Introductions to EllipticExp

Introduction to to elliptic exp and elliptic log

General

The elliptic exp and elliptic log appeared in the article of D. Masser (1975). These functions and the derivative of elliptic exp were implemented in the technical computing software *Mathematica* as part of the implementation of numerous mathematical functions that were used for elliptic and number theory functions.

Definitions of the elliptic exp and elliptic log

The elliptic exponent $\text{eexp}(z; a, b)$, its derivative $\text{eexp}'_z(z; a, b)$, and the elliptic logarithm $\text{elog}(z_1, z_2; a, b)$ are defined by the following formulas:

$$\text{eexp}(z; a, b) = \{x, y\} /; \left(z = \text{elog}(x, y; a, b) /; y^2 - x\left(x^2 + a\,x + b\right) = 0\right)$$

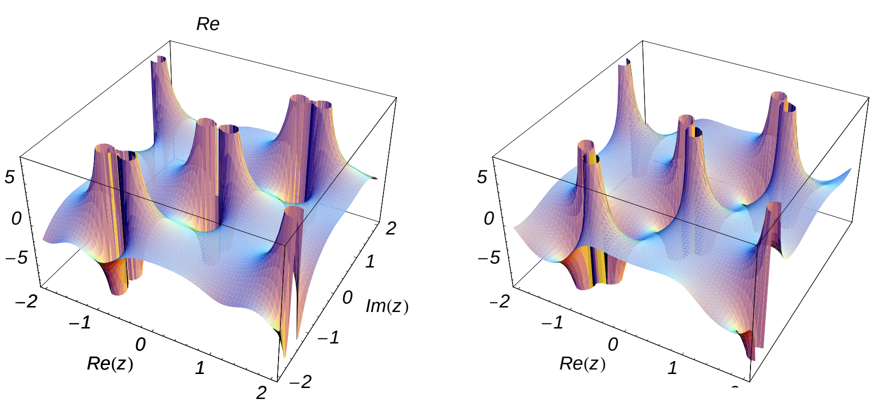
$$\text{eexp}'_z(z; a, b) = \frac{\partial\,\text{eexp}(z; a, b)}{\partial z}$$

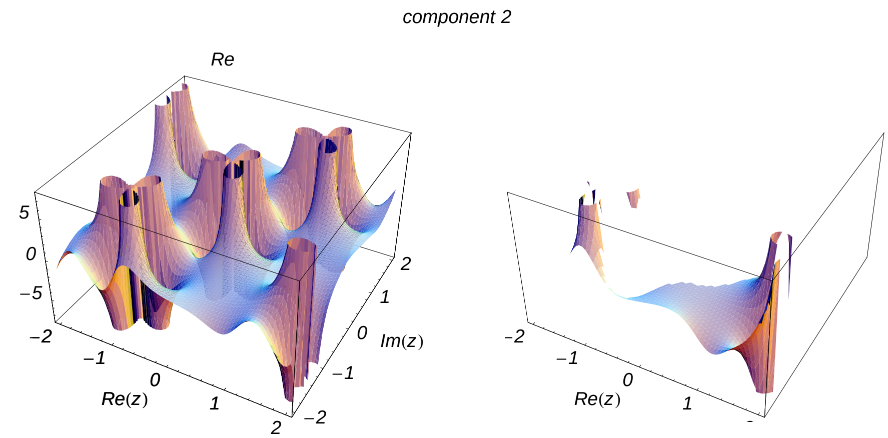
$$\text{elog}(z_1, z_2; a, b) = \frac{\sqrt{z_2^2}}{2\,z_2} \int_\infty^{z_1} \frac{1}{\sqrt{t^3 + a\,t^2 + b\,t}}\,dt /; z_1^3 + a\,z_1^2 + b\,z_1 - z_2^2 = 0 \bigwedge a \in \mathbb{R} \bigwedge b \in \mathbb{R}.$$

A quick look at the elliptic exp and elliptic log

Here is a quick look at the two components of the elliptic exponent and its derivative. All of the following graphics use the parameters $\{a, b\} = \{0.86 + 2.88\,i, 1.05 - 1.152\,i\}$. The double periodicity of the function and the poles of order 2 to 4 are clearly visible.

component 1



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{- GraphicsArray -, - GraphicsArray -}
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The last pair of graphics shows the elliptic logarithm over the complex *z*-plane. Compared with the direct function, it is relatively structureless.

