Introductions to EllipticTheta4

Introduction to the Jacobi theta functions

General

The basic achievements in studying infinite series were made in the 18th and 19th centuries when mathematicians investigated issues regarding the convergence of different types of series. In particular, they found that the famous geometrical series:

$$\sum_{k=0}^{\infty} q^k$$

converges inside the unit circle $|z| < 1$ to the function $1/(1-q)$, but can be analytically extended outside this circle by the formulas $-\sum_{k=0}^{\infty} q^{-k-1} \,/;\, |z| > 1$ and $\sum_{k=0}^{\infty} (q-q_0)^k \,/;\, c_k == (1-q_0)^{-k-1} \bigwedge |q-q_0| < |1-q_0|$. The sums of these two series produce the same function $1/(1-q)$. But restrictions on convergence for all three series strongly depend on the distance between the center of expansion q_0 and the nearest singular point 1 (where the function $1/(1-q)$ has a first-order pole).

The properties of the series:

$$\sum_{k=0}^{\infty} q^{k^2}$$

lead to similar results, which attracted the interest of J. Bernoulli (1713), L. Euler, J. Fourier, and other researchers. They found that this series cannot be analytically continued outside the unit circle $|z| < 1$ because its boundary $|z| == 1$ has not one, but an infinite set of dense singular points. This boundary was called the natural boundary of analyticity of the corresponding function, which is defined as the sum of the previous series.

Special contributions to the theoretical development of these series were made by C. G. J. Jacobi (1827), who introduced the elliptic amplitude $\text{am}(z \mid m)$ and studied the twelve elliptic functions $\text{cd}(z \mid m)$, $\text{cn}(z \mid m)$, $\text{cs}(z \mid m)$, $\text{dc}(z \mid m)$, $\text{dn}(z \mid m)$, $\text{ds}(z \mid m)$, $\text{nc}(z \mid m)$, $\text{nd}(z \mid m)$, $\text{ns}(z \mid m)$, $\text{sc}(z \mid m)$, $\text{sd}(z \mid m)$, $\text{sn}(z \mid m)$. All these functions later were named for Jacobi. C. G. J. Jacobi also introduced four basic theta functions, which can be expressed through the following series:

$$u(w, q) = \sum_{k=-\infty}^{\infty} q^{k^2 + w k}.$$

These Jacobi elliptic theta functions notated by the symbols $\vartheta_1(z, q)$, $\vartheta_2(z, q)$, $\vartheta_3(z, q)$, and $\vartheta_4(z, q)$ have the following representations:

$$\vartheta_1(z, q) = -i \sqrt[4]{q} \sum_{k=-\infty}^{\infty} (-1)^k q^{k(k+1)} e^{(2k+1) i z} = -i \sqrt[4]{q} e^{i z} u\left(\frac{2 i z + \pi i}{\log(q)} + 1, q\right)$$

$$\vartheta_2(z, q) = \sqrt[4]{q} \sum_{k=-\infty}^{\infty} q^{k(k+1)} e^{(2n+1) i z} = \sqrt[4]{q} e^{i z} u\left(\frac{2 i z}{\log(q)} + 1, q\right)$$

$$\vartheta_3(z, q) = \sum_{n=-\infty}^{\infty} q^{n^2} e^{2 n i z} = u\left(\frac{2 i z}{\log(q)}, q\right)$$

$$\vartheta_4(z, q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} e^{2 n i z} = u\left(\frac{2 i z + \pi i}{\log(q)}, q\right).$$

A more detailed theory of elliptic theta functions was developed by C. W. Borchardt (1838), K. Weierstrass (1862–1863), and others. Many relations in the theory of elliptic functions include derivatives of the theta functions with respect to the variable z: $\vartheta_1'(z, q)$, $\vartheta_2'(z, q)$, $\vartheta_3'(z, q)$, and $\vartheta_4'(z, q)$, which cannot be expressed through other special functions. For this reason, *Mathematica* includes not only four well-known theta functions, but also their derivatives.

Definitions of Jacobi theta functions

The Jacobi elliptic theta functions $\vartheta_1(z, q)$, $\vartheta_2(z, q)$, $\vartheta_3(z, q)$, and $\vartheta_4(z, q)$, and their derivatives with respect to z: $\vartheta_1'(z, q)$, $\vartheta_2'(z, q)$, $\vartheta_3'(z, q)$, and $\vartheta_4'(z, q)$ are defined by the following formulas:

$$\vartheta_1(z, q) = 2 \sqrt[4]{q} \sum_{k=0}^{\infty} (-1)^k q^{k(k+1)} \sin((2k+1) z) \;/;\; |q| < 1$$

$$\vartheta_2(z, q) = 2 \sqrt[4]{q} \sum_{k=0}^{\infty} q^{k(k+1)} \cos((2k+1) z) \;/;\; |q| < 1$$

$$\vartheta_3(z, q) = 2 \sum_{k=1}^{\infty} q^{k^2} \cos(2 k z) + 1 \;/;\; |q| < 1$$

$$\vartheta_4(z, q) = 1 + 2 \sum_{k=1}^{\infty} (-1)^k q^{k^2} \cos(2 k z) \;/;\; |q| < 1$$

$$\vartheta_1'(z, q) = 2 \sqrt[4]{q} \sum_{k=0}^{\infty} (-1)^k q^{k(k+1)} (2k+1) \cos((2k+1) z) \;/;\; |q| < 1$$

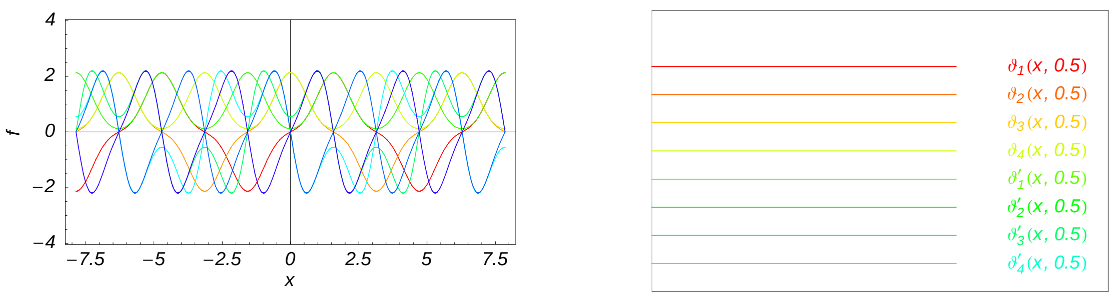
$$\vartheta_2'(z, q) == -2 \sqrt[4]{q} \sum_{k=0}^{\infty} q^{k(k+1)} (2k+1) \sin((2k+1) z) /; |q| < 1$$

$$\vartheta_3'(z, q) == -4 \sum_{k=1}^{\infty} q^{k^2} k \sin(2kz) /; |q| < 1$$

$$\vartheta_4'(z, q) == -4 \sum_{k=1}^{\infty} (-1)^k k q^{k^2} \sin(2kz) /; |q| < 1.$$

A quick look at the Jacobi theta functions

Here is a quick look at the graphics for the Jacobi theta functions along the real axis for $q = 1/2$.



Connections within the group of Jacobi theta functions and with other function groups

Representations through related equivalent functions

The elliptic theta functions $\vartheta_1(z, q)$, $\vartheta_2(z, q)$, $\vartheta_3(z, q)$, and $\vartheta_4(z, q)$ can be represented through the Weierstrass sigma functions by the following
formulas:

$$\vartheta_1(z, q) == \frac{\pi}{\omega_1} \sqrt[4]{q} \exp\left(-\frac{2 \eta_1 \omega_1 z^2}{\pi^2}\right) \left(\prod_{n=1}^{\infty} \left(1 - q^{2n}\right)\right)^3 \sigma\left(\frac{2 \omega_1 z}{\pi}; g_2, g_3\right) /;$$

$$\{\omega_1, \omega_3\} == \{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\} \bigwedge \eta_1 == \zeta(\omega_1; g_2, g_3) \bigwedge q == \exp\left(\frac{\pi i \omega_3}{\omega_1}\right)$$

$$\vartheta_2(z, q) == 2 \sqrt[4]{q} \left(\prod_{n=1}^{\infty} \left(1 - q^{2n}\right)\right) \left(\prod_{n=1}^{\infty} \left(1 + q^{2n}\right)\right)^2 \exp\left(-\frac{2 \eta_1 \omega_1 z^2}{\pi^2}\right) \sigma_1(u; g_2, g_3) /;$$

$$\{\omega_1, \omega_3\} == \{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\} \bigwedge \eta_1 == \zeta(\omega_1; g_2, g_3) \bigwedge q == \exp\left(\frac{\pi i \omega_3}{\omega_1}\right)$$

$$\vartheta_3(z,q) = \left(\prod_{n=1}^{\infty}\left(1-q^{2n}\right)\right)\left(\prod_{n=1}^{\infty}\left(1+q^{2n-1}\right)\right)^2 \exp\left(-\frac{2\,\eta_1\,\omega_1\,z^2}{\pi^2}\right)\sigma_2\left(\frac{2\,\omega_1\,z}{\pi};\,g_2,\,g_3\right) /;$$

$$\{\omega_1,\,\omega_3\} = \{\omega_1(g_2,\,g_3),\,\omega_3(g_2,\,g_3)\} \bigwedge \eta_1 = \zeta(\omega_1;\,g_2,\,g_3) \bigwedge q = \exp\left(\frac{\pi\,i\,\omega_3}{\omega_1}\right)$$

$$\vartheta_4(z,q) = \left(\prod_{n=1}^{\infty}\left(1-q^{2n}\right)\right)\left(\prod_{n=1}^{\infty}\left(1-q^{2n-1}\right)\right)^2 \exp\left(-\frac{2\,\eta_1\,\omega_1\,z^2}{\pi^2}\right)\sigma_3\left(\frac{2\,\omega_1\,z}{\pi};\,g_2,\,g_3\right) /;$$

$$\{\omega_1,\,\omega_3\} = \{\omega_1(g_2,\,g_3),\,\omega_3(g_2,\,g_3)\} \bigwedge \eta_1 = \zeta(\omega_1;\,g_2,\,g_3) \bigwedge q = \exp\left(\frac{\pi\,i\,\omega_3}{\omega_1}\right),$$

where ω_1, ω_3 are the Weierstrass half-periods and $\zeta(z;\,g_2,\,g_3)$ is the Weierstrass zeta function.

