Introductions to Erf

Introduction to the probability integrals and inverses

General

The probability integral (error function) $\mathrm{erf}(z)$ has a long history beginning with the articles of A. de Moivre (1718–1733) and P.-S. Laplace (1774) where it was expressed through the following integral:

$$\int e^{-t^2}\, dt.$$

Later C. Kramp (1799) used this integral for the definition of the complementary error function $\mathrm{erfc}(z)$. P.-S. Laplace (1812) derived an asymptotic expansion of the error function.

The probability integrals were so named because they are widely applied in the theory of probability, in both normal and limit distributions.

To obtain, say, a normal distributed random variable from a uniformly distributed random variable, the inverse of the error function, namely $\mathrm{erf}^{-1}(z)$ is needed. The inverse was systematically investigated in the second half of the twentieth century, especially by J. R. Philip (1960) and A. J. Strecok (1968).

Definitions of probability integrals and inverses

The probability integral (error function) $\mathrm{erf}(z)$, the generalized error function $\mathrm{erf}(z_1, z_2)$, the complementary error function $\mathrm{erfc}(z)$, the imaginary error function $\mathrm{erfi}(z)$, the inverse error function $\mathrm{erf}^{-1}(z)$, the inverse of the generalized error function $\mathrm{erf}^{-1}(z_1, z_2)$, and the inverse complementary error function $\mathrm{erfc}^{-1}(z)$ are defined through the following formulas:

$$\mathrm{erf}(z) == \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2}\, dt$$

$$\mathrm{erf}(z_1, z_2) == \mathrm{erf}(z_2) - \mathrm{erf}(z_1)$$

$$\mathrm{erfc}(z) == \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-t^2}\, dt$$

$$\mathrm{erfi}(z) == \frac{2}{\sqrt{\pi}} \int_0^z e^{t^2}\, dt$$

$$z == \mathrm{erf}(w) \,/;\, w == \mathrm{erf}^{-1}(z)$$

$$z_2 == \mathrm{erf}(z_1, w) \,/;\, w == \mathrm{erf}^{-1}(z_1, z_2)$$

$$z == \mathrm{erfc}(w) \,/;\, w == \mathrm{erfc}^{-1}(z).$$

These seven functions are typically called probability integrals and their inverses.

Instead of using definite integrals, the three univariate error functions can be defined through the following infinite series.

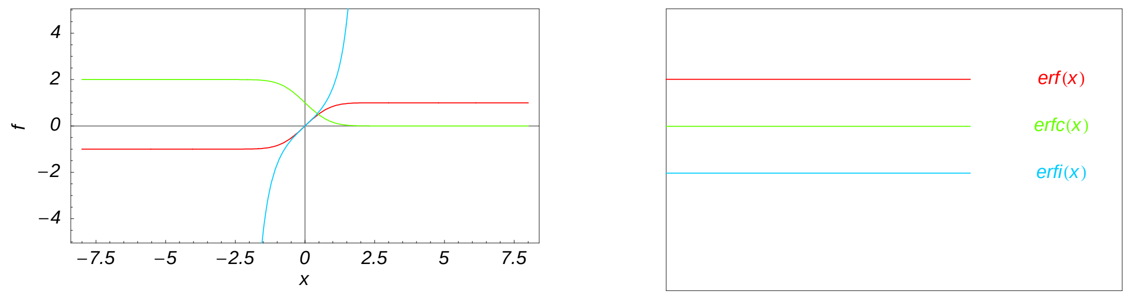
$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{k!\,(2k+1)}$$

$$\operatorname{erfc}(z) = 1 - \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{k!\,(2k+1)}$$

$$\operatorname{erfi}(z) = \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{z^{2k+1}}{k!\,(2k+1)}.$$

A quick look at the probability integrals and inverses

Here is a quick look at the graphics for the probability integrals and inverses along the real axis.

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      ListContourPlot, ListDensityPlot};

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     PlotStyle -> {{Hue[0.8 (k - 1)/Length[funs]], Thickness[0.002]}},
     FrameTicks -> {Automatic,  Automatic, None, None},
                       FrameLabel -> {x, f},
     AspectRatio -> 1/2, PlotRange -> {All, 1.01 {-5, 5}}],
                    {k, Length[funs]}, {x0, 0, 0}];
trigGraphic = Show[tab];
legend1 = Graphics[
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      {k, 1, Length[funs]}],
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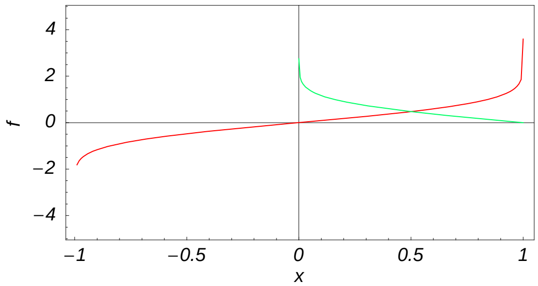

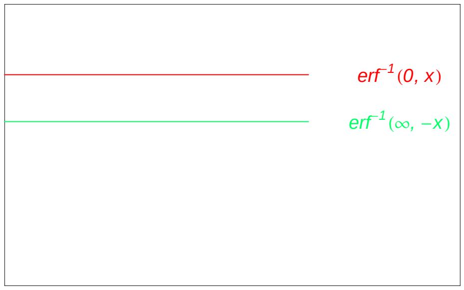

Connections within the group of probability integrals and inverses and with other function groups

Representations through more general functions

The probability integrals $\mathrm{erf}(z)$, $\mathrm{erf}(z_1, z_2)$, $\mathrm{erfc}(z)$, and $\mathrm{erfi}(z)$ are the particular cases of two more general functions: hypergeometric and Meijer G functions.