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Show[GraphicsArray[
ParametricPlot3D[{x, y, #[ EllipticLog[{x + I y, Sqrt[b*(x + I y) + a*(x + I y) ^2 +
(x + I y) ^3]}, {a, b}]],
                    {EdgeForm[]}}, {x, -2, 2}, {y, -2, 2},
                    PlotPoints -> 72, BoxRatios -> {1, 1, 0.6}, PlotLabel -> #,
                    DisplayFunction -> Identity,
                   PlotRange -> {All, All, {-8, 8}},
                   AxesLabel -> {StyleForm[TraditionalForm[Re[z]]],
                                 StyleForm[TraditionalForm[Im[z]]], None}]& /@
                   {Re, Im}]]
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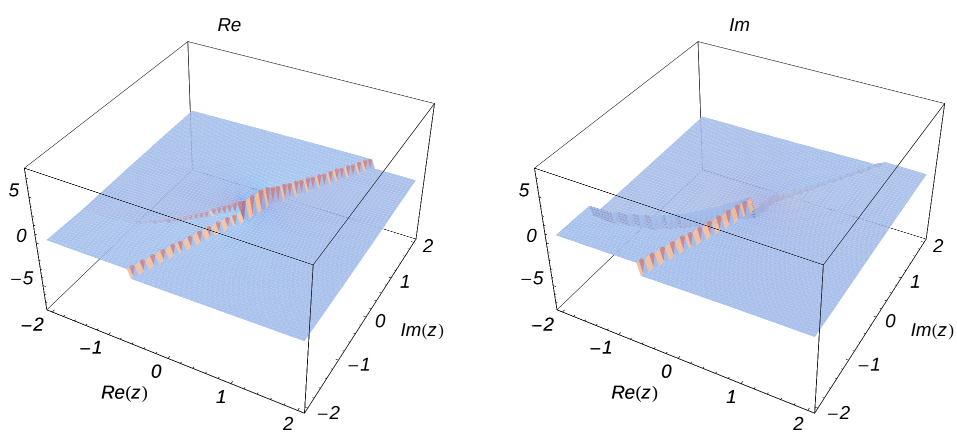


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Connections within the group of elliptic exp and elliptic log and with other function groups

Representations through more general functions

The elliptic logarithm elog$(z_1, z_2; a, b)$ is the particular case of the hypergeometric function of two variables (Appell function F_1):

$$\operatorname{elog}(z_1, z_2; a, b) = -\frac{\sqrt{z_2^2}}{\sqrt{z_1}\; z_2} F_1\left(\frac{1}{2}; \frac{1}{2}, \frac{1}{2}; \frac{3}{2}; \frac{-a-\sqrt{a^2-4b}}{2 z_1}, \frac{\sqrt{a^2-4b}-a}{2 z_1}\right) /; z_1^3 + a z_1^2 + b z_1 - z_2^2 = 0.$$

Representations through related equivalent functions

The elliptic exponent eexp(z; a, b) is connected with Jacobi amplitude by the following formula:

$$x = \frac{1}{2}\left(a+\sqrt{a^2-4b}\right)\cot^2\left(\operatorname{am}\left(-\frac{\sqrt{2}\; z}{\sqrt{\frac{a-\sqrt{a^2-4b}}{b}}} \,\middle|\, \frac{2\sqrt{a^2-4b}}{a+\sqrt{a^2-4b}}\right)\right) /; \{x, y\} = \operatorname{eexp}(z; a, b).$$

