Introductions to InverseWeierstrassP

Introduction to the Weierstrass functions and inverses

General

Historical remarks

The Weierstrass elliptic functions are identified with the famous mathematicians N. H. Abel (1827) and K. Weierstrass (1855, 1862). In the year 1849, C. Hermite first used the notation $\wp$ for the basic Weierstrass doubly periodic function with only one double pole. The sigma and zeta Weierstrass functions were introduced in the works of F. G. Eisenstein (1847) and K. Weierstrass (1855, 1862, 1895).

The Weierstrass elliptic and related functions can be defined as inversions of elliptic integrals like $\mathcal{I} = \int_\infty^z \frac{1}{\sqrt{4\,t^3 - g_2\,t - g_3}}\,dt$ and $z = z(\mathcal{I};\, g_2,\, g_3)$. Such integrals were investigated in the works of L. Euler (1761) and J.-L. Lagrange (1769), who basically introduced the functions that are known today as the inverse Weierstrass functions.

Periodic functions

An analytic function $f(z)$ is called periodic if there exists a complex constant $\rho \neq 0$, such that $f(z + \rho) == f(z) /; z \in \mathbb{C}$. The number ρ (with a minimal possible value of $|\rho|$) is called the period of the function $f(z)$.

Examples of well-known singly periodic functions are the exponential functions, all the trigonometric and hyperbolic functions: e^z, sin(z), cos(z), csc(z), sec(z), tan(z), cot(z), sinh(z), cosh(z), csch(z), sech(z), tanh(z), and coth(z), which have periods $\rho = 2\,\pi\,i$, $\rho = 2\,\pi$, $\rho = \pi$, $\rho = 2\,\pi\,i$, and $\rho = \pi\,i$. The study of such functions can be restricted to any period-strip $\{z_0 + \alpha\,\rho /;\, 0 \leq \alpha < 1 \bigwedge z_0 \in \mathbb{C}\}$, because outside this strip, the values of these functions coincide with their corresponding values inside the strip.

Nonconstant analytic functions over the field of complex numbers cannot have more than two independent periods. So, generically, periodic functions can satisfy the following relations:

$$f(z + n\,\rho) == f(z) /;\, n \in \mathbb{Z}$$

$$f(z + m\,\rho_1 + n\,\rho_2) == f(z) /;\, \{m,\, n\} \in \mathbb{Z} \bigwedge \mathrm{Im}\left(\frac{\rho_1}{\rho_2}\right) \neq 0,$$

where ρ, ρ_1, and ρ_2 are periods (basic primitive periods). The condition $\mathrm{Im}\left(\frac{\rho_1}{\rho_2}\right) \neq 0$ for doubly periodic functions implies the existence of a period-parallelogram $\{z_0 + \alpha_1 \rho_1 + \alpha_2 \rho_2 /; 0 \leq \alpha_1 < 1 \bigwedge 0 \leq \alpha_1 < 1 \bigwedge z_0 \in \mathbb{C}\}$, which is the analog of the period-strip $\{z_0 + \alpha \rho /; 0 \leq \alpha < 1 \bigwedge z_0 \in \mathbb{C}\}$ for singly periodic functions with period ρ.

In the case $z_0 == 0 \bigwedge \mathrm{Im}\left(\frac{\rho_1}{\rho_2}\right) > 0$, this parallelogram is called the basic fundamental period-parallelogram: $\mathrm{P}_{0,0} == \{\alpha_1 \rho_1 + \alpha_2 \rho_2 /; 0 \leq \alpha_1 < 1 \bigwedge 0 \leq \alpha_2 < 1\}$. The two line segments $\{\alpha_i \rho_i /; 0 \leq \alpha_i < 1\} /; i == 1, 2$ lying on the boundary of the period-parallelogram and beginning from the origin 0 belong to $\mathrm{P}_{0,0}$. The region $\mathrm{P}_{0,0}$ includes only one corner point 0 from four points lying at the boundary of the parallelogram with corners in $\{0, \rho_1, \rho_1 + \rho_2, \rho_2\}$.
Sometimes the convention $\mathrm{P}_{0,0} == \{\alpha_1 \rho_1 + \alpha_2 \rho_2 /; -1/2 \leq \alpha_1 < 1/2 \bigwedge -1/2 \leq \alpha_2 < 1/2\}$ is used.

The set of all such period-parallelograms:

$$\mathrm{P}_{m,n} == \{m \rho_1 + n \rho_2 + \alpha_1 \rho_1 + \alpha_2 \rho_2 /; 0 \leq \alpha_1 < 1 \bigwedge 0 \leq \alpha_2 < 1 \bigwedge m \in \mathbb{Z} \bigwedge n \in \mathbb{Z}\}$$

covers all complex planes: $\mathbb{C} == \{\mathrm{P}_{m,n} \mid -\infty < m, n < \infty \bigwedge m \in \mathbb{Z} \bigwedge n \in \mathbb{Z}\}$.