The ratios of two different elliptic theta functions $\vartheta_1(z,q)$, $\vartheta_2(z,q)$, $\vartheta_3(z,q)$, and $\vartheta_4(z,q)$ can be expressed through corresponding elliptic Jacobi functions with
power factors by the following formulas:

$$\frac{\vartheta_1(z,\,q(m))}{\vartheta_2(z,\,q(m))} = \frac{1}{(1-m)^{-1/4}}\,\mathrm{sc}\left(\frac{2\,K(m)\,z}{\pi}\,\middle|\,m\right)$$

$$\frac{\vartheta_1(z,\,q(m))}{\vartheta_3(z,\,q(m))} = \sqrt[4]{m\,(1-m)}\;\mathrm{sd}\left(\frac{2\,K(m)\,z}{\pi}\,\middle|\,m\right)$$

$$\frac{\vartheta_1\,(z,\,q\,(m))}{\vartheta_4\,(z,\,q\,(m))} = \sqrt[4]{m}\;\mathrm{sn}\left(\frac{2\,K(m)\,z}{\pi}\,\middle|\,m\right)$$

$$\frac{\vartheta_2(z,\,q(m))}{\vartheta_1(z,\,q(m))} = \frac{1}{\sqrt[4]{1-m}}\,\mathrm{cs}\left(\frac{2\,K(m)\,z}{\pi}\,\middle|\,m\right)$$

$$\frac{\vartheta_2(z,\,q(m))}{\vartheta_3(z,\,q(m))} = \sqrt[4]{m}\;\mathrm{cd}\left(\frac{2\,K(m)\,z}{\pi}\,\middle|\,m\right)$$

$$\frac{\vartheta_2(z,\,q(m))}{\vartheta_4(z,\,q(m))} = \frac{\sqrt[4]{m}}{\sqrt[4]{1-m}}\,\mathrm{cn}\left(\frac{2\,K(m)\,z}{\pi}\,\middle|\,m\right)$$

$$\frac{\vartheta_3(z,\,q(m))}{\vartheta_1(z,\,q(m))} = \frac{1}{(m\,(1-m))^{1/4}}\,\mathrm{ds}\left(\frac{2\,K(m)\,z}{\pi}\,\middle|\,m\right)$$

$$\frac{\vartheta_3(z,\,q(m))}{\vartheta_2(z,\,q(m))} = \frac{1}{\sqrt[4]{m}}\,\mathrm{dc}\left(\frac{2\,K(m)\,z}{\pi}\,\middle|\,m\right)$$

$$\frac{\vartheta_3(z,\,q(m))}{\vartheta_4(z,\,q(m))} = \frac{1}{\sqrt[4]{1-m}}\,\mathrm{dn}\left(\frac{2\,K(m)\,z}{\pi}\,\middle|\,m\right)$$

$$\frac{\vartheta_4(z,\,q(m))}{\vartheta_1(z,\,q(m))} = \frac{1}{\sqrt[4]{m}}\,\mathrm{ns}\left(\frac{2\,K(m)\,z}{\pi}\,\middle|\,m\right)$$

$$\frac{\vartheta_4(z, q(m))}{\vartheta_2(z, q(m))} = \frac{\sqrt[4]{1-m}}{\sqrt[4]{m}} \operatorname{nc}\left(\frac{2 K(m) z}{\pi} \middle| m\right)$$

$$\frac{\vartheta_4(z, q(m))}{\vartheta_3(z, q(m))} = \sqrt[4]{1-m} \operatorname{nd}\left(\frac{2 K(m) z}{\pi} \middle| m\right),$$

where $q(m)$ is an elliptic nome and $K(m)$ is a complete elliptic integral.

Representations through other Jacobi theta functions

Each of the theta functions $\vartheta_1(z, q)$, $\vartheta_2(z, q)$, $\vartheta_3(z, q)$, and $\vartheta_4(z, q)$ can be represented through the other theta functions by the following
formulas:

$$\vartheta_1(z, q) = (-1)^{m-1} \vartheta_2\left(z + \frac{\pi}{2}(2m+1), q\right) /; m \in \mathbb{Z}$$

$$\vartheta_1(z, q) = -i (-1)^m e^{i(2m+1)z} q^{\left(m+\frac{1}{2}\right)^2} \vartheta_3\left(z + \frac{1}{2}(\pi - i(2m+1)\log(q)), q\right) /; m \in \mathbb{Z}$$

$$\vartheta_1(z, q) = i (-1)^m e^{-(2m+1) i z} q^{\left(m+\frac{1}{2}\right)^2} \vartheta_4\left(z + \frac{i \log(q)}{2}(2m+1), q\right) /; m \in \mathbb{Z}$$

$$\vartheta_2(z, q) = (-1)^m \vartheta_1\left(z + \frac{\pi}{2}(2m+1), q\right) /; m \in \mathbb{Z}$$

$$\vartheta_2(z, q) = e^{-i(2m+1)z} q^{\left(m+\frac{1}{2}\right)^2} \vartheta_3\left(z + \frac{i \log(q)}{2}(2m+1), q\right) /; m \in \mathbb{Z}$$

$$\vartheta_2(z, q) = e^{-i(2m+1)z} q^{\left(m+\frac{1}{2}\right)^2} \vartheta_4\left(z + \frac{2m+1}{2}(i \log(q) + \pi), q\right)$$

$$\vartheta_1(z, q) = -e^{-i(2m+1)z} q^{\left(m+\frac{1}{2}\right)^2} \vartheta_1\left(z - \frac{1}{2}(\pi - i(2m+1)\log(q)), q\right) /; m \in \mathbb{Z}$$

$$\vartheta_3(z, q) = e^{i(2m+1)z} q^{\left(m+\frac{1}{2}\right)^2} \vartheta_2\left(z - \frac{i \log(q)}{2}(2m+1), q\right) /; m \in \mathbb{Z}$$

$$\vartheta_3(z, q) = \vartheta_4\left(z + \frac{\pi}{2}(2m+1), q\right) /; m \in \mathbb{Z}$$

$$\vartheta_4(z, q) = i (-1)^m e^{-i(2m+1)z} q^{\left(m+\frac{1}{2}\right)^2} \vartheta_1\left(z + \frac{i \log(q)}{2}(2m+1), q\right) /; m \in \mathbb{Z}$$

$$\vartheta_4(z, q) = -i e^{i(2m+1)z} q^{\left(m+\frac{1}{2}\right)^2} \vartheta_2\left(z - \frac{2m+1}{2}(i \log(q) + \pi), q\right)$$

$$\vartheta_4(z, q) = \vartheta_3\left(z + \frac{\pi}{2}(2m+1), q\right) /; m \in \mathbb{Z}.$$

The derivatives of the theta functions $\vartheta_1(z, q)$, $\vartheta_2(z, q)$, $\vartheta_3(z, q)$, and $\vartheta_4(z, q)$ can also be expressed through the other theta functions and their
derivatives by the following formulas:

$$\vartheta_1'(z, q) = (-1)^{m+n} e^{-2 i n z} q^{n^2} \left(\vartheta_1'(z + \pi m + i n \log(q), q) - 2 i n \vartheta_1(z + \pi m + i n \log(q), q)\right) /; \{m, n\} \in \mathbb{Z}$$

$$\vartheta_1'(z, q) = (-1)^{m-1} \vartheta_2'\left(z + \frac{1}{2} \pi (2 m + 1), q\right) /; m \in \mathbb{Z}$$

$$\vartheta_1'(z, q) = (-1)^m e^{i (2 m+1) z} q^{\left(m+\frac{1}{2}\right)^2} \left((2 m + 1) \vartheta_3\left(z + \frac{1}{2} (\pi - i (2 m + 1) \log(q)), q\right) - i \vartheta_3'\left(z + \frac{1}{2} (\pi - i (2 m + 1) \log(q)), q\right)\right) /; m \in \mathbb{Z}$$

$$\vartheta_1'(z, q) = (-1)^m e^{-i (2 m+1) z} q^{m^2+m+\frac{1}{4}} \left((2 m + 1) \vartheta_4\left(z + \frac{1}{2} i (2 m + 1) \log(q), q\right) + i \vartheta_4'\left(z + \frac{1}{2} i (2 m + 1) \log(q), q\right)\right) /; m \in \mathbb{Z}$$

$$\vartheta_2'(z, q) = (-1)^m \vartheta_1'\left(\frac{1}{2} \pi (2 m + 1) + z, q\right) /; m \in \mathbb{Z}$$

$$\vartheta_2'(z, q) = (-1)^m e^{-2 i n z} q^{n^2} \left(-2 i n \vartheta_2(z + \pi m + i n \log(q), q) + \vartheta_2'(z + \pi m + i n \log(q), q)\right) /; \{m, n\} \in \mathbb{Z}$$

$$\vartheta_2'(z, q) = e^{-i (2 m+1) z} q^{m^2+m+\frac{1}{4}} \left(\vartheta_3'\left(z + \frac{1}{2} i (2 m + 1) \log(q), q\right) - i (2 m + 1) \vartheta_3\left(z + \frac{1}{2} i (2 m + 1) \log(q), q\right)\right) /; m \in \mathbb{Z}$$

$$\vartheta_2'(z, q) = e^{-i (2 m+1) z} q^{m^2+m+\frac{1}{4}} \left(\vartheta_4'\left(z + \frac{1}{2} (2 m + 1) (i \log(q) + \pi), q\right) - i (2 m + 1) \vartheta_4\left(z + \frac{1}{2} (2 m + 1) (i \log(q) + \pi), q\right)\right) /; m \in \mathbb{Z}$$

$$\vartheta_3'(z, q) = i e^{-i (2 m+1) z} q^{\left(m+\frac{1}{2}\right)^2} \left((2 m + 1) \vartheta_1\left(z + \frac{1}{2} i (2 m + 1) \log(q) - \frac{\pi}{2}, q\right) + i \vartheta_1'\left(z + \frac{1}{2} i (2 m + 1) \log(q) - \frac{\pi}{2}, q\right)\right) /; m \in \mathbb{Z}$$

$$\vartheta_3'(z, q) = e^{i (2 m+1) z} q^{m^2+m+\frac{1}{4}} \left(i (2 m + 1) \vartheta_2\left(z - \frac{1}{2} i (2 m + 1) \log(q), q\right) + \vartheta_2'\left(z - \frac{1}{2} i (2 m + 1) \log(q), q\right)\right) /; m \in \mathbb{Z}$$

$$\vartheta_3'(z, q) = e^{-2 i n z} q^{n^2} \left(-2 i n \vartheta_3(z + m \pi + i n \log(q), q) + \vartheta_3'(z + m \pi + i n \log(q), q)\right) /; \{m, n\} \in \mathbb{Z} \bigwedge q = e^{i \pi \tau}$$

$$\vartheta_3'(z, q) = \vartheta_4'\left(z + \frac{1}{2} \pi (2 m + 1), q\right) /; m \in \mathbb{Z}$$

$$\vartheta_4'(z, q) = (-1)^m e^{-i (2 m+1) z} q^{m^2+m+\frac{1}{4}} \left((2 m + 1) \vartheta_1\left(z + \frac{1}{2} i (2 m + 1) \log(q), q\right) + i \vartheta_1'\left(z + \frac{1}{2} i (2 m + 1) \log(q), q\right)\right) /; m \in \mathbb{Z}$$

$$\vartheta_4'(z, q) = e^{i (2 m+1) z} q^{m^2+m+\frac{1}{4}} \left((2 m + 1) \vartheta_2\left(z - \frac{1}{2} (2 m + 1) (i \log(q) + \pi), q\right) - i \vartheta_2'\left(z - \frac{1}{2} (2 m + 1) (i \log(q) + \pi), q\right)\right) /; m \in \mathbb{Z}$$

$$\vartheta_4'(z, q) = \vartheta_3'\left(\frac{1}{2} \pi (2 m + 1) + z, q\right) /; m \in \mathbb{Z}$$

$$\vartheta_4'(z, q) = (-1)^n e^{-2 i n z} q^{n^2} \left(-2 i n \vartheta_4(z + \pi m + i n \log(q), q) + \vartheta_4'(z + \pi m + i n \log(q), q)\right) /; \{m, n\} \in \mathbb{Z} \bigwedge q = e^{i \pi \tau}.$$

The best-known properties and formulas for the Jacobi theta functions

Values for real arguments

For real values of the arguments z, q (with $-1 < q < 1$), the values of the Jacobi theta functions $\vartheta_3(z, q)$, $\vartheta_4(z, q)$, $\vartheta_3'(z, q)$, and $\vartheta_4'(z, q)$ are real.

