Csc}\left[z+\frac{\pi k}{n}\right]$$

$$2^{-1+n}\,\text{Csc}[n\,z]$$

$$\prod_{k=1}^{n-1}\text{Csc}\left[\frac{k\,\pi}{n}\right]$$

$$\frac{2^{-1+n}}{n}$$

Indefinite integration

Mathematica can calculate a huge number of doable indefinite integrals that contain the cosecant function. The results can contain special functions. Here are some examples.

$$\int \mathbf{Csc[z]}\, dz$$

$$-\text{Log}\left[\text{Cos}\left[\frac{z}{2}\right]\right] + \text{Log}\left[\text{Sin}\left[\frac{z}{2}\right]\right]$$

$$\int \frac{1}{\sqrt{\mathbf{Csc[z]}}}\, dz$$

$$-\frac{2\,\text{EllipticE}\left[\frac{1}{2}\left(\frac{\pi}{2} - z\right), 2\right]}{\sqrt{\text{Csc}[z]}\,\sqrt{\text{Sin}[z]}}$$

$$\int \mathbf{Csc[z]^a}\, dz$$

$$-\text{Cos}[z]\,\text{Csc}[z]^{-1+a}\,\text{Hypergeometric2F1}\left[\frac{1}{2}, \frac{1+a}{2}, \frac{3}{2}, \text{Cos}[z]^2\right]\left(\text{Sin}[z]^2\right)^{\frac{1}{2}(-1+a)}$$

$$\int \frac{1}{a + b\,\mathbf{Csc[z]}}\, dz \text{ // Simplify}$$

$$\frac{z - \frac{2\,b\,\text{ArcTan}\left[\frac{a+b\,\text{Tan}\left[\frac{z}{2}\right]}{\sqrt{-a^2+b^2}}\right]}{\sqrt{-a^2+b^2}}}{a}$$

Definite integration

Mathematica can calculate wide classes of definite integrals that contain the cosecant function. Here are some examples.

In[321]:= $$\int_0^{\pi/2} \sqrt{\mathbf{Csc[t]}}\, dt$$

Out[321]= $$\frac{2\sqrt{\pi}\,\text{Gamma}\left[\frac{5}{4}\right]}{\text{Gamma}\left[\frac{3}{4}\right]}$$

In[322]:= $$\int_0^{\frac{\pi}{2}} \frac{t}{\sqrt{\mathbf{Csc[t]}}}\, dt$$

Out[322]= $\frac{1}{36\,\text{Gamma}\left[\frac{5}{4}\right]}\left(\sqrt{\pi}\left(6\,\pi\,\text{Gamma}\left[\frac{7}{4}\right]+\text{Gamma}\left[\frac{5}{4}\right]^2\left(\frac{9\,\text{HypergeometricPFQ}\left[\left\{\frac{1}{2},\frac{1}{2},\frac{5}{4}\right\},\left\{\frac{3}{2},\frac{7}{4}\right\},1\right]}{\text{Gamma}\left[\frac{7}{4}\right]}-\frac{8\,\text{HypergeometricPFQ}\left[\left\{\frac{1}{2},\frac{3}{4},\frac{5}{4}\right\},\left\{\frac{7}{4},\frac{7}{4}\right\},1\right]}{\text{Gamma}\left[\frac{3}{4}\right]}\right)\right)\right)$

Limit operation

Mathematica can calculate limits that contain the cosecant function. Here are some examples.

```
Limit[z Csc[3 z], z → 0]
```

$\frac{1}{3}$

```
Limit[z Csc[2 Sqrt[z^2]], z → 0, Direction → 1]
```

$-\frac{1}{2}$

```
Limit[z Csc[2 Sqrt[z^2]], z → 0, Direction → -1]
```

$\frac{1}{2}$

Solving equations

The next inputs solve two equations that contain the cosecant function. Because of the multivalued nature of the inverse cosecant function, a printed message indicates that only some of the possible solutions are returned.

```
Solve[Csc[z]^2 + 2 Csc[z + Pi/6] == 4 == 4, z]
```

Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found.

```
{{z → ArcCos[Root[-11 - 24 #1 + 76 #1^2 + 56 #1^3 - 128 #1^4 - 32 #1^5 + 64 #1^6 &, 1]]},
 {z → -ArcCos[Root[-11 - 24 #1 + 76 #1^2 + 56 #1^3 - 128 #1^4 - 32 #1^5 + 64 #1^6 &, 2]]},
 {z → ArcCos[Root[-11 - 24 #1 + 76 #1^2 + 56 #1^3 - 128 #1^4 - 32 #1^5 + 64 #1^6 &, 3]]},
 {z → ArcCos[Root[-11 - 24 #1 + 76 #1^2 + 56 #1^3 - 128 #1^4 - 32 #1^5 + 64 #1^6 &, 4]]},
 {z → -ArcCos[Root[-11 - 24 #1 + 76 #1^2 + 56 #1^3 - 128 #1^4 - 32 #1^5 + 64 #1^6 &, 5]]},
 {z → -ArcCos[Root[-11 - 24 #1 + 76 #1^2 + 56 #1^3 - 128 #1^4 - 32 #1^5 + 64 #1^6 &, 6]]}}
```

```
Solve[Csc[x] == a, x]
```

Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found.

```
{{x → ArcCsc[a]}}
```

A complete solution of the previous equation can be obtained using the function `Reduce`.

```
Reduce[Csc[x] == a, x] // InputForm
```

```
// InputForm = C[1] ∈ Integers && a ≠ 0 &&
  (x == Pi - ArcSin[a^(-1)] + 2*Pi*C[1] || x == ArcSin[a^(-1)] + 2*Pi*C[1])
```

Solving differential equations

Here is a nonlinear first-order differential equation that is obeyed by the cosecant function.

```
In[67]:= w'[z]^2 - w[z]^4 + w[z]^2 == 0 /. w → Csc // Simplify

Out[67]= True
```

Mathematica can find the general solution of this differential equation. In doing so, the generically multivariate inverse of a function is encountered, and a message is issued that a solution branch is potentially missed.

```
In[65]:= DSolve[{w'[z]^2 - w[z]^4 + w[z]^2 == 0}, w[z], z]

Solve::tdep :
 The equations appear to involve the variables to be solved for in an essentially non-algebraic way. More…

Solve::tdep :
 The equations appear to involve the variables to be solved for in an essentially non-algebraic way. More…
```

$$\text{Out[65]= } \left\{\left\{w[z] \to \text{InverseFunction}\left[-\frac{\text{ArcTan}\left[\frac{1}{\sqrt{-1+\#1^2}}\right] \#1 \sqrt{-1+\#1^2}}{\sqrt{-\#1^2+\#1^4}} \&\right][-z+C[1]]\right\},\right.$$

$$\left.\left\{w[z] \to \text{InverseFunction}\left[-\frac{\text{ArcTan}\left[\frac{1}{\sqrt{-1+\#1^2}}\right] \#1 \sqrt{-1+\#1^2}}{\sqrt{-\#1^2+\#1^4}} \&\right][z+C[1]]\right\}\right\}$$

Plotting

Mathematica has built-in functions for 2D and 3D graphics. Here are some examples.

$$\text{In[77]:= Plot}\left[\text{Csc}\left[\sum_{k=0}^{3} z^k\right], \left\{z, -\frac{2\pi}{3}, \frac{2\pi}{3}\right\}\right];$$

```
In[78]:= Plot3D[Re[Csc[x + ⅈ y]], {x, -π, π}, {y, 0, 2},
   PlotPoints → 240, PlotRange → {-5, 5},
   ClipFill → None, Mesh → False, AxesLabel → {"x", "y", None}];
```

$$\text{In[79]:= ContourPlot}\left[\text{Arg}\left[\text{Csc}\left[\frac{1}{x + i\, y}\right]\right], \left\{x, -\frac{1}{2}, \frac{1}{2}\right\}, \left\{y, -\frac{1}{2}, \frac{1}{2}\right\},\right.$$

```
   PlotPoints → 400, PlotRange → {-π, π}, FrameLabel → {"x", "y", None, None},
   ColorFunction → Hue, ContourLines → False, Contours → 200];
```

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