Any doubly periodic function $\mathcal{E}(z)$ is called an elliptic function. The set of numbers $m \rho_1 + n \rho_2 /; \{m, n\} \in \mathbb{Z}$ is called the period-lattice for elliptic function $\mathcal{E}(z)$.

An elliptic function $\mathcal{E}(z)$, which does not have poles in the period-parallelogram, is equal to a constant (Liouville's theorem).

Nonconstant elliptic (doubly periodic) functions $\mathcal{E}(z)$ cannot be entire functions. This is not the case for singly periodic functions, for example, $\sin(z)$ is entire function.

Any nonconstant elliptic function $\mathcal{E}(z)$ has at least two simple poles or at least one double pole in any period -parallelogram. The sum of all its residues at the poles inside a period-parallelogram is zero.

The numbers of the zeros and poles of a nonconstant elliptic function $\mathcal{E}(z)$ in a fundamental period-parallelogram P are finite.

The number of the zeros of $\mathcal{E}(z) - A$, where A is any complex number, in a fundamental period-parallelogram $\mathrm{P}_{0,0}$ does not depend on the value A and coincides with number s of the poles $b_1, b_2, \ldots, b_s$ counted according to their multiplicity (s is called the order of the elliptic function $\mathcal{E}(z)$).

The simplest elliptic function has order 2.

Let $a_1, a_2, \ldots, a_r$ (and $b_1, b_2, \ldots, b_s$) be the zeros (and poles) of a nonconstant elliptic function $\mathcal{E}(z)$ in a fundamental period-parallelogram $P_{0,0}$, both listed one or more times according to their multiplicity. Then the following hold:

$r == s$

$$\sum_{j=1}^{r} a_j - \sum_{k=1}^{s} b_k == \mu\ \rho_1 + \nu\,\rho_2 \ /;\ a_j \in P_{0,0} \bigwedge b_k \in P_{0,0} \bigwedge \mathcal{E}(a_j) == 0 \bigwedge 1/\mathcal{E}(b_k) == 0 \bigwedge \mu \in \mathbb{Z} \bigwedge \nu \in \mathbb{Z}.$$

So, the number of zeros of a nonconstant elliptic function $\mathcal{E}(z)$ in the fundamental period-parallelogram $P_{0,0}$ is equal to the number of poles there and counted according to their multiplicity. The sum of zeros of a nonconstant elliptic function $\mathcal{E}(z)$ in the fundamental period-parallelogram $P_{0,0}$ differs from the sum of its poles by a period $\mu\ \rho_1 + \nu\,\rho_2$, where $\mu \in \mathbb{Z} \bigwedge \nu \in \mathbb{Z}$ and the values of μ, ν depend on the function $\mathcal{E}(z)$.

All elliptic functions $\mathcal{E}(z)$ satisfy a common fundamental property, which generalizes addition, duplication, and multiple angle properties for trigonometric and hyperbolic functions (like $\sin(z_1 + z_2)$, $\sin(n\,z) \ /;\ n \in \mathbb{N}^+$). It can be formulated as the following:

$$\mathcal{E}\left(\textstyle\sum_{k=1}^{n} z_k\right).$$

It can also be expressed as an algebraic function of $\mathcal{E}(z_k) \ /;\ 1 \le k \le n$.

In other words, there exists an irreducible polynomial $C(t_1, t_2, \ldots, t_{n+1})$ in $n+1$ variables with constant coefficients, for which the following relation holds:

$$C\left(\mathcal{E}(z_1), \mathcal{E}(z_2), \ldots, \mathcal{E}(z_n), \mathcal{E}\left(\sum_{k=1}^{n} z_k\right)\right) == 0.$$

And conversely, among all smooth functions, only elliptic functions and their degenerations have algebraic addition theorems.

The simplest elliptic functions (with order 2) can be divided into two classes:

(1) Functions that at the period-parallelogram $P_{0,0}$ have only a double pole with residue zero (e.g., the Weierstrass elliptic functions $\wp(z; g_2, g_3)$).

(2) Functions that in the period-parallelogram $P_{0,0}$ have only two simple poles with residues, which are equal in absolute value but opposite in sign (e.g., Jacobian elliptic functions $\mathrm{sn}(z \mid m)$ etc.).

Any elliptic function $\mathcal{E}(z)$ with periods ρ_1 and ρ_2 can be expressed as a rational function of the Weierstrassian elliptic
functions $\wp(z; g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3))$ and their derivative $\wp'(z; g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3))$ with the same periods $\rho_1 \equiv 2\,\omega_1,\ \rho_2 \equiv 2\,\omega_3$.

