Introductions to Floor

Introduction to the rounding and congruence functions

General

The rounding and congruence functions have a long history that is closely related to the history of number theory. Many calculations use rounding of the floating-point and rational numbers to the closest smaller or larger integers. J. Nemorarius (1237) was one of the first mathematicians to use the quotient of two numbers m and n in a modern sense, but the word *quotient* appeared for the first time around 1250 in the writings of Meister Gernadus.

Special notations for rounding and congruence functions were introduced much later. C. F. Gauss (1801) suggested the symbol mod $(k = m \bmod n)$ for the notation of the property that the ratio $(m-k)/n$ is an integer. He observed that k and m are the congruent modulo n. The number n is called modulus.

C. F. Gauss (1808) and J. Liouville (1838) widely used the floor and round functions in their investigations. They and other mathematicians used different and sometimes confusing notations for those functions. The modern notations of $\lfloor z \rfloor$ and $\lceil z \rceil$ for floor and ceiling functions, respectively, were suggested by K. E. Iverson (1962). The notation $\lfloor z \rceil$ for the rounding function was proposed by J. Hastad (1988).

Definitions of the rounding and congruence functions

The rounding and congruence functions include seven basic functions. They all deal with the separation of integer or fractional parts from real and complex numbers: the floor function (entire part function) $\lfloor z \rfloor$, the nearest integer function (round) $\lfloor z \rceil$, the ceiling function (least integer) $\lceil z \rceil$, the integer part $\operatorname{int}(z)$, the fractional part $\operatorname{frac}(z)$, the modulo function (congruence) $m \bmod n$, and the integer part of the quotient (quotient or integer division) $\operatorname{quotient}(m, n)$.

The floor function (entire function) $\lfloor z \rfloor$ can be considered as the basic function of this group. The other six functions can be uniquely defined through the floor function.

Floor

For real z, the floor function $\lfloor z \rfloor$ is the greatest integer less than or equal to z.

For arbitrary complex z, the function $\lfloor z \rfloor$ can be described (or defined) by the following formulas:

$$\lfloor x \rfloor = n \;/;\; x \in \mathbb{R} \land n \in \mathbb{Z} \land n \leq x < n + 1$$

$$\lfloor z \rfloor = \lfloor \operatorname{Re}(z) \rfloor + i \lfloor \operatorname{Im}(z) \rfloor.$$

Examples: $\lfloor 3.2 \rfloor = 3$, $\lfloor 3 \rfloor = 3$, $\lfloor -0.2 \rfloor = -1$, $\lfloor -2.3 \rfloor = -3$, $\left\lfloor \frac{2}{3} \right\rfloor = 0$, $\lfloor -\pi \rfloor = -4$, $\left\lfloor -4 - \frac{5}{3} i \right\rfloor = -4 - 2 i$, $\left\lfloor \frac{5}{2} \right\rfloor = 2$, $\left\lfloor \frac{7}{2} \right\rfloor = 3$.

Round

For real z, the rounding function $\lfloor z \rceil$ is the integer closest to z (if $z \neq \pm \frac{1}{2}, \pm \frac{3}{2}, \ldots$).

For arbitrary z, the round function $\lfloor z \rceil$ can be described (or defined) by the following formulas:

$$\lfloor x \rceil = n \,/;\, x \in \mathbb{R} \bigwedge n \in \mathbb{Z} \bigwedge |x - n| < \frac{1}{2}$$

$$\lfloor z \rceil = \lfloor \mathrm{Re}(z) \rceil + i \lfloor \mathrm{Im}(z) \rceil$$

$$\left\lfloor n + \frac{1}{2} \right\rceil = n \,/;\, \frac{n}{2} \in \mathbb{Z}$$

$$\left\lfloor n + \frac{1}{2} \right\rceil = n + 1 \,/;\, \frac{n+1}{2} \in \mathbb{Z}.$$

Examples: $\lfloor 3.2 \rceil = 3$, $\lfloor 3 \rceil = 3$, $\lfloor -0.2 \rceil = 0$, $\lfloor -2.3 \rceil = -2$, $\left\lfloor \frac{2}{3} \right\rceil = 1$, $\lfloor -\pi \rceil = -3$, $\left\lfloor -4 - \frac{5}{3} i \right\rceil = -4 - 2 i$, $\left\lfloor \frac{5}{2} \right\rceil = 2$, $\left\lfloor \frac{7}{2} \right\rceil = 4$.

Ceiling

For real z, the ceiling function $\lceil z \rceil$ is the smallest integer greater than or equal to z.

For arbitrary z, the function $\lceil z \rceil$ can be described (or defined) by the following formulas:

$$\lceil x \rceil = n \,/;\, x \in \mathbb{R} \land n \in \mathbb{Z} \land n - 1 < x \leq n$$

$$\lceil z \rceil = \lceil \mathrm{Re}(z) \rceil + i \lceil \mathrm{Im}(z) \rceil.$$

Examples: $\lceil 3.2 \rceil = 4$, $\lceil 3 \rceil = 3$, $\lceil -0.2 \rceil = 0$, $\lceil -2.3 \rceil = -2$, $\left\lceil \frac{2}{3} \right\rceil = 1$, $\lceil -\pi \rceil = -3$, $\left\lceil -4 - \frac{5}{3} i \right\rceil = -4 - i$, $\left\lceil \frac{5}{2} \right\rceil = 3$, $\left\lceil \frac{7}{2} \right\rceil = 4$.

Integer part

For real z, the function integer part $\mathrm{int}(z)$ is the integer part of z.

For arbitrary z, the function $\mathrm{int}(z)$ can be described (or defined) by the following formulas:

$$\mathrm{int}(x) = n \,/;\, x \in \mathbb{R} \bigwedge n \in \mathbb{Z} \bigwedge 0 \leq \mathrm{sgn}(x)\,(x - n) < 1 \bigwedge x \neq 0$$

$$\mathrm{int}(z) = \mathrm{int}(\mathrm{Re}(z)) + i\, \mathrm{int}(\mathrm{Im}(z)).$$

Examples: $\mathrm{int}(3.2) = 3$, $\mathrm{int}(3) = 3$, $\mathrm{int}(-0.2) = 0$, $\mathrm{int}(-2.3) = -2$, $\mathrm{int}\left(\frac{2}{3}\right) = 0$, $\mathrm{int}(-\pi) = -3$, $\mathrm{int}\left(-4 - \frac{5}{3} i\right) = -4 - i$, $\mathrm{int}\left(\frac{5}{2}\right) = 2$, $\mathrm{int}\left(\frac{7}{2}\right) = 3$.

Fractional part

For real z, the function fractional part $\mathrm{frac}(z)$ is the fractional part of z.

For arbitrary z, the function $\mathrm{frac}(z)$ can be described (or defined) by the following formulas:

$$\mathrm{frac}(x) = x - n \,/;\, x \in \mathbb{R} \bigwedge n \in \mathbb{Z} \bigwedge 0 \leq \mathrm{sgn}(x)\,(x - n) < 1 \bigwedge x \neq 0$$

$\text{frac}(z) = \text{frac}(\text{Re}(z)) + i\,\text{frac}(\text{Im}(z))$.

Examples: $\text{frac}(3.2) = 0.2$, $\text{frac}(3) = 0$, $\text{frac}(-0.2) = -0.2$, $\text{frac}(-2.3) = -0.3$, $\text{frac}\left(\frac{2}{3}\right) = \frac{2}{3}$, $\text{frac}(-\pi) = 3 - \pi$, $\text{frac}\left(-4 - \frac{5}{3}\,i\right) = -\frac{2\,i}{3}$, $\text{frac}\left(\frac{5}{2}\right) = \frac{1}{2}$, $\text{frac}\left(\frac{7}{2}\right) = \frac{1}{2}$.

Mod

For complex n and m, the mod function $m \bmod n$ is the remainder of the division of m by n. The sign of $m \bmod n$ for real m, n is always the same as the sign of n.

The mod function $m \bmod n$ can be described (or defined) by the following formula:

$$m \bmod n = m - n \left\lfloor \frac{m}{n} \right\rfloor.$$

The functional property $m \bmod n = n \left(\frac{m}{n} \bmod 1\right) = n \left(\frac{m}{n} - \left\lfloor \frac{m}{n} \right\rfloor\right)$ makes the behavior of $m \bmod n$ similar to the behavior of $\left\lfloor \frac{m}{n} \right\rfloor$.

Examples: $5 \bmod 2 = 1$, $8 \bmod 3 = 2$, $-5 \bmod 3 = 1$, $(7\,\pi) \bmod 3 = -21 + 7\,\pi$, $(27 - 3\,i) \bmod 4 = 3 + i$, $\text{frac}(-\pi) = 3 - \pi$, $(2.7 - 3\,i) \bmod 5 = 2.7 + 2\,i$.

Quotient

For complex n and m, the integer part of the quotient (quotient) function $\text{quotient}(m, n)$ is the integer quotient of m and n.

The quotient function $\text{quotient}(m, n)$ can be described (or defined) by the following formula:

$$\text{quotient}(m, n) = \left\lfloor \frac{m}{n} \right\rfloor.$$

Examples: $\text{quotient}(5, 2) = 2$, $\text{quotient}(13, 3) = 4$, $\text{quotient}(-4, 3) = -2$, $\text{quotient}(\pi, 2) = 1$, $\text{quotient}(27 - 3\,i, 5) = 5 - i$, $\text{quotient}(-\pi, 2) = -2$, $\text{quotient}(2.7 - 3\,i, 5) = -i$.

Connections within the group of rounding and congruence functions and with other function groups

Representations through related functions

The rounding and congruence functions $\lfloor z \rceil$, $\lfloor z \rfloor$, $\lceil z \rceil$, $\text{int}(z)$, $\text{frac}(z)$, $m \bmod n$, and $\text{quotient}(m, n)$ have numerous representations through related functions, which are shown in the following tables, where the symbol $\chi_{\mathbb{A}}(a)$ means the characteristic function of a set $\mathbb{A}$ (having the value 1 when its argument a is an element of the specified set $\mathbb{A}$, and a value of 0 otherwise):

x /; $x \in \mathbb{R}$ (or (m, n) /; $m \in \mathbb{R} \land n \in \mathbb{R}$)	**Floor**	**Round**
Floor		$\lfloor x \rfloor == \left\lfloor x - \frac{1}{2} \right\rceil$ /; $\frac{x+1}{2} \notin \mathbb{Z}$ $\lfloor x \rfloor == \left\lfloor x - \frac{1}{2} \right\rceil + 1$ /; $\frac{x+1}{2} \in \mathbb{Z}$ $\lfloor x \rfloor == \left\lfloor x - \frac{1}{2} \right\rceil + \chi_{\mathbb{Z}}\left(\frac{x+1}{2}\right)$
Round	$\lfloor x \rceil == \left\lfloor x + \frac{1}{2} \right\rfloor$ /; $\frac{2x-1}{4} \notin \mathbb{Z}$ $\lfloor x \rceil == \left\lfloor x - \frac{1}{2} \right\rfloor$ /; $\frac{2x-1}{4} \in \mathbb{Z}$ $\lfloor x \rceil == \left\lfloor x + \frac{1}{2} \right\rfloor - \chi_{\mathbb{Z}}\left(\frac{2x-1}{4}\right)$	
Ceiling	$\lceil x \rceil == \lfloor x \rfloor + 1$ /; $x \notin \mathbb{Z}$ $\lceil x \rceil == \lfloor x \rfloor$ /; $x \in \mathbb{Z}$ $\lceil x \rceil == \lfloor x \rfloor - \theta(\chi_{\mathbb{Z}}(x) - 1) + 1$	$\lceil x \rceil == \left\lfloor x + \frac{1}{2} \right\rceil$ /; $\frac{x+1}{2} \notin \mathbb{Z}$ $\lceil x \rceil == \left\lfloor x + \frac{1}{2} \right\rceil - 1$ /; $\frac{x+1}{2} \in \mathbb{Z}$ $\lceil x \rceil == \left\lfloor x + \frac{1}{2} \right\rceil - \chi_{\mathbb{Z}}\left(\frac{x+1}{2}\right)$
IntegerPart	$\text{int}(x) == \lfloor x \rfloor$ /; $x > 0 \lor x \in \mathbb{Z}$ $\text{int}(x) == \lfloor x \rfloor + 1$ /; $x < 0 \bigwedge x \notin \mathbb{Z}$ $\text{int}(x) == \lfloor x \rfloor + 1 - \text{sgn}(\chi_{\mathbb{Z}}(x) + \theta(x))$	$\text{int}(x) == \left\lfloor x - \frac{1}{2} \right\rceil$ /; $x \geq 0 \bigwedge \frac{x+1}{2} \notin \mathbb{Z}$ $\text{int}(x) == \left\lfloor x - \frac{1}{2} \right\rceil + 1$ /; $x < 0 \bigvee \frac{x+1}{2} \in \mathbb{Z}$ $\text{int}(x) == \left\lfloor x - \frac{1}{2} \right\rceil + 1 + \chi_{\mathbb{Z}}\left(\frac{x+1}{2}\right) - \text{sgn}(\chi_{\mathbb{Z}}(x) + \theta(x))$
FractionalPart	$\text{frac}(x) == x - \lfloor x \rfloor$ /; $x > 0 \lor x \in \mathbb{Z}$ $\text{frac}(x) == x - \lfloor x \rfloor - 1$ /; $x < 0 \bigwedge x \notin \mathbb{Z}$ $\text{frac}(x) == x - \lfloor x \rfloor - 1 + \text{sgn}(\chi_{\mathbb{Z}}(x) + \theta(x))$	$\text{frac}(x) == x - \left\lfloor x - \frac{1}{2} \right\rceil$ /; $x \geq 0 \bigwedge \frac{x+1}{2} \notin \mathbb{Z}$ $\text{frac}(x) == x - \left\lfloor x - \frac{1}{2} \right\rceil - 1$ /; $x < 0 \bigvee \frac{x+1}{2} \in \mathbb{Z}$ $\text{frac}(x) == x - \left\lfloor x - \frac{1}{2} \right\rceil - 1 - \chi_{\mathbb{Z}}\left(\frac{x+1}{2}\right) + \text{sgn}(\chi_{\mathbb{Z}}(x) + \theta(x$
Mod	$m \bmod n == m - n \left\lfloor \frac{m}{n} \right\rfloor$	$m \bmod n == m - n \left\lfloor \frac{m}{n} - \frac{1}{2} \right\rceil$ /; $\frac{m+n}{2n} \notin \mathbb{Z}$ $m \bmod n == m - n - n \left\lfloor \frac{m}{n} - \frac{1}{2} \right\rceil$ /; $\frac{m+n}{2n} \in \mathbb{Z}$ $m \bmod n == m - n\left(\chi_{\mathbb{Z}}\left(\frac{m+n}{2n}\right) + \left\lfloor \frac{m}{n} - \frac{1}{2} \right\rceil\right)$
Quotient	$\text{quotient}(m, n) == \left\lfloor \frac{m}{n} \right\rfloor$	$\text{quotient}(m, n) == \left\lfloor \frac{m}{n} - \frac{1}{2} \right\rceil$ /; $\frac{m+n}{2n} \notin \mathbb{Z}$ $\text{quotient}(m, n) == \left\lfloor \frac{m}{n} - \frac{1}{2} \right\rceil + 1$ /; $\frac{m+n}{2n} \in \mathbb{Z}$ $\text{quotient}(m, n) == \left\lfloor \frac{m}{n} - \frac{1}{2} \right\rceil + \chi_{\mathbb{Z}}\left(\frac{m+n}{2n}\right)$

