Introductions to FresnelS

Introduction to the Fresnel integrals

General

The Fresnel integrals appeared in the works by A. J. Fresnel (1798, 1818, 1826) who investigated an optical problem. Later K. W. Knochenhauer (1839) found series representations of these integrals. N. Nielsen (1906) studied various properties of these integrals.

Different authors used the same notations $S(z)$ and $C(z)$, but with slightly different definitions.

Definitions of Fresnel integrals

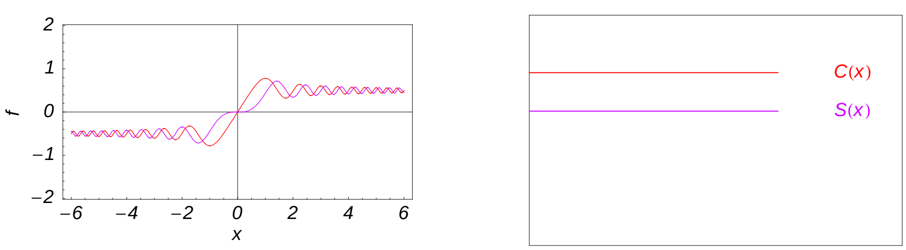
The Fresnel integrals $S(z)$ and $C(z)$ are defined as values of the following definite integrals:

$$S(z) = \int_0^z \sin\left(\frac{\pi t^2}{2}\right) dt$$

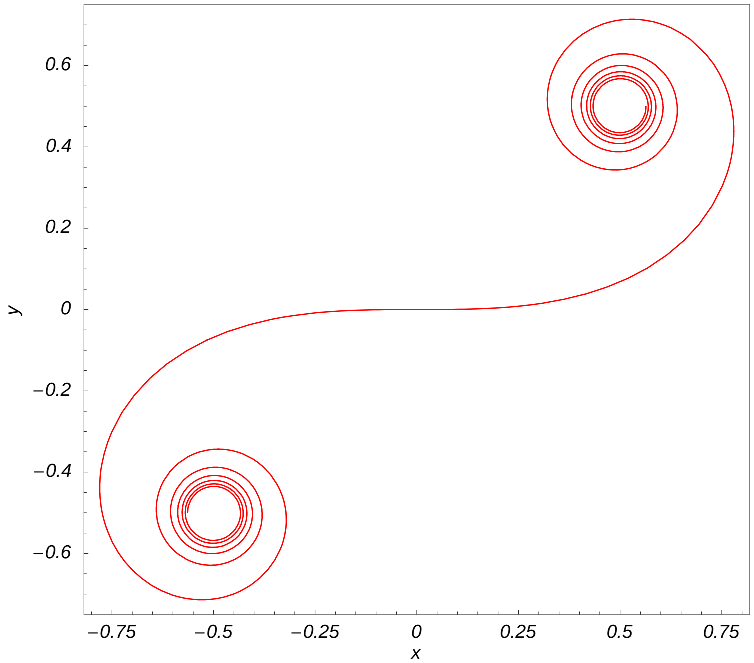
$$C(z) = \int_0^z \cos\left(\frac{\pi t^2}{2}\right) dt.$$

A quick look at the Fresnel integrals

Here is a quick look at the graphics for the Fresnel integrals along the real axis.



The parametrically described curve $\{C(s), S(s)\}$ with s ranging over a subset of the real axis gives the following characteristic spiral.



Connections within the group of Fresnel integrals and with other function groups

Representations through more general functions

The Fresnel integrals $S(z)$ and $C(z)$ are particular cases of the more general functions: hypergeometric and Meijer G functions.

For example, they can be represented through regularized hypergeometric functions ${}_1\tilde{F}_2$:

$$S(z) = \frac{\pi z^3}{6} \, {}_1F_2\left(\frac{3}{4}; \frac{3}{2}, \frac{7}{4}; -\frac{\pi^2 z^4}{16}\right)$$

$$C(z) = z \, {}_1F_2\left(\frac{1}{4}; \frac{1}{2}, \frac{5}{4}; -\frac{\pi^2 z^4}{16}\right).$$

These two integrals can also be expressed through generalized and classical Meijer G functions:

$$S(z) = \frac{1}{2} G_{1,3}^{1,1}\left(\frac{\sqrt{\pi}\, z}{2}, \frac{1}{4} \,\middle|\, \begin{matrix} 1 \\ \frac{3}{4}, 0, \frac{1}{4} \end{matrix}\right)$$

$$C(z) = \frac{1}{2} G_{1,3}^{1,1}\left(\frac{\sqrt{\pi}\, z}{2}, \frac{1}{4} \,\middle|\, \begin{matrix} 1 \\ \frac{1}{4}, 0, \frac{3}{4} \end{matrix}\right)$$

$$S(z) = \frac{\pi}{\sqrt{2}\, \left(z^2\right)^{3/4} (-z)^{3/4}} z^{9/4} G_{1,3}^{1,0}\left(-\frac{\pi^2 z^4}{16} \,\middle|\, \begin{matrix} 1 \\ \frac{3}{4}, \frac{1}{4}, 0 \end{matrix}\right)$$

$$C(z) == \frac{\pi z^{3/4}}{\sqrt{2}\,\sqrt[4]{z^2}\,\sqrt[4]{-z}}\, G_{1,3}^{1,0}\!\left(-\frac{\pi^2}{16} z^4 \,\middle|\, \begin{matrix} 1 \\ \frac{1}{4}, \frac{3}{4}, 0 \end{matrix}\right).$$

The first two formulas are simpler than the last two classical representations (which include factors like $z^{\nu+1} \left(z^2\right)^{-\frac{\nu+1}{2}}$).