For example, they can be represented through the confluent hypergeometric functions ${}_1F_1$ and U:

$$\mathrm{erf}(z) = \frac{2z}{\sqrt{\pi}}\, {}_1F_1\left(\frac{1}{2}; \frac{3}{2}; -z^2\right)$$

$$\mathrm{erf}(z) = \frac{z}{\sqrt{z^2}}\left(1 - \frac{1}{\sqrt{\pi}}\, e^{-z^2}\, U\left(\frac{1}{2}, \frac{1}{2}, z^2\right)\right)$$

$$\mathrm{erf}(z_1, z_2) = \frac{2z_2}{\sqrt{\pi}}\, {}_1F_1\left(\frac{1}{2}; \frac{3}{2}; -z_2^2\right) - \frac{2z_1}{\sqrt{\pi}}\, {}_1F_1\left(\frac{1}{2}; \frac{3}{2}; -z_1^2\right)$$

$$\mathrm{erf}(z_1, z_2) = \frac{z_2}{\sqrt{z_2^2}}\left(1 - \frac{1}{\sqrt{\pi}}\, e^{-z_2^2}\, U\left(\frac{1}{2}, \frac{1}{2}, z_2^2\right)\right) - \frac{z_1}{\sqrt{z_1^2}}\left(1 - \frac{1}{\sqrt{\pi}}\, e^{-z_1^2}\, U\left(\frac{1}{2}, \frac{1}{2}, z_1^2\right)\right)$$

$$\mathrm{erfc}(z) = 1 - \frac{2z}{\sqrt{\pi}}\, {}_1F_1\left(\frac{1}{2}; \frac{3}{2}; -z^2\right)$$

$$\mathrm{erfc}(z) = \frac{z}{\sqrt{z^2}}\left(\frac{1}{\sqrt{\pi}}\, e^{-z^2}\, U\left(\frac{1}{2}, \frac{1}{2}, z^2\right) - 1\right) + 1$$

$$\mathrm{erfi}(z) = \frac{2z}{\sqrt{\pi}}\, {}_1F_1\left(\frac{1}{2}; \frac{3}{2}; z^2\right)$$

$$\mathrm{erfi}(z) = \frac{z}{\sqrt{-z^2}}\left(1 - \frac{1}{\sqrt{\pi}}\, e^{z^2}\, U\left(\frac{1}{2}, \frac{1}{2}, -z^2\right)\right).$$

Representations of the probability integrals $\mathrm{erf}(z)$, $\mathrm{erf}(z_1, z_2)$, $\mathrm{erfc}(z)$, and $\mathrm{erfi}(z)$ through classical Meijer G functions are rather simple:

$$\operatorname{erf}(z) = \frac{z}{\sqrt{\pi}} G_{1,2}^{1,1}\left(z^2 \,\middle|\, \begin{matrix} \frac{1}{2} \\ 0, -\frac{1}{2} \end{matrix}\right)$$

$$\operatorname{erf}(z_1, z_2) = \frac{1}{\sqrt{\pi}} \left(z_2 \, G_{1,2}^{1,1}\left(z_2^2 \,\middle|\, \begin{matrix} \frac{1}{2} \\ 0, -\frac{1}{2} \end{matrix}\right) - z_1 \, G_{1,2}^{1,1}\left(z_1^2 \,\middle|\, \begin{matrix} \frac{1}{2} \\ 0, -\frac{1}{2} \end{matrix}\right)\right)$$

$$\operatorname{erfc}(z) = 1 - \frac{z}{\sqrt{\pi}} G_{1,2}^{1,1}\left(z^2 \,\middle|\, \begin{matrix} \frac{1}{2} \\ 0, -\frac{1}{2} \end{matrix}\right)$$

$$\operatorname{erfi}(z) = \frac{z}{\sqrt{\pi}} G_{1,2}^{1,1}\left(-z^2 \,\middle|\, \begin{matrix} \frac{1}{2} \\ 0, -\frac{1}{2} \end{matrix}\right).$$