The Weierstrass elliptic function $\wp(z; g_2, g_3)$ arises as a solution to the following ordinary nonlinear differential equation:

$$w''(z) \equiv -\frac{g_2}{2} + 6\,w(z)^2 \;/;\; w(z) \equiv \wp(a+z; g_2, g_3).$$

Definitions of Weierstrass functions and inverses

The Weierstrass elliptic $\wp$ function $\wp(z; g_2, g_3)$, its derivative $\wp'(z; g_2, g_3)$, the Weierstrass sigma function $\sigma(z; g_2, g_3)$, associated Weierstrass sigma functions $\sigma_n(z, g_2, g_3) \;/;\; n \in \{1, 2, 3\}$, Weierstrass zeta function $\zeta(z; g_2, g_3)$, inverse elliptic Weierstrass $\wp$ function $\wp^{-1}(z; g_2, g_3)$, and generalized inverse Weierstrass $\wp$ function $\wp^{-1}(z_1, z_2; g_2, g_3)$ are defined by the following formulas:

$$\wp(z; g_2, g_3) \equiv \frac{1}{z^2} + \sum_{\substack{m,n=-\infty\\ \{m,n\}\neq\{0,0\}}}^{\infty} \frac{1}{(z - 2\,m\,\omega_1 - 2\,n\,\omega_3)^2} - \frac{1}{(2\,m\,\omega_1 + 2\,n\,\omega_3)^2} \;/;\; \{\omega_1, \omega_3\} \equiv$$

$$\{\omega_1(g_2, g_3), \omega_3\,(g_2, g_3)\} \equiv \left\{ i \left(\frac{60}{g_2} \sum_{\substack{m,n=-\infty\\ \{m,n\}\neq\{0,0\}}}^{\infty} \frac{1}{(2\,m + 2\,n\,t)^4} \right)^{1/4},\ i\,t \left(\frac{60}{g_2} \sum_{\substack{m,n=-\infty\\ \{m,n\}\neq\{0,0\}}}^{\infty} \frac{1}{(2\,m + 2\,n\,t)^4} \right)^{1/4} \right\} \;/;\; J(t) \equiv \frac{g_2^3}{g_2^3 - 27\,g_3^2}$$

$$\wp'(z; g_2, g_3) \equiv -2 \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{(z - 2\,m\,\omega_1 - 2\,n\,\omega_3)^3} \;/;\; \{\omega_1, \omega_3\} \equiv \{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\}$$

$$\sigma(z; g_2, g_3) \equiv z \prod_{\substack{m,n=-\infty\\ \{m,n\}\neq\{0,0\}}}^{\infty} \left(1 - \frac{z}{2\,m\,\omega_1 + 2\,n\,\omega_3}\right) \exp\left(\frac{z^2}{2\,(2\,m\,\omega_1 + 2\,n\,\omega_3)^2} + \frac{z}{2\,m\,\omega_1 + 2\,n\,\omega_3} \right) \;/;\; \{\omega_1, \omega_3\} \equiv \{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\}$$

$$\sigma_n(z, g_2, g_3) \equiv \frac{e^{-\eta_n z}\,\sigma(z + \omega_n; g_2, g_3)}{\sigma(\omega_n; g_2, g_3)} \;/;$$
$$\{\omega_1, \omega_2, \omega_3\} \equiv \{\omega_1(g_2, g_3), -\omega_1(g_2, g_3) - \omega_3(g_2, g_3), \omega_3(g_2, g_3)\} \wedge \eta_n \equiv \zeta(\omega_n; g_2, g_3) \wedge n \in \{1, 2, 3\}$$

$$\zeta(z; g_2, g_3) \equiv \frac{1}{z} + \sum_{\substack{m,n=-\infty\\ \{m,n\}\neq\{0,0\}}}^{\infty} \frac{z}{(2\,m\,\omega_1 + 2\,n\,\omega_3)^2} + \frac{1}{2\,m\,\omega_1 + 2\,n\,\omega_3} + \frac{1}{z - 2\,m\,\omega_1 - 2\,n\,\omega_3} \;/;\; \{\omega_1, \omega_3\} \equiv \{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\}$$

$$z \equiv \wp(w; g_2, g_3) \;/;\; w \equiv \wp^{-1}(z; g_2, g_3)$$

$$\wp^{-1}(z; g_2, g_3) \equiv \int_{\infty}^{z} \frac{1}{\sqrt{4\,t^3 - g_2\,t - g_3}}\,dt \;/;\; z \in \mathbb{R} \bigwedge \mathrm{Re}\left(4\,z^3 - g_2\,z - g_3\right) > 0$$

$$z_1 \equiv \wp(w; g_2, g_3) \;/;\; w \equiv \wp^{-1}(z_1, z_2; g_2, g_3) \bigwedge z_2 \equiv \sqrt{4\,z_1^3 - g_2\,z_1 - g_3}$$

$$\wp^{-1}(z_1, z_2; g_2, g_3) == \int_{\infty}^{z_1} \frac{1}{\sqrt{4 t^3 - g_2 t - g_3}} \, dt \,/;\, z_2 == \sqrt{4 z_1^3 - g_2 z_1 - g_3} \, .$$

The function $\wp^{-1}(z_1, z_2; g_2, g_3)$ is the unique value of u for which $z_1 = \wp(u; g_2, g_3)$ and $z_2 = \wp'(u; g_2, g_3)$. For the existence of $\wp^{-1}(z_1, z_2; g_2, g_3)$, the values z_1 and z_2 must be related by $z_2^2 = 4 z_1^3 - g_2 z_1 - g_3$.