x /; $x \in \mathbb{R}$ **(or (m, n) /; $m \in \mathbb{R} \land n \in \mathbb{R}$)**	**Mod**	**Quotient**
Floor	$\lfloor x \rfloor = x - x \bmod 1$	$\lfloor x \rfloor = \text{quotient}(x, 1)$
Round	$\lfloor x \rceil = x - \text{frac}\left(x + \frac{1}{2}\right) + \frac{1}{2}$ /; $\frac{2x-1}{4} \notin \mathbb{Z}$ $\lfloor x \rceil = x - \text{frac}\left(x + \frac{1}{2}\right) - \frac{1}{2}$ /; $\frac{2x-1}{4} \in \mathbb{Z}$ $\lfloor x \rceil = x - \left(x + \frac{1}{2}\right) \bmod 1 + \frac{1}{2} - \chi_{\mathbb{Z}}\left(\frac{2x-1}{4}\right)$	$\lfloor x \rceil = \text{quotient}\left(x + \frac{1}{2}, 1\right)$ /; $\frac{2x-1}{4} \notin \mathbb{Z}$ $\lfloor x \rceil = \text{quotient}\left(x + \frac{1}{2}, 1\right) - 1$ /; $\frac{2x-1}{4} \in \mathbb{Z}$ $\lfloor x \rceil = \text{quotient}\left(x + \frac{1}{2}, 1\right) - \chi_{\mathbb{Z}}\left(\frac{2x-1}{4}\right)$
Ceiling	$\lceil x \rceil = x + -x \bmod 1$	$\lceil x \rceil = -\text{quotient}(-x, 1)$
IntegerPart	$\text{int}(x) = x - x \bmod 1$ /; $x > 0 \lor x \in \mathbb{Z}$ $\text{int}(x) = x + 1 - x \bmod 1$ /; $x < 0 \bigwedge x \notin \mathbb{Z}$ $\text{int}(x) = x + 1 - x \bmod 1 - \text{sgn}(\chi_{\mathbb{Z}}(x) + \theta(x))$	$\text{int}(x) = \text{quotient}(x, 1)$ /; $x > 0 \lor x \in \mathbb{Z}$ $\text{int}(x) = \text{quotient}(x, 1) + 1$ /; $x < 0 \bigwedge x \notin \mathbb{Z}$ $\text{int}(x) = \text{quotient}(x, 1) + 1 - \text{sgn}(\chi_{\mathbb{Z}}(x) + \theta(x))$
FractionalPart	$\text{frac}(x) = x \bmod 1$ /; $x > 0 \lor x \in \mathbb{Z}$ $\text{frac}(x) = x \bmod 1 - 1$ /; $x < 0 \bigwedge x \notin \mathbb{Z}$ $\text{frac}(x) = x \bmod 1 - 1 + \text{sgn}(\chi_{\mathbb{Z}}(x) + \theta(x))$	$\text{frac}(x) = x - \text{quotient}(x, 1)$ /; $x > 0 \lor x \in \mathbb{Z}$ $\text{frac}(x) = x - \text{quotient}(x, 1) - 1$ /; $x < 0 \bigwedge x \notin \mathbb{Z}$ $\text{frac}(x) = x - \text{quotient}(x, 1) - 1 + \text{sgn}(\chi_{\mathbb{Z}}(x) + \theta(x)$
Mod		$m \bmod n = m - n\,\text{quotient}(m, n)$
Quotient	$\text{quotient}(m, n) = \frac{m - m \bmod n}{n}$	

z (or (m, n))	**Floor**	**Round**	**Ceiling**
Floor		$\left\lfloor z - \frac{1+i}{2} \right\rceil + \chi_{\mathbb{Z}}\left(\frac{\mathrm{Re}(z)+1}{2}\right) + i\,\chi_{\mathbb{Z}}\left(\frac{\mathrm{Im}(z)+1}{2}\right)$	$\lfloor z \rfloor == \lceil z \rceil + \theta(\chi_{\mathbb{Z}}(\mathrm{Re}(z)) - 1) - i\,\theta(-\chi_{\mathbb{Z}}(\mathrm{Im}(z))) - 1$ $\lfloor z \rfloor == -\lceil -z \rceil$
Round	$\lfloor z \rceil == \left\lfloor z + \frac{1+i}{2} \right\rfloor - i\,\chi_{\mathbb{Z}}\left(\frac{2\,\mathrm{Im}(z)-1}{4}\right) - \chi_{\mathbb{Z}}\left(\frac{2\,\mathrm{Re}(z)-1}{4}\right)$		$\lfloor z \rceil == \left\lceil z - \frac{1+i}{2} \right\rceil + \chi_{\mathbb{Z}}\left(\frac{2\,\mathrm{Re}(z)+1}{4}\right) + i\,\chi_{\mathbb{Z}}\left(\frac{2\,\mathrm{Im}(z)+1}{4}\right)$
Ceiling	$\lceil z \rceil == \lfloor z \rfloor - \theta(\chi_{\mathbb{Z}}(\mathrm{Re}(z)) - 1) + i\,\theta(-\chi_{\mathbb{Z}}(\mathrm{Im}(z))) + 1$ $\lceil z \rceil == -\lfloor -z \rfloor$	$\lceil z \rceil == \left\lfloor z + \frac{1+i}{2} \right\rceil - \chi_{\mathbb{Z}}\left(\frac{\mathrm{Re}(z)+1}{2}\right) - i\,\chi_{\mathbb{Z}}\left(\frac{\mathrm{Im}(z)+1}{2}\right)$	
IntegerPart	$\mathrm{int}(z) == \lfloor z \rfloor + 1 + i - \mathrm{sgn}(\chi_{\mathbb{Z}}(\mathrm{Re}(z)) + \theta(\mathrm{Re}(z))) - i\,\mathrm{sgn}(\chi_{\mathbb{Z}}(\mathrm{Im}(z)) + \theta(\mathrm{Im}(z)))$	$\mathrm{int}(z) == \left\lfloor z - \frac{1+i}{2} \right\rceil + 1 + i + \chi_{\mathbb{Z}}\left(\frac{\mathrm{Re}(z)+1}{2}\right) + i\,\chi_{\mathbb{Z}}\left(\frac{\mathrm{Im}(z)+1}{2}\right) - \mathrm{sgn}(\chi_{\mathbb{Z}}(\mathrm{Re}(z)) + \theta(\mathrm{Re}(z))) - i\,\mathrm{sgn}(\chi_{\mathbb{Z}}(\mathrm{Im}(z)) + \theta(\mathrm{Im}(z)))$	$\mathrm{int}(z) == \lceil z \rceil - \mathrm{sgn}(\chi_{\mathbb{Z}}(\mathrm{Re}(z)) + \theta(\mathrm{Re}(z))) + i\,(-\mathrm{sgn}(\chi_{\mathbb{Z}}(\mathrm{Im}(z)) + \theta(\mathrm{Im}(z))) - \theta(-\chi_{\mathbb{Z}}(\mathrm{Im}(z))) + \theta(\chi_{\mathbb{Z}}(\mathrm{Re}(z)) - 1) + 1)$
FractionalPart	$\mathrm{frac}(z) == z - \lfloor z \rfloor - 1 - i + \mathrm{sgn}(\chi_{\mathbb{Z}}(\mathrm{Re}(z)) + \theta(\mathrm{Re}(z))) + i\,\mathrm{sgn}(\chi_{\mathbb{Z}}(\mathrm{Im}(z)) + \theta(\mathrm{Im}(z)))$	$\mathrm{frac}(z) == z - \left\lfloor z - \frac{1+i}{2} \right\rceil - 1 - i - \chi_{\mathbb{Z}}\left(\frac{\mathrm{Re}(z)+1}{2}\right) - i\,\chi_{\mathbb{Z}}\left(\frac{\mathrm{Im}(z)+1}{2}\right) + \mathrm{sgn}(\chi_{\mathbb{Z}}(\mathrm{Re}(z)) + \theta(\mathrm{Re}(z))) + i\,\mathrm{sgn}(\chi_{\mathbb{Z}}(\mathrm{Im}(z)) + \theta(\mathrm{Im}(z)))$	$\mathrm{frac}(z) == z - \lceil z \rceil + \mathrm{sgn}(\chi_{\mathbb{Z}}(\mathrm{Re}(z)) + \theta(\mathrm{Re}(z))) - i\,(1 - \mathrm{sgn}(\chi_{\mathbb{Z}}(\mathrm{Im}(z)) + \theta(\mathrm{Im}(z))) - \theta(-\chi_{\mathbb{Z}}(\mathrm{Im}(z))) + \theta(\chi_{\mathbb{Z}}(\mathrm{Re}(z)) - 1))$
Mod	$m \bmod n == m - n\left\lfloor \frac{m}{n} \right\rfloor$	$m \bmod n == m + n\left(\left\lfloor \frac{1+i}{2} - \frac{m}{n} \right\rceil - \chi_{\mathbb{Z}}\left(\frac{1}{2}\left(\mathrm{Re}\left(\frac{m}{n}\right) + 1\right)\right) - i\,\chi_{\mathbb{Z}}\left(\frac{1}{2}\left(\mathrm{Im}\left(\frac{m}{n}\right) + 1\right)\right)\right)$	$m \bmod n == m + n - n\left\lceil \frac{m}{n} \right\rceil - n\,\theta\left(\chi_{\mathbb{Z}}\left(\mathrm{Re}\left(\frac{m}{n}\right)\right) - 1\right) + n\,i\,\theta\left(-\chi_{\mathbb{Z}}\left(\mathrm{Im}\left(\right.\right.\right.$ $m \bmod n == m + n\left\lceil -\frac{m}{n} \right\rceil$
Quotient	$\mathrm{quotient}(m, n) == \left\lfloor \frac{m}{n} \right\rfloor$	$\mathrm{quotient}(m, n) == \left\lfloor \frac{m}{n} - \frac{1+i}{2} \right\rceil + \chi_{\mathbb{Z}}\left(\frac{1}{2}\left(\mathrm{Re}\left(\frac{m}{n}\right) + 1\right)\right) + i\,\chi_{\mathbb{Z}}\left(\frac{1}{2}\left(\mathrm{Im}\left(\frac{m}{n}\right) + 1\right)\right)$	$\mathrm{quotient}(m, n) == \left\lceil \frac{m}{n} \right\rceil + \theta\left(\chi_{\mathbb{Z}}\left(\mathrm{Re}\left(\frac{m}{n}\right)\right) - 1\right.$ $i\,\theta\left(-\chi_{\mathbb{Z}}\left(\mathrm{Im}\left(\frac{m}{n}\right)\right)\right) - 1$ $\mathrm{quotient}(m, n) == -\left\lceil -\frac{m}{n} \right\rceil$

z (or (m, n))	**Mod**	**Quotient**
Floor	$\lfloor z \rfloor == z - z \bmod 1$	$\lfloor z \rfloor == \mathrm{quotient}(z, 1)$
Round	$\lfloor z \rceil == \frac{1+i}{2} + z - \left(\frac{1+i}{2} + z\right) \bmod 1 - i\,\chi_{\mathbb{Z}}\left(\frac{2\,\mathrm{Im}(z)-1}{4}\right) - \chi_{\mathbb{Z}}\left(\frac{2\,\mathrm{Re}(z)-1}{4}\right)$	$\lfloor z \rceil == \mathrm{quotient}\left(z + \frac{1+i}{2}, 1\right) - \chi_{\mathbb{Z}}\left(\frac{2\,\mathrm{Re}(z)-1}{4}\right) - i\,\chi_{\mathbb{Z}}$
Ceiling	$\lceil z \rceil == z + -z \bmod 1$	$\lceil z \rceil == -\mathrm{quotient}(-z, 1)$
IntegerPart	$\mathrm{int}(z) == z + 1 + i - z \bmod 1 - i\,\mathrm{sgn}(\chi_{\mathbb{Z}}(\mathrm{Im}(z)) + \theta(\mathrm{Im}(z))) - \mathrm{sgn}(\chi_{\mathbb{Z}}(\mathrm{Re}(z)) + \theta(\mathrm{Re}(z)))$	$\mathrm{int}(z) == \mathrm{quotient}(z, 1) + 1 + i - i\,\mathrm{sgn}(\chi_{\mathbb{Z}}(\mathrm{Im}(z)$ $\theta(\mathrm{Im}(z))) - \mathrm{sgn}(\chi_{\mathbb{Z}}(\mathrm{Re}(z)) + \theta(\mathrm{Re}(z)))$
FractionalPart	$\mathrm{frac}(z) == z \bmod 1 - 1 - i + i\,\mathrm{sgn}(\chi_{\mathbb{Z}}(\mathrm{Im}(z)) + \theta(\mathrm{Im}(z))) + \mathrm{sgn}(\chi_{\mathbb{Z}}(\mathrm{Re}(z)) + \theta(\mathrm{Re}(z)))$	$\mathrm{frac}(z) == z - \mathrm{quotient}(z, 1) - 1 - i + i\,\mathrm{sgn}(\chi_{\mathbb{Z}}(\mathrm{I}$ $\mathrm{sgn}(\chi_{\mathbb{Z}}(\mathrm{Re}(z)) + \theta(\mathrm{Re}(z)))$
Mod		$m \bmod n == m - n\,\mathrm{quotient}(m, n)$
Quotient	$\mathrm{quotient}(m, n) == \frac{m - m \bmod n}{n}$	

The rounding and congruence functions $\lfloor z \rfloor$, $\lceil z \rceil$, int(z), frac(z), $m \bmod n$, and quotient(m, n) can also be represented through elementary functions by the following formulas:

$$\lfloor z \rfloor = z + \frac{\tan^{-1}(\cot(\pi z))}{\pi} - \frac{1}{2} \;/;\; z \in \mathbb{R} \bigwedge z \notin \mathbb{Z}$$

$$\lceil z \rceil = z + \frac{\tan^{-1}(\cot(\pi z))}{\pi} + \frac{1}{2} \;/;\; z \in \mathbb{R} \bigwedge z \notin \mathbb{Z}$$

$$\operatorname{int}(z) = z + \frac{\tan^{-1}(\cot(\pi z))}{\pi} - \operatorname{sgn}(\theta(z)) + \frac{1}{2} \;/;\; z \in \mathbb{R} \bigwedge z \notin \mathbb{Z}$$

$$\operatorname{frac}(z) = -\frac{\tan^{-1}(\cot(\pi z))}{\pi} + \operatorname{sgn}(\theta(z)) - \frac{1}{2} \;/;\; z \in \mathbb{R} \bigwedge z \notin \mathbb{Z}$$

$$m \bmod n = \frac{n}{2} - \frac{n}{\pi} \tan^{-1}\left(\cot\left(\frac{\pi m}{n}\right)\right) \;/;\; \frac{m}{n} \in \mathbb{R} \bigwedge \frac{m}{n} \notin \mathbb{Z}$$

$$\operatorname{quotient}(m, n) = \frac{m}{n} + \frac{1}{\pi} \tan^{-1}\left(\cot\left(\frac{\pi m}{n}\right)\right) - \frac{1}{2} \;/;\; \frac{m}{n} \in \mathbb{R} \bigwedge \frac{m}{n} \notin \mathbb{Z}.$$