The factor z in the last four formulas can be removed by changing the classical Meijer G functions to the generalized one:

$$\operatorname{erf}(z) = \frac{1}{\sqrt{\pi}} G_{1,2}^{1,1}\left(z, \frac{1}{2} \,\middle|\, \begin{matrix} 1 \\ \frac{1}{2}, 0 \end{matrix}\right)$$

$$\operatorname{erf}(z_1, z_2) = \frac{1}{\sqrt{\pi}} \left(G_{1,2}^{1,1}\left(z_2, \frac{1}{2} \,\middle|\, \begin{matrix} 1 \\ \frac{1}{2}, 0 \end{matrix}\right) - G_{1,2}^{1,1}\left(z_1, \frac{1}{2} \,\middle|\, \begin{matrix} 1 \\ \frac{1}{2}, 0 \end{matrix}\right)\right)$$

$$\operatorname{erfc}(z) = \frac{1}{\sqrt{\pi}} G_{1,2}^{2,0}\left(z, \frac{1}{2} \,\middle|\, \begin{matrix} 1 \\ 0, \frac{1}{2} \end{matrix}\right)$$

$$\operatorname{erfi}(z) = -\frac{i}{\sqrt{\pi}} G_{1,2}^{1,1}\left(i\, z, \frac{1}{2} \,\middle|\, \begin{matrix} 1 \\ \frac{1}{2}, 0 \end{matrix}\right).$$

The probability integrals $\operatorname{erf}(z)$, $\operatorname{erf}(z_1, z_2)$, $\operatorname{erfc}(z)$, and $\operatorname{erfi}(z)$ are the particular cases of the incomplete gamma function, regularized incomplete gamma function, and exponential integral E:

$$\operatorname{erf}(z) = \frac{\sqrt{z^2}}{z} \left(1 - \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{1}{2}, z^2\right)\right)$$

$$\operatorname{erf}(z) = \frac{\sqrt{z^2}}{z} \left(1 - Q\left(\frac{1}{2}, z^2\right)\right)$$

$$\operatorname{erf}(z) = \frac{\sqrt{z^2}}{z} - \frac{z}{\sqrt{\pi}} E_{\frac{1}{2}}(z^2)$$

$$\operatorname{erf}(z_1, z_2) = \frac{\sqrt{z_2^2}}{z_2} \left(1 - \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{1}{2}, z_2^2\right)\right) - \frac{\sqrt{z_1^2}}{z_1} \left(1 - \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{1}{2}, z_1^2\right)\right)$$

$$\operatorname{erf}(z_1, z_2) = \frac{\sqrt{z_2^2}}{z_2} \left(1 - Q\left(\frac{1}{2}, z_2^2\right)\right) - \frac{\sqrt{z_1^2}}{z_1} \left(1 - Q\left(\frac{1}{2}, z_1^2\right)\right)$$

$$\operatorname{erf}(z_1, z_2) = \frac{z_1}{\sqrt{\pi}} E_{\frac{1}{2}}(z_1^2) - \frac{z_2}{\sqrt{\pi}} E_{\frac{1}{2}}(z_2^2) + \frac{\sqrt{z_2^2}}{z_2} - \frac{\sqrt{z_1^2}}{z_1}$$

$$\operatorname{erfc}(z) = 1 - \frac{\sqrt{z^2}}{z}\left(1 - \frac{1}{\sqrt{\pi}}\Gamma\left(\frac{1}{2}, z^2\right)\right)$$

$$\operatorname{erfc}(z) = 1 - \frac{\sqrt{z^2}}{z}\left(1 - Q\left(\frac{1}{2}, z^2\right)\right)$$

$$\operatorname{erfc}(z) = 1 - \frac{\sqrt{z^2}}{z} + \frac{z}{\sqrt{\pi}} E_{\frac{1}{2}}(z^2)$$

$$\operatorname{erfi}(z) = \frac{\sqrt{-z^2}}{z}\left(\frac{1}{\sqrt{\pi}}\Gamma\left(\frac{1}{2}, -z^2\right) - 1\right)$$

$$\operatorname{erfi}(z) = \frac{\sqrt{-z^2}}{z}\left(Q\left(\frac{1}{2}, -z^2\right) - 1\right)$$

$$\operatorname{erfi}(z) = -\frac{\sqrt{-z^2}}{z} - \frac{z}{\sqrt{\pi}} E_{\frac{1}{2}}(-z^2).$$

Representations through related equivalent functions

The probability integrals $\operatorname{erf}(z)$, $\operatorname{erfc}(z)$, and $\operatorname{erfi}(z)$ can be represented through Fresnel integrals by the following formulas:

$$\operatorname{erf}(z) = (1 + i)\left(C\left(\frac{(1 - i)\,z}{\sqrt{\pi}}\right) - i\,S\left(\frac{(1 - i)\,z}{\sqrt{\pi}}\right)\right)$$

$$\operatorname{erfc}(z) = 1 - (1 + i)\left(C\left(\frac{(1 - i)\,z}{\sqrt{\pi}}\right) - i\,S\left(\frac{(1 - i)\,z}{\sqrt{\pi}}\right)\right)$$

$$\operatorname{erfi}(z) = (1 - i)\left(C\left(\frac{(1 + i)\,z}{\sqrt{\pi}}\right) - i\,S\left(\frac{(1 + i)\,z}{\sqrt{\pi}}\right)\right).$$