Representations through related equivalent functions

The Fresnel integrals $S(z)$ and $C(z)$ can be represented through a combination of probability integrals $\operatorname{erf}(z_1) + c_1 \operatorname{erf}(z_2)$ with corresponding values of z_1, z_2, and c_1:

$$S(z) == \frac{1+i}{4}\left(\operatorname{erf}\left(\frac{1+i}{2}\sqrt{\pi}\, z\right) - i \operatorname{erf}\left(\frac{1-i}{2}\sqrt{\pi}\, z\right)\right)$$

$$C(z) == \frac{1-i}{4}\left(\operatorname{erf}\left(\frac{1+i}{2}\sqrt{\pi}\, z\right) + i \operatorname{erf}\left(\frac{1-i}{2}\sqrt{\pi}\, z\right)\right).$$

The best-known properties and formulas for Fresnel integrals

Real values for real arguments

For real values of argument z, the values of the Fresnel integrals $S(z)$ and $C(z)$ are real.

Simple values at zero and infinity

The Fresnel integrals $S(z)$ and $C(z)$ have simple values for arguments $z == 0$ and $z == \pm\infty$:

$$S(0) == 0 \quad S(\infty) == \frac{1}{2} \quad S(-\infty) == -\frac{1}{2}$$

$$C(0) == 0 \quad C(\infty) == \frac{1}{2} \quad C(-\infty) == -\frac{1}{2}.$$

Analyticity

The Fresnel integrals $S(z)$ and $C(z)$ are defined for all complex values of z, and they are analytical functions of z over the whole complex z-plane and do not have branch cuts or branch points. They are entire functions with an essential singular point at $z = \tilde{\infty}$.

Periodicity

The Fresnel integrals $S(z)$ and $C(z)$ do not have periodicity.

Parity and symmetry

The Fresnel integrals $S(z)$ and $C(z)$ are odd functions and have mirror symmetry:

$$S(-z) == -S(z) \quad S(\bar{z}) == \overline{S(z)}$$

$$C(-z) == -C(z) \quad C(\bar{z}) == \overline{C(z)}.$$

Series representations

The Fresnel integrals $S(z)$ and $C(z)$ have rather simple series representations at the origin:

$$S(z) \propto \frac{\pi}{6} z^3 \left(1 - \frac{\pi^2 z^4}{56} + \frac{\pi^4 z^8}{7040} - \ldots\right) /; (z \to 0)$$

$$C(z) \propto z \left(1 - \frac{\pi^2 z^4}{40} + \frac{\pi^4 z^8}{3456} - \ldots\right) /; (z \to 0).$$

These series converge at the whole z-plane and their symbolic forms are the following:

$$S(z) = z^3 \sum_{k=0}^{\infty} \frac{2^{-2k-1} \pi^{2k+1} \left(-z^4\right)^k}{(4k+3)(2k+1)!}$$

$$C(z) = z \sum_{k=0}^{\infty} \frac{2^{-2k} \pi^{2k} \left(-z^4\right)^k}{(4k+1)(2k)!}.$$

Interestingly, closed-form expressions for the truncated version of the Taylor series at the origin can be expressed through the generalized hypergeometric function ${}_2F_2$, for example:

$$C(z) = F_\infty(z) /; \left(\left(F_n(z) = z \sum_{k=0}^{n} \frac{2^{-2k} \pi^{2k} \left(-z^4\right)^k}{(4k+1)(2k)!} = C(z) + \frac{(-1)^n 4^{-n-1} \pi^{2(n+1)} z^{4n+5}}{(4n+5)(2n+2)!} \, {}_2F_3\left(1, n + \frac{5}{4}; n + \frac{3}{2}, n+2, n + \frac{9}{4}; -\frac{1}{16} \pi^2 z^4\right)\right) \bigwedge n \in \mathbb{N}\right).$$

Asymptotic series expansions

The asymptotic behavior of the Fresnel integrals $S(z)$ and $C(z)$ can be described by the following formulas (only the main terms of asymptotic expansion are given):

$$S(z) \propto \frac{\sqrt[4]{z^4}}{2z} - \frac{1}{\pi z} \cos\left(\frac{\pi z^2}{2}\right)\left(1 + O\left(\frac{1}{z^4}\right)\right) - \frac{1}{\pi^2 z^3} \sin\left(\frac{\pi z^2}{2}\right)\left(1 + O\left(\frac{1}{z^4}\right)\right) /; (|z| \to \infty)$$

$$C(z) \propto \frac{\left(z^4\right)^{3/4}}{2z^3} - \frac{1}{\pi^2 z^3} \cos\left(\frac{\pi z^2}{2}\right)\left(1 + O\left(\frac{1}{z^4}\right)\right) + \frac{1}{\pi z} \sin\left(\frac{\pi z^2}{2}\right)\left(1 + O\left(\frac{1}{z^4}\right)\right) /; (|z| \to \infty).$$