The previous nine functions are typically called Weierstrass elliptic functions. The last two functions are called inverse elliptic Weierstrass functions.

Despite the commonly used naming convention, only the Weierstrass function $\wp(z; g_2, g_3)$ and its derivative $\wp'(z; g_2, g_3)$ are elliptic functions because only these functions are doubly periodic. The other Weierstrass functions $\sigma(z; g_2, g_3)$, $\sigma_n(z, g_2, g_3) \,/;\, n \in \{1, 2, 3\}$, and $\zeta(z; g_2, g_3)$ are not elliptic functions because they are only quasi-periodic functions with respect to z. But historically they are also placed into the class of elliptic functions.

The Weierstrass half-periods $\{\omega_1, \omega_3\} == \{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\}$ and the invariants $\{g_2, g_3\} == \{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\}$, the Weierstrass $\wp$ function values at half-periods $\{e_1, e_2, e_3\} == \{e_1(g_2, g_3), e_2(g_2, g_3), e_3(g_2, g_3)\}$, and the Weierstrass zeta function values at half-periods $\{\eta_1, \eta_2, \eta_3\} == \{\eta_1(g_2, g_3), \eta_2(g_2, g_3), \eta_3(g_2, g_3)\}$ are defined by the following formulas. The description of the Weierstrass functions follows the notations used throughout. The left-hand sides indicate that ω_1 and ω_3 are either independent variables or depend on g_2 and g_3, or vice versa:

$$\{\omega_1, \omega_3\} == \{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\} == \left\{ i \left(\frac{60}{g_2} \sum_{\substack{m, n=-\infty \\ \{m,n\} \neq \{0,0\}}}^{\infty} \frac{1}{(2m + 2nt)^4} \right)^{1/4}, \, i\, t \left(\frac{60}{g_2} \sum_{\substack{m, n=-\infty \\ \{m,n\} \neq \{0,0\}}}^{\infty} \frac{1}{(2m + 2nt)^4} \right)^{1/4} \right\} \,/;\, J(t) == \frac{g_2^3}{g_2^3 - 27 g_3^2}$$

$$\{g_2, g_3\} == \{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\} == \left\{ 60 \sum_{\substack{m, n=-\infty \\ \{m,n\} \neq \{0,0\}}}^{\infty} \frac{1}{(2m\omega_1 + 2n\omega_3)^4}, \, 140 \sum_{\substack{m, n=-\infty \\ \{m,n\} \neq \{0,0\}}}^{\infty} \frac{1}{(2m\omega_1 + 2n\omega_3)^6} \right\} \,/;\, \mathrm{Im}\left(\frac{\omega_3}{\omega_1}\right) > 0$$

$$\{e_1, e_2, e_3\} == \{e_1(g_2, g_3), e_2(g_2, g_3), e_3(g_2, g_3)\} == \{\wp(\omega_1; g_2, g_3), \wp(\omega_2; g_2, g_3), \wp(\omega_3; g_2, g_3)\} \,/;\, \{\omega_1, \omega_3\} == \{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\} \bigwedge \omega_2 == -\omega_1 - \omega_3$$