The best-known properties and formulas of the number theory functions

Simple values at zero

The rounding and congruence functions $\lfloor z \rfloor$, $\lfloor z \rceil$, $\lceil z \rceil$, int(z), and frac(z) have zero values at zero:

$$\lfloor 0 \rfloor = 0$$

$$\lfloor 0 \rceil = 0$$

$$\lceil 0 \rceil = 0$$

$$\operatorname{int}(0) = 0$$

$$\operatorname{frac}(0) = 0.$$

Specific values for specialized variables

The values of five rounding and congruence functions $\lfloor z \rfloor$, $\lfloor z \rceil$, $\lceil z \rceil$, int(z), and frac(z) at some fixed points or for specialized variables and infinities are shown in the following table:

z	$\lfloor z \rfloor$	$\lfloor z \rceil$	$\lceil z \rceil$	$\mathrm{int}(z)$	$\mathrm{frac}(z)$
0	0	0	0	0	0
1	1	1	1	1	0
-1	-1	-1	-1	-1	0
i	i	i	i	i	0
$-i$	$-i$	$-i$	$-i$	$-i$	0
$\frac{1}{2}$	0	0	1	0	$\frac{1}{2}$
$-\frac{1}{2}$	-1	0	0	0	$-\frac{1}{2}$
$\frac{i}{2}$	0	0	i	0	$\frac{i}{2}$
$-\frac{i}{2}$	$-i$	0	0	0	$-\frac{i}{2}$
$\frac{3}{2}$	1	2	2	1	$\frac{1}{2}$
$-\frac{3}{2}$	-2	-2	-1	-1	$-\frac{1}{2}$
$\frac{3i}{2}$	i	$2i$	$2i$	i	$\frac{i}{2}$
$-\frac{3i}{2}$	$-2i$	$-2i$	$-i$	$-i$	$-\frac{i}{2}$
$\frac{23}{10}$	2	2	3	2	$\frac{3}{10}$
-3	-3	-3	-3	-3	0
$-\pi$	-4	-3	-3	-3	$3-\pi$
$-\frac{27}{10}$	-3	-3	-2	-2	$-\frac{7}{10}$
-3.4	-4	-3	-3	-3	$-0, 4$
$\frac{23}{10} - i e$	$2-3i$	$2-3i$	$3-2i$	$2-2i$	$\frac{3}{10}-(e-2)\,i$
∞	∞	∞	∞	∞	$\mathrm{frac}(\infty) \in (0, 1)$
$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$\mathrm{frac}(-\infty) \in (-1, 0)$
$i\infty$	$i\infty$	$i\infty$	$i\infty$	$i\infty$	$\mathrm{frac}(i\infty) \in (0, i)$
$-i\infty$	$-i\infty$	$-i\infty$	$-i\infty$	$-i\infty$	$\mathrm{frac}(-i\infty) \in (-i, 0)$
$\tilde{\infty}$	$\tilde{\infty}$	$\tilde{\infty}$	$\tilde{\infty}$	$\tilde{\infty}$	$\mathrm{frac}(\tilde{\infty}) \in (0, 1)$
$n /; n \in \mathbb{Z}$	$\lfloor n \rfloor == n$	$\lfloor n \rceil == n$	$\lceil n \rceil == n$	$\mathrm{int}(n) == n$	$\mathrm{frac}(n) == 0$
$i n /; n \in \mathbb{Z}$	$\lfloor i n \rfloor == i n$	$\lfloor i n \rceil == i n$	$\lceil i n \rceil == n$	$\mathrm{int}(i n) == i n$	$\mathrm{frac}(i n) == 0$
$x + i y /; x \in \mathbb{R} \land y \in \mathbb{R}$	$\lfloor x + i y \rfloor == \lfloor x \rfloor + i \lfloor y \rfloor$	$\lfloor x + i y \rceil == \lfloor x \rceil + i \lfloor y \rceil$	$\lceil x + i y \rceil == \lceil x \rceil + i \lceil y \rceil$	$\mathrm{int}(x + i y) == \mathrm{int}(x) + i\,\mathrm{int}(y)$	$\mathrm{frac}(x + i y) == \mathrm{frac}(x) + i\,\mathrm{frac}(y)$
∞	∞	∞	∞	∞	$\mathrm{frac}(\infty) \in (0, 1)$
$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$\mathrm{frac}(-\infty) \in (-1, 0)$
$i\infty$	$i\infty$	$i\infty$	$i\infty$	$i\infty$	$\mathrm{frac}(i\infty) \in (0, i)$
$-i\infty$	$-i\infty$	$-i\infty$	$-i\infty$	$-i\infty$	$\mathrm{frac}(-i\infty) \in (-i, 0)$
$\tilde{\infty}$	$\tilde{\infty}$	$\tilde{\infty}$	$\tilde{\infty}$	$\tilde{\infty}$	$\mathrm{frac}(\tilde{\infty}) \in (0, 1)$

The values of mod function $m \bmod n$, and $\mathrm{quotient}(m, n)$ at some fixed points or for specialized variables are shown here:

$m \setminus n$	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	n
0	0	0	0	0	0	0	0	0	0	0	$0 \bmod n = 0 /; n \neq 0$
1	0	1	1	1	1	1	1	1	1	1	$1 \bmod n = n+1 /; -n \in \mathbb{N}^+$ $1 \bmod n = 1 /; n \in \mathbb{Z} \land n > 1$
2	0	0	2	2	2	2	2	2	2	2	$2 \bmod n = 2 /; n \in \mathbb{Z} \land n > 2$
3	0	1	0	3	3	3	3	3	3	3	$3 \bmod n = 3 /; n \in \mathbb{Z} \land n > 3$
4	0	0	1	0	4	4	4	4	4	4	$4 \bmod n = 4 /; n \in \mathbb{Z} \land n > 4$
5	0	1	2	1	0	5	5	5	5	5	$5 \bmod n = 5 /; n \in \mathbb{Z} \bigwedge n > 5$
6	0	0	0	2	1	0	6	6	6	6	$6 \bmod n = 6 /; n \in \mathbb{Z} \land n > 6$
7	0	1	1	3	2	1	0	7	7	7	$7 \bmod n = 7 /; n \in \mathbb{Z} \land n > 7$
8	0	0	2	0	3	2	1	0	8	8	$8 \bmod n = 8 /; n \in \mathbb{Z} \land n > 8$
9	0	1	0	1	4	3	2	1	0	9	$9 \bmod n = 9 /; n \in \mathbb{Z} \land n > 9$
10	0	0	1	2	0	4	3	2	1	0	$10 \bmod n = 10 /; n \in \mathbb{Z} \land n > 10$
m	$m \bmod 1 = 0 /;$ $m \in \mathbb{Z}$										$n \bmod n = 0$
m											$(2n) \bmod n = 0$

$m \setminus n$	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	n
0	0	0	0	0	0	0	0	0	0	0	$\text{quotient}(0, n) = 0 /; n \neq 0$
1	1	0	0	0	0	0	0	0	0	0	$\text{quotient}(1, n) = -1 /; n \in \mathbb{Z} \land n < 0$ $\text{quotient}(1, n) = 0 /; n \in \mathbb{Z} \land n > 1$
2	2	1	0	0	0	0	0	0	0	0	$\text{quotient}(2, n) = 0 /; n \in \mathbb{Z} \land n > 2$
3	3	1	1	0	0	0	0	0	0	0	$\text{quotient}(3, n) = 0 /; n \in \mathbb{Z} \land n > 3$
4	4	2	1	1	0	0	0	0	0	0	$\text{quotient}(4, n) = 0 /; n \in \mathbb{Z} \land n > 4$
5	5	2	1	1	1	0	0	0	0	0	$\text{quotient}(5, n) = 0 /; n \in \mathbb{Z} \bigwedge n > 5$
6	6	3	2	1	1	1	0	0	0	0	$\text{quotient}(6, n) = 0 /; n \in \mathbb{Z} \land n > 6$
7	7	3	2	1	1	1	1	0	0	0	$\text{quotient}(7, n) = 0 /; n \in \mathbb{Z} \land n > 7$
8	8	4	2	2	1	1	1	1	0	0	$\text{quotient}(8, n) = 0 /; n \in \mathbb{Z} \land n > 8$
9	9	4	3	2	1	1	1	1	1	0	$\text{quotient}(9, n) = 0 /; n \in \mathbb{Z} \land n > 9$
10	10	5	3	2	2	1	1	1	1	1	$\text{quotient}(10, n) = 0 /; n \in \mathbb{Z} \land n > 10$
m	$\text{quotient}(m, 1) =$ $m /; m \in \mathbb{Z}$										$\text{quotient}(n, n) = 1$
m											$\text{quotient}(2n, n) = 2$

$$m \bmod n = m /; m \in \mathbb{N} \land n \in \mathbb{Z} \land m < n$$

$$m \bmod n = m - n /; m \in \mathbb{N}^+ \bigwedge n \in \mathbb{N}^+ \bigwedge n \leq m < 2n$$

$$m \bmod n = m - k\, n /; m \in \mathbb{N}^+ \bigwedge n \in \mathbb{N}^+ \bigwedge k \in \mathbb{N}^+ \bigwedge k\, n \leq m < (k+1)\, n$$

$$(p-1)! \bmod p = p - 1 /; p \in \mathbb{P}$$

$$\binom{2p-1}{p-1} \bmod p^3 = 1 /; p \in \mathbb{P} \bigwedge p > 3$$

$$|B_{2n}| \bmod 1 = \delta_{\frac{n+1}{2} \bmod 1, 0} + (-1)^n \left(\left(\sum_{k=3}^{2n+1} \frac{1}{k} \chi_{\mathbb{Z}}\left(\frac{2n}{k-1} \right) \chi_{\mathbb{P}}(k) + \frac{1}{2} \right) \bmod 1 \right)$$

$$\text{quotient}(m, n) = 0 /; m \in \mathbb{N} \bigwedge n \in \mathbb{N}^+ \bigwedge m < n$$

$\text{quotient}(m, n) = 1 /; m \in \mathbb{N}^+ \bigwedge n \in \mathbb{N}^+ \bigwedge n \leq m < 2n$

$\text{quotient}(m, n) = k /; m \in \mathbb{N}^+ \bigwedge n \in \mathbb{N}^+ \bigwedge k \in \mathbb{N}^+ \bigwedge k n \leq m < (k+1) n$

$$\text{quotient}((p-1)!, p) = \frac{(p-1)! + 1 - p}{p} /; p \in \mathbb{P}$$

$$\text{quotient}\left(\binom{2p-1}{p-1}, p^3\right) = \frac{1}{p^3}\left(\binom{2p-1}{p-1} - 1\right) /; p \in \mathbb{P} \bigwedge p > 3.$$

Analyticity

All seven rounding and congruence functions (floor function $\lfloor z \rfloor$, round function $\lfloor z \rceil$, ceiling function $\lceil z \rceil$, integer part $\text{int}(z)$, fractional part $\text{frac}(z)$, mod function $m \bmod n$, and the quotient function $\text{quotient}(m, n)$) are not analytical functions. They are defined for all complex values of their arguments $z \in \mathbb{C}$ and $(m, n) \in \mathbb{C}^2$. The functions $\lfloor z \rfloor$, $\lfloor z \rceil$, $\lceil z \rceil$, and $\text{int}(z)$ are piecewise constant functions and the functions $\text{frac}(z)$, $m \bmod n$, and $\text{quotient}(m, n)$ are piecewise continuous functions.

Periodicity

The rounding and congruence functions $\lfloor z \rfloor$, $\lfloor z \rceil$, $\lceil z \rceil$, $\text{int}(z)$, $\text{frac}(z)$, and $\text{quotient}(m, n)$ are not periodic functions.

$m \bmod n$ is a periodic function with respect to m with period n:

$(m+n) \bmod n = m \bmod n$

$(m + k n) \bmod n = m \bmod n /; k \in \mathbb{Z}.$

Parity and symmetry

Four rounding and congruence functions (round function $\lfloor z \rceil$, integer part $\text{int}(z)$, fractional part $\text{frac}(z)$, and mod function $m \bmod n$) are odd functions. The quotient function $\text{quotient}(m, n)$ is an even function:

$\lfloor -z \rceil = -\lfloor z \rceil$

$\text{int}(-z) = -\text{int}(z)$

$\text{frac}(-z) = -\text{frac}(z)$

$-m \bmod -n = -(m \bmod n)$

$\text{quotient}(-m, -n) = \text{quotient}(m, n).$

The rounding and congruence functions $\lfloor z \rfloor$, $\lfloor z \rceil$, $\lceil z \rceil$, $\text{int}(z)$, and $\text{frac}(z)$ have the following mirror symmetry:

$\lfloor \bar{z} \rfloor = \overline{\lfloor z \rceil} - i\,(1 - \chi_{\mathbb{Z}}(\text{Im}(z)))$

$\lfloor \bar{z} \rceil = \overline{\lfloor z \rceil}$

$\lceil \bar{z} \rceil = \overline{\lceil z \rceil} + i\,(1 - \chi_{\mathbb{Z}}(\text{Im}(z)))$

$\text{int}(\bar{z}) = \overline{\text{int}(z)}$

$\text{frac}(\bar{z}) = \overline{\text{frac}(z)}.$

Sets of discontinuity

The floor and ceiling functions $\lfloor z \rfloor$ and $\lceil z \rceil$ are piecewise constant functions with unit jumps on the lines $\operatorname{Re}(z) = k \bigvee \operatorname{Im}(z) = l /; k, l \in \mathbb{Z}$.

The functions $\lfloor z \rfloor$ (and $\lceil z \rceil$) are continuous from the right (from the left) on the intervals $(k - i\infty, k + i\infty)$, $k \in \mathbb{Z}$, and from above (from below) on the intervals $(i k - \infty, i k + \infty)$, $k \in \mathbb{Z}$.

The function $\lfloor z \rceil$ is a piecewise constant function with unit jumps on the lines $\operatorname{Re}(z) + \frac{1}{2} = k \bigvee \operatorname{Im}(z) + \frac{1}{2} = l /; k, l \in \mathbb{Z}$.
The function $\lfloor z \rceil$ is continuous from the right on the intervals $\left(2k - \frac{1}{2} - i\infty, 2k - \frac{1}{2} + i\infty\right)$, $k \in \mathbb{Z}$, and from the left on the intervals $\left(2k + \frac{1}{2} - i\infty, 2k + \frac{1}{2} + i\infty\right)$, $k \in \mathbb{Z}$.
The function $\lfloor z \rceil$ is continuous from above on the intervals $\left(-\infty + 2 i k - \frac{i}{2}, \infty + 2 i k - \frac{i}{2}\right)$, $k \in \mathbb{Z}$, and from below on the intervals $\left(-\infty + 2 i k + \frac{i}{2}, \infty + 2 i k + \frac{i}{2}\right)$, $k \in \mathbb{Z}$.

The function $\operatorname{int}(z)$ (and $\operatorname{frac}(z)$) is a piecewise constant (continuous) function with unit jumps on the lines $\operatorname{Re}(z) = k \bigvee \operatorname{Im}(z) = l /; k, l \in \mathbb{Z}, k \neq 0, l \neq 0$.

The functions $\operatorname{int}(z)$ and $\operatorname{frac}(z)$ are continuous from the right on the intervals $(k - i\infty, k + i\infty)$, $k \in \mathbb{N}^+$, and from the left on the intervals $(-k - i\infty, -k + i\infty)$, $k \in \mathbb{N}^+$.

The functions $\operatorname{int}(z)$ and $\operatorname{frac}(z)$ are continuous from above on the intervals $(-\infty + i k, \infty + i k)$, $k \in \mathbb{N}^+$, and from below on the intervals $(-\infty - i k, \infty - i k)$, $k \in \mathbb{N}^+$.