Representations through other probability integrals and inverses

The probability integrals and their inverses $\operatorname{erf}(z)$, $\operatorname{erf}(z_1, z_2)$, $\operatorname{erfc}(z)$, $\operatorname{erfi}(z)$, $\operatorname{erf}^{-1}(z)$, $\operatorname{erf}^{-1}(z_1, z_2)$, and $\operatorname{erfc}^{-1}(z)$ are interconnected by the following formulas:

$$\operatorname{erf}(z) = \operatorname{erf}(0, z)$$

$$\operatorname{erf}(z) = 1 - \operatorname{erfc}(z)$$

$$\operatorname{erf}(z) = -i \operatorname{erfi}(i\,z)$$

$$\operatorname{erf}(\operatorname{erf}^{-1}(z)) = z$$

$$\mathrm{erf}\left(\mathrm{erf}^{-1}(0, z)\right) == z$$

$$\mathrm{erf}\left(z_1, \mathrm{erf}^{-1}(z_1, z_2)\right) == z_2$$

$$\mathrm{erf}\left(\mathrm{erfc}^{-1}(1-z)\right) == z$$

$$\mathrm{erfc}(z) == \mathrm{erf}(z, \infty)$$

$$\mathrm{erfc}(z) == 1 - \mathrm{erf}(z)$$

$$\mathrm{erfc}(z) == 1 + i\, \mathrm{erfi}(i\, z)$$

$$\mathrm{erfc}\left(\mathrm{erf}^{-1}(1-z)\right) == z$$

$$\mathrm{erfc}\left(\mathrm{erf}^{-1}(\infty, -z)\right) == z$$

$$\mathrm{erfc}\left(\mathrm{erfc}^{-1}(z)\right) == z$$

$$\mathrm{erfi}(z) == i\, \mathrm{erf}(i\, z, 0)$$

$$\mathrm{erfi}(z) == i\, \mathrm{erfc}(i\, z) - i$$

$$\mathrm{erfi}(z) == -i\, \mathrm{erf}(i\, z)$$

$$\mathrm{erfi}\left(i\, \mathrm{erf}^{-1}(z)\right) == i\, z$$

$$\mathrm{erfi}\left(i\, \mathrm{erf}^{-1}(0, z)\right) == i\, z$$

$$\mathrm{erfi}\left(i\, \mathrm{erfc}^{-1}(1-z)\right) == i\, z$$

$$\mathrm{erf}^{-1}(z) == \mathrm{erf}^{-1}(0, z)$$

$$\mathrm{erf}^{-1}(z) == \mathrm{erfc}^{-1}(1-z)$$

$$\mathrm{erfc}^{-1}(z) == \mathrm{erf}^{-1}(\infty, -z)$$

$$\mathrm{erfc}^{-1}(z) == \mathrm{erf}^{-1}(1-z).$$

The best-known properties and formulas for probability integrals and inverses

Real values for real arguments

For real values of argument z, the values of the probability integrals $\mathrm{erf}(z)$, $\mathrm{erf}(z_1, z_2)$, $\mathrm{erfc}(z)$, and $\mathrm{erfi}(z)$ are real. For real arguments $-1 < z < 1$, the values of the inverse error function $\mathrm{erf}^{-1}(z)$ are real; for real arguments $-1 < z_1 < 1$, $-1 < z_2 < 1$, the values of the inverse of the generalized error function $\mathrm{erf}^{-1}(z_1, z_2)$ are real; and for real arguments $0 < z < 2$, the values of the inverse complementary error function $\mathrm{erfc}^{-1}(z)$ are real.

Simple values at zero and one

The probability integrals $\mathrm{erf}(z)$, $\mathrm{erf}(z_1, z_2)$, $\mathrm{erfc}(z)$, and $\mathrm{erfi}(z)$, and their inverses $\mathrm{erf}^{-1}(z)$, $\mathrm{erf}^{-1}(z_1, z_2)$, and $\mathrm{erfc}^{-1}(z)$ have simple values for zero or unit arguments:

$\text{erf}(0) = 0$

$\text{erf}(0, 0) = 0$

$\text{erfc}(0) = 1$

$\text{erfi}(0) = 0$

$\text{erf}^{-1}(0) = 0$

$\text{erf}^{-1}(1) = \infty$

$\text{erf}^{-1}(0, 0) = 0$

$\text{erf}^{-1}(0, 1) = \infty$

$\text{erf}^{-1}(1, 0) = 1$

$\text{erfc}^{-1}(0) = \infty$

$\text{erfc}^{-1}(1) = 0.$

Simple values at infinity

The probability integrals $\text{erf}(z)$, $\text{erfc}(z)$, and $\text{erfi}(z)$ have simple values at infinity:

$\text{erf}(\infty) = 1$

$\text{erfc}(\infty) = 0$

$\text{erfi}(\infty) = \infty.$

Specific values for specialized arguments

In cases when z_1 or z_2 is equal to 0 or ∞, the generalized error function $\text{erf}(z_1, z_2)$ and its inverse $\text{erf}^{-1}(z_1, z_2)$ can be expressed through the probability integrals $\text{erf}(z)$, $\text{erfc}(z)$, or their inverses by the following formulas:

$\text{erf}(z, 0) = -\text{erf}(z)$

$\text{erf}(0, z) = \text{erf}(z)$

$\text{erf}(z, \infty) = \text{erfc}(z)$

$\text{erf}(\infty, z) = \text{erf}(z) - 1$

$\text{erf}^{-1}(0, z) = \text{erf}^{-1}(z)$

$\text{erf}^{-1}(\infty, z) = \text{erfc}^{-1}(-z).$

Analyticity

The probability integrals $\text{erf}(z)$, $\text{erfc}(z)$, and $\text{erfi}(z)$, and their inverses $\text{erf}^{-1}(z)$, and $\text{erfc}^{-1}(z)$ are defined for all complex values of z, and they are analytical functions of z over the whole complex z-plane. The probability integrals $\text{erf}(z)$, $\text{erfc}(z)$, and $\text{erfi}(z)$ are entire functions with an essential singular point at $z = \tilde{\infty}$, and they do not have branch cuts or branch points.

The generalized error function $\operatorname{erf}(z_1, z_2)$ is an analytical function of z_1 and z_2, which is defined in $\mathbb{C}^2$. For fixed z_1, it is an entire function of z_2. For fixed z_2, it is an entire function of z_1. It does not have branch cuts or branch points. The inverse of the generalized error function $\operatorname{erf}^{-1}(z_1, z_2)$ is an analytical function of z_1 and z_2, which is defined in $\mathbb{C}^2$.

Poles and essential singularities

The probability integrals $\operatorname{erf}(z)$, $\operatorname{erfc}(z)$, and $\operatorname{erfi}(z)$ have only one singular point at $z = \tilde{\infty}$. It is an essential singular point.

The generalized error function $\operatorname{erf}(z_1, z_2)$ has singular points at $z_1 = \tilde{\infty}$ and $z_2 == \tilde{\infty}$. They are essential singular points.

Periodicity

The probability integrals $\operatorname{erf}(z)$, $\operatorname{erf}(z_1, z_2)$, $\operatorname{erfc}(z)$, and $\operatorname{erfi}(z)$, and their inverses $\operatorname{erf}^{-1}(z)$, $\operatorname{erf}^{-1}(z_1, z_2)$, and $\operatorname{erfc}^{-1}(z)$ do not have periodicity.

Parity and symmetry

The probability integrals $\operatorname{erf}(z)$, $\operatorname{erf}(z_1, z_2)$, and $\operatorname{erfi}(z)$ are odd functions and have mirror symmetry:

$$\operatorname{erf}(-z) == -\operatorname{erf}(z) \quad \operatorname{erf}(\bar{z}) == \overline{\operatorname{erf}(z)}$$

$$\operatorname{erf}(-z_1, -z_2) == -\operatorname{erf}(z_1, z_2) \quad \operatorname{erf}(\overline{z_1}, \overline{z_2}) == \overline{\operatorname{erf}(z_1, z_2)}$$

$$\operatorname{erfi}(-z) == -\operatorname{erfi}(z) \quad \operatorname{erfi}(\bar{z}) == \overline{\operatorname{erfi}(z)}\,.$$

The generalized error function $\operatorname{erf}(z_1, z_2)$ has permutation symmetry:

$$\operatorname{erf}(z_1, z_2) == -\operatorname{erf}(z_2, z_1).$$

The complementary error function $\operatorname{erfc}(z)$ has mirror symmetry:

$$\operatorname{erfc}(\bar{z}) == \overline{\operatorname{erfc}(z)}.$$

Series representations

The probability integrals $\operatorname{erf}(z)$, $\operatorname{erf}(z_1, z_2)$, $\operatorname{erfc}(z)$, and $\operatorname{erfi}(z)$, and their inverses $\operatorname{erf}^{-1}(z)$ and $\operatorname{erfc}^{-1}(z)$ have the following series expansions:

$$\operatorname{erf}(z) \propto \frac{2}{\sqrt{\pi}} \left(z - \frac{z^3}{3} + \frac{z^5}{10} - \ldots \right) /;\ (z \to 0)$$

$$\operatorname{erf}(z) == \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{k!\,(2k+1)}$$

$$\operatorname{erf}(z_1, z_2) \propto \frac{2}{\sqrt{\pi}} \left(z_2 - \frac{z_2^3}{3} + \frac{z_2^5}{10} - \ldots \right) - \frac{2}{\sqrt{\pi}} \left(z_1 - \frac{z_1^3}{3} + \frac{z_1^5}{10} - \ldots \right) /;\ (z_1 \to 0) \land (z_2 \to 0)$$

$$\text{erf}(z_1, z_2) == \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k \left(z_2^{2k+1} - z_1^{2k+1}\right)}{k!\,(2k+1)}$$

$$\text{erfc}(z) \propto 1 - \frac{2}{\sqrt{\pi}} \left(z - \frac{z^3}{3} + \frac{z^5}{10} - \ldots\right) /; (z \to 0)$$

$$\text{erfc}(z) == 1 - \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{k!\,(2k+1)}$$