For real values of the arguments z, q (with $0 \leq q < 1$), the values of the Jacobi theta functions $\vartheta_1(z, q)$, $\vartheta_2(z, q)$, $\vartheta_1'(z, q)$, and $\vartheta_2'(z, q)$ are real.

Simple values at zero

All Jacobi theta functions $\vartheta_1(z, q)$, $\vartheta_2(z, q)$, $\vartheta_3(z, q)$, $\vartheta_4(z, q)$, $\vartheta_1'(z, q)$, $\vartheta_2'(z, q)$, $\vartheta_3'(z, q)$, and $\vartheta_4'(z, q)$ have the following simple values at the origin point:

$$\vartheta_1(0, 0) == 0 \quad \vartheta_2(0, 0) == 0 \quad \vartheta_3(0, 0) == 1 \quad \vartheta_4(0, 0) == 1$$
$$\vartheta_1'(0, 0) == 0 \quad \vartheta_2'(0, 0) == 0 \quad \vartheta_3'(0, 0) == 0 \quad \vartheta_4'(0, 0) == 0.$$

Specific values for specialized parameter

All Jacobi theta functions $\vartheta_1(z, q)$, $\vartheta_2(z, q)$, $\vartheta_3(z, q)$, $\vartheta_4(z, q)$, $\vartheta_1'(z, q)$, $\vartheta_2'(z, q)$, $\vartheta_3'(z, q)$, and $\vartheta_4'(z, q)$ have the following simple values if $q == 0$:

$$\vartheta_1(z, 0) == 0 \quad \vartheta_2(z, 0) == 0 \quad \vartheta_3(z, 0) == 1 \quad \vartheta_4(z, 0) == 1$$

$$\vartheta_1'(z, 0) == 0 \quad \vartheta_2'(z, 0) == 0 \quad \vartheta_3'(z, 0) == 0 \quad \vartheta_4'(z, 0) == 0.$$

At the points $z == 0$ and $z == \frac{\pi}{2}$, all theta functions $\vartheta_1(z, q)$, $\vartheta_2(z, q)$, $\vartheta_3(z, q)$, $\vartheta_4(z, q)$, $\vartheta_1'(z, q)$, $\vartheta_2'(z, q)$, $\vartheta_3'(z, q)$, and $\vartheta_4'(z, q)$ can be expressed through the Dedekind eta function $\eta(w)$ /; $w == -i \log(q)/\pi$ or a composition of the complete elliptic function K and the inverse elliptic nome $K\left(q^{-1}(q)\right)$ by the following formulas:

$$\vartheta_1(0, q) == 0 \qquad \vartheta_2(0, q) == \frac{2}{\eta\left(-\frac{i \log(q)}{\pi}\right)} \eta\left(-\frac{2 i \log(q)}{\pi}\right)^2$$

$$\vartheta_3(0, q) == \sqrt{\frac{2}{\pi}} \sqrt{K\left(q^{-1}(q)\right)} \qquad \vartheta_4(0, q) == \frac{1}{\eta\left(-\frac{i \log(q)}{\pi}\right)} \eta\left(-\frac{i \log(q)}{2 \pi}\right)^2$$

$$\vartheta_1'(0, q) == 2 \eta\left(-\frac{i \log(q)}{\pi}\right)^3 \qquad \vartheta_2'(0, q) == 0$$
$$\vartheta_3'(0, q) == 0 \qquad \vartheta_4'(0, q) == 0$$

$$\vartheta_1\left(\frac{\pi}{2}, q\right) == \sqrt{\frac{2}{\pi}} \sqrt[4]{q^{-1}(q)} \sqrt{K\left(q^{-1}(q)\right)} \qquad \vartheta_2\left(\frac{\pi}{2}, q\right) == 0$$

$$\vartheta_3\left(\frac{\pi}{2}, q\right) == \sqrt{\frac{2}{\pi}} \sqrt[4]{1 - q^{-1}(q)} \sqrt{K\left(q^{-1}(q)\right)} \qquad \vartheta_4\left(\frac{\pi}{2}, q\right) == \frac{1}{\eta\left(-\frac{2 i \log(q)}{\pi}\right)^2 \eta\left(-\frac{i \log(q)}{2 \pi}\right)^2} \eta\left(-\frac{i \log(q)}{\pi}\right)^5$$

$$\vartheta_1'\left(\frac{\pi}{2}, q\right) == 0 \qquad \vartheta_2'\left(\frac{\pi}{2}, q\right) == -2 \eta\left(-\frac{i \log(q)}{\pi}\right)^3$$
$$\vartheta_3'\left(\frac{\pi}{4}, q\right) == -4 \eta\left(-\frac{4 i \log(q)}{\pi}\right)^3 \qquad \vartheta_4'\left(\frac{\pi}{4}, q\right) == 4 \eta\left(-\frac{4 i \log(q)}{\pi}\right)^3.$$

The previous relations can be generalized for the cases $z == \pi m$ and $z == \pi/2 + \pi m$, where $m \in \mathbb{Z}$:

$$\vartheta_1(m \pi, q) == 0 \text{ /; } m \in \mathbb{Z} \quad \vartheta_2(m \pi, q) == \frac{2 (-1)^m}{\eta\left(-\frac{i \log(q)}{\pi}\right)} \eta\left(-\frac{2 i \log(q)}{\pi}\right)^2 \text{ /; } m \in \mathbb{Z}$$

$$\vartheta_3(m\pi, q) = \frac{1}{\eta\left(-\frac{2 i \log(q)}{\pi}\right)^2 \eta\left(-\frac{i \log(q)}{2\pi}\right)^2} \eta\left(-\frac{i \log(q)}{\pi}\right)^5 /; m \in \mathbb{Z} \quad \vartheta_4(m\pi, q) = \frac{1}{\eta\left(-\frac{i \log(q)}{\pi}\right)} \eta\left(-\frac{i \log(q)}{2\pi}\right)^2 /; m \in \mathbb{Z}$$

$$\vartheta_1\left(\pi\left(m + \frac{1}{2}\right), q\right) = (-1)^m \sqrt{\frac{2}{\pi}} \sqrt[4]{q^{-1}(q)} \sqrt{K\left(q^{-1}(q)\right)} /; m \in \mathbb{Z} \quad \vartheta_2\left(\pi\left(m + \frac{1}{2}\right), q\right) = 0 /; m \in \mathbb{Z}$$

$$\vartheta_3\left(\pi\left(m + \frac{1}{2}\right), q\right) = \sqrt{\frac{2}{\pi}} \sqrt[4]{1 - q^{-1}(q)} \sqrt{K\left(q^{-1}(q)\right)} /; m \in \mathbb{Z}$$

$$\vartheta_4\left(\pi\left(m + \frac{1}{2}\right), q\right) = \frac{1}{\eta\left(-\frac{2 i \log(q)}{\pi}\right)^2 \eta\left(-\frac{i \log(q)}{2\pi}\right)^2} \eta\left(-\frac{i \log(q)}{\pi}\right)^5 /; m \in \mathbb{Z}$$

$$\vartheta_1'(\pi m, q) = 2(-1)^m \eta\left(-\frac{i \log(q)}{\pi}\right)^3 /; m \in \mathbb{Z} \quad \vartheta_2'(m\pi, q) = 0 /; m \in \mathbb{Z} \quad \vartheta_3'\left(\frac{\pi m}{2}, q\right) = 0 /; m \in \mathbb{Z} \quad \vartheta_4'\left(\frac{m\pi}{2}, q\right) = 0 /; m \in \mathbb{Z}$$

$$\vartheta_1'\left(\pi m + \frac{\pi}{2}, q\right) = 0 /; m \in \mathbb{Z} \quad \vartheta_2'\left(\pi m + \frac{\pi}{2}, q\right) = 2(-1)^{m-1} \eta\left(-\frac{i \log(q)}{\pi}\right)^3 /; m \in \mathbb{Z}$$

$$\vartheta_3'\left(\frac{\pi m}{2} + \frac{\pi}{4}, q\right) = 4(-1)^{m-1} \eta\left(-\frac{4 i \log(q)}{\pi}\right)^3 /; m \in \mathbb{Z} \quad \vartheta_4'\left(\frac{\pi m}{2} + \frac{\pi}{4}, q\right) = 4(-1)^m \eta\left(-\frac{4 i \log(q)}{\pi}\right)^3 /; m \in \mathbb{Z}.$$

Analyticity

All Jacobi theta functions $\vartheta_1(z, q)$, $\vartheta_2(z, q)$, $\vartheta_3(z, q)$, $\vartheta_4(z, q)$, $\vartheta_1'(z, q)$, $\vartheta_2'(z, q)$, $\vartheta_3'(z, q)$, and $\vartheta_4'(z, q)$ are analytic functions of z and q for $z, q \in \mathbb{C}$ and $|q| < 1$.

Poles and essential singularities

All Jacobi theta functions $\vartheta_1(z, q)$, $\vartheta_2(z, q)$, $\vartheta_3(z, q)$, $\vartheta_4(z, q)$, $\vartheta_1'(z, q)$, $\vartheta_2'(z, q)$, $\vartheta_3'(z, q)$, and $\vartheta_4'(z, q)$ do not have poles and essential singularities inside of the unit circle $|q| < 1$.

Branch points and branch cuts

For fixed z, the functions $\vartheta_1(z, q)$, $\vartheta_2(z, q)$, $\vartheta_1'(z, q)$, and $\vartheta_2'(z, q)$ have one branch point: $q = 0$. (The point $q = -1$ is the branch cut endpoint.)