The functions $m \bmod n$ and $\operatorname{quotient}(m, n)$ are piecewise continuous functions with jumps on the curves $\operatorname{Re}\left(\frac{m}{n}\right) = k \bigvee \operatorname{Im}\left(\frac{m}{n}\right) = l /; k, l \in \mathbb{Z}$. The functional properties $m \bmod n = n\left(\frac{m}{n} \bmod 1\right) = n\left(\frac{m}{n} - \left\lfloor \frac{m}{n} \right\rfloor\right)$ and $\operatorname{quotient}(m, n) = \operatorname{quotient}\left(\frac{m}{n}, 1\right) = \left\lfloor \frac{m}{n} \right\rfloor$ make the behavior of that functions similar to the behavior of floor function $\left\lfloor \frac{m}{n} \right\rfloor$.

The previous described properties can be described in more detail by the formulas from the following table:

	$z \pm \epsilon$ (or $m \pm \epsilon$) /; $\epsilon \to +0$	$z \pm i\, \epsilon$ (or $m \pm i\, \epsilon$) /; $\epsilon \to +0$
Floor	$\lim_{\epsilon \to +0} \lfloor z \pm \epsilon \rfloor == \lfloor z \rfloor - \frac{1 \mp 1}{2}$ /; $\operatorname{Re}(z) \in \mathbb{Z}$	$\lim_{\epsilon \to +0} \lfloor z \pm i\, \epsilon \rfloor == \lfloor z \rfloor - \frac{1 \mp 1}{2}\, i$ /; $\operatorname{Im}(z) \in \mathbb{Z}$
Round	$\lim_{\epsilon \to +0} \lfloor z \pm \epsilon \rceil == \lfloor z \rceil$ /; $\frac{1}{4}(2 \operatorname{Re}(z) \pm 1) \in \mathbb{Z}$ $\lim_{\epsilon \to +0} \lfloor z \pm \epsilon \rceil == \lfloor z \rceil \pm 1$ /; $\frac{1}{4} \operatorname{Re}(2 z \mp 1) \in \mathbb{Z}$	$\lim_{\epsilon \to +0} \lfloor z \pm i\, \epsilon \rceil == \lfloor z \rceil$ /; $\frac{1}{4}(2 \operatorname{Im}(z) \pm 1) \in \mathbb{Z}$ $\lim_{\epsilon \to +0} \lfloor z \pm i\, \epsilon \rceil == \lfloor z \rceil \pm i$ /; $\frac{1}{4}(2 \operatorname{Im}(z) \mp 1) \in \mathbb{Z}$
Ceiling	$\lim_{\epsilon \to +0} \lceil z \pm \epsilon \rceil == \lceil z \rceil + \frac{1 \pm 1}{2}$ /; $\operatorname{Re}(z) \in \mathbb{Z}$	$\lim_{\epsilon \to +0} \lceil z \pm i\, \epsilon \rceil == \lceil z \rceil + \frac{1 \pm 1}{2}\, i$ /; $\operatorname{Im}(z) \in \mathbb{Z}$
IntegerPart	$\lim_{\epsilon \to +0} \operatorname{int}(z \pm \epsilon) == \operatorname{int}(z)$ /; $\pm \operatorname{Re}(z) \in \mathbb{N}^+$ $\lim_{\epsilon \to +0} \operatorname{int}(z \pm \epsilon) == \operatorname{int}(z) \pm 1$ /; $\mp \operatorname{Re}(z) \in \mathbb{N}^+$	$\lim_{\epsilon \to +0} \operatorname{int}(z \pm i\, \epsilon) == \operatorname{int}(z)$ /; $\pm \operatorname{Im}(z) \in \mathbb{N}^+$ $\lim_{\epsilon \to +0} \operatorname{int}(z \pm i\, \epsilon) == \operatorname{int}(z) \pm i$ /; $\mp \operatorname{Im}(z) \in \mathbb{N}^+$
FractionalPart	$\lim_{\epsilon \to +0} \operatorname{frac}(z \pm \epsilon) == \operatorname{frac}(z)$ /; $\pm \operatorname{Re}(z) \in \mathbb{N}^+$ $\lim_{\epsilon \to +0} \operatorname{frac}(z \pm \epsilon) == \operatorname{frac}(z) \mp 1$ /; $\mp \operatorname{Re}(z) \in \mathbb{N}^+$	$\lim_{\epsilon \to +0} \operatorname{frac}(z \pm i\, \epsilon) == \operatorname{frac}(z)$ /; $\pm \operatorname{Im}(z) \in \mathbb{N}^+$ $\lim_{\epsilon \to +0} \operatorname{frac}(z \pm i\, \epsilon) == \operatorname{frac}(z) \mp i$ /; $\mp \operatorname{Im}(z) \in \mathbb{N}^+$
Mod	$\lim_{\epsilon \to +0} (m \pm \epsilon) \bmod n == m \bmod n + n\, \frac{1 \mp 1}{2}$ /; $\operatorname{Re}\left(\frac{m}{n}\right) \in \mathbb{Z} \bigwedge n \in \mathbb{R} \bigwedge n > 0$	$\lim_{\epsilon \to +0} (m \pm i\, \epsilon) \bmod n == m \bmod n + i\, n\, \frac{1 \mp 1}{2}$ /; $\operatorname{Im}\left(\frac{m}{n}\right) \in \mathbb{Z} \bigwedge n \in \mathbb{R} \bigwedge n > 0$
Quotient	$\lim_{\epsilon \to +0} \operatorname{quotient}(m \pm \epsilon, n) == \operatorname{quotient}(m, n) - \frac{1 \mp 1}{2}$ /; $\operatorname{Re}\left(\frac{m}{n}\right) \in \mathbb{Z} \bigwedge n \in \mathbb{R} \bigwedge n > 0$	$\lim_{\epsilon \to +0} \operatorname{quotient}(m \pm i\, \epsilon, n) == \operatorname{quotient}(m, n) - i\, \frac{1 \mp 1}{2}$ /; $\operatorname{Im}\left(\frac{m}{n}\right) \in \mathbb{Z} \bigwedge n \in \mathbb{R} \bigwedge n > 0$

Series representations

The rounding and congruence functions $\lfloor z \rfloor$, $\lfloor z \rceil$, $\lceil z \rceil$, int(z), frac(z), $m \bmod n$, and quotient(m, n) have the following series representations:

	x	$\frac{m}{n}$
Floor	$\lfloor x \rfloor == x - \frac{1}{2} + \frac{1}{\pi} \sum_{k=1}^{\infty} \frac{\sin(2 \pi k x)}{k}$ /; $x \in \mathbb{R} \wedge x \notin \mathbb{Z}$	$\left\lfloor \frac{m}{n} \right\rfloor == \frac{m}{n} + \frac{1}{2 n} \sum_{k=1}^{n-1} \sin\left(\frac{2 \pi k m}{n}\right) \cot\left(\frac{\pi k}{n}\right) - \frac{1}{2}$ /; $m \in \mathbb{Z}$ /
Round	$\lfloor x \rceil == x + \frac{1}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^k \sin(2 \pi k x)}{k}$ /; $x \in \mathbb{R} \bigwedge x + \frac{1}{2} \notin \mathbb{Z}$	$\left\lfloor \frac{m}{n} + \frac{1}{2} \right\rceil == \frac{m}{n} + \frac{1}{2 n} \sum_{k=1}^{n-1} \sin\left(\frac{2 \pi k m}{n}\right) \cot\left(\frac{\pi k}{n}\right) + \frac{1}{2}$ /; $m \in$
Ceiling	$\lceil x \rceil == x + \frac{1}{2} + \frac{1}{\pi} \sum_{k=1}^{\infty} \frac{\sin(2 \pi k x)}{k}$ /; $x \in \mathbb{R} \wedge x \notin \mathbb{Z}$	$\left\lceil \frac{m}{n} \right\rceil == \frac{m}{n} + \frac{1}{2 n} \sum_{k=1}^{n-1} \sin\left(\frac{2 \pi k m}{n}\right) \cot\left(\frac{\pi k}{n}\right) + \frac{1}{2}$ /; $m \in \mathbb{Z}$ /
IntegerPart	$\operatorname{int}(x) == x + \frac{1}{\pi} \sum_{k=1}^{\infty} \frac{\sin(2 \pi k x)}{k} - \theta(x) + \frac{1}{2}$ /; $x \in \mathbb{R} \wedge x \notin \mathbb{Z}$	$\operatorname{int}\left(\frac{m}{n}\right) == \frac{m}{n} - \frac{1}{2} \operatorname{sgn}\left(\frac{m}{n}\right) + \frac{1}{2 n} \sum_{k=1}^{n-1} \sin\left(\frac{2 \pi k m}{n}\right) \cot\left(\frac{\pi k}{n}\right)$
FractionalPart	$\operatorname{frac}(x) == -\frac{1}{\pi} \sum_{k=1}^{\infty} \frac{\sin(2 \pi k x)}{k} + \theta(x) - \frac{1}{2}$ /; $x \in \mathbb{R} \wedge x \notin \mathbb{Z}$	$\operatorname{frac}\left(\frac{m}{n}\right) == \frac{1}{2} \operatorname{sgn}\left(\frac{m}{n}\right) - \frac{1}{2 n} \sum_{k=1}^{n-1} \sin\left(\frac{2 \pi k m}{n}\right) \cot\left(\frac{\pi k}{n}\right)$ /; m
Mod		$m \bmod n == \frac{n}{2} - \frac{n}{\pi} \sum_{k=1}^{\infty} \frac{1}{k} \sin\left(\frac{2 \pi k m}{n}\right)$ /; $\frac{m}{n} \in \mathbb{Q} \bigwedge \frac{m}{n} \notin$ $m \bmod n == \frac{n}{2} - \frac{1}{2} \sum_{k=1}^{n-1} \sin\left(\frac{2 \pi k m}{n}\right) \cot\left(\frac{\pi k}{n}\right)$ /; $m \in \mathbb{Z} \bigwedge$
Quotient		$\operatorname{quotient}(m, n) == \frac{m}{n} + \frac{1}{\pi} \sum_{k=1}^{\infty} \frac{1}{k} \sin\left(\frac{2 \pi k m}{n}\right) - \frac{1}{2}$ /; $\frac{m}{n} \in$ $\operatorname{quotient}(m, n) == \frac{m}{n} + \frac{1}{2 n} \sum_{k=1}^{n-1} \sin\left(\frac{2 \pi k m}{n}\right) \cot\left(\frac{\pi k}{n}\right) - \frac{1}{2}$

Transformations and argument simplifications (arguments involving basic arithmetic operations)

The values of rounding and congruence functions $\lfloor z \rfloor$, $\lfloor z \rceil$, $\lceil z \rceil$, int(z), and frac(z) at the points $-z$, $\pm i\, z$, $z + n$ /; $n \in \mathbb{Z}$ can also be represented by the following formulas:

z	$-z$	$i\,z$	$-i\,z$	$z+n$
$\lfloor z\rfloor$	$\lfloor -z\rfloor = -\lfloor z\rfloor \,/;\, \operatorname{Re}(z)\in\mathbb{Z}\land\operatorname{Im}(z)\in\mathbb{Z}$ $\lfloor -z\rfloor = -\lfloor z\rfloor - \operatorname{sgn}(\lvert\operatorname{Re}(z)\rvert) - i\,\operatorname{sgn}(\lvert\operatorname{Im}(z)\rvert) \,/;$ $\operatorname{Re}(z)\notin\mathbb{Z}\bigwedge\operatorname{Im}(z)\notin\mathbb{Z}$ $\lfloor -z\rfloor = -\lfloor z\rfloor + \chi_{\mathbb{Z}}(z) - 1 \,/;\, z\in\mathbb{R}$ $\lfloor -z\rfloor = -\lfloor z\rfloor - i\,(1-\chi_{\mathbb{Z}}(\operatorname{Im}(z)))\operatorname{sgn}(\lvert\operatorname{Im}(z)\rvert) -$ $(1-\chi_{\mathbb{Z}}(\operatorname{Re}(z)))\operatorname{sgn}(\lvert\operatorname{Re}(z)\rvert)$	$\lfloor i\,z\rfloor = i\lfloor z\rfloor + \chi_{\mathbb{Z}}(\operatorname{Im}(z)) - 1$ $\lfloor i\,z\rfloor = \lfloor -\operatorname{Im}(z)\rfloor + i\lfloor\operatorname{Re}(z)\rfloor$	$\lfloor -i\,z\rfloor = -i\lfloor z\rfloor - i\,(1-\chi_{\mathbb{Z}}(\operatorname{Re}(z)))$ $\lfloor -i\,z\rfloor = \lfloor\operatorname{Im}(z)\rfloor + i\lfloor -\operatorname{Re}(z)\rfloor$	$\lfloor z+$
$\lfloor z\rceil$	$\lfloor -z\rceil = -\lfloor z\rceil$	$\lfloor i\,z\rceil = i\lfloor z\rceil$	$\lfloor -i\,z\rceil = -i\lfloor z\rceil$	$\lfloor z+$ Re(
$\lceil z\rceil$	$\lceil -z\rceil = -\lceil z\rceil \,/;\, \operatorname{Re}(z)\in\mathbb{Z}\land\operatorname{Im}(z)\in\mathbb{Z}$ $\lceil -z\rceil = -\lceil z\rceil + i\,\operatorname{sgn}(\lvert\operatorname{Im}(z)\rvert) + \operatorname{sgn}(\lvert\operatorname{Re}(z)\rvert) \,/;$ $\operatorname{Re}(z)\notin\mathbb{Z}\bigwedge\operatorname{Im}(z)\notin\mathbb{Z}$ $\lceil -z\rceil = -\lceil z\rceil + i\,(1-\chi_{\mathbb{Z}}(\operatorname{Im}(z)))\operatorname{sgn}(\lvert\operatorname{Im}(z)\rvert) +$ $(1-\chi_{\mathbb{Z}}(\operatorname{Re}(z)))\operatorname{sgn}(\lvert\operatorname{Re}(z)\rvert)$	$\lceil i\,z\rceil = i\lceil z\rceil - \chi_{\mathbb{Z}}(\operatorname{Im}(z)) + 1$ $\lceil i\,z\rceil = -\lfloor\operatorname{Im}(z)\rfloor + i\lceil\operatorname{Re}(z)\rceil$	$\lceil -i\,z\rceil = -i\lceil z\rceil + i\,(1-\chi_{\mathbb{Z}}(\operatorname{Re}(z)))$ $\lceil -i\,z\rceil = \lceil\operatorname{Im}(z)\rceil - i\lfloor\operatorname{Re}(z)\rfloor$	$\lceil z+$
$\operatorname{int}(z)$	$\operatorname{int}(-z) = -\operatorname{int}(z)$	$\operatorname{int}(i\,z) = i\operatorname{int}(z)$	$\operatorname{int}(-i\,z) = -i\operatorname{int}(z)$	int (n
$\operatorname{frac}(z)$	$\operatorname{frac}(-z) = -\operatorname{frac}(z)$	$\operatorname{frac}(i\,z) = i\operatorname{frac}(z)$	$\operatorname{frac}(-i\,z) = -i\operatorname{frac}(z)$	frac (

The values of the functions $m \bmod n$ and $\operatorname{quotient}(m, n)$ at the points $(\pm m, -n)$, $(-m, n)$, $(i\,m, i\,n)$, $(\pm i\,m, n)$, $(m, \pm i\,n)$, and $\left(\frac{m}{n}, 1\right)$ have the following representations:

(m, n)	$m \bmod n$	$\operatorname{quotient}(m, n)$
$(-m, -n)$	$(-m) \bmod (-n) = -(m \bmod n)$	$\operatorname{quotient}(-m, -n) = \operatorname{quotient}(m, n)$
$(m, -n)$	$m \bmod -n = m \bmod n + \chi_{\mathbb{Z}}\left(\frac{m}{n}\right) n - n \,/;\, m\in\mathbb{R}\land n\in\mathbb{R}$ $m \bmod -n = m \bmod n - n\left(1-\chi_{\mathbb{Z}}\left(\operatorname{Re}\left(\frac{m}{n}\right)\right)\right)\operatorname{sgn}\left(\left\lvert\operatorname{Re}\left(\frac{m}{n}\right)\right\rvert\right) -$ $i\,n\left(1-\chi_{\mathbb{Z}}\left(\operatorname{Im}\left(\frac{m}{n}\right)\right)\right)\operatorname{sgn}\left(\left\lvert\operatorname{Im}\left(\frac{m}{n}\right)\right\rvert\right)$	$\operatorname{quotient}(m, -n) = -\operatorname{quotient}(m, n) + \chi_{\mathbb{Z}}\left(\frac{m}{n}\right) - 1 \,/;\, m$ $\operatorname{quotient}(m, -n) = -\operatorname{quotient}(m, n) - \left(1-\chi_{\mathbb{Z}}\left(\operatorname{Re}\left(\frac{m}{n}\right)\right.\right.$ $i\left(1-\chi_{\mathbb{Z}}\left(\operatorname{Im}\left(\frac{m}{n}\right)\right)\right)\operatorname{sgn}\left(\left\lvert\operatorname{Im}\left(\frac{m}{n}\right)\right\rvert\right)$
$(-m, n)$	$-m \bmod n = n - m \bmod n \,/;\, m\in\mathbb{R}\bigwedge n\in\mathbb{R}\bigwedge\frac{m}{n}\notin\mathbb{Z}$ $-m \bmod n = -\chi_{\mathbb{Z}}\left(\frac{m}{n}\right) n + n - m \bmod n \,/;\, m\in\mathbb{R}\land n\in\mathbb{R}$ $-m \bmod n = -(m \bmod n) + n\left(1-\chi_{\mathbb{Z}}\left(\operatorname{Re}\left(\frac{m}{n}\right)\right)\right)\operatorname{sgn}\left(\left\lvert\operatorname{Re}\left(\frac{m}{n}\right)\right\rvert\right) +$ $i\,n\left(1-\chi_{\mathbb{Z}}\left(\operatorname{Im}\left(\frac{m}{n}\right)\right)\right)\operatorname{sgn}\left(\left\lvert\operatorname{Im}\left(\frac{m}{n}\right)\right\rvert\right)$	$\operatorname{quotient}(-m, n) = \chi_{\mathbb{Z}}\left(\frac{m}{n}\right) - 1 - \operatorname{quotient}(m, n) \,/;\, m\in$ $\operatorname{quotient}(-m, n) = -\operatorname{quotient}(m, n) - \left(1-\chi_{\mathbb{Z}}\left(\operatorname{Re}\left(\frac{m}{n}\right)\right.\right.$ $i\left(1-\chi_{\mathbb{Z}}\left(\operatorname{Im}\left(\frac{m}{n}\right)\right)\right)\operatorname{sgn}\left(\left\lvert\operatorname{Im}\left(\frac{m}{n}\right)\right\rvert\right)$
$(i\,m, i\,n)$	$(i\,m) \bmod (i\,n) = i\,(m \bmod n)$	$\operatorname{quotient}(i\,m, i\,n) = \operatorname{quotient}(m, n)$
$(i\,m, n)$	$(i\,m) \bmod n = n - n\,\chi_{\mathbb{Z}}\left(\operatorname{Im}\left(\frac{m}{n}\right)\right) + i\,(m \bmod n)$	$\operatorname{quotient}(i\,m, n) = i\operatorname{quotient}(m, n) + \chi_{\mathbb{Z}}\left(\operatorname{Im}\left(\frac{m}{n}\right)\right) - 1$
$(-i\,m, n)$	$(-i\,m) \bmod n = -i\,n\left(\chi_{\mathbb{Z}}\left(\operatorname{Re}\left(\frac{m}{n}\right)\right) - 1\right) - i\,(m \bmod n)$	$\operatorname{quotient}(-i\,m, n) = -i\operatorname{quotient}(m, n) + i\left(\chi_{\mathbb{Z}}\left(\operatorname{Re}\left(\frac{m}{n}\right)\right.\right.$
$(m, i\,n)$	$m \bmod (i\,n) = m \bmod n + \left(\chi_{\mathbb{Z}}\left(\operatorname{Re}\left(\frac{m}{n}\right)\right) - 1\right) n$	$\operatorname{quotient}(m, i\,n) = -i\operatorname{quotient}(m, n) + i\left(\chi_{\mathbb{Z}}\left(\operatorname{Re}\left(\frac{m}{n}\right)\right)\right.$
$(m, -i\,n)$	$m \bmod (-i\,n) = m \bmod n + \left(\chi_{\mathbb{Z}}\left(\operatorname{Im}\left(\frac{m}{n}\right)\right) - 1\right) i\,n$	$\operatorname{quotient}(m, -i\,n) = \chi_{\mathbb{Z}}\left(\operatorname{Im}\left(\frac{m}{n}\right)\right) + i\operatorname{quotient}(m, n) -$
$\left(\frac{m}{n}, 1\right)$	$\frac{m}{n} \bmod 1 = \frac{m \bmod n}{n}$	$\operatorname{quotient}\left(\frac{m}{n}, 1\right) = \operatorname{quotient}(m, n)$

Transformations and argument simplifications (arguments involving related functions)

Compositions of rounding and congruence functions $\lfloor z \rfloor$, $\lfloor z \rceil$, $\lceil z \rceil$, int(z), frac(z), $m \bmod n$, and quotient(m, n) with the rounding and congruence functions in many cases lead to very simple zero results:

z	Floor	Round	Ceiling	IntegerPart
Floor	$\lfloor \lfloor z \rfloor \rfloor = \lfloor z \rfloor$ $\lfloor z - \lfloor z \rfloor \rfloor = 0$	$\lfloor \lfloor z \rceil \rfloor = \lfloor z \rceil$	$\lfloor \lceil z \rceil \rfloor = \lceil z \rceil$	$\lfloor \operatorname{int}(z) \rfloor = \operatorname{int}(z)$
Round	$\lfloor \lfloor z \rfloor \rceil = \lfloor z \rfloor$	$\lfloor \lfloor z \rceil \rceil = \lfloor z \rceil$ $\lfloor z - \lfloor z \rceil \rceil = 0$	$\lfloor \lceil z \rceil \rceil = \lceil z \rceil$	$\lfloor \operatorname{int}(z) \rceil = \operatorname{int}(z)$
Ceiling	$\lceil \lfloor z \rfloor \rceil = \lfloor z \rfloor$	$\lceil \lfloor z \rceil \rceil = \lfloor z \rceil$	$\lceil \lceil z \rceil \rceil = \lceil z \rceil$ $\lceil z - \lceil z \rceil \rceil = 0$	$\lceil \operatorname{int}(z) \rceil = \operatorname{int}(z)$
IntegerPart	$\operatorname{int}(\lfloor z \rfloor) = \lfloor z \rfloor$	$\operatorname{int}(\lfloor z \rceil) = \lfloor z \rceil$	$\operatorname{int}(\lceil z \rceil) = \lceil z \rceil$	$\operatorname{int}(\operatorname{int}(z)) = \operatorname{int}(z)$ $\operatorname{int}(z - \operatorname{int}(z)) = 0$
FractionalPart	$\operatorname{frac}(\lfloor z \rfloor) = 0$	$\operatorname{frac}(\lfloor z \rceil) = 0$	$\operatorname{frac}(\lceil z \rceil) = 0$	$\operatorname{frac}(\operatorname{int}(z)) = 0$
Mod	$\lfloor m \rfloor \bmod n = \lfloor m \rfloor - n \left\lfloor \frac{\lfloor m \rfloor}{n} \right\rfloor$ $\lfloor m \rfloor \bmod 1 = 0$	$\lfloor m \rceil \bmod n = \lfloor m \rceil - n \left\lfloor \frac{\lfloor m \rceil}{n} \right\rfloor$ $\lfloor m \rceil \bmod 1 = 0$	$\lceil m \rceil \bmod n = \lceil m \rceil - n \left\lfloor \frac{\lceil m \rceil}{n} \right\rfloor$ $\lceil m \rceil \bmod 1 = 0$	$\operatorname{int}(m) \bmod n = \operatorname{int}(m) - n$ $\operatorname{int}(m) \bmod 1 = 0$
Quotient	$\operatorname{quotient}(\lfloor m \rfloor, n) = \left\lfloor \frac{\lfloor m \rfloor}{n} \right\rfloor$ $\operatorname{quotient}(\lfloor m \rfloor, 1) = \lfloor m \rfloor$	$\operatorname{quotient}(\lfloor m \rceil, n) = \left\lfloor \frac{\lfloor m \rceil}{n} \right\rfloor$ $\operatorname{quotient}(\lfloor m \rceil, 1) = \lfloor m \rceil$	$\operatorname{quotient}(\lceil m \rceil, n) = \left\lfloor \frac{\lceil m \rceil}{n} \right\rfloor$ $\operatorname{quotient}(\lceil m \rceil, 1) = \lceil m \rceil$	$\operatorname{quotient}(\operatorname{int}(m), n) = \lfloor \operatorname{int}($ $\operatorname{quotient}(\operatorname{int}(m), 1) = \operatorname{int}($

$$\left\lfloor \frac{\lfloor n\, x \rfloor}{n} \right\rfloor = \lfloor x \rfloor \;/;\; x \in \mathbb{R} \land n \in \mathbb{Z}$$

$$\left\lceil \frac{\lceil n\, x \rceil}{n} \right\rceil = \lceil x \rceil \;/;\; x \in \mathbb{R} \land n \in \mathbb{Z}.$$

(m, n)	Mod	Quotient
Floor	$\lfloor m \bmod n \rfloor = \left\lfloor m - n \left\lfloor \frac{m}{n} \right\rfloor \right\rfloor$ $\lfloor m \bmod 1 \rfloor = 0$	$\lfloor \operatorname{quotient}(m, n) \rfloor = \left\lfloor \frac{m}{n} \right\rfloor$ $\left\lfloor \frac{m}{n} - \operatorname{quotient}(m, n) \right\rfloor = 0$
Round	$\lfloor m \bmod n \rceil = \left\lfloor m - n \left\lfloor \frac{m}{n} \right\rfloor \right\rceil$	$\lfloor \operatorname{quotient}(m, n) \rceil = \left\lfloor \frac{m}{n} \right\rfloor$
Ceiling	$\lceil m \bmod n \rceil = \left\lceil m - n \left\lfloor \frac{m}{n} \right\rfloor \right\rceil$	$\lceil \operatorname{quotient}(m, n) \rceil = \left\lfloor \frac{m}{n} \right\rfloor$
IntegerPart	$\operatorname{int}(m \bmod n) = \operatorname{int}\left(m - n \left\lfloor \frac{m}{n} \right\rfloor\right)$	$\operatorname{int}(\operatorname{quotient}(m, n)) = \left\lfloor \frac{m}{n} \right\rfloor$
FractionalPart	$\operatorname{frac}(m \bmod n) = \operatorname{frac}\left(m - n \left\lfloor \frac{m}{n} \right\rfloor\right)$	$\operatorname{frac}(\operatorname{quotient}(m, n)) = 0$
Mod	$(m \bmod n) \bmod n = m \bmod n$ $(m \bmod n) \bmod n = m - n \left\lfloor \frac{m}{n} \right\rfloor$	$\operatorname{quotient}(m, n) \bmod n = \left\lfloor \frac{m}{n} \right\rfloor - n \left\lfloor \frac{1}{n} \left\lfloor \frac{m}{n} \right\rfloor \right\rfloor$ $\operatorname{quotient}(m, 1) \bmod 1 = 0$
Quotient	$\operatorname{quotient}(m \bmod n, n) = \left\lfloor \frac{m \bmod n}{n} \right\rfloor$ $\operatorname{quotient}(m \bmod 1, 1) = 0$	$\operatorname{quotient}(\operatorname{quotient}(m, n), n) = \left\lfloor \frac{1}{n} \left\lfloor \frac{m}{n} \right\rfloor \right\rfloor$ $\operatorname{quotient}(\operatorname{quotient}(m, 1), 1) = \lfloor m \rfloor$

Addition formulas

The rounding and congruence functions $\lfloor z \rfloor$, $\lfloor z \rceil$, $\lceil z \rceil$, int(z), and frac(z) satisfy the following addition formulas:

	$z+n$	z_1+z_2
Floor	$\lfloor z+n\rfloor = \lfloor z\rfloor + n \;/;\; n\in\mathbb{Z}$	$\lfloor z_1+z_2\rfloor = \lfloor z_1+z_2-\lfloor z_1\rfloor-\lfloor z_2\rfloor\rfloor+\lfloor z_1\rfloor+\lfloor z_2\rfloor$
Round	$\lfloor z+n\rceil = \lfloor z\rceil + n \;/;\; n\in\mathbb{Z} \bigwedge \mathrm{Re}(z)+\frac{1}{2}\notin\mathbb{Z} \bigwedge \mathrm{Im}(z)+\frac{1}{2}\notin\mathbb{Z}$	
Ceiling	$\lceil z+n\rceil = \lceil z\rceil + n \;/;\; n\in\mathbb{Z}$	$\lceil z_1+z_2\rceil = \lceil z_1\rceil+\lceil z_2\rceil+\lceil z_1+z_2-\lceil z_1\rceil-\lceil z_2\rceil\rceil$
IntegerPart	$\mathrm{int}(z+n) = \mathrm{int}(z)+n-\theta(z+n)+\theta(z) \;/;\; n\in\mathbb{Z} \bigwedge z\notin\mathbb{Z}$	
FractionalPart	$\mathrm{frac}(z+n) = \mathrm{frac}(z)+\theta(z+n)-\theta(z) \;/;\; n\in\mathbb{Z} \bigwedge z\notin\mathbb{Z}$	

$(m+k\,n) \bmod n = m \bmod n \;/;\; k\in\mathbb{Z}$

$\mathrm{quotient}(m+k\,n,\, n) = \mathrm{quotient}(m,\, n)+k \;/;\; k\in\mathbb{Z}.$

Multiple arguments

The rounding and congruence functions $\lfloor z\rfloor$, $\lceil z\rceil$, int(z), frac(z), $m \bmod n$, and quotient(m, n) have the following relations for multiple arguments:

	$n\,z$ (or $k\,m$)
Floor	$\lfloor n\,z\rfloor = n\lfloor z\rfloor + \sum_{k=0}^{n-1} k\,\theta\left(z \bmod 1-\frac{k}{n}\right)\left(1-\theta\left(z \bmod 1-\frac{k+1}{n}\right)\right) \;/;\; n\in\mathbb{N}\land z\in\mathbb{R}$
Round	
Ceiling	$\lceil n\,z\rceil = n\lceil z\rceil - \sum_{k=0}^{n-1} k\,\theta\left(-\frac{k}{n}-z \bmod 1+1\right)\left(1-\theta\left(-\frac{k+1}{n}-z \bmod 1+1\right)\right) \;/;\; n\in\mathbb{N}\land z\in\mathbb{R}$
IntegerPart	$\mathrm{int}(n\,z) = \mathrm{int}(z)\,n + n\,\mathrm{sgn}(\chi_{\mathbb{Z}}(z)+\theta(z)) - \mathrm{sgn}(\chi_{\mathbb{Z}}(n\,z)+\theta(z)) + \sum_{k=0}^{n-1} k\,\theta\left(z \bmod 1-\frac{k}{n}\right)\left(1-\theta\left(z \bmod 1-\frac{k+1}{n}\right)\right)-n+1 \;/;\; n\in\mathbb{N}\land z\in\mathbb{R}$
FractionalPart	$\mathrm{frac}(n\,z) = \mathrm{frac}(z)\,n - n\,\mathrm{sgn}(\chi_{\mathbb{Z}}(z)+\theta(z)) + \mathrm{sgn}(\chi_{\mathbb{Z}}(n\,z)+\theta(z)) - \sum_{k=0}^{n-1} k\,\theta\left(z \bmod 1-\frac{k}{n}\right)\left(1-\theta\left(z \bmod 1-\frac{k+1}{n}\right)\right)+n-1 \;/;\; n\in\mathbb{N}\land z\in\mathbb{R}$
Mod	$(k\,m) \bmod n = k\,(m \bmod n) - n\sum_{j=0}^{k-1} j\,\theta\left(\frac{m}{n}-\frac{j}{k}-\mathrm{quotient}(m,\,n)\right)\left(1-\theta\left(\frac{m}{n}-\frac{j+1}{k}-\mathrm{quotient}(m,\,n)\right)\right) \;/;\; k\in\mathbb{N} \bigwedge \frac{m}{n}\in\mathbb{R}$
Quotient	$\mathrm{quotient}(k\,m,\, n) = k\,\mathrm{quotient}(m,\,n) + \sum_{j=0}^{k-1} j\left(\theta\left(\frac{m}{n} \bmod 1-\frac{j}{k}\right)\left(1-\theta\left(\frac{m}{n} \bmod 1-\frac{j+1}{k}\right)\right)\right) \;/;\; k\in\mathbb{N} \bigwedge \frac{m}{n}\in\mathbb{R}$

Sums of the direct function

Sums of the floor and ceiling functions $\lfloor z\rfloor$ and $\lceil z\rceil$ satisfy the following relations:

$$\lfloor z_1\rfloor+\lfloor z_2\rfloor = \lfloor z_1+z_2\rfloor-\lfloor z_1+z_2-\lfloor z_1\rfloor-\lfloor z_2\rfloor\rfloor$$

$$\sum_{k=0}^{n-1}\left\lfloor\frac{k\,m+x}{n}\right\rfloor = \sum_{k=0}^{m-1}\left\lfloor\frac{k\,n+x}{m}\right\rfloor \;/;\; x\in\mathbb{R} \bigwedge n\in\mathbb{N}^+ \bigwedge m\in\mathbb{N}^+$$

$$\lceil z_1\rceil+\lceil z_2\rceil = \lceil z_1+z_2\rceil-\lceil z_1+z_2-\lceil z_1\rceil-\lceil z_2\rceil\rceil$$

$$\sum_{k=0}^{n-1}\left\lceil\frac{x-k\,m}{n}\right\rceil = \sum_{k=0}^{m-1}\left\lceil\frac{x-k\,n}{m}\right\rceil \;/;\; x\in\mathbb{R} \bigwedge n\in\mathbb{N}^+ \bigwedge m\in\mathbb{N}^+.$$

Identities

All rounding and congruence functions satisfy numerous identities, for example:

$$\left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{n+2}{6} \right\rfloor + \left\lfloor \frac{n+4}{6} \right\rfloor = \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n+3}{6} \right\rfloor /; n \in \mathbb{Z}$$

$$\left\lfloor \sqrt{n + \frac{1}{2}} + \frac{1}{2} \right\rfloor = \left\lfloor \sqrt{n + \frac{1}{4}} + \frac{1}{2} \right\rfloor /; n \in \mathbb{Z}$$

$$\left\lfloor \sqrt{n} + \sqrt{n+1} \right\rfloor = \left\lfloor \sqrt{4n+2} \right\rfloor /; n \in \mathbb{Z}$$

$$(a + c) \bmod n = (b + d) \bmod n /; a \bmod n = b \bmod n \land c \bmod n = d \bmod n \land a \in \mathbb{R} \land b \in \mathbb{R} \land c \in \mathbb{R} \land d \in \mathbb{R} \land n \in \mathbb{R}.$$

Complex characteristics

Complex characteristics (real and imaginary parts $\operatorname{Re}(z)$ and $\operatorname{Im}(z)$, absolute value $|z|$, argument $\operatorname{Arg}(z)$, complex conjugate $\bar{z}$, and signum sgn(z)) of the rounding and congruence functions can be represented in the forms shown in the following tables:

z	**Floor**	**Round**	**Ceiling**	**IntegerPart**
Re	$\operatorname{Re}(\lfloor z \rfloor) = \lfloor \operatorname{Re}(z) \rfloor$	$\operatorname{Re}(\lfloor z \rceil) = \lfloor \operatorname{Re}(z) \rceil$	$\operatorname{Re}(\lceil z \rceil) = \lceil \operatorname{Re}(z) \rceil$	$\operatorname{Re}(\operatorname{int}(z)) = \operatorname{int}(\operatorname{Re}(z))$
Im	$\operatorname{Im}(\lfloor z \rfloor) = \lfloor \operatorname{Im}(z) \rfloor$	$\operatorname{Im}(\lfloor z \rceil) = \lfloor \operatorname{Im}(z) \rceil$	$\operatorname{Im}(\lceil z \rceil) = \lceil \operatorname{Im}(z) \rceil$	$\operatorname{Im}(\operatorname{int}(z)) = \operatorname{int}(\operatorname{Im}(z))$
Abs	$\lvert\lfloor z \rfloor\rvert = \sqrt{\lfloor \operatorname{Im}(z) \rfloor^2 + \lfloor \operatorname{Re}(z) \rfloor^2}$	$\lvert\lfloor z \rceil\rvert = \sqrt{\lfloor \operatorname{Re}(z) \rceil^2 + \lfloor \operatorname{Im}(z) \rceil^2}$	$\lvert\lceil z \rceil\rvert = \sqrt{\lceil \operatorname{Im}(z) \rceil^2 + \lceil \operatorname{Re}(z) \rceil^2}$	$\lvert\operatorname{int}(z)\rvert = \sqrt{\operatorname{int}(\operatorname{Im}(z))^2 + \operatorname{int}(\operatorname{Re}(z))^2}$
Arg	$\operatorname{Arg}(\lfloor z \rfloor) = \tan^{-1}(\lfloor \operatorname{Re}(z) \rfloor, \lfloor \operatorname{Im}(z) \rfloor)$	$\operatorname{Arg}(\lfloor z \rceil) = \tan^{-1}(\lfloor \operatorname{Re}(z) \rceil, \lfloor \operatorname{Im}(z) \rceil)$	$\operatorname{Arg}(\lceil z \rceil) = \tan^{-1}(\lceil \operatorname{Re}(z) \rceil, \lceil \operatorname{Im}(z) \rceil)$	$\operatorname{Arg}(\operatorname{int}(z)) = \tan^{-1}(\operatorname{int}(\operatorname{Re}(z)), \operatorname{int}(\operatorname{Im}(z)))$
Conjugate	$\overline{\lfloor z \rfloor} = \lfloor \operatorname{Re}(z) \rfloor - i \lfloor \operatorname{Im}(z) \rfloor$	$\overline{\lfloor z \rceil} = \lfloor \operatorname{Re}(z) \rceil - i \lfloor \operatorname{Im}(z) \rceil$	$\overline{\lceil z \rceil} = \lceil \operatorname{Re}(z) \rceil + i \lceil \operatorname{Im}(z) \rceil$	$\overline{\operatorname{int}(z)} = \operatorname{int}(\operatorname{Re}(z)) - i \operatorname{int}(\operatorname{Im}($
Sign	$\operatorname{sgn}(\lfloor z \rfloor) = \frac{\lfloor z \rfloor}{\lvert\lfloor z \rfloor\rvert}$	$\operatorname{sgn}(\lfloor z \rceil) = \frac{\lfloor z \rceil}{\lvert\lfloor z \rceil\rvert}$	$\operatorname{sgn}(\lceil z \rceil) = \frac{\lceil z \rceil}{\lvert\lceil z \rceil\rvert}$	$\operatorname{sgn}(\operatorname{int}(z)) = \frac{\operatorname{int}(z)}{\lvert\operatorname{int}(z)\rvert}$

$x + i y /; x \in \mathbb{R} \land y \in \mathbb{R}$	**Floor**	**Round**	**Ceiling**	**IntegerPart**
Re	$\operatorname{Re}(\lfloor x + i y \rfloor) = \lfloor x \rfloor$	$\operatorname{Re}(\lfloor x + i y \rceil) = \lfloor x \rceil$	$\operatorname{Re}(\lceil x + i y \rceil) = \lceil x \rceil$	$\operatorname{Re}(\operatorname{int}(x + i y)) = \operatorname{int}($
Im	$\operatorname{Im}(\lfloor x + i y \rfloor) = \lfloor y \rfloor$	$\operatorname{Im}(\lfloor x + i y \rceil) = \lfloor y \rceil$	$\operatorname{Im}(\lceil x + i y \rceil) = \lceil y \rceil$	$\operatorname{Im}(\operatorname{int}(x + i y)) = \operatorname{int}($
Abs	$\lvert\lfloor x + i y \rfloor\rvert = \sqrt{\lfloor x \rfloor^2 + \lfloor y \rfloor^2}$	$\lvert\lfloor x + i y \rceil\rvert = \sqrt{\lfloor x \rceil^2 + \lfloor y \rceil^2}$	$\lceil x + i y \rceil\rvert = \sqrt{\lceil x \rceil^2 + \lceil y \rceil^2}$	$\lvert\operatorname{int}(x + i y)\rvert = \sqrt{\operatorname{int}(x)}$
Arg	$\operatorname{Arg}(\lfloor x + i y \rfloor) = \tan^{-1}(\lfloor x \rfloor, \lfloor y \rfloor)$	$\operatorname{Arg}(\lfloor x + i y \rceil) = \tan^{-1}(\lfloor x \rceil, \lfloor y \rceil)$	$\operatorname{Arg}(\lceil x + i y \rceil) = \tan^{-1}(\lceil x \rceil, \lceil y \rceil)$	$\operatorname{Arg}(\operatorname{int}(x + i y)) = \tan^{-1}(\operatorname{int}(x), \operatorname{int}(y))$
Conjugate	$\overline{\lfloor x + i y \rfloor} = \lfloor x \rfloor - i \lfloor y \rfloor$	$\overline{\lfloor x + i y \rceil} = \lfloor x \rceil - i \lfloor y \rceil$	$\overline{\lceil x + i y \rceil} = \lceil x \rceil - i \lceil y \rceil$	$\overline{\operatorname{int}(x + i y)} = \operatorname{int}(x) - i$
Sign	$\operatorname{sgn}(\lfloor x + i y \rfloor) = \frac{\lfloor x + i y \rfloor}{\lvert\lfloor x + i y \rfloor\rvert}$	$\operatorname{sgn}(\lceil x + i y \rceil) = \frac{\lfloor x \rceil + i \lfloor y \rceil}{\sqrt{\lfloor x \rceil^2 + \lfloor y \rceil^2}}$	$\operatorname{sgn}(\lceil x + i y \rceil) = \frac{\lceil x + i y \rceil}{\lvert\lceil x + i y \rceil\rvert}$	$\operatorname{sgn}(\operatorname{int}(x + i y)) = \frac{\text{int}}{\sqrt{\text{in}}}$