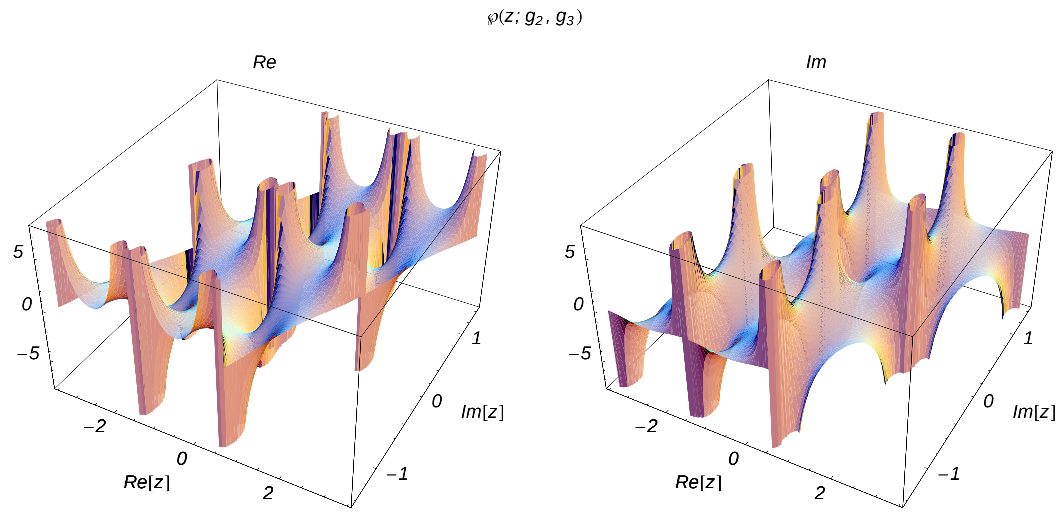
$$\{\eta_1, \eta_2, \eta_3\} == \{\eta_1(g_2, g_3), \eta_2(g_2, g_3), \eta_3(g_2, g_3)\} == \{\zeta(\omega_1; g_2, g_3), \zeta(\omega_2; g_2, g_3), \zeta(\omega_3; g_2, g_3)\} \,/;\, \{\omega_1, \omega_3\} == \{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\} \bigwedge \omega_2 == -\omega_1 - \omega_3,$$

where $J(t)$ is the Klein invariant modular function, $\wp(z; g_2, g_3)$ is the Weierstrass elliptic $\wp$ function, and $\zeta(z; g_2, g_3)$ denotes the Weierstrass zeta function.

A quick look at the Weierstrass functions and inverses

Here is a quick look at the graphics for the Weierstrass functions and
inverses. All of the following graphics use the half-periods $\{\omega_1, \omega_3\} = \left\{1, (1 + i) / \sqrt{2}\right\}$.

The next pair of graphics shows the Weierstrass ℘ function over the complex *z*-plane. The double periodicity of the function and the poles of
order 2 are clearly visible.

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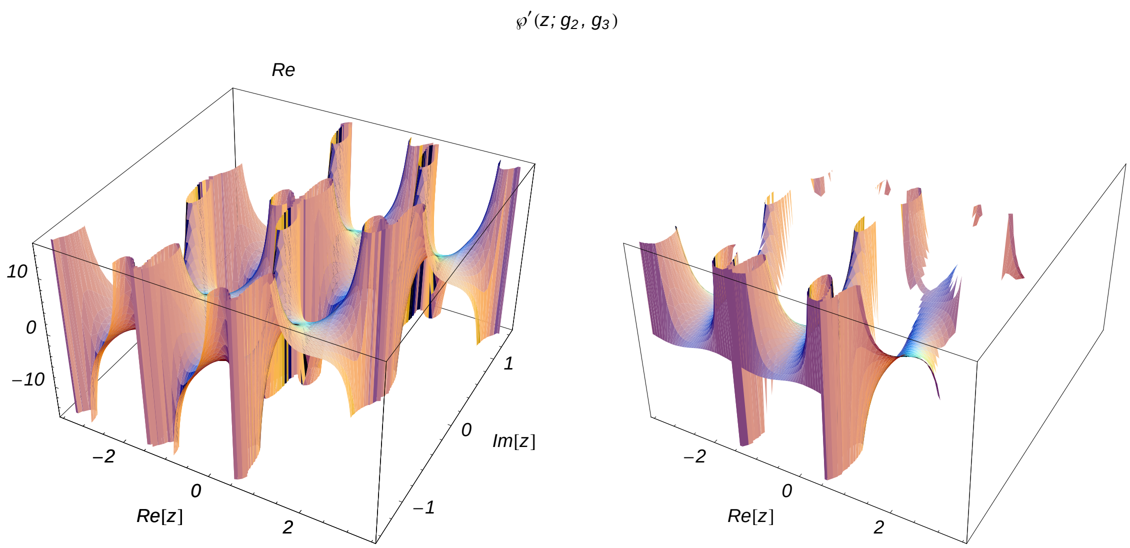
- GraphicsArray -
```

The next pair of graphics shows the derivative of the Weierstrass ℘ function over the complex *z*-plane. The double periodicity of the function and the poles of
order 3 are clearly visible.

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- GraphicsArray -
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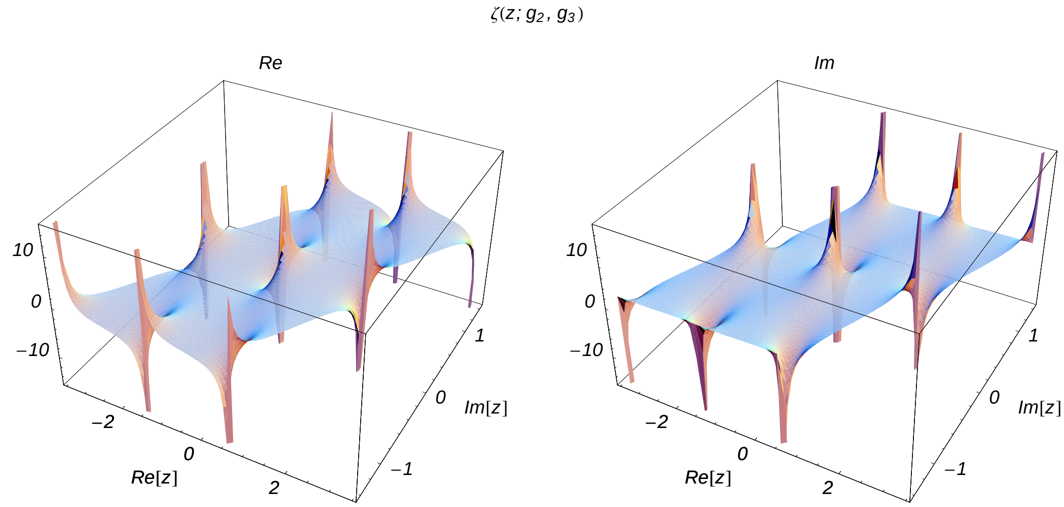
The next pair of graphics shows the Weierstrass zeta function over the complex *z*-plane. The pseudo-double periodicity of the function and the poles of order 1 are clearly visible.



- GraphicsArray -

The next pair of graphics shows the Weierstrass sigma function over the complex *z*-plane.