For fixed z, the functions $\vartheta_1(z, q)$, $\vartheta_2(z, q)$, $\vartheta_1'(z, q)$, and $\vartheta_2'(z, q)$ are the single-valued functions inside the unit circle of the complex q-plane, cut along the interval $(-1, 0)$, where they are continuous from above:

$$\lim_{\epsilon \to +0} \vartheta_1(z, q + i\epsilon) = \vartheta_1(z, q) /; -1 < q < 0 \quad \lim_{\epsilon \to +0} \vartheta_1(z, q - i\epsilon) = -i\,\vartheta_1(z, q) /; -1 < q < 0$$

$$\lim_{\epsilon \to +0} \vartheta_2(z, q + i\epsilon) = \vartheta_2(z, q) /; -1 < q < 0 \quad \lim_{\epsilon \to +0} \vartheta_2(z, q - i\epsilon) = -i\,\vartheta_2(z, q) /; -1 < q < 0$$

$$\lim_{\epsilon \to +0} \vartheta_1'(z, q + i\epsilon) = \vartheta_1'(z, q) /; -1 < q < 0 \quad \lim_{\epsilon \to +0} \vartheta_1'(z, q - i\epsilon) = -i\,\vartheta_1'(z, q) /; -1 < q < 0$$

$$\lim_{\epsilon \to +0} \vartheta_2'(z, q + i\epsilon) = \vartheta_2'(z, q) /; -1 < q < 0 \quad \lim_{\epsilon \to +0} \vartheta_2'(z, q - i\epsilon) = -i\,\vartheta_2'(z, q) /; -1 < q < 0.$$

For fixed q, the functions $\vartheta_1(z, q)$, $\vartheta_2(z, q)$, $\vartheta_1'(z, q)$, and $\vartheta_2'(z, q)$ do not have branch points and branch cuts with respect to z.

The functions $\vartheta_3(z, q)$, $\vartheta_4(z, q)$, $\vartheta_3'(z, q)$, and $\vartheta_4'(z, q)$ do not have branch points and branch cuts.

Natural boundary of analyticity

The unit circle $|q| == 1$ is the natural boundary of the region of analyticity for all Jacobi theta functions $\vartheta_1(z, q)$, $\vartheta_2(z, q)$, $\vartheta_3(z, q)$, $\vartheta_4(z, q)$, $\vartheta_1'(z, q)$, $\vartheta_2'(z, q)$, $\vartheta_3'(z, q)$, and $\vartheta_4'(z, q)$.

Periodicity

The Jacobi theta functions $\vartheta_1(z, q)$ and $\vartheta_2(z, q)$ are the periodic functions with respect to z with period 2π and a quasi-period $i \log(q)$:

$$\vartheta_1(z + 2\pi, q) == \vartheta_1(z, q) \quad \vartheta_1(z + i \log(q), q) == -\frac{e^{2 i z}}{q} \vartheta_1(z, q)$$

$$\vartheta_2(z + 2\pi, q) == \vartheta_2(z, q) \quad \vartheta_2(z + i \log(q), q) == \frac{e^{2 i z}}{q} \vartheta_2(z, q).$$

The Jacobi theta functions $\vartheta_3(z, q)$ and $\vartheta_4(z, q)$ are the periodic functions with respect to z with period π and a quasi-period $i \log(q)$:

$$\vartheta_3(z + \pi, q) == \vartheta_3(z, q) \quad \vartheta_3(z + i \log(q), q) == \frac{e^{2 i z}}{q} \vartheta_3(z, q)$$

$$\vartheta_4(z + \pi, q) == \vartheta_4(z, q) \quad \vartheta_4(z + i \log(q), q) == -\frac{e^{2 i z}}{q} \vartheta_4(z, q).$$

The Jacobi theta functions $\vartheta_1'(z, q)$ and $\vartheta_2'(z, q)$ are the periodic functions with respect to z with period 2π:

$$\vartheta_1'(z + \pi, q) == -\vartheta_1'(z, q) \quad \vartheta_1'(z + 2\pi, q) == \vartheta_1'(z, q)$$

$$\vartheta_2'(z + \pi, q) == -\vartheta_2'(z, q) \quad \vartheta_2'(z + 2\pi, q) == \vartheta_2'(z, q).$$

The Jacobi theta functions $\vartheta_3'(z, q)$ and $\vartheta_4'(z, q)$ are the periodic functions with respect to z with period π:

$$\vartheta_3'(z + \pi, q) == \vartheta_3'(z, q)$$

$$\vartheta_4'(z + \pi, q) == \vartheta_4'(z, q).$$

The previous formulas are the particular cases of the following general relations that reflect the periodicity and quasi-periodicity of the theta functions by variable z:

$$\vartheta_1(z + m\pi + n\pi\tau, q) == (-1)^{m+n} q^{-n^2} e^{-2 n z i} \vartheta_1(z, q) /; \{m, n\} \in \mathbb{Z} \bigwedge q == e^{i \pi \tau}$$

$$\vartheta_2(z + m\pi + n\pi\tau, q) == (-1)^{m} q^{-n^2} e^{-2 n z i} \vartheta_2(z, q) /; \{m, n\} \in \mathbb{Z} \bigwedge q == e^{i \pi \tau}$$

$$\vartheta_3(z + m\pi + n\pi\tau, q) == q^{-n^2} e^{-2 n z i} \vartheta_3(z, q) /; \{m, n\} \in \mathbb{Z} \bigwedge q == e^{i \pi \tau}$$

$$\vartheta_4(z + m\pi + n\pi\tau, q) == (-1)^{n} q^{-n^2} e^{-2 n z i} \vartheta_4(z, q) /; \{m, n\} \in \mathbb{Z} \bigwedge q == e^{i \pi \tau}$$

$$\vartheta_1'(z + m\pi, q) == (-1)^m \vartheta_1'(z, q) /; m \in \mathbb{Z}$$

$$\vartheta_2'(z + m\pi, q) == (-1)^m \vartheta_2'(z, q) /; m \in \mathbb{Z}$$

$$\vartheta_3'(z + m\pi, q) == \vartheta_3'(z, q) /; m \in \mathbb{Z}$$

$$\vartheta_4'(z + m\pi, q) == \vartheta_4'(z, q) /; m \in \mathbb{Z}.$$

Parity and symmetry

All Jacobi theta functions $\vartheta_1(z, q)$, $\vartheta_2(z, q)$, $\vartheta_3(z, q)$, $\vartheta_4(z, q)$, $\vartheta_1'(z, q)$, $\vartheta_2'(z, q)$, $\vartheta_3'(z, q)$, and $\vartheta_4'(z, q)$ have mirror symmetry:

$$\vartheta_1(\bar{z}, \overline{q}) = \overline{\vartheta_1(z, q)} \quad \vartheta_2(\bar{z}, \overline{q}) = \overline{\vartheta_2(z, q)} \quad \vartheta_3(\bar{z}, \overline{q}) = \overline{\vartheta_3(z, q)} \quad \vartheta_4(\bar{z}, \overline{q}) = \overline{\vartheta_4(z, q)}$$

$$\vartheta_1'(\bar{z}, \overline{q}) = \overline{\vartheta_1'(z, q)} \quad \vartheta_2'(\bar{z}, \overline{q}) = \overline{\vartheta_2'(z, q)} \quad \vartheta_3'(\bar{z}, \overline{q}) = \overline{\vartheta_3'(z, q)} \quad \vartheta_4'(\bar{z}, \overline{q}) = \overline{\vartheta_4'(z, q)}.$$

The Jacobi theta functions $\vartheta_1(z, q)$, $\vartheta_2'(z, q)$, $\vartheta_3'(z, q)$, and $\vartheta_4'(z, q)$ are odd functions with respect to z:

$$\vartheta_1(-z, q) = -\vartheta_1(z, q) \quad \vartheta_2'(-z, q) = -\vartheta_2'(z, q) \quad \vartheta_3'(-z, q) = -\vartheta_3'(z, q) \quad \vartheta_4'(-z, q) = -\vartheta_4'(z, q).$$

The other Jacobi theta functions $\vartheta_2(z, q)$, $\vartheta_3(z, q)$, $\vartheta_4(z, q)$, and $\vartheta_1'(z, q)$ are even functions with respect to z:

$$\vartheta_2(-z, q) = \vartheta_2(z, q) \quad \vartheta_3(-z, q) = \vartheta_3(z, q) \quad \vartheta_4(-z, q) = \vartheta_4(z, q) \quad \vartheta_1'(-z, q) = \vartheta_1'(z, q).$$

The Jacobi theta functions $\vartheta_1(z, q)$, $\vartheta_1'(z, q)$, $\vartheta_2(z, q)$, and $\vartheta_2'(z, q)$ satisfy the following parity type relations with respect to q:

$$\vartheta_1(z, -q) = \exp\left(-\frac{i\pi}{4}\operatorname{sgn}(\operatorname{Im}(q))\right)\vartheta_1(z, q) \quad \vartheta_2(z, -q) = \exp\left(-\frac{i\pi}{4}\operatorname{sgn}(\operatorname{Im}(q))\right)\vartheta_2(z, q)$$

$$\vartheta_1'(z, -q) = \exp\left(-\frac{i\pi}{4}\operatorname{sgn}(\operatorname{Im}(q))\right)\vartheta_1'(z, q) \quad \vartheta_2'(z, -q) = \exp\left(-\frac{i\pi}{4}\operatorname{sgn}(\operatorname{Im}(q))\right)\vartheta_2'(z, q).$$

The Jacobi theta functions $\vartheta_3(z, q)$, $\vartheta_3'(z, q)$, $\vartheta_4(z, q)$, and $\vartheta_4'(z, q)$ with argument $-q$ can be self-transformed by the following relations:

$$\vartheta_3(z, -q) = \vartheta_4(z, q) \quad \vartheta_4(z, -q) = \vartheta_3(z, q) \quad \vartheta_3'(z, -q) = \vartheta_4'(z, q) \quad \vartheta_4'(z, -q) = \vartheta_3'(z, q).$$

q-series representations

All Jacobi elliptic theta functions $\vartheta_1(z, q)$, $\vartheta_2(z, q)$, $\vartheta_3(z, q)$, and $\vartheta_4(z, q)$, and their derivatives $\vartheta_1'(z, q)$, $\vartheta_2'(z, q)$, $\vartheta_3'(z, q)$, and $\vartheta_4'(z, q)$ have the following series expansions, which can be called q-series representations:

$$\vartheta_1(z, q) = 2\sqrt[4]{q}\sum_{k=0}^{\infty}(-1)^k q^{k(k+1)}\sin((2k+1)z) = -i\sqrt[4]{q}\sum_{k=-\infty}^{\infty}(-1)^k q^{k(k+1)} e^{(2k+1)iz} \;/;\; |q| < 1$$

$$\vartheta_2(z, q) = 2\sqrt[4]{q}\sum_{k=0}^{\infty}q^{k(k+1)}\cos((2k+1)z) = \sqrt[4]{q}\sum_{k=-\infty}^{\infty}q^{k(k+1)} e^{(2n+1)iz} \;/;\; |q| < 1$$