The previous formulas are valid in any directions of approaching point z to infinity ($|z| \to \infty$). In particular cases when $|\mathrm{Arg}(z)| < \pi$ and $\mathrm{Re}(z) \neq 0$, the formulas can be simplified to the following relations:

$$S(z) \propto \frac{z}{2z} - \frac{1}{\pi z} \cos\left(\frac{\pi z^2}{2}\right)\left(1 + O\left(\frac{1}{z^4}\right)\right) - \frac{1}{\pi^2 z^3} \sin\left(\frac{\pi z^2}{2}\right)\left(1 + O\left(\frac{1}{z^4}\right)\right) /; |\mathrm{Arg}(z)| < \pi \bigwedge \mathrm{Re}(z) \neq 0 \bigwedge (|z| \to \infty)$$

$$C(z) \propto \frac{z^3}{2z^3} - \frac{1}{\pi^2 z^3} \cos\left(\frac{\pi z^2}{2}\right)\left(1 + O\left(\frac{1}{z^4}\right)\right) + \frac{1}{\pi z} \sin\left(\frac{\pi z^2}{2}\right)\left(1 + O\left(\frac{1}{z^4}\right)\right) /; |\mathrm{Arg}(z)| < \pi \bigwedge \mathrm{Re}(z) \neq 0 \bigwedge \; \bigwedge (|z| \to \infty).$$

Integral representations

The Fresnel integrals $S(z)$ and $C(z)$ have the following simple integral representations through sine or cosine that directly follow from the definition of these integrals:

$$S(z) = \frac{1}{\sqrt{2\pi}} \int_0^{\frac{\pi z^2}{2}} \frac{\sin(t)}{\sqrt{t}} \, dt$$

$$C(z) = \frac{1}{\sqrt{2\pi}} \int_0^{\frac{\pi z^2}{2}} \frac{\cos(t)}{\sqrt{t}} \, dt.$$

Transformations

The argument of the Fresnel integrals $S(z)$ and $C(z)$ with square root arguments can sometimes be simplified:

$$S\left(\sqrt{z^2}\right) = \frac{(z^2)^{3/2}}{z^3} S(z)$$

$$C\left(\sqrt{z^2}\right) = \frac{\sqrt{z^2}}{z} C(z).$$

Simple representations of derivatives

The derivatives of the Fresnel integrals $S(z)$ and $C(z)$ are the sine or cosine functions with simple arguments:

$$\frac{\partial S(z)}{\partial z} = \sin\left(\frac{\pi z^2}{2}\right)$$

$$\frac{\partial C(z)}{\partial z} = \cos\left(\frac{\pi z^2}{2}\right).$$

The symbolic derivatives of the n^{th} order have the following representations:

$$\frac{\partial^n S(z)}{\partial z^n} = 2^{2n-\frac{11}{2}} \pi^{5/2} z^{3-n} \, {}_3\tilde{F}_4\left(\frac{3}{4}, 1, \frac{5}{4}; 1-\frac{n}{4}, \frac{5-n}{4}, \frac{6-n}{4}, \frac{7-n}{4}; -\frac{\pi^2 z^4}{16}\right) /; n \in \mathbb{N}^+$$

$$\frac{\partial^n C(z)}{\partial z^n} = 2^{2n-\frac{3}{2}} \pi^{3/2} z^{1-n} \, {}_3\tilde{F}_4\left(\frac{1}{4}, \frac{3}{4}, 1; \frac{2-n}{4}, \frac{3-n}{4}, 1-\frac{n}{4}, \frac{5-n}{4}; -\frac{\pi^2 z^4}{16}\right) /; n \in \mathbb{N}.$$

Simple of differential equations

The Fresnel integrals $S(z)$ and $C(z)$ satisfy the following third-order linear ordinary differential equation:

$$z\, w^{(3)}(z) - w''(z) + \pi^2 z^3 w'(z) = 0 /; w(z) = c_1 S(z) + c_2 C(z) + c_3.$$

They can be represented as partial solutions of the previous equation under the following corresponding initial conditions:

$$z\, w^{(3)}(z) - w''(z) + \pi^2 z^3 w'(z) = 0 /; w(z) = C(z) \bigwedge w(0) = 0 \bigwedge w'(0) = 1 \bigwedge w^{(3)}(0) = 0$$

$$z\, w^{(3)}(z) - w''(z) + \pi^2 z^3 w'(z) = 0 /; w(z) = C(z) \bigwedge w(0) = 0 \bigwedge w'(0) = 1 \bigwedge w^{(3)}(0) = 0.$$

Applications of Fresnel integrals

Applications of Fresnel integrals include Fraunhofer diffraction, asymptotics of Weyl sums, and railway and freeway constructions.

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