$$\text{erfi}(z) \propto \frac{2}{\sqrt{\pi}} \left(z + \frac{z^3}{3} + \frac{z^5}{10} + \ldots\right) /; (z \to 0)$$

$$\text{erfi}(z) == \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{z^{2k+1}}{k!\,(2k+1)}$$

$$\text{erf}^{-1}(z) == \frac{1}{2} \sqrt{\pi} \left(z + \frac{\pi z^3}{12} + O\left(z^5\right)\right)$$

$$\text{erf}^{-1}(z) == \sum_{k=0}^{\infty} \frac{c_k}{2k+1} \left(\frac{\sqrt{\pi}}{2} z\right)^{2k+1} /; c_0 == 1 \bigwedge c_k == \sum_{m=0}^{k-1} \frac{c_m\, c_{k-1-m}}{(m+1)(2m+1)}$$

$$\text{erf}^{-1}(z_1, z_2) \propto \text{erf}^{-1}(z_2) + e^{\text{erf}^{-1}(z_2)^2} z_1 + e^{2\,\text{erf}^{-1}(z_2)^2} \text{erf}^{-1}(z_2)\, z_1^2 + O\left(z_1^3\right)$$

$$\text{erf}^{-1}(z_1, z_2) \propto z_1 + \frac{1}{2} e^{z_1^2} \sqrt{\pi}\, z_2 + \frac{\pi z_1}{4} e^{2 z_1^2} z_2^2 + O\left(z_2^3\right)$$

$$\text{erfc}^{-1}(z) == \frac{\sqrt{\pi}}{2} \left(-(z-1) - \frac{1}{12} \pi (z-1)^3 + O\left((z-1)^5\right)\right)$$

$$\text{erfc}^{-1}(z) == -\sum_{k=0}^{\infty} \frac{c_k}{2k+1} \left(\frac{\sqrt{\pi}}{2} (z-1)\right)^{2k+1} /; c_0 == 1 \bigwedge c_k == \sum_{m=0}^{k-1} \frac{c_m\, c_{k-1-m}}{(m+1)(2m+1)}.$$

The series for functions $\text{erf}(z)$, $\text{erf}(z_1, z_2)$, $\text{erfc}(z)$, and $\text{erfi}(z)$ converge for all complex values of their arguments.

Interestingly, closed-form expressions for the truncated version of the Taylor series at the origin can be expressed through generalized hypergeometric function ${}_2F_2$, for example:

$$\text{erf}(z) == F_\infty(z) /; \left(\left(F_n(z) == \frac{2z}{\sqrt{\pi}} \sum_{k=0}^{n} \frac{(-1)^k z^{2k}}{(2k+1)\,k!} == \text{erf}(z) + \frac{(-1)^n\, 2\, z^{2n+3}}{\sqrt{\pi}\,(2n+3)\,(n+1)!}\, {}_2F_2\left(1, n+\frac{3}{2}; n+2, n+\frac{5}{2}; -z^2\right)\right) \bigwedge n \in \mathbb{N}\right).$$

Asymptotic series expansions

The asymptotic behavior of the probability integrals $\text{erf}(z)$, $\text{erfc}(z)$, and $\text{erfi}(z)$ can be described by the following formulas (only the main terms of the asymptotic expansion are given):

$$\text{erf}(z) \propto \frac{\sqrt{z^2}}{z} - \frac{1}{\sqrt{\pi}\, z} e^{-z^2} \left(1 + O\left(\frac{1}{z^2}\right)\right) /; (|z| \to \infty)$$

$$\operatorname{erfc}(z) \propto 1 - \frac{\sqrt{z^2}}{z} + \frac{1}{\sqrt{\pi}\, z} e^{-z^2} \left(1 + O\left(\frac{1}{z^2}\right)\right) /; (|z| \to \infty)$$

$$\operatorname{erfi}(z) \propto \frac{z}{\sqrt{-z^2}} + \frac{1}{\sqrt{\pi}\, z} e^{z^2} \left(1 + O\left(\frac{1}{z^2}\right)\right) /; (|z| \to \infty).$$

The previous formulas are valid in any direction approaching infinity ($|z| \to \infty$). In particular cases, these formulas can be simplified to the following relations:

$$\operatorname{erf}(z) \propto 1 - \frac{1}{\sqrt{\pi}\, z} e^{-z^2} \left(1 + O\left(\frac{1}{z^2}\right)\right) /; (|z| \to \infty) \land \operatorname{Re}(z) \geq 0$$

$$\operatorname{erfc}(z) \propto \frac{1}{\sqrt{\pi}\, z} e^{-z^2} \left(1 + O\left(\frac{1}{z^2}\right)\right) /; (|z| \to \infty) \land \operatorname{Re}(z) \geq 0$$

$$\operatorname{erfi}(z) \propto i + \frac{1}{\sqrt{\pi}\, z} e^{z^2} \left(1 + O\left(\frac{1}{z^2}\right)\right) /; (|z| \to \infty) \land \operatorname{Im}(z) \geq 0.$$