$$\vartheta_2(0, q) = 2\sqrt[4]{q}\sum_{k=0}^{\infty}q^{k(k+1)}$$

$$\vartheta_3(z, q) = 2\sum_{k=1}^{\infty}q^{k^2}\cos(2kz) + 1 = \sum_{n=-\infty}^{\infty}q^{n^2} e^{2niz} \;/;\; |q| < 1$$

$$\vartheta_3(0, q) = 1 + 2\sum_{n=1}^{\infty}q^{n^2}$$

$$\vartheta_4(z, q) == 1 + 2\sum_{k=1}^{\infty}(-1)^k q^{k^2} \cos(2\,k\,z) == \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} e^{2\,n\,i\,z} \;/;\; |q| < 1$$

$$\vartheta_4(0, q) == 1 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2}$$

$$\vartheta_1'(z, q) == 2\sqrt[4]{q} \sum_{k=0}^{\infty}(-1)^k q^{k\,(k+1)}\,(2\,k+1)\cos((2\,k+1)\,z) == \sqrt[4]{q} \sum_{k=-\infty}^{\infty} (-1)^k\,(2\,k+1)\,q^{k\,(k+1)}\, e^{(2\,k+1)\,i\,z} \;/;\; |q| < 1$$

$$\vartheta_1'(0, q) == 2\sqrt[4]{q} \sum_{n=0}^{\infty}(-1)^n\,(2\,n+1)\,q^{n\,(n+1)}$$

$$\vartheta_2'(z, q) == -2\sqrt[4]{q} \sum_{k=0}^{\infty} q^{k\,(k+1)}\,(2\,k+1)\sin((2\,k+1)\,z) == i \sum_{k=-\infty}^{\infty} q^{\left(k+\frac{1}{2}\right)^2}\,(2\,k+1)\, e^{(2\,k+1)\,i\,z} \;/;\; |q| < 1$$

$$\vartheta_3'(z, q) == -4\sum_{k=1}^{\infty} q^{k^2}\,k\sin(2\,k\,z) == 2\,i \sum_{k=-\infty}^{\infty} q^{k^2}\,k\, e^{2\,k\,i\,z} \;/;\; |q| < 1$$

$$\vartheta_4'(z, q) == -4\sum_{k=1}^{\infty}(-1)^k\,k\,q^{k^2}\sin(2\,k\,z) == 2\,i \sum_{k=-\infty}^{\infty} (-1)^k q^{k^2}\,k\, e^{2\,k\,i\,z} \;/;\; |q| < 1.$$

Other series representations

The theta functions $\vartheta_1(z, q)$, $\vartheta_2(z, q)$, $\vartheta_3(z, q)$, and $\vartheta_4(z, q)$, and their derivatives $\vartheta_1'(z, q)$, $\vartheta_2'(z, q)$, $\vartheta_3'(z, q)$, and $\vartheta_4'(z, q)$ can also be represented through the following series:

$$\vartheta_1(z, q) == -i\exp\left(-\frac{i\,z^2}{\pi\,\tau}\right)\sum_{n=-\infty}^{\infty}(-1)^n \exp\left(\pi\,\tau\,i\left(n+\frac{1}{2}+\frac{z}{\pi\,\tau}\right)^2\right) \;/;\; q == e^{i\,\pi\,\tau}$$

$$\vartheta_2(z, q) == \exp\left(-\frac{i\,z^2}{\pi\,\tau}\right)\sum_{n=-\infty}^{\infty}\exp\left(i\,\pi\,\tau\left(n+\frac{1}{2}+\frac{z}{\pi\,\tau}\right)^2\right) \;/;\; q == e^{i\,\pi\,\tau}$$

$$\vartheta_3(u, q) == \exp\left(-\frac{i\,u^2}{\pi\,\tau}\right)\sum_{n=-\infty}^{\infty}\exp\left(i\,\pi\,\tau\left(n+\frac{u}{\pi\,\tau}\right)^2\right) \;/;\; q == e^{i\,\pi\,\tau}$$

$$\vartheta_4(z, q) == \exp\left(-\frac{i\,z^2}{\pi\,\tau}\right)\sum_{n=-\infty}^{\infty}(-1)^n\exp\left(i\,\pi\,\tau\left(n+\frac{z}{\pi\,\tau}\right)^2\right) \;/;\; q == e^{i\,\pi\,\tau}$$

$$\vartheta_1'(z, q) == -\frac{2\,i^{3/2}}{\tau^{3/2}}\sum_{n=-\infty}^{\infty}(-1)^n\left(\frac{z}{\pi}+n-\frac{1}{2}\right)\exp\left(-\frac{i\,\pi}{\tau}\left(\frac{z}{\pi}+n-\frac{1}{2}\right)^2\right) \;/;\; q == e^{i\,\pi\,\tau}$$

$$\vartheta_2'(z, q) == -\frac{2\,i^{3/2}}{\tau^{3/2}}\sum_{n=-\infty}^{\infty}(-1)^n\left(\frac{z}{\pi}+n\right)\exp\left(-\frac{\pi\,i}{\tau}\left(\frac{z}{\pi}+n\right)^2\right) \;/;\; q == e^{i\,\pi\,\tau}$$

$$\vartheta_3'(z, q) == -\frac{2\,i^{3/2}}{\tau^{3/2}}\sum_{n=-\infty}^{\infty}\left(n+\frac{z}{\pi}\right)\exp\left(-\frac{\pi\,i}{\tau}\left(\frac{z}{\pi}+n\right)^2\right) \;/;\; q == e^{i\,\pi\,\tau}$$

$$\vartheta_4'(z,q) = -\frac{2\, i^{3/2}}{\tau^{3/2}} \sum_{n=-\infty}^{\infty} \left(n + \frac{z}{\pi} - \frac{1}{2}\right) \exp\left(-\frac{\pi i}{\tau} \left(\frac{z}{\pi} + n - \frac{1}{2}\right)^2\right) /;\ q = e^{i \pi \tau}.$$

Product representations

The theta functions can be represented through infinite products, for example:

$$\vartheta_1(z,q) = 2 \sqrt[4]{q}\ \sin(z) \prod_{k=1}^{\infty} \left(1 - q^{2k}\right) \left(1 - 2 q^{2k} \cos(2z) + q^{4k}\right)$$

$$\vartheta_2(0,q) = 2 \sqrt[4]{q} \prod_{n=1}^{\infty} \left(1 - q^{2n}\right) \left(1 + q^{2n}\right)^2$$

$$\vartheta_2(z,q) = 2 \sqrt[4]{q}\ \cos(z) \prod_{k=1}^{\infty} \left(1 - q^{2k}\right) \left(1 + 2 q^{2k} \cos(2z) + q^{4k}\right)$$

$$\vartheta_3(0,q) = \prod_{n=1}^{\infty} \left(1 - q^{2n}\right) \left(1 + q^{2n-1}\right)^2$$

$$\vartheta_3(z,q) = \prod_{k=1}^{\infty} \left(1 - q^{2k}\right) \left(1 + 2 q^{2k-1} \cos(2z) + q^{4k-2}\right)$$

$$\vartheta_4(0,q) = \prod_{n=1}^{\infty} \left(1 - q^{2n}\right) \left(1 - q^{2n-1}\right)^2$$

$$\vartheta_4(z,q) = \prod_{k=1}^{\infty} \left(1 - q^{2j}\right) \left(1 - 2 q^{2k-1} \cos(2z) + q^{4k-2}\right)$$

$$\vartheta_1'(0,q) = 2 \sqrt[4]{q} \left(\prod_{n=1}^{\infty} \left(1 - q^{2n}\right)\right)^3.$$

Transformations

The theta functions $\vartheta_1(z,q)$, $\vartheta_2(z,q)$, $\vartheta_3(z,q)$, and $\vartheta_4(z,q)$ satisfy numerous relations that can provide transformations of their
arguments, for example:

$$\vartheta_1\left(\frac{z}{\tau}, e^{-\frac{i\pi}{\tau}}\right) = -i \sqrt[4]{e^{-\frac{i\pi}{\tau}}}\ \sqrt{-i\tau}\ \exp\left(\frac{i z^2}{\pi \tau} + \frac{i \pi}{4 \tau}\right) \vartheta_1(z,q) /;\ q = e^{i \pi \tau}$$

$$\vartheta_2\left(\frac{z}{\tau}, e^{-\frac{i\pi}{\tau}}\right) = \frac{\sqrt{\tau}}{\sqrt{i}} \exp\left(\frac{i z^2}{\pi \tau}\right) \vartheta_4(z,q) /;\ q = e^{i \pi \tau}$$

$$\vartheta_3\left(\frac{z}{\tau}, e^{-\frac{i\pi}{\tau}}\right) = \frac{\sqrt{\tau}}{\sqrt{i}} \exp\left(\frac{i z^2}{\pi \tau}\right) \vartheta_3(z, q) \;/;\; q = e^{i \pi \tau}$$

$$\vartheta_4\left(\frac{z}{\tau}, e^{-\frac{i\pi}{\tau}}\right) = \frac{\sqrt{\tau}}{\sqrt{i}} \exp\left(\frac{i z^2}{\pi \tau}\right) \vartheta_2(z, q) \;/;\; q = e^{i \pi \tau}.$$

Among those transformations, several kinds can be combined into specially named groups:

n^{th} root of q:

$$\vartheta_j\left(z, q^{1/n}\right) = \left(\prod_{r=1}^{\infty} \frac{1 - q^{\frac{2r}{n}}}{\left(1 - q^{2r}\right)^n}\right) \prod_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} \vartheta_j\left(z + \frac{i k \log(q)}{n}, q\right) \;/;\; \frac{n+1}{2} \in \mathbb{N}^+ \bigwedge j \in \{1, 2, 3, 4\}.$$