```
Show[GraphicsArray[

ParametricPlot3D[{Re[s ω1 + t ω3], Im[s ω1 + t
ω3], #[ WeierstrassSigma[s ω1 + t ω3, {g2, g3}] ],

                  {EdgeForm[]}}, {s, -3.5, 3.5}, {t, -3.5, 3.5},

                  PlotPoints -> 72, BoxRatios -> {1, 1, 0.6}, PlotLabel -> #,

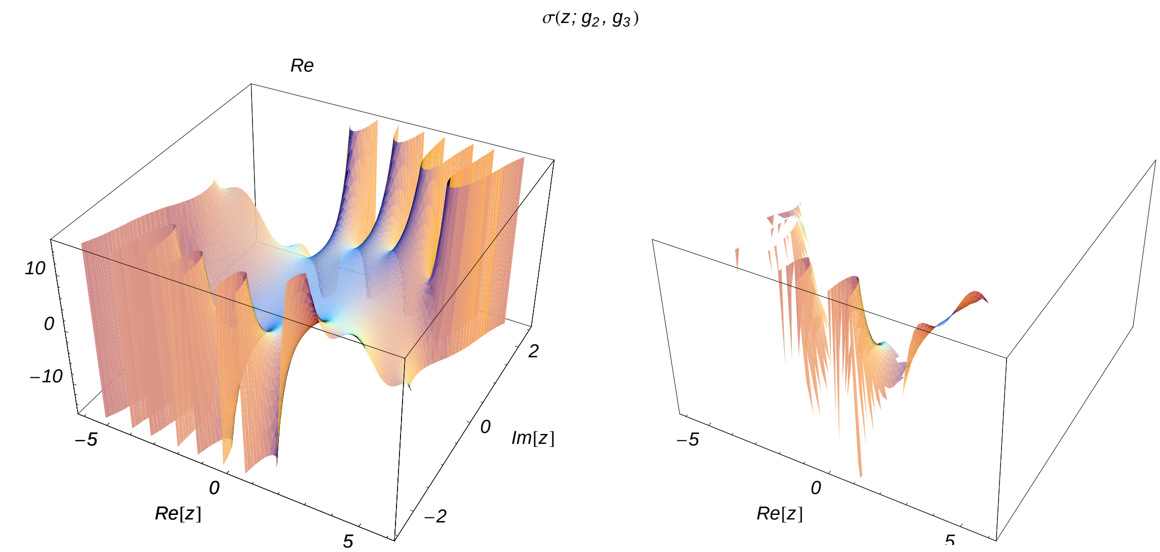
                  DisplayFunction -> Identity,