The elliptic exponent eexp(z; a, b) and elliptic logarithm elog(z_1, z_2; a, b) can be expressed through direct and inverse Weierstrass functions by the following formulas:

$$x = \sqrt[3]{4}\; \wp\left(\sqrt[3]{2}\; z; \sqrt[3]{4}\left(\frac{a^2}{3}-b\right), \frac{a b}{3} - \frac{2 a^3}{27}\right) - \frac{a}{3} /; \{x, y\} = \operatorname{eexp}(z; a, b)$$

$$\operatorname{elog}\left(z_1, \sqrt{z_1^3 + a z_1^2 + b z_1}; a, b\right) = \frac{1}{\sqrt[3]{2}} \wp^{-1}\left(\frac{1}{6}\sqrt[3]{2}\;(a + 3 z_1); \sqrt[3]{4}\left(\frac{a^2}{3}-b\right), \frac{a b}{3} - \frac{2 a^3}{27}\right).$$

The elliptic logarithm elog(z_1, z_2; a, b) has the following representation through incomplete elliptic integral F:

$$\operatorname{elog}\left(z_1, \sqrt{z_1^3 + a z_1^2 + b z_1}; a, b\right) = -\frac{1}{2}\sqrt{\frac{2\left(a-\sqrt{a^2-4b}\right)}{b}}\; F\left(\cot^{-1}\left(\sqrt{\frac{2 z_1}{a+\sqrt{a^2-4b}}}\right) \,\middle|\, \frac{2\sqrt{a^2-4b}}{a+\sqrt{a^2-4b}}\right).$$

Relations to inverse functions

The elliptic logarithm elog(z_1, z_2; a, b) is the inverse function to the elliptic exponent eexp(z; a, b) and its derivative $\operatorname{eexp}_z'(z; a, b)$. Relations between them are described by the following formulas:

$$\operatorname{eexp}(\operatorname{elog}(z_1, z_2; a, b); a, b) = \{z_1, z_2\} /; z_1^3 + a z_1^2 + b z_1 - z_2^2 = 0$$

$$\xi = 2 z_2 /; z_1^3 + a z_1^2 + b z_1 - z_2^2 = 0 /; \{\xi, \eta\} = \operatorname{eexp}_z'(\operatorname{elog}(z_1, z_2; a, b); a, b).$$

The best-known properties and formulas for elliptic exp and elliptic log

Values at zero

The elliptic exponent eexp(z; a, b), its derivative $\operatorname{eexp}_z'(z; a, b)$, and the elliptic logarithm elog(z_1, z_2; a, b) have the following values at the origin point:

$$\operatorname{elog}(0, 0; 0, 0) = \tilde{\infty}$$

$$\operatorname{eexp}(0; 0, 0) = \{\tilde{\infty}, \tilde{\infty}\}$$

$$\operatorname{eexp}_z'(0; 0, 0) = \{\tilde{\infty}, \tilde{\infty}\}.$$

Specific values for specialized parameter

The elliptic exponent eexp(z; a, b) has the following value at the specialized point z:

$$\text{eexp}\left(-\frac{\sqrt{y^2}}{\sqrt{x}\ y} F_1\left(\frac{1}{2}; \frac{1}{2}, \frac{1}{2}; \frac{3}{2}; \frac{-a-\sqrt{a^2-4b}}{2x}, \frac{\sqrt{a^2-4b}-a}{2x}\right); a, b\right) = \{x, y\} /; y^2 - x\left(x^2 + a x + b\right) = 0.$$

Analyticity

The elliptic exponent eexp(z; a, b) and its derivative $\text{eexp}_z'(z; a, b)$ are vector-valued functions of z, a, and b, which are analytic in each component, and they are defined over $\mathbb{C}^3$.

The elliptic logarithm elog(z_1, z_2; a, b) is an analytical function of z_1, z_2, a, b, which is defined in $\mathbb{C}^4$.

The elliptic exponent eexp(z; a, b), its derivative $\text{eexp}_z'(z; a, b)$, and the elliptic logarithm elog(z_1, z_2; a, b) have complicated branch cuts.

Poles and essential singularities

The elliptic logarithm elog(z_1, z_2; a, b) does not have poles and essential singularities.

Periodicity

The elliptic exponent eexp(z; a, b), its derivative $\text{eexp}_z'(z; a, b)$, and the elliptic logarithm elog(z_1, z_2; a, b) do not have periodicity.