$(m,\ n)$	Mod	Quotient
Re	$\mathrm{Re}\,(m \bmod n) == \mathrm{Re}(m) + \left\lfloor \frac{\mathrm{Im}(m)\,\mathrm{Re}(n)-\mathrm{Im}(n)\,\mathrm{Re}(m)}{\mathrm{Im}(n)^2+\mathrm{Re}(n)^2} \right\rfloor \mathrm{Im}(n) - \left\lfloor \frac{\mathrm{Im}(m)\,\mathrm{Im}(n)+\mathrm{Re}(m)\,\mathrm{Re}(n)}{\mathrm{Im}(n)^2+\mathrm{Re}(n)^2} \right\rfloor \mathrm{Re}(n)$	$\mathrm{Re}\,(\mathrm{quotient}(m,\ n)) == \left\lfloor \frac{\mathrm{Re}(m)\,\mathrm{Re}(n)+\mathrm{Im}(m)\,\mathrm{Im}(n)}{\mathrm{Im}(n)^2+\mathrm{Re}(n)^2} \right\rfloor$
Im	$\mathrm{Im}\,(m \bmod n) == \mathrm{Im}(m) - \left\lfloor \frac{\mathrm{Im}(m)\,\mathrm{Im}(n)+\mathrm{Re}(m)\,\mathrm{Re}(n)}{\mathrm{Im}(n)^2+\mathrm{Re}(n)^2} \right\rfloor \mathrm{Im}(n) - \left\lfloor \frac{\mathrm{Im}(m)\,\mathrm{Re}(n)-\mathrm{Im}(n)\,\mathrm{Re}(m)}{\mathrm{Im}(n)^2+\mathrm{Re}(n)^2} \right\rfloor \mathrm{Re}(n)$	$\mathrm{Im}\,(\mathrm{quotient}(m,\ n)) == \left\lfloor \frac{\mathrm{Im}(m)\,\mathrm{Re}(n)-\mathrm{Im}(n)\,\mathrm{Re}(m)}{\mathrm{Im}(n)^2+\mathrm{Re}(n)^2} \right\rfloor$
Abs	$\lvert m \bmod n \rvert == \sqrt{\left(\left(\mathrm{Im}(m) - \left\lfloor \frac{\mathrm{Im}(m)\,\mathrm{Im}(n)+\mathrm{Re}(m)\,\mathrm{Re}(n)}{\mathrm{Im}(n)^2+\mathrm{Re}(n)^2} \right\rfloor \mathrm{Im}(n) - \left\lfloor \frac{\mathrm{Im}(m)\,\mathrm{Re}(n)-\mathrm{Im}(n)\,\mathrm{Re}(m)}{\mathrm{Im}(n)^2+\mathrm{Re}(n)^2} \right\rfloor \mathrm{Re}(n)\right)^2 + \left(\left\lfloor \frac{\mathrm{Im}(m)\,\mathrm{Re}(n)-\mathrm{Im}(n)\,\mathrm{Re}(m)}{\mathrm{Im}(n)^2+\mathrm{Re}(n)^2} \right\rfloor \mathrm{Im}(n) + \mathrm{Re}(m) - \left\lfloor \frac{\mathrm{Im}(m)\,\mathrm{Im}(n)+\mathrm{Re}(m)\,\mathrm{Re}(n)}{\mathrm{Im}(n)^2+\mathrm{Re}(n)^2} \right\rfloor \mathrm{Re}(n)\right)^2\right)}$	$\lvert \mathrm{quotient}(m,\ n) \rvert ==$ $\sqrt{\left\lfloor \frac{\mathrm{Re}(m)\,\mathrm{Re}(n)+\mathrm{Im}(m)\,\mathrm{Im}(n)}{\mathrm{Im}(n)^2+\mathrm{Re}(n)^2} \right\rfloor^2 \ldots}$
Arg	$\mathrm{Arg}\,(m \bmod n) == \tan^{-1}\left(\mathrm{Re}(m) + \left\lfloor \frac{\mathrm{Im}(m)\,\mathrm{Re}(n)-\mathrm{Im}(n)\,\mathrm{Re}(m)}{\mathrm{Im}(n)^2+\mathrm{Re}(n)^2} \right\rfloor \mathrm{Im}(n) - \left\lfloor \frac{\mathrm{Im}(m)\,\mathrm{Im}(n)+\mathrm{Re}(m)\,\mathrm{Re}(n)}{\mathrm{Im}(n)^2+\mathrm{Re}(n)^2} \right\rfloor \mathrm{Re}(n),\ \mathrm{Im}(m) - \left\lfloor \frac{\mathrm{Im}(m)\,\mathrm{Im}(n)+\mathrm{Re}(m)\,\mathrm{Re}(n)}{\mathrm{Im}(n)^2+\mathrm{Re}(n)^2} \right\rfloor \mathrm{Im}(n) - \left\lfloor \frac{\mathrm{Im}(m)\,\mathrm{Re}(n)-\mathrm{Im}(n)\,\mathrm{Re}(m)}{\mathrm{Im}(n)^2+\mathrm{Re}(n)^2} \right\rfloor \mathrm{Re}(n)\right)$	$\mathrm{Arg}\,(\mathrm{quotient}(m,\ n)) ==$ $\tan^{-1}\left(\left\lfloor \frac{\mathrm{Im}(m)\,\mathrm{Im}(n)+\mathrm{Re}(m)\,\mathrm{Re}(n)}{\mathrm{Im}(n)^2+\mathrm{Re}(n)^2} \ldots\right.\right.$
Conjugate	$\overline{m \bmod n} == \mathrm{Re}(m) + \left\lfloor \frac{\mathrm{Im}(m)\,\mathrm{Re}(n)-\mathrm{Im}(n)\,\mathrm{Re}(m)}{\mathrm{Im}(n)^2+\mathrm{Re}(n)^2} \right\rfloor \mathrm{Im}(n) - \left\lfloor \frac{\mathrm{Im}(m)\,\mathrm{Im}(n)+\mathrm{Re}(m)\,\mathrm{Re}(n)}{\mathrm{Im}(n)^2+\mathrm{Re}(n)^2} \right\rfloor \mathrm{Re}(n) - i\left(\mathrm{Im}(m) - \left\lfloor \frac{\mathrm{Im}(m)\,\mathrm{Im}(n)+\mathrm{Re}(m)\,\mathrm{Re}(n)}{\mathrm{Im}(n)^2+\mathrm{Re}(n)^2} \right\rfloor \mathrm{Im}(n) - \left\lfloor \frac{\mathrm{Im}(m)\,\mathrm{Re}(n)-\mathrm{Im}(n)\,\mathrm{Re}(m)}{\mathrm{Im}(n)^2+\mathrm{Re}(n)^2} \right\rfloor \mathrm{Re}(n)\right)$	$\overline{\mathrm{quotient}(m,\ n)} ==$ $\left\lfloor \frac{\mathrm{Im}(m)\,\mathrm{Im}(n)+\mathrm{Re}(m)\,\mathrm{Re}(n)}{\mathrm{Im}(n)^2+\mathrm{Re}(n)^2} \right\rfloor - i \left\lfloor \ldots\right.$
Sign	$\mathrm{sgn}\,(m \bmod n) == \left(\left\lfloor \frac{\mathrm{Im}(m)\,\mathrm{Re}(n)-\mathrm{Im}(n)\,\mathrm{Re}(m)}{\mathrm{Im}(n)^2+\mathrm{Re}(n)^2} \right\rfloor \mathrm{Im}(n) + \mathrm{Re}(m) - \left\lfloor \frac{\mathrm{Im}(m)\,\mathrm{Im}(n)+\mathrm{Re}(m)\,\mathrm{Re}(n)}{\mathrm{Im}(n)^2+\mathrm{Re}(n)^2} \right\rfloor \mathrm{Re}(n) + i\left(\mathrm{Im}(m) - \left\lfloor \frac{\mathrm{Im}(m)\,\mathrm{Im}(n)+\mathrm{Re}(m)\,\mathrm{Re}(n)}{\mathrm{Im}(n)^2+\mathrm{Re}(n)^2} \right\rfloor \mathrm{Im}(n) - \left\lfloor \frac{\mathrm{Im}(m)\,\mathrm{Re}(n)-\mathrm{Im}(n)\,\mathrm{Re}(m)}{\mathrm{Im}(n)^2+\mathrm{Re}(n)^2} \right\rfloor \mathrm{Re}(n)\right)\right) \Big/ \left(\sqrt{\left(\left(\mathrm{Im}(m) - \left\lfloor \frac{\mathrm{Im}(m)\,\mathrm{Im}(n)+\mathrm{Re}(m)\,\mathrm{Re}(n)}{\mathrm{Im}(n)^2+\mathrm{Re}(n)^2} \right\rfloor \mathrm{Im}(n) - \left\lfloor \frac{\mathrm{Im}(m)\,\mathrm{Re}(n)-\mathrm{Im}(n)\,\mathrm{Re}(m)}{\mathrm{Im}(n)^2+\mathrm{Re}(n)^2} \right\rfloor \mathrm{Re}(n)\right)^2 + \left(\left\lfloor \frac{\mathrm{Im}(m)\,\mathrm{Re}(n)-\mathrm{Im}(n)\,\mathrm{Re}(m)}{\mathrm{Im}(n)^2+\mathrm{Re}(n)^2} \right\rfloor \mathrm{Im}(n) + \mathrm{Re}(m) - \left\lfloor \frac{\mathrm{Im}(m)\,\mathrm{Im}(n)+\mathrm{Re}(m)\,\mathrm{Re}(n)}{\mathrm{Im}(n)^2+\mathrm{Re}(n)^2} \right\rfloor \mathrm{Re}(n)\right)^2\right)}\right)$ $\mathrm{sgn}(m \bmod n) == \mathrm{sgn}(n)\ /;\ m \in \mathbb{R} \land n \in \mathbb{R}$	$\mathrm{sgn}\,(\mathrm{quotient}(m,\ n)) == \ldots$ $\mathrm{sgn}(\mathrm{quotient}(m,\ n)) == \mathrm{sgn}\left(\ldots\right.$

Differentiation

Derivatives of the rounding and congruence functions $\lfloor z \rfloor$, $\lfloor z \rceil$, $\lceil z \rceil$, int(z), frac(z), $m \bmod n$, and quotient(m, n) can be evaluated in the classical and distributional sense. In the last case, all variables should be real and results include the Dirac delta function. All rounding and congruence functions also have fractional derivatives. All these derivatives can be represented as shown in the following tables:

	$\frac{\partial}{\partial z}$ (in classical sense)	$\frac{\partial}{\partial x}$ (in distributional sense for real x)	$\frac{\partial^\alpha}{\partial z^\alpha}$ (in fractional sense)
Floor	$\frac{\partial \lfloor z \rfloor}{\partial z} == 0$	$\frac{\partial \lfloor x \rfloor}{\partial x} == \sum_{k=-\infty}^{\infty} \delta(x-k)$	$\frac{\partial^\alpha \lfloor z \rfloor}{\partial z^\alpha} == \frac{\lfloor z \rfloor\, z^{-\alpha}}{\Gamma(1-\alpha)}$
Round	$\frac{\partial \lfloor z \rceil}{\partial z} == 0$	$\frac{\partial \lfloor x \rceil}{\partial x} == \sum_{k=-\infty}^{\infty} \delta\left(x-k+\frac{1}{2}\right)$	$\frac{\partial^\alpha \lfloor z \rceil}{\partial z^\alpha} == \frac{\lfloor z \rceil\, z^{-\alpha}}{\Gamma(1-\alpha)}$
Ceiling	$\frac{\partial \lceil z \rceil}{\partial z} == 0$	$\frac{\partial \lceil x \rceil}{\partial x} == \sum_{k=-\infty}^{\infty} \delta\,(x-k)$	$\frac{\partial^\alpha \lceil z \rceil}{\partial z^\alpha} == \frac{\lceil z \rceil\, z^{-\alpha}}{\Gamma(1-\alpha)}$
IntegerPart	$\frac{\partial \mathrm{int}(z)}{\partial z} == 0$	$\frac{\partial \mathrm{int}(x)}{\partial x} == \sum_{k=-\infty}^{\infty} \delta_{k,0}\, \delta(x-k)$	$\frac{\partial^\alpha \mathrm{int}(z)}{\partial z^\alpha} == \frac{\mathrm{int}(z)\, z^{-\alpha}}{\Gamma(1-\alpha)}$
FractionalPart	$\frac{\partial \mathrm{frac}(x)}{\partial x} == 1$	$\frac{\partial \mathrm{frac}(x)}{\partial x} == x - \sum_{k=-\infty}^{\infty} \delta_{k,0}\, \delta\,(x-k)$	$\frac{\partial^\alpha \mathrm{frac}(z)}{\partial z^\alpha} == \frac{\alpha\, z^{1-\alpha}}{\Gamma(2-\alpha)} + \frac{\mathrm{frac}(z)\, z^{-\alpha}}{\Gamma(1-\alpha)}$

	$\frac{\partial}{\partial m}$ (in classical sense)	$\frac{\partial}{\partial n}$ (in classical sense)	$\frac{\partial}{\partial m}$ (in distributional sense for real m, n)	$\frac{\partial}{\partial n}$ (in distributional sense for real m, n)	$\frac{\partial^\alpha}{\partial m^\alpha}$ (in fractional sense
Mod	$\frac{\partial(m \bmod n)}{\partial m} == 1$	$\frac{\partial(m \bmod n)}{\partial n} == -\left\lfloor \frac{m}{n} \right\rfloor$	$\frac{\partial(m \bmod n)}{\partial m} == m + \sum_{k=-\infty}^{\infty} \delta(m - k\,n)$	$\frac{\partial(m \bmod n)}{\partial n} == \operatorname{sgn}(n) \left(\operatorname{int}\left(\frac{m}{n}\right) - \frac{m}{n^2} \sum_{k=-\infty}^{\infty} \delta_{k,0}\, \delta\left(\frac{m}{n} - k\right)\right)$	$\frac{\partial^\alpha (m \bmod n)}{\partial m^\alpha} == \frac{\alpha\, m^{1-\alpha}}{\Gamma(2-\alpha)} + \frac{(m \bmod n)\, m}{\Gamma(1-\alpha)}$
Quotient	$\frac{\partial \operatorname{quotient}(m,n)}{\partial m} == 0$	$\frac{\partial \operatorname{quotient}(m,n)}{\partial n} == 0$	$\frac{\partial \operatorname{quotient}(m,n)}{\partial m} == \operatorname{sgn}(n) \sum_{k=-\infty}^{\infty} \delta(m - k\,n)$	$\frac{\partial \operatorname{quotient}(m,n)}{\partial n} == -\frac{\operatorname{sgn}(m)\, m}{n^2} \sum_{k=-\infty}^{\infty} \delta_{k,0}\, \delta\left(\frac{m}{n} - k\right)$	$\frac{\partial^\alpha \operatorname{quotient}(m,n)}{\partial m^\alpha} == \frac{\operatorname{quotient}(m,n)\, m^{-\alpha}}{\Gamma(1-\alpha)}$

Indefinite integration

Simple indefinite integrals of the rounding and congruence functions $\lfloor z \rfloor$, $\lfloor z \rceil$, $\lceil z \rceil$, int(z), frac(z), $m \bmod n$, and quotient(m, n) have the following representations:

	$\int f(z)\, dz$
Floor	$\int \lfloor z \rfloor\, dz == z \lfloor z \rfloor$
Round	$\int \lfloor z \rceil\, dz == z\ \lfloor z \rceil$
Ceiling	$\int \lceil z \rceil\, dz == z \lceil z \rceil$
IntegerPart	$\int \operatorname{int}(z)\, dz == z \operatorname{int}(z)$
FractionalPart	$\int \operatorname{frac}(z)\, dz == z \operatorname{frac}(z) - \frac{z^2}{2}$
Mod	$\int z \bmod n\, dz == z\,(z \bmod n) - \frac{z^2}{2}$ $\int m \bmod z\, dz == \frac{1}{2}\, z\,(m + m \bmod z)$
Quotient	$\int \operatorname{quotient}(z, n)\, dz == z \operatorname{quotient}(z, n)$ $\int \operatorname{quotient}(m, z)\, dz == z \operatorname{quotient}(m, z)$

Definite integration

Some definite integrals of the rounding and congruence functions $\lfloor z \rfloor$, $\lfloor z \rceil$, $\lceil z \rceil$, int(z), frac(z), $m \bmod n$, and quotient(m, n) can be evaluated and are shown in the following table:

$a \in \mathbb{R}$	$\int_0^n f(t)\,dt\ /;\ n \in \mathbb{N}$	$\int_0^a f(t)\,dt$	$\int_0^a t^{\alpha-1}\ f(t)\,dt$
Floor	$\int_0^n \lfloor t \rfloor\,dt == \frac{n(n-1)}{2}$	$\int_0^a \lfloor t \rfloor\,dt == \frac{1}{2}(2a - \lfloor a \rfloor - 1)\lfloor a \rfloor$	$\int_0^a t^{\alpha-1} \lfloor t \rfloor\,dt ==$ $\frac{1}{\alpha}(\lfloor a \rfloor a^{\alpha} - \zeta(-\alpha) + \zeta(-\alpha, \lfloor$
Round	$\int_0^n \lfloor t \rceil\,dt == \frac{n^2}{2}$	$\int_0^a \lfloor t \rceil\,dt == \frac{1}{2}(2a - \lfloor a \rceil)\lfloor a \rceil$	$\int_0^a t^{\alpha-1} \lfloor t \rceil\,dt == \frac{1}{\alpha}\left(\left\lfloor a + \frac{1}{2} \right\rfloor a^{\alpha}\right.$ $(1 - 2^{-\alpha})\,\zeta(-\alpha) + \zeta\left(-\right.$
Ceiling	$\int_0^n \lceil t \rceil\,dt == \frac{n(n+1)}{2}$	$\int_0^a \lceil t \rceil\,dt == \frac{1}{2}(2a - \lceil a \rceil + 1)\lceil a \rceil$	$\int_0^a t^{\alpha-1} \lceil t \rceil\,dt == \frac{\lceil a \rceil a^{\alpha} - \zeta(-\alpha) + \zeta(-}{\alpha}$
IntegerPart	$\int_0^n \mathrm{int}(t)\,dt == \frac{n(n-1)}{2}\ /;\ n \in \mathbb{N}$	$\int_0^a \mathrm{int}(t)\,dt == \frac{1}{2}(2a - \mathrm{int}(a) - 1)\,\mathrm{int}(a)$	$\int_0^a t^{\alpha-1}\,\mathrm{int}(t)\,dt == \frac{\mathrm{int}(a)\,a^{\alpha} - \zeta(-\alpha)}{}$
FractionalPart	$\int_0^n \mathrm{frac}(t)\,dt == \frac{n}{2}\ /;\ n \in \mathbb{N}$	$\int_0^a \mathrm{frac}(t)\,dt == \frac{1}{2}\left(\mathrm{frac}(a)^2 - \mathrm{frac}(a) + a\right)$	$\int_0^a t^{\alpha-1}\,\mathrm{frac}(t)\,dt == \frac{a^{\alpha+1}}{\alpha+1} - \frac{1}{\alpha}(($ $\zeta(-\alpha) + \zeta(-\alpha, a -$ $\mathrm{Re}(\alpha) > -1$
Mod		$\int_0^a t \bmod n\,dt == \frac{1}{2}\left((a \bmod n)^2 - n\,(a \bmod n) + a\,n\right)$ $\int_0^a m \bmod t\,dt == \frac{1}{2}\left(-\psi^{(1)}\left(\frac{a+m-m \bmod a}{a}\right) m^2 + a\,m + a\,(m \bmod a)\right)$	$\int_0^a t^{\alpha-1}\,(t \bmod n)\,dt == \frac{a^{\alpha+1}}{\alpha+1} -$ $n^{\alpha+1}\,\zeta(-\alpha) + n^{\alpha+1}$ $\mathrm{Re}(\alpha) > -1$ $\int_0^a t^{\alpha-1}\,(m \bmod t)\,dt == \frac{1}{\alpha^2+\alpha}\left(\right.$ $m^{\alpha+1}\,\alpha\,\zeta\left(\alpha + 1, \frac{a+m}{}\right.$
Quotient		$\int_0^a \mathrm{quotient}(t, n)\,dt == \frac{1}{2}\,\mathrm{quotient}(a, n)\,(2a - n - n\,\mathrm{quotient}(a, n))$	$\int_0^a t^{\alpha-1}\,\mathrm{quotient}(t, n)\,dt == \frac{1}{\alpha}\,($ $n^{\alpha}\,(\zeta(-\alpha, \mathrm{quotient}(a,$ $\int_0^a t^{\alpha-1}\,\mathrm{quotient}(m, t)\,dt ==$ $\frac{\mathrm{quotient}(m,a)\,a^{\alpha} + m^{\alpha}\,\zeta(\alpha, \mathrm{quotient}(m}{\alpha}$ $\mathrm{Re}(\alpha) > 1$