Multiple angle formulas:

$$\vartheta_1(n z, q^n) = \frac{\sqrt[4]{q^n}}{q^{n/4}} \left(\prod_{r=1}^{\infty} \frac{1 - q^{2nr}}{\left(1 - q^{2r}\right)^n}\right) \prod_{r=0}^{n-1} \vartheta_1\left(z + \frac{\pi r}{n}, q\right) \;/;\; n \in \mathbb{N}^+$$

$$\vartheta_1(n z, q^n) = (-1)^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{\sqrt[4]{q^n}}{q^{n/4}} \left(\prod_{r=1}^{\infty} \frac{1 - q^{2nr}}{\left(1 - q^{2r}\right)^n}\right) \prod_{r=\left\lfloor -\frac{n-1}{2} \right\rfloor}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \vartheta_1\left(z + \frac{r\pi}{n}, q\right) \;/;\; n \in \mathbb{N}^+$$

$$\vartheta_2(n z, q^n) = (-1)^{\frac{n-1}{2}} \left(\prod_{r=1}^{\infty} \frac{1 - q^{2nr}}{\left(1 - q^{2r}\right)^n}\right) \prod_{r=0}^{n-1} \vartheta_2\left(z + \frac{\pi r}{n}, q\right) \;/;\; \frac{n+1}{2} \in \mathbb{N}^+$$

$$\vartheta_2(n z, q^n) = \left(\prod_{r=1}^{\infty} \frac{1 - q^{2nr}}{\left(1 - q^{2r}\right)^n}\right) \prod_{r=-\frac{n-1}{2}}^{\frac{n-1}{2}} \vartheta_2\left(z + \frac{\pi r}{n}, q\right) \;/;\; n \in \mathbb{N}^+$$

$$\vartheta_3(n z, q^n) = \left(\prod_{r=1}^{\infty} \frac{1 - q^{2nr}}{\left(1 - q^{2r}\right)^n}\right) \prod_{r=0}^{n-1} \vartheta_3\left(z + \frac{r\pi}{n}, q\right) \;/;\; \frac{n+1}{2} \in \mathbb{N}^+$$

$$\vartheta_3(n z, q^n) = \left(\prod_{r=1}^{\infty} \frac{1 - q^{2nr}}{\left(1 - q^{2r}\right)^n}\right) \prod_{r=-\frac{n-1}{2}}^{\frac{n-1}{2}} \vartheta_3\left(z + \frac{r\pi}{n}, q\right) \;/;\; n \in \mathbb{N}^+$$

$$\vartheta_4(n z, q^n) = \left(\prod_{r=1}^{\infty} \frac{1 - q^{2nr}}{\left(1 - q^{2r}\right)^n}\right) \prod_{r=0}^{n-1} \vartheta_4\left(z + \frac{\pi r}{n}, q\right) \;/;\; n \in \mathbb{N}^+$$

$$\vartheta_4(n z, q^n) = \left(\prod_{r=1}^{\infty} \frac{1 - q^{2nr}}{\left(1 - q^{2r}\right)^n}\right) \prod_{r=\left\lfloor -\frac{n-1}{2} \right\rfloor}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \vartheta_4\left(z + \frac{\pi r}{n}, q\right) \;/;\; n \in \mathbb{N}^+.$$

Double-angle formulas (which are not particular cases of the previous group), for example:

$$\vartheta_1(2z, q) = \frac{2\,\vartheta_1(z, q)\,\vartheta_2(z, q)\,\vartheta_3(z, q)\,\vartheta_4(z, q)}{\vartheta_2(0, q)\,\vartheta_3(0, q)\,\vartheta_4(0, q)}.$$

$$\vartheta_2(2z, q) = \frac{\vartheta_2(z, q)^4 - \vartheta_1(z, q)^4}{\vartheta_2(0, q)^3}.$$

$$\vartheta_3(2z, q) = \frac{\vartheta_3(z, q)^4 + \vartheta_1(z, q)^4}{\vartheta_3(0, q)^3}.$$

$$\vartheta_4(2z, q) = \frac{\vartheta_4(z, q)^4 - \vartheta_1(z, q)^4}{\vartheta_4(0, q)^3}.$$

Landen's transformation:

$$\frac{\vartheta_3(z, q)\,\vartheta_4(z, q)}{\vartheta_4(2z, q^2)} = \frac{\vartheta_3(0, q)\,\vartheta_4(0, q)}{\vartheta_4(0, q^2)}$$

$$\frac{\vartheta_2(z, q)\,\vartheta_1(z, q)}{\vartheta_1(2z, q^2)} = \frac{\vartheta_3(0, q)\,\vartheta_4(0, q)}{\vartheta_4(0, q^2)}.$$

Identities

The theta functions at $z = 0$ satisfy numerous modular identities of the form $p(\vartheta_1(0, q^{e_{1,1}}) \ldots, \vartheta_4(0, q^{e_{4,n}})) = 0$, where the $e_{i,j}$ are positive integers and p is a multivariate polynomials over the integers, for example:

$$\left(\frac{3\,\vartheta_2(0, q^9)}{\vartheta_2(0, q)} - 1\right)^3 = \frac{9\,\vartheta_2(0, q^3)^4}{\vartheta_2(0, q)^4} - 1$$

$$\frac{\vartheta_3(0, q^3)^4}{\vartheta_3(0, q^9)^4} = \left(\frac{\vartheta_3(0, q)}{\vartheta_3(0, q^9)} - 1\right)^3 + 1$$

$$\left(\frac{3\,\vartheta_4(0, q^9)}{\vartheta_4(0, q)} - 1\right)^3 = \frac{9\,\vartheta_4(0, q^3)^4}{\vartheta_4(0, q)^4} - 1.$$

Among the numerous identities for theta functions, several kinds can be joined into specially named groups:

Relations involving squares:

$$\vartheta_2(0, q)^2\, \vartheta_3(z, q)^2 - \vartheta_4(0, q)^2\, \vartheta_1(z, q)^2 == \vartheta_3(0, q)^2\, \vartheta_2(z, q)^2$$

$$\vartheta_2(z, q)^2\, \vartheta_3(0, q)^2 + \vartheta_4(0, q)^2\, \vartheta_1(z, q)^2 == \vartheta_2(0, q)^2\, \vartheta_3(z, q)^2$$

$$\vartheta_2(0, q)^2\, \vartheta_2(z, q)^2 + \vartheta_4(0, q)^2\, \vartheta_4(z, q)^2 == \vartheta_3(0, q)^2\, \vartheta_3(z, q)^2$$

$$\vartheta_2(0, q)^2\, \vartheta_1(z, q)^2 + \vartheta_4(0, q)^2\, \vartheta_3(z, q)^2 == \vartheta_3(0, q)^2\, \vartheta_4(z, q)^2$$

$$\vartheta_3(0, q)^2\, \vartheta_1(z, q)^2 + \vartheta_4(0, q)^2\, \vartheta_2(z, q)^2 == \vartheta_2(0, q)^2\, \vartheta_4(z, q)^2.$$

Relations involving quartic powers:

$$\vartheta_2(0, q)^4 + \vartheta_4(0, q)^4 == \vartheta_3(0, q)^4$$

$$\vartheta_1(z, q)^4 + \vartheta_3(z, q)^4 == \vartheta_2(z, q)^4 + \vartheta_4(z, q)^4.$$

Relations between the four theta functions where the first argument is zero, for example:

$$\vartheta_1'(0, q) == \vartheta_2(0, q)\, \vartheta_3(0, q)\, \vartheta_4(0, q).$$

Addition formulas:

$$\vartheta_1(x+y, q)\, \vartheta_1(x-y, q) == \frac{\vartheta_1(x, q)^2\, \vartheta_2(y, q)^2 - \vartheta_2(x, q)^2\, \vartheta_1(y, q)^2}{\vartheta_2(0, q)^2} == \frac{\vartheta_4(x, q)^2\, \vartheta_3(y, q)^2 - \vartheta_3(x, q)^2\, \vartheta_4(y, q)^2}{\vartheta_2(0, q)^2}$$

$$\vartheta_1(x+y, q)\, \vartheta_1(x-y, q) == \frac{\vartheta_1(x, q)^2\, \vartheta_3(y, q)^2 - \vartheta_3(x, q)^2\, \vartheta_1(y, q)^2}{\vartheta_3(0, q)^2} == \frac{\vartheta_4(x, q)^2\, \vartheta_2(y, q)^2 - \vartheta_2(x, q)^2\, \vartheta_4(y, q)^2}{\vartheta_3(0, q)^2}$$

$$\vartheta_1(x+y, q)\, \vartheta_1(x-y, q) == \frac{\vartheta_1(x, q)^2\, \vartheta_4(y, q)^2 - \vartheta_4(x, q)^2\, \vartheta_1(y, q)^2}{\vartheta_4(0, q)^2} == \frac{\vartheta_3(x, q)^2\, \vartheta_2(y, q)^2 - \vartheta_2(x, q)^2\, \vartheta_3(y, q)^2}{\vartheta_4(0, q)^2}$$

$$\vartheta_2(x+y, q)\, \vartheta_2(x-y, q) == \frac{\vartheta_2(x, q)^2\, \vartheta_2(y, q)^2 - \vartheta_1(x, q)^2\, \vartheta_1(y, q)^2}{\vartheta_2(0, q)^2} == \frac{\vartheta_3(x, q)^2\, \vartheta_3(y, q)^2 - \vartheta_4(x, q)^2\, \vartheta_4(y, q)^2}{\vartheta_2(0, q)^2}$$

$$\vartheta_2(x+y, q)\, \vartheta_2(x-y, q) == \frac{\vartheta_2(x, q)^2\, \vartheta_3(y, q)^2 - \vartheta_4(x, q)^2\, \vartheta_1(y, q)^2}{\vartheta_3(0, q)^2} == \frac{\vartheta_3(x, q)^2\, \vartheta_2(y, q)^2 - \vartheta_1(x, q)^2\, \vartheta_4(y, q)^2}{\vartheta_3(0, q)^2}$$

$$\vartheta_2(x+y, q)\, \vartheta_2(x-y, q) == \frac{\vartheta_2(x, q)^2\, \vartheta_4(y, q)^2 - \vartheta_3(x, q)^2\, \vartheta_1(y, q)^2}{\vartheta_4(0, q)^2} == \frac{\vartheta_4(x, q)^2\, \vartheta_2(y, q)^2 - \vartheta_1(x, q)^2\, \vartheta_3(y, q)^2}{\vartheta_4(0, q)^2}$$

$$\vartheta_3(x+y, q)\, \vartheta_3(x-y, q) == \frac{\vartheta_3(x, q)^2\, \vartheta_2(y, q)^2 + \vartheta_4(x, q)^2\, \vartheta_1(y, q)^2}{\vartheta_2(0, q)^2} == \frac{\vartheta_2(x, q)^2\, \vartheta_3(y, q)^2 + \vartheta_1(x, q)^2\, \vartheta_4(y, q)^2}{\vartheta_2(0, q)^2}$$

$$\vartheta_3(x+y,q)\,\vartheta_3(x-y,q) = \frac{\vartheta_3(x,q)^2\,\vartheta_3(y,q)^2+\vartheta_1(x,q)^2\,\vartheta_1(y,q)^2}{\vartheta_3(0,q)^2} = \frac{\vartheta_2(x,q)^2\,\vartheta_2(y,q)^2+\vartheta_4(x,q)^2\,\vartheta_4(y,q)^2}{\vartheta_3(0,q)^2}$$

$$\vartheta_3(x+y,q)\,\vartheta_3(x-y,q) = \frac{\vartheta_3(x,q)^2\,\vartheta_4(y,q)^2-\vartheta_2(x,q)^2\,\vartheta_1(y,q)^2}{\vartheta_4(0,q)^2} = \frac{\vartheta_4(x,q)^2\,\vartheta_3(y,q)^2-\vartheta_1(x,q)^2\,\vartheta_2(y,q)^2}{\vartheta_4(0,q)^2}$$

$$\vartheta_4(x+y,q)\,\vartheta_4(x-y,q) = \frac{\vartheta_3(x,q)^2\,\vartheta_1(y,q)^2+\vartheta_4(x,q)^2\,\vartheta_2(y,q)^2}{\vartheta_2(0,q)^2} = \frac{\vartheta_1(x,q)^2\,\vartheta_3(y,q)^2+\vartheta_2(x,q)^2\,\vartheta_4(y,q)^2}{\vartheta_2(0,q)^2}$$

$$\vartheta_4(x+y,q)\,\vartheta_4(x-y,q) = \frac{\vartheta_4(x,q)^2\,\vartheta_3(y,q)^2+\vartheta_2(x,q)^2\,\vartheta_1(y,q)^2}{\vartheta_3(0,q)^2} = \frac{\vartheta_1(x,q)^2\,\vartheta_2(y,q)^2+\vartheta_3(x,q)^2\,\vartheta_4(y,q)^2}{\vartheta_3(0,q)^2}$$

$$\vartheta_4(x+y,q)\,\vartheta_4(x-y,q) = \frac{\vartheta_4(x,q)^2\,\vartheta_4(y,q)^2-\vartheta_1(x,q)^2\,\vartheta_1(y,q)^2}{\vartheta_4(0,q)^2} = \frac{\vartheta_3(x,q)^2\,\vartheta_3(y,q)^2-\vartheta_2(x,q)^2\,\vartheta_2(y,q)^2}{\vartheta_4(0,q)^2}.$$