Parity and symmetry

The elliptic exponent eexp(z; a, b), its derivative $\text{eexp}_z'(z; a, b)$, and the elliptic logarithm elog(z_1, z_2; a, b) have mirror symmetry:

$$\text{eexp}\left(\bar{z}; \bar{a}, \bar{b}\right) = \overline{\text{eexp}(z; a, b)}$$

$$\text{eexp}_z'\left(\bar{z}; \bar{a}, \bar{b}\right) = \overline{\text{eexp}_z'(z; a, b)}$$

$$\text{elog}\left(\overline{z_1}, \overline{z_2}; \bar{a}, \bar{b}\right) = \overline{\text{elog}(z_1, z_2; a, b)}.$$

Integral representations

The elliptic logarithm elog(z_1, z_2; a, b) has the following integral representation:

$$\text{elog}(z_1, z_2; a, b) = \frac{\sqrt{z_2^2}}{2 z_2} \int_{\infty}^{z_1} \frac{1}{\sqrt{t^3 + a t^2 + b t}}\, dt /; z_1^3 + a z_1^2 + b z_1 - z_2^2 = 0 \bigwedge a \in \mathbb{R} \bigwedge b \in \mathbb{R}.$$

Identities

The elliptic exponent eexp(z; a, b) satisfies the following identities including the complete elliptic integral $K(w)$:

$$\operatorname{eexp}(z;\, a,\, b) = \operatorname{eexp}\left(z - n\sqrt{8}\,\sqrt{\frac{a-\sqrt{a^2-4b}}{b}}\; K\left(\frac{2\sqrt{a^2-4b}}{a+\sqrt{a^2-4b}}\right);\, a,\, b\right) /;\, n \in \mathbb{Z}$$

$$\operatorname{eexp}(z;\, a,\, b) = \operatorname{eexp}\left(z - i\, n\sqrt{8}\,\sqrt{\frac{a-\sqrt{a^2-4b}}{b}}\; K\left(1-\frac{2\sqrt{a^2-4b}}{a+\sqrt{a^2-4b}}\right);\, a,\, b\right) /;\, n \in \mathbb{Z}.$$

Simple representations of derivatives

The first derivatives of elliptic exponent eexp$(z;\, a,\, b)$ and the elliptic logarithm elog$(z_1,\, z_2;\, a,\, b)$ have the following representations:

$$\frac{\partial \operatorname{eexp}(z;\, a,\, b)}{\partial z} = \operatorname{eexp}'_z(z;\, a,\, b)$$

$$\frac{\partial \operatorname{elog}(z_1,\, z_2;\, a,\, b)}{\partial z_1} = \frac{1}{2\, z_2}.$$

Differential equations

The elliptic exponent eexp$(z;\, a,\, b)$, its derivative eexp$'_z(z;\, a,\, b)$, and the elliptic logarithm elog$(z_1,\, z_2;\, a,\, b)$ satisfy the following ordinary nonlinear differential equations:

$$4\, w(z)^3 + 4\, a\, w(z)^2 + 4\, b\, w(z) - w'(z)^2 = 0 /;\, \{w(z),\, v(z)\} = \operatorname{eexp}(z;\, a,\, b)$$

$$27\, w(z)^4 + 8\, a\left(2\, a^2 + 9\, b\right) w(z)^2 + 64\, b^3 - 2\, w'(z)^3 - 16\, a^2\, b^2 - 12\, b\, w'(z)^2 = 0 /;\, w(z) = \operatorname{eexp}'_z(z;\, a,\, b)$$

$$2\, z\, (b + z\, (a + z))\, w''(z) + (b + z\, (2\, a + 3\, z))\, w'(z) = 1 /;\, w(z) = \operatorname{elog}(z,\, z_2;\, a,\, b) \bigwedge z^3 + a\, z^2 + b\, z - z_2^2 = 0$$

$$4\left(z^3 + a\, z^2 + b\, z\right) w'(z)^2 = 1 /;\, w(z) = \operatorname{elog}(z,\, z_2;\, a,\, b).$$

Applications of the elliptic exp and elliptic log

Applications of the elliptic exponent eexp$(z;\, a,\, b)$ and elliptic logarithm elog$(z_1,\, z_2;\, a,\, b)$ include number theory and solutions for the cubic and quartic Diophantine equations.

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