Integral transforms

All Fourier transforms of the rounding and congruence functions $\lfloor z \rfloor$, $\lfloor z \rceil$, $\lceil z \rceil$, int(z), frac(z), $m \bmod n$, and quotient(m, n) can be evaluated in a distributional sense and include the Dirac delta function:

	$\mathrm{F}_t[f(t)](z)$	$\mathrm{Fc}_t[f(t)](z)$	$\mathrm{Fs}_t[f(t)](z)$
Floor	$-\sqrt{\frac{\pi}{2}}\ \delta(z) + \frac{i}{\sqrt{2\pi}} \sum_{k=1}^{\infty} \frac{\delta(2k\pi - z) - \delta(2\pi k + z)}{k} - i\sqrt{2\pi}\ \delta'(z)$	$\mathcal{F}c_t[\lfloor t \rfloor](z) == -\frac{1}{\sqrt{2\pi}\ z} \cot\left(\frac{z}{2}\right) - \sqrt{\frac{\pi}{2}}\ \delta(z)$	$\mathcal{F}s_t[\lfloor t \rfloor](z) ==$ $\frac{1}{\sqrt{2\pi}} \sum_{k=}^{\infty}$
Round	$\mathcal{F}_t[\lfloor t \rceil](z) == \frac{i}{\sqrt{2\pi}} \sum_{k=1}^{\infty} \frac{(-1)^k (\delta(2k\pi - z) - \delta(2\pi k + z))}{k} - i\sqrt{2\pi}\ \delta'(z)$	$\mathcal{F}c_t[\lfloor t \rceil](z) == -\frac{1}{\sqrt{2\pi}\ z} \csc\left(\frac{z}{2}\right)$	$\mathcal{F}s_t[\lfloor t \rceil](z) ==$
Ceiling	$\mathcal{F}_t[\lceil t \rceil](z) == \sqrt{\frac{\pi}{2}}\ \delta(z) + \frac{i}{\sqrt{2\pi}} \sum_{k=1}^{\infty} \frac{\delta(2k\pi - z) - \delta(2\pi k + z)}{k} - i\sqrt{2\pi}\ \delta'(z)$	$\mathcal{F}c_t[\lceil t \rceil](z) == \sqrt{\frac{\pi}{2}}\ \delta(z) - \frac{1}{\sqrt{2\pi}\ z} \cot\left(\frac{z}{2}\right)$	$\mathcal{F}s_t[\lceil t \rceil](z) ==$ $\frac{1}{\sqrt{2\pi}} \sum_{k=}^{\infty}$
IntegerPart	$\mathcal{F}_t[\mathrm{int}(t)](z) == -\frac{i}{\sqrt{2\pi}\ z} + \frac{i}{\sqrt{2\pi}} \sum_{k=1}^{\infty} \frac{\delta(2k\pi - z) - \delta(2\pi k + z)}{k} - i\sqrt{2\pi}\ \delta'(z)$	$\mathcal{F}c_t[\mathrm{int}(t)](z) == -\frac{1}{\sqrt{2\pi}\ z} \cot\left(\frac{z}{2}\right) - \sqrt{\frac{\pi}{2}}\ \delta(z)$	$\mathcal{F}s_t[\mathrm{int}(t)](z)$ $\frac{1}{\sqrt{2\pi}} \sum_{k=}^{\infty}$
FractionalPart	$\mathcal{F}_t[\mathrm{frac}(t)](z) == \frac{i}{\sqrt{2\pi}\ z} - \frac{i}{\sqrt{2\pi}} \sum_{k=1}^{\infty} \frac{\delta(2k\pi - z) - \delta(2\pi k + z)}{k}$	$\mathcal{F}c_t[\mathrm{frac}(t)](z) == \frac{1}{\sqrt{2\pi}\ z^2} \left(z \cot\left(\frac{z}{2}\right) - 2\right) + \sqrt{\frac{\pi}{2}}\ \delta(z)$	$\mathcal{F}s_t[\mathrm{frac}(t)]($ $\frac{1}{\sqrt{2\pi}} \sum_{k=}^{\infty}$
Mod	$\mathcal{F}_t[t \bmod n](z) == n\sqrt{\frac{\pi}{2}}\ \delta(z) - \frac{i\,n}{\sqrt{2\pi}} \sum_{k=1}^{\infty} \frac{1}{k} \left(\delta\left(\frac{2k\pi}{n} - z\right) - \delta\left(\frac{2\pi k}{n} + z\right)\right)$	$\mathcal{F}c_t[t \bmod n](z) == \frac{1}{\sqrt{2\pi}\ z^2} \left(n\,z \cot\left(\frac{n\,z}{2}\right) - 2\right) + \sqrt{\frac{\pi}{2}}\ n\,\delta(z)$	$\mathcal{F}s_t[t \bmod n]$ $\frac{n}{\sqrt{2\pi}} \sum_{k=}^{\infty}$
Quotient	$\mathcal{F}_t[\mathrm{quotient}(t, n)](z) == -\frac{i\sqrt{2\pi}}{n}\ \delta'(z) - \sqrt{\frac{\pi}{2}}\ \delta(z) + \frac{i}{\sqrt{2\pi}} \sum_{k=1}^{\infty} \frac{1}{k} \left(\delta\left(\frac{2k\pi}{n} - z\right) - \delta\left(\frac{2\pi k}{n} + z\right)\right)$	$\mathcal{F}c_t[\mathrm{quotient}(t, n)](z) == -\frac{1}{\sqrt{2\pi}\ z} \cot\left(\frac{n\,z}{2}\right) - \sqrt{\frac{\pi}{2}}\ \delta(z)$	$\mathcal{F}s_t[\mathrm{quotient}($ $\frac{1}{\sqrt{2\pi}} \sum_{k=}^{\infty}$

Laplace and Mellin integral transforms of the rounding and congruence functions $\lfloor z \rfloor$, $\lfloor z \rceil$, $\lceil z \rceil$, int(z), frac(z), $m \bmod n$, and quotient(m, n) can be evaluated in the classical sense:

	$\mathrm{L}_t[f(t)](z)$	$\mathrm{M}_t[f(t)](z)$
Floor	$\mathcal{L}_t[\lfloor t \rfloor](z) == \frac{1}{(e^z - 1)\,z}$ /; $\mathrm{Re}(z) > 0$	$\mathcal{M}_t[\lfloor t \rfloor](z) == -\frac{\zeta(-z)}{z}$ /; $\mathrm{Re}(z) < -1$
Round	$\mathcal{L}_t[\lfloor t \rceil](z) == \frac{e^{z/2}}{(e^z - 1)\,z}$ /; $\mathrm{Re}(z) > 0$	$\mathcal{M}_t[\lfloor t \rceil](z) == -\frac{1}{z}\left(\zeta\left(-z, \frac{3}{2}\right) + 2^{-z}\right)$ /; $\mathrm{Re}(z) < -1$
Ceiling	$\mathcal{L}_t[\lceil t \rceil](z) == \frac{e^z}{(e^z - 1)\,z}$ /; $\mathrm{Re}(z) > 0$	
IntegerPart	$\mathcal{L}_t[\mathrm{int}(t)](z) == \frac{1}{(e^z - 1)\,z}$ /; $\mathrm{Re}(z) > 0$	$\mathcal{M}_t[\mathrm{int}(t)](z) == -\frac{\zeta(-z)}{z}$ /; $\mathrm{Re}(z) < -1$
FractionalPart	$\mathcal{L}_t[\mathrm{frac}(t)](z) == \frac{1}{z^2}\left(1 - \frac{z}{e^z - 1}\right)$ /; $\mathrm{Re}(z) > 0$	$\mathcal{L}_t[\mathrm{frac}(t)](z) == \frac{1}{z^2}\left(1 - \frac{z}{e^z - 1}\right)$ /; $\mathrm{Re}(z) > 0$
Mod	$\mathcal{L}_t[t \bmod n](z) == \frac{1}{z^2}\left(1 - \frac{n\,z}{e^{n\,z} - 1}\right)$ /; $\mathrm{Re}(n\,z) > 0$	$\mathcal{M}_t[t \bmod n](z) == \frac{n^{z+1}\,\zeta(-z)}{z}$ /; $-1 < \mathrm{Re}(z) < 0$ $\mathcal{M}_t[m \bmod t](z) == -\frac{m^{z+1}\,\zeta(z+1)}{z+1}$ /; $-1 < \mathrm{Re}(z) < 0$
Quotient	$\mathcal{L}_t[\mathrm{quotient}(t, n)](z) == \frac{1}{(e^{n\,z} - 1)\,z}$ /; $\mathrm{Re}(n\,z) > 0$	$\mathcal{M}_t[\mathrm{quotient}(t, n)](z) == -\frac{n^z\,\zeta(-z)}{z}$ /; $\mathrm{Re}(z) < -1$ $\mathcal{M}_t[\mathrm{quotient}(m, t)](z) == \frac{m^z\,\zeta(z)}{z}$ /; $\mathrm{Re}(z) > 1$

Summation

Sometimes finite and infinite sums including rounding and congruence functions have rather simple representations, for example:

$$\sum_{k=0}^{y} \left\lfloor \frac{k}{y} + x \right\rfloor == \lfloor x\, y + (\lceil y \rceil - y) \lfloor x + 1 \rfloor \rfloor \;/;\; x \in \mathbb{R} \bigwedge y \in \mathbb{R} \bigwedge y > 0$$

$$\sum_{k=1}^{p-1} \left\lfloor \frac{k\, q}{p} \right\rfloor == \frac{1}{2} (p-1)(q-1) \;/;\; p \in \mathbb{N}^+ \bigwedge q \in \mathbb{N}^+ \bigwedge \gcd(p, q) == 1$$

$$\sum_{k=1}^{n-1} \left(\frac{k}{n} - \left\lfloor \frac{k}{n} \right\rfloor - \frac{1}{2}\right)\left(\frac{k\, m}{n} - \left\lfloor \frac{m\, k}{n} \right\rfloor - \frac{1}{2}\right) == \frac{1}{4\, n} \sum_{k=1}^{n-1} \cot\left(\frac{k\, \pi}{n}\right) \cot\left(\frac{m\, k\, \pi}{n}\right) \;/;\; m \in \mathbb{N}^+ \bigwedge n \in \mathbb{N}^+ \bigwedge \gcd(m, n) == 1$$

$$\sum_{l=1}^{p-1} \sum_{k=1}^{p-1} \left\lfloor \frac{k\, l}{p} \right\rfloor == \left(\frac{p-1}{2}\right)^2 (p-2) \;/;\; p \in \mathbb{P}$$

$$\sum_{k=1}^{\frac{p-1}{2}} \left\lfloor \frac{q\, k}{p} \right\rfloor + \sum_{k=1}^{\frac{q-1}{2}} \left\lfloor \frac{p\, k}{q} \right\rfloor == \frac{1}{4} (p-1)(q-1) \;/;\; p \in \mathbb{N}^+ \bigwedge q \in \mathbb{N}^+ \bigwedge \gcd(p, q) == 1$$

$$\sum_{k=0}^{y} \left\lceil x - \frac{k}{y} \right\rceil == \lceil x\, y - \lceil x - 1 \rceil (y + \lceil -y \rceil) \rceil \;/;\; x \in \mathbb{R} \bigwedge y \in \mathbb{R} \bigwedge 0 < x < 1 \bigwedge 0 < y < 1$$

$$\sum_{k=0}^{n-1} \left\lceil \frac{1}{p} \left\lfloor \frac{k}{m} \right\rfloor \right\rceil == \left(n - m - \frac{p\, m}{2} \left\lfloor \frac{n-m}{p\, m} \right\rfloor\right) \left\lceil \frac{1}{p} \left\lfloor \frac{n}{m} \right\rfloor \right\rceil \;/;\; n \in \mathbb{N}^+ \bigwedge m \in \mathbb{N}^+ \bigwedge p \in \mathbb{N}^+$$

$$\sum_{n=1}^{\infty} \frac{\left\lfloor \frac{n}{m} \right\rfloor}{k^n} == \frac{k}{(k-1)(k^m - 1)} \;/;\; k \in \mathbb{N}^+ \bigwedge m \in \mathbb{N}^+.$$

Zeros

Zeros of rounding and congruence functions are given as follows:

$$\lfloor z \rfloor == 0 \;/;\; 0 \leq \mathrm{Re}(z) < 1 \land 0 \leq \mathrm{Im}(z) < 1$$

$$\lfloor z \rceil == 0 \;/;\; -\frac{1}{2} \leq \mathrm{Re}(z) \leq \frac{1}{2} \bigwedge -\frac{1}{2} \leq \mathrm{Im}(z) \leq \frac{1}{2}$$

$$\lceil z \rceil == 0 \;/;\; -1 < \mathrm{Re}(z) \leq 0 \land -1 < \mathrm{Im}(z) \leq 0$$

$$\mathrm{int}(z) == 0 \;/;\; |\mathrm{Re}(z)| < 1 \land |\mathrm{Im}(z)| < 1$$

$$\mathrm{frac}(z) == 0 \;/;\; \mathrm{Re}(z) \in \mathbb{Z} \lor \mathrm{Im}(z) \in \mathbb{Z}$$

$$m \bmod n == 0 \;/;\; m == 0 \land n \neq 0$$

$$\mathrm{quotient}(m, n) == 0 \;/;\; 0 \leq \mathrm{Re}\left(\frac{m}{n}\right) < 1 \bigwedge 0 \leq \mathrm{Im}\left(\frac{m}{n}\right) < 1.$$

Applications of the rounding and congruence functions

All rounding and congruence functions are used throughout mathematics, the exact sciences, and engineering.

Copyright