                 PlotRange -> {All, All, {-16, 16}}, AxesLabel -> {Re[z],
Im[z], None}]& /@

                 {Re, Im}],

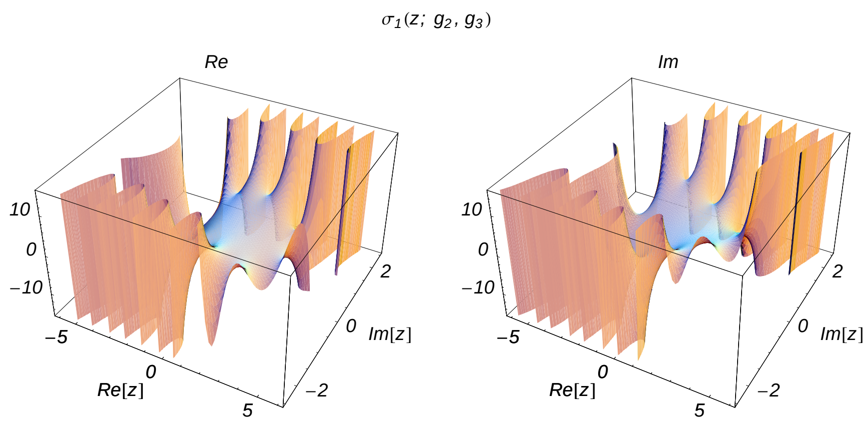
                 PlotLabel -> StyleForm[TraditionalForm @