Triple addition formulas, for example:

$$\vartheta_3(x+y+z,q)\,\vartheta_3(x,q)\,\vartheta_3(y,q)\,\vartheta_3(z,q)-\vartheta_2(x+y+z,q)\,\vartheta_2(x,q)\,\vartheta_2(y,q)\,\vartheta_2(z,q) = \vartheta_1(x,q)\,\vartheta_1(y,q)\,\vartheta_1(z,q)\,\vartheta_1(x+y+z,q)+\vartheta_4(x,q)\,\vartheta_4(y,q)\,\vartheta_4(z,q)\,\vartheta_4(x+y+z,q)$$

$$\vartheta_1(z,q)^4+\vartheta_3(z,q)^4 = \vartheta_2(z,q)^4+\vartheta_4(z,q)^4.$$

Representations of derivatives

The derivatives of the Jacobi theta functions $\vartheta_1(z,q)$, $\vartheta_2(z,q)$, $\vartheta_3(z,q)$, and $\vartheta_4(z,q)$, and their derivatives $\vartheta_1'(z,q)$, $\vartheta_2'(z,q)$, $\vartheta_3'(z,q)$, and $\vartheta_4'(z,q)$ with respect to variable z can be expressed by the following formulas:

$$\frac{\partial\vartheta_1(z,q)}{\partial z} = \vartheta_1'(z,q) = 2\sqrt[4]{q}\sum_{k=0}^{\infty}(-1)^k q^{k(k+1)}(2k+1)\cos((2k+1)z)\;/;\;|q|<1$$

$$\frac{\partial\vartheta_2(z,q)}{\partial z} = \vartheta_2'(z,q) = -2\sqrt[4]{q}\sum_{k=0}^{\infty}q^{k(k+1)}(2k+1)\sin((2k+1)z)\;/;\;|q|<1$$

$$\frac{\partial\vartheta_3(z,q)}{\partial z} = \vartheta_3'(z,q) = -4\sum_{k=1}^{\infty}q^{k^2}k\sin(2kz)\;/;\;|q|<1$$

$$\frac{\partial\vartheta_4(z,q)}{\partial z} = \vartheta_4'(z,q) = -4\sum_{k=1}^{\infty}(-1)^k k\,q^{k^2}\sin(2kz)\;/;\;|q|<1$$

$$\frac{\partial\vartheta_1'(z,q)}{\partial z} = -2\sqrt[4]{q}\sum_{k=0}^{\infty}(-1)^k q^{k(k+1)}(2k+1)^2\sin((2k+1)z)\;/;\;|q|<1$$

$$\frac{\partial\vartheta_2'(z,q)}{\partial z} = -2\sqrt[4]{q}\sum_{k=0}^{\infty}q^{k(k+1)}(2k+1)^2\cos((2k+1)z)\;/;\;|q|<1$$

$$\frac{\partial\vartheta_3'(z,q)}{\partial z} = -8\sum_{k=1}^{\infty}q^{k^2}k^2\cos(2kz)\;/;\;|q|<1$$

$$\frac{\partial \vartheta_4'(z,q)}{\partial z} = 8\sum_{k=1}^{\infty}(-1)^{k-1} k^2 q^{k^2} \cos(2kz) \text{ /; } |q|<1.$$

The derivatives of the Jacobi theta functions $\vartheta_1(z,q)$, $\vartheta_2(z,q)$, $\vartheta_3(z,q)$, and $\vartheta_4(z,q)$, and their derivatives $\vartheta_1'(z,q)$, $\vartheta_2'(z,q)$, $\vartheta_3'(z,q)$, and $\vartheta_4'(z,q)$ with respect to variable q can be expressed by the following formulas:

$$\frac{\partial \vartheta_1(z,q)}{\partial q} = \frac{\vartheta_1(z,q)}{4q} + 2\sum_{k=1}^{\infty}(-1)^k k(k+1) q^{k(k+1)-\frac{3}{4}} \sin((2k+1)z) \text{ /; } |q|<1$$

$$\frac{\partial \vartheta_2(z,q)}{\partial q} = \frac{\vartheta_2(z,q)}{4q} + 2\sum_{k=1}^{\infty} k(k+1) q^{k(k+1)-\frac{3}{4}} \cos((2k+1)z) \text{ /; } |q|<1$$

$$\frac{\partial \vartheta_3(z,q)}{\partial q} = 2\sum_{k=1}^{\infty} q^{k^2-1} k^2 \cos(2kz) \text{ /; } |q|<1$$

$$\frac{\partial \vartheta_4(z,q)}{\partial q} = 2\sum_{k=1}^{\infty}(-1)^k k^2 q^{k^2-1} \cos(2kz) \text{ /; } |q|<1$$

$$\frac{\partial \vartheta_1'(z,q)}{\partial q} = 2q^{-\frac{3}{4}}\sum_{k=1}^{\infty}(-1)^k q^{k(k+1)} k(k+1)(2k+1)\cos((2k+1)z) + \frac{\vartheta_1'(z,q)}{4q} \text{ /; } |q|<1$$

$$\frac{\partial \vartheta_2'(z,q)}{\partial q} = \frac{\vartheta_2'(z,q)}{4q} - 2q^{-\frac{3}{4}}\sum_{k=1}^{\infty} q^{k(k+1)} k(k+1)(2k+1)\sin((2k+1)z) \text{ /; } |q|<1$$

$$\frac{\partial \vartheta_3'(z,q)}{\partial q} = -4\sum_{k=1}^{\infty} q^{k^2-1} k^3 \sin(2kz) \text{ /; } |q|<1$$

$$\frac{\partial \vartheta_4'(z,q)}{\partial q} = 4\sum_{k=1}^{\infty}(-1)^{k-1} k^3 q^{k^2-1} \sin(2kz) \text{ /; } |q|<1.$$

The n^{th}-order derivatives of the Jacobi theta functions $\vartheta_1(z,q)$, $\vartheta_2(z,q)$, $\vartheta_3(z,q)$, and $\vartheta_4(z,q)$, and their derivatives $\vartheta_1'(z,q)$, $\vartheta_2'(z,q)$, $\vartheta_3'(z,q)$, and $\vartheta_4'(z,q)$ with respect to variable z can be expressed by the following formulas:

$$\frac{\partial^n \vartheta_1(z,q)}{\partial z^n} = 2\sqrt[4]{q}\sum_{k=0}^{\infty}(-1)^k q^{k(k+1)}(2k+1)^n \sin\left(\frac{\pi n}{2} + (2k+1)z\right) \text{ /; } |q|<1 \land n \in \mathbb{N}^+$$

$$\frac{\partial^n \vartheta_2(z,q)}{\partial z^n} = 2\sqrt[4]{q}\sum_{k=0}^{\infty} q^{k(k+1)}(2k+1)^n \cos\left(\frac{\pi n}{2} + (2k+1)z\right) \text{ /; } |q|<1 \land n \in \mathbb{N}^+$$

$$\frac{\partial^n \vartheta_3(z,q)}{\partial z^n} = 2^{n+1}\sum_{k=0}^{\infty} q^{k^2} k^n \cos\left(\frac{\pi n}{2} + 2kz\right) \text{ /; } |q|<1 \land n \in \mathbb{N}^+$$

$$\frac{\partial^n \vartheta_4(z,q)}{\partial z^n} = 2^{n+1}\sum_{k=1}^{\infty}(-1)^k q^{k^2} k^n \cos\left(\frac{\pi n}{2} + 2kz\right) \text{ /; } |q|<1 \land n \in \mathbb{N}^+$$

$$\frac{\partial^n \vartheta_1'(z, q)}{\partial z^n} = 2\sqrt[4]{q}\sum_{k=0}^{\infty}(-1)^k q^{k(k+1)}(2k+1)^{n+1}\cos\left(\frac{\pi n}{2}+(2k+1)z\right) /; |q|<1 \bigwedge n \in \mathbb{N}^+$$

$$\frac{\partial^n \vartheta_2'(z, q)}{\partial z^n} = -2\sqrt[4]{q}\sum_{k=0}^{\infty} q^{k(k+1)}(2k+1)^{n+1}\sin\left(\frac{\pi n}{2}+(2k+1)z\right) /; |q|<1 \bigwedge n \in \mathbb{N}^+$$

$$\frac{\partial^n \vartheta_3'(z, q)}{\partial z^n} = -2^{n+2}\sum_{k=1}^{\infty} q^{k^2} k^{n+1}\sin\left(\frac{\pi n}{2}+2kz\right) /; |q|<1 \bigwedge n \in \mathbb{N}^+$$

$$\frac{\partial^n \vartheta_4'(z, q)}{\partial z^n} = 2^{n+2}\sum_{k=1}^{\infty}(-1)^{k-1} q^{k^2} k^{n+1}\sin\left(\frac{\pi n}{2}+2kz\right) /; |q|<1 \bigwedge n \in \mathbb{N}^+.$$