Integral representations

The probability integrals $\operatorname{erf}(z)$, $\operatorname{erf}(z_1, z_2)$, $\operatorname{erfc}(z)$, and $\operatorname{erfi}(z)$ can also be represented through the following equivalent integrals:

$$\operatorname{erf}(x) = \frac{2}{\pi} \int_0^\infty \frac{e^{-t^2} \sin(2\, x\, t)}{t} \, dt /; x \in \mathbb{R}$$

$$\operatorname{erf}(z_1, z_2) = \frac{2}{\sqrt{\pi}} \int_{z_1}^{z_2} e^{-t^2} \, dt$$

$$\operatorname{erfc}(x) = 1 - \frac{2}{\pi} \int_0^\infty \frac{e^{-t^2} \sin(2\, x\, t)}{t} \, dt /; x \in \mathbb{R}$$

$$\operatorname{erfi}(x) = \frac{2}{\sqrt{\pi}} e^{x^2} \int_0^\infty e^{-t^2} \sin(2\, x\, t) \, dt /; x \in \mathbb{R}$$

$$\operatorname{erfi}(x) = -\frac{1}{\pi} e^{x^2} \mathcal{P} \int_{-\infty}^\infty \frac{e^{-t^2}}{t - x} \, dt /; x \in \mathbb{R}.$$

The symbol $\mathcal{P}$ in the preceding integral means that the integral evaluates as the Cauchy principal value: $\mathcal{P} \int_a^b \frac{f(t)}{t-x} \, dt = \lim_{\epsilon \to 0} \left(\int_a^{x-\epsilon} \frac{f(t)}{t-x} \, dt + \int_{x+\epsilon}^b \frac{f(t)}{t-x} \, dt \right) /; a < x < b.$

Transformations

If the arguments of the probability integrals $\operatorname{erf}(z)$, $\operatorname{erfc}(z)$, and $\operatorname{erfi}(z)$ contain square roots, the arguments can sometimes be simplified:

$$\operatorname{erf}\left(\sqrt{z^2}\right) = \frac{\sqrt{z^2}}{z} \operatorname{erf}(z)$$

$$\operatorname{erfc}\left(\sqrt{z^2}\right) = 1 - \frac{\sqrt{z^2}}{z} \operatorname{erf}(z)$$

$$\operatorname{erfi}\left(\sqrt{z^2}\right) = \frac{\sqrt{z^2}}{z} \operatorname{erfi}(z).$$

Representations of derivatives

The derivative of the probability integrals $\operatorname{erf}(z)$, $\operatorname{erf}(z_1, z_2)$, $\operatorname{erfc}(z)$, and $\operatorname{erfi}(z)$, and their inverses $\operatorname{erf}^{-1}(z)$, $\operatorname{erf}^{-1}(z_1, z_2)$, and $\operatorname{erfc}^{-1}(z)$ have simple representations through elementary functions:

$$\frac{\partial \operatorname{erf}(z)}{\partial z} = \frac{2\, e^{-z^2}}{\sqrt{\pi}}$$

$$\frac{\partial \operatorname{erf}(z_1, z_2)}{\partial z_1} = -\frac{2\, e^{-z_1^2}}{\sqrt{\pi}}$$

$$\frac{\partial \operatorname{erf}(z_1, z_2)}{\partial z_2} = \frac{2\, e^{-z_2^2}}{\sqrt{\pi}}$$

$$\frac{\partial \operatorname{erfc}(z)}{\partial z} = -\frac{2\, e^{-z^2}}{\sqrt{\pi}}$$

$$\frac{\partial \operatorname{erfi}(z)}{\partial z} = \frac{2\, e^{z^2}}{\sqrt{\pi}}$$

$$\frac{\partial \operatorname{erf}^{-1}(z)}{\partial z} = \frac{\sqrt{\pi}}{2}\, e^{\operatorname{erf}^{-1}(z)^2}$$

$$\frac{\partial \operatorname{erf}^{-1}(z_1, z_2)}{\partial z_1} = e^{\operatorname{erf}^{-1}(z_1, z_2)^2 - z_1^2}$$

$$\frac{\partial \operatorname{erf}^{-1}(z_1, z_2)}{\partial z_2} = \frac{\sqrt{\pi}}{2}\, e^{\operatorname{erf}^{-1}(z_1, z_2)^2}$$

$$\frac{\partial \operatorname{erfc}^{-1}(z)}{\partial z} = -\frac{\sqrt{\pi}}{2}\, e^{\operatorname{erfc}^{-1}(z)^2}.$$