The n^{th}-order derivatives of Jacobi theta functions $\vartheta_1(z, q)$, $\vartheta_2(z, q)$, $\vartheta_3(z, q)$, and $\vartheta_4(z, q)$, and their derivatives $\vartheta_1'(z, q)$, $\vartheta_2'(z, q)$, $\vartheta_3'(z, q)$, and $\vartheta_4'(z, q)$ with respect to variable q can be expressed by the following formulas:

$$\frac{\partial^n \vartheta_1(z, q)}{\partial q^n} = 2q^{\frac{1}{4}-n}\sum_{k=0}^{\infty}(-1)^k q^{k(k+1)}\left(k(k+1)-n+\frac{5}{4}\right)_n \sin((2k+1)z) /; |q|<1 \bigwedge n \in \mathbb{N}^+$$

$$\frac{\partial^n \vartheta_2(z, q)}{\partial q^n} = 2q^{\frac{1}{4}-n}\sum_{k=0}^{\infty} q^{k(k+1)}\left(k(k+1)-n+\frac{5}{4}\right)_n \cos((2k+1)z) /; |q|<1 \bigwedge n \in \mathbb{N}^+$$

$$\frac{\partial^n \vartheta_3(z, q)}{\partial q^n} = 2\sum_{k=1}^{\infty} q^{k^2-n}\left(k^2-n+1\right)_n \cos(2kz) /; |q|<1 \bigwedge n \in \mathbb{N}^+$$

$$\frac{\partial^n \vartheta_4(z, q)}{\partial q^n} = 2\sum_{k=1}^{\infty}(-1)^k q^{k^2-n}\left(k^2-n+1\right)_n \cos(2kz) /; |q|<1 \bigwedge n \in \mathbb{N}^+$$

$$\frac{\partial^n \vartheta_1(z, q)}{\partial q^n} = 2q^{\frac{1}{4}-n}\sum_{k=0}^{\infty}(-1)^k q^{k(k+1)}(2k+1)\left(k(k+1)-n+\frac{5}{4}\right)_n \cos((2k+1)z) /; |q|<1 \bigwedge n \in \mathbb{N}^+$$

$$\frac{\partial^n \vartheta_2(z, q)}{\partial q^n} = -2q^{\frac{1}{4}-n}\sum_{k=0}^{\infty} q^{k(k+1)}(2k+1)\left(k(k+1)-n+\frac{5}{4}\right)_n \sin((2k+1)z) /; |q|<1 \bigwedge n \in \mathbb{N}^+$$

$$\frac{\partial^n \vartheta_3'(z, q)}{\partial q^n} = -4\sum_{k=1}^{\infty} q^{k^2-n} k\left(k^2-n+1\right)_n \sin(2kz) /; |q|<1 \bigwedge n \in \mathbb{N}^+$$

$$\frac{\partial^n \vartheta_4'(z, q)}{\partial q^n} = 4\sum_{k=1}^{\infty}(-1)^{k-1} q^{k^2-n} k\left(k^2-n+1\right)_n \sin(2kz) /; |q|<1 \bigwedge n \in \mathbb{N}^+.$$

Integration

The indefinite integrals of the Jacobi theta functions $\vartheta_1(z, q)$, $\vartheta_2(z, q)$, $\vartheta_3(z, q)$, and $\vartheta_4(z, q)$, and their derivatives $\vartheta_1'(z, q)$, $\vartheta_2'(z, q)$, $\vartheta_3'(z, q)$, and $\vartheta_4'(z, q)$ with respect to variable z can be expressed by the following formulas:

$$\int \vartheta_1(z, q)\, dz = -2\sqrt[4]{q}\sum_{k=0}^{\infty}\frac{(-1)^k q^{k(k+1)}}{2k+1}\cos((2k+1)z) /; |q|<1$$

$$\int \vartheta_2(z, q)\, dz == 2\sqrt[4]{q} \sum_{k=0}^{\infty} \frac{(-1)^k q^{k(k+1)}}{2k+1} \sin((2k+1)z) \text{ /; } |q| < 1$$

$$\int \vartheta_3(z, q)\, dz == z + \sum_{k=1}^{\infty} \frac{q^{k^2} \sin(2kz)}{k} \text{ /; } |q| < 1$$

$$\int \vartheta_4(z, q)\, dz == z + \sum_{k=1}^{\infty} \frac{(-1)^k q^{k^2} \sin(2kz)}{k} \text{ /; } |q| < 1$$

$$\int \vartheta_1'(z, q)\, dz == \vartheta_1(z, q)$$

$$\int \vartheta_2'(z, q)\, dz == \vartheta_2(z, q)$$

$$\int \vartheta_3'(z, q)\, dz == \vartheta_3(z, q)$$

$$\int \vartheta_4'(z, q)\, dz == \vartheta_4(z, q).$$

The first four sums cannot be expressed in closed form through the named functions.

The indefinite integrals of the Jacobi theta functions $\vartheta_1(z, q)$, $\vartheta_2(z, q)$, $\vartheta_3(z, q)$, and $\vartheta_4(z, q)$, and their derivatives $\vartheta_1'(z, q)$, $\vartheta_2'(z, q)$, $\vartheta_3'(z, q)$, and $\vartheta_4'(z, q)$ with respect to variable q can be expressed by the following formulas:

$$\int \vartheta_1(z, q)\, dq == 2 \sum_{k=0}^{\infty} \frac{(-1)^k q^{k(k+1)+\frac{5}{4}} \sin((2k+1)z)}{k(k+1)+\frac{5}{4}} \text{ /; } |q| < 1$$

$$\int \vartheta_2(z, q)\, dq == 2 \sum_{k=0}^{\infty} \frac{q^{k(k+1)+\frac{5}{4}} \cos((2k+1)z)}{k(k+1)+\frac{5}{4}} \text{ /; } |q| < 1$$

$$\int \vartheta_3(z, q)\, dq == q + 2 \sum_{k=1}^{\infty} \frac{q^{k^2+1} \cos(2kz)}{k^2+1} \text{ /; } |q| < 1$$

$$\int \vartheta_4(z, q)\, dq == q + 2 \sum_{k=1}^{\infty} \frac{(-1)^k q^{k^2+1} \cos(2kz)}{k^2+1} \text{ /; } |q| < 1$$

$$\int \vartheta_1'(z, q)\, dq == 2 \sum_{k=0}^{\infty} \frac{(-1)^k q^{(k+1)k+\frac{5}{4}} (2k+1) \cos((2k+1)z)}{(k+1)k+\frac{5}{4}} \text{ /; } |q| < 1$$

$$\int \vartheta_2'(z, q)\, dq == -2 \sum_{k=0}^{\infty} \frac{q^{(k+1)k+\frac{5}{4}} (2k+1) \sin((2k+1)z)}{(k+1)k+\frac{5}{4}} \text{ /; } |q| < 1$$

$$\int \vartheta_3'(z, q)\, dq = -4 \sum_{k=1}^{\infty} \frac{k\, q^{k^2+1} \sin(2\, k\, z)}{k^2+1} /;\ |q| < 1$$

$$\int \vartheta_4'(z, q)\, dq = 4 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}\, k\, q^{k^2+1} \sin(2\, k\, z)}{k^2+1} /;\ |q| < 1.$$

Partial differential equations

The elliptic theta functions $\vartheta_1(z, q)$, $\vartheta_2(z, q)$, $\vartheta_3(z, q)$, and $\vartheta_4(z, q)$, and their derivatives $\vartheta_1'(z, q)$, $\vartheta_2'(z, q)$, $\vartheta_3'(z, q)$, and $\vartheta_4'(z, q)$ satisfy the one-dimensional heat equations:

$$\frac{\partial \vartheta_j(z, q)}{\partial \tau} = -\frac{\pi\, i}{4} \frac{\partial^2 \vartheta_j(z, q)}{\partial z^2} /;\ q = e^{i\, \pi\, \tau} \bigwedge j \in \{1, 2, 3, 4\}$$

$$\frac{\partial \vartheta_j'(z, q)}{\partial \tau} = -\frac{\pi\, i}{4} \frac{\partial^2 \vartheta_j'(z, q)}{\partial z^2} /;\ q = e^{i\, \pi\, \tau} \bigwedge j \in \{1, 2, 3, 4\}.$$

The elliptic theta functions $\vartheta_1(z, q)$, $\vartheta_2(z, q)$, $\vartheta_3(z, q)$, and $\vartheta_4(z, q)$, and their derivatives $\vartheta_1'(z, q)$, $\vartheta_2'(z, q)$, $\vartheta_3'(z, q)$, and $\vartheta_4'(z, q)$ satisfy the following second-order partial differential equations:

$$4\, q \frac{\partial \vartheta_j(z, q)}{\partial q} + \frac{\partial^2 \vartheta_j(z, q)}{\partial z^2} = 0 /;\ j \in \{1, 2, 3, 4\}$$

$$4\, q \frac{\partial \vartheta_j'(z, q)}{\partial q} + \frac{\partial^2 \vartheta_j'(z, q)}{\partial z^2} = 0 /;\ j \in \{1, 2, 3, 4\}.$$

Zeros

The Jacobi theta functions $\vartheta_1(z, q)$, $\vartheta_2(z, q)$, $\vartheta_3(z, q)$, and $\vartheta_4(z, q)$, and their derivatives $\vartheta_1'(z, q)$, $\vartheta_2'(z, q)$, $\vartheta_3'(z, q)$, and $\vartheta_4'(z, q)$ are equal to zero in the following points:

$$\vartheta_1(z, 0) = 0 \quad \vartheta_2(z, 0) = 0 \quad \vartheta_1'(z, 0) = 0 \quad \vartheta_2'(z, 0) = 0 \quad \vartheta_3'(z, 0) = 0 \quad \vartheta_4'(z, 0) = 0$$

$$\vartheta_1(m\, \pi + n\, \tau, q) = 0 /;\ \{m, n\} \in \mathbb{Z} \bigwedge q = e^{i\, \pi\, \tau}$$

$$\vartheta_2\left((2\, m+1) \frac{\pi}{2} + n\, \pi\, \tau, q\right) = 0 /;\ \{m, n\} \in \mathbb{Z} \bigwedge q = e^{i\, \pi\, \tau}$$

$$\vartheta_3\left((2\, m+1) \frac{\pi}{2} + (2\, n+1) \frac{\pi\, \tau}{2}, q\right) = 0 /;\ \{m, n\} \in \mathbb{Z} \bigwedge q = e^{i\, \pi\, \tau}$$

$$\vartheta_4\left(m\, \pi + (2\, n+1) \frac{\pi\, \tau}{2}, q\right) = 0 /;\ \{m, n\} \in \mathbb{Z} \bigwedge q = e^{i\, \pi\, \tau}$$

$$\vartheta_1'\left(\frac{\pi}{2} + m\, \pi, q\right) = 0 /;\ m \in \mathbb{Z}$$

$$\vartheta_2'(m\, \pi, q) = 0 /;\ m \in \mathbb{Z}$$

$$\vartheta_3'\left(\frac{\pi m}{2}, q\right) == 0 \,/;\, m \in \mathbb{Z}$$

$$\vartheta_4'\left(\frac{\pi m}{2}, q\right) == 0 \,/;\, m \in \mathbb{Z}.$$

Applications of Jacobi theta functions

Applications of the Jacobi theta functions include the analytic solution of the heat equation, square potential well problems in quantum mechanics, Wannier functions in solid state physics, conformal mapping of periodic regions, gravitational physics, quantum cosmology, coding theory, sphere packings, crystal lattice calculations, and study of the fractional quantum Hall effect.

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