The symbolic n^{th}-order derivatives from the probability integrals $\operatorname{erf}(z)$, $\operatorname{erf}(z_1, z_2)$, $\operatorname{erfc}(z)$, and $\operatorname{erfi}(z)$ have the following simple representations through the regularized generalized hypergeometric function ${}_2\tilde{F}_2$:

$$\frac{\partial^n \operatorname{erf}(z)}{\partial z^n} = 2^n z^{1-n}\, {}_2\tilde{F}_2\left(\frac{1}{2}, 1; 1 - \frac{n}{2}, \frac{3-n}{2}; -z^2\right) /; n \in \mathbb{N}$$

$$\frac{\partial^n \operatorname{erf}(z_1, z_2)}{\partial z_1^n} = -2^n z_1^{1-n}\, {}_2\tilde{F}_2\left(\frac{1}{2}, 1; 1 - \frac{n}{2}, \frac{3-n}{2}; -z_1^3\right) /; n \in \mathbb{N}$$

$$\frac{\partial^n \operatorname{erf}(z_1, z_2)}{\partial z_2^n} = 2^n z_2^{1-n} \, {}_2\tilde{F}_2\left(\frac{1}{2}, 1; 1 - \frac{n}{2}, \frac{3-n}{2}; -z_2^3\right) /; n \in \mathbb{N}$$

$$\frac{\partial^n \operatorname{erfc}(z)}{\partial z^n} = -2^n z^{1-n} \, {}_2\tilde{F}_2\left(\frac{1}{2}, 1; 1 - \frac{n}{2}, \frac{3-n}{2}; -z^2\right) /; n \in \mathbb{N}$$

$$\frac{\partial^n \operatorname{erfi}(z)}{\partial z^n} = 2^n z^{1-n} \, {}_2\tilde{F}_2\left(\frac{1}{2}, 1; 1 - \frac{n}{2}, \frac{3-n}{2}; z^2\right) /; n \in \mathbb{N}.$$

But the symbolic n^{th}-order derivatives from the inverse probability integrals $\operatorname{erf}^{-1}(z)$, $\operatorname{erf}^{-1}(z_1, z_2)$, and $\operatorname{erfc}^{-1}(z)$ have very complicated structures in which the regularized generalized hypergeometric function ${}_2\tilde{F}_2$ appears in the multidimensional sums, for example:

$$\frac{\partial^n \operatorname{erf}^{-1}(z)}{\partial z^n} = \operatorname{erf}^{-1}(z)\, \delta_n + \frac{\pi^{n/2}}{2^n} e^{n \operatorname{erf}^{-1}(z)^2} \sum_{j_2=0}^{n} \dots \sum_{j_n=0}^{n} \delta_{\sum_{i=2}^{n}(i-1)\, j_i, n-1} \, (-1)^{\sum_{i=2}^{n} j_i} \left(n + \sum_{i=2}^{n} j_i - 1\right)! \prod_{i=2}^{n} \frac{1}{j_i!} \left(\frac{2^{i-1} e^{\operatorname{erf}^{-1}(z)^2} \sqrt{\pi} \, \operatorname{erf}^{-1}(z)^{1-i}}{i!}\right)^{j_i} {}_2\tilde{F}_2\left(\frac{1}{2}, 1; 1 - \frac{i}{2}, \frac{3-i}{2}; -\operatorname{erf}^{-1}(z)^2\right)^{j_i} /; n \in \mathbb{N}.$$

Differential equations

The probability integrals $\operatorname{erf}(z)$, $\operatorname{erf}(z_1, z_2)$, $\operatorname{erfc}(z)$, and $\operatorname{erfi}(z)$ satisfy the following second-order linear differential equations:

$$w''(z) + 2 z\, w'(z) = 0 /; w(z) = \operatorname{erf}(z) \bigwedge w(0) = 0 \bigwedge w'(0) = \frac{2}{\sqrt{\pi}}$$

$$w''(z_1) + 2 z_1\, w'(z_1) = 0 /; w(z_1) = c_1 \operatorname{erf}(z_1, z_2) + c_2$$

$$w''(z_2) + 2 z_2\, w'(z_2) = 0 /; w(z_2) = c_1 \operatorname{erf}(z_1, z_2) + c_2$$

$$w''(z) + 2 z\, w'(z) = 0 /; w(z) = \operatorname{erfc}(z) \bigwedge w(0) = 1 \bigwedge w'(0) = -\frac{2}{\sqrt{\pi}}$$

$$w''(z) - 2 z\, w'(z) = 0 /; w(z) = \operatorname{erfi}(z) \bigwedge w(0) = 0 \bigwedge w'(0) = \frac{2}{\sqrt{\pi}},$$

where c_1 and c_2 are arbitrary constants.

The inverses of the probability integrals $\operatorname{erf}^{-1}(z)$, $\operatorname{erf}^{-1}(z_1, z_2)$, and $\operatorname{erfc}^{-1}(z)$ satisfy the following ordinary second-order nonlinear differential equations:

$$w''(z) - 2 w(z)\, w'(z)^2 = 0 /; w(z) = \operatorname{erf}^{-1}(z)$$

$$w''(z_2) - 2 w(z_2)\, w'(z_2)^2 = 0 /; w(z_2) = \operatorname{erf}^{-1}(z_1, z_2)$$

$$w''(z) - 2 w(z)\, w'(z)^2 = 0 /; w(z) = \operatorname{erfc}^{-1}(z).$$

Applications of probability integrals and inverses

Applications of probability integrals include solutions of linear partial differential equations, probability theory, Monte Carlo simulations, and the Ewald method for calculating electrostatic lattice constants.

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