                               WeierstrassSigma[z, {Subscript[g, 2],
Subscript[g, 3]}]]]
```

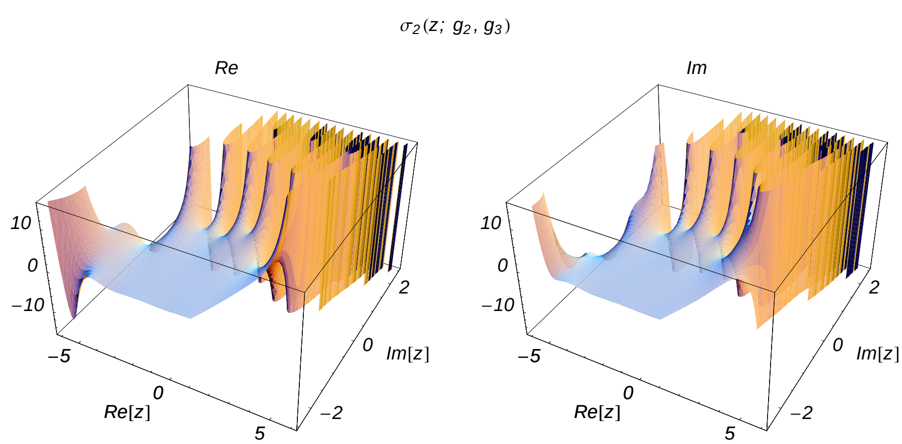


- GraphicsArray -

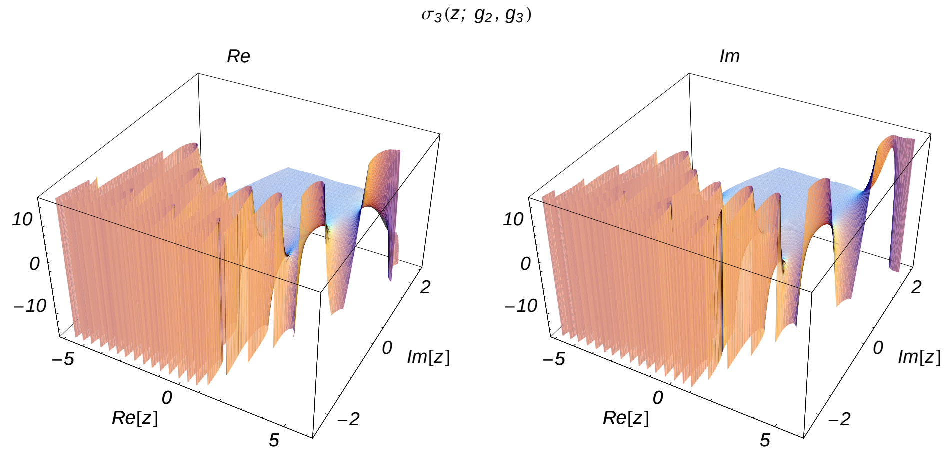
The next three pairs of graphics show the associated Weierstrass sigma functions over the complex *z*-plane.



- GraphicsArray -



- GraphicsArray -



- GraphicsArray -

The last pair of graphics shows the inverse of the Weierstrass ℘ function over the complex z-plane. Compared with the direct function, it is relatively
structureless.

- GraphicsArray -

Connections within the group of Weierstrass functions and inverses and with other function groups

Representations through more general functions

The Weierstrass elliptic ℘ function $\wp(z; g_2, g_3)$ and its inverse $\wp^{-1}(z; g_2, g_3)$ can be represented through the more general hypergeometric Appell F_1 function of two variables by the following formulas:

$$\wp(z; g_2, g_3) == w \,/;\, z == -\frac{1}{\sqrt{w-e_1}} F_1\left(\frac{1}{2}; \frac{1}{2}, \frac{1}{2}; \frac{3}{2}; \frac{e_2-e_1}{w-e_1}, \frac{e_3-e_1}{w-e_1}\right) \bigwedge 4 w^3 - g_2 w - g_3 == 4 (w-e_1)(w-e_2)(w-e_3)$$

$$\wp^{-1}(z; g_2, g_3) == -\frac{1}{\sqrt{z-r_1}} F_1\left(\frac{1}{2}; \frac{1}{2}, \frac{1}{2}; \frac{3}{2}; \frac{r_2-r_1}{z-r_1}, \frac{r_3-r_1}{z-r_1}\right) /;$$
$$4 z^3 - g_2 z - g_3 == 4 (z-r_1)(z-r_2)(z-r_3) \bigwedge \mathrm{Re}\left(4 z^3 - g_2 z - g_3\right) > 0.$$

Representations through related equivalent functions

The Weierstrass functions $\wp(z; g_2, g_3)$, $\wp'(z; g_2, g_3)$, $\sigma(z; g_2, g_3)$, $\sigma_n(z, g_2, g_3) /; n \in \{1, 2, 3\}$, $\zeta(z; g_2, g_3)$, $\wp^{-1}(z; g_2, g_3)$, and $\wp^{-1}(z_1, z_2; g_2, g_3)$ can be represented through some related equivalent functions, for example, through Jacobi functions:

$$\wp(z; g_2, g_3) == (e_1 - e_3)\, \mathrm{ns}\left(\sqrt{e_1 - e_3}\; z \,\middle|\, \frac{e_2-e_3}{e_1-e_3}\right)^2 + e_3$$

$$\wp(z; g_2, g_3) == e_3 + \frac{e_1 - e_3}{\operatorname{sn}\left(z \sqrt{e_1 - e_3} \,\middle|\, m\right)^2} \;/;\; m == \lambda\left(\frac{\omega_3}{\omega_1}\right)$$

$$\wp'(z; g_2, g_3) == -2\,(e_1 - e_3)^{3/2}\, \frac{\operatorname{cn}\left(\sqrt{e_1 - e_3}\; z \,\middle|\, m\right) \operatorname{dn}\left(\sqrt{e_1 - e_3}\; z \,\middle|\, m\right)}{\operatorname{sn}\left(\sqrt{e_1 - e_3}\; z \,\middle|\, m\right)^3} \;/;\; m == \lambda\left(\frac{\omega_3}{\omega_1}\right),$$

where $\lambda\left(\frac{\omega_3}{\omega_1}\right)$ is modular lambda function, or through theta functions:

$$\wp(z; g_2, g_3) == e_i + \frac{\pi^2}{4\,\omega_1^2} \left(\frac{\vartheta_1'(0, q)\,\vartheta_{i+1}\left(\frac{\pi z}{2\,\omega_1}, q\right)}{\vartheta_{i+1}(0, q)\,\vartheta_1\left(\frac{\pi z}{2\,\omega_1}, q\right)}\right)^2 \;/;\; i \in \{1, 2, 3\} \bigwedge q == \exp\left(\frac{\pi\, i\, \omega_3}{\omega_1}\right)$$

$$\wp(z; g_2, g_3) == \frac{\pi^2\, \vartheta_1^{(0,2,0)}(1, 0, q)}{12\, \omega_1\, \vartheta_1'(0, q)} - \frac{\partial^2 \log\left(\vartheta_1\left(\frac{\pi z}{2\,\omega_1}, q\right)\right)}{\partial z^2} \;/;\; q == \exp\left(\frac{\pi\, i\, \omega_3}{\omega_1}\right)$$

$$\wp'(z; g_2, g_3) == -\frac{\pi^3}{4\,\omega_1^3}\, \frac{\vartheta_2\left(\frac{\pi z}{2\,\omega_1}, q\right) \vartheta_3\left(\frac{\pi z}{2\,\omega_1}, q\right) \vartheta_4\left(\frac{\pi z}{2\,\omega_1}, q\right) \vartheta_1'(0, q)^3}{\vartheta_2(0, q)\, \vartheta_3(0, q)\, \vartheta_4(0, q)\, \vartheta_1\left(\frac{\pi z}{2\,\omega_1}, q\right)^3} \;/;\; q == \exp\left(\frac{\pi\, i\, \omega_3}{\omega_1}\right)$$

$$\sigma(z; g_2, g_3) == \frac{\omega_1}{\pi}\, \frac{1}{\sqrt[4]{q}} \left(\prod_{n=1}^{\infty} \frac{1}{1 - q^{2n}}\right)^3 \exp\left(\frac{\eta_1 z^2}{2\,\omega_1}\right) \vartheta_1\left(\frac{\pi z}{2\,\omega_1}, q\right) \;/;\; q == \exp\left(\frac{\pi\, i\, \omega_3}{\omega_1}\right)$$

$$\sigma(z; g_2, g_3) == \frac{2\,\omega_1}{\pi} \exp\left(\frac{\eta_1 z^2}{2\,\omega_1}\right) \frac{\vartheta_1\left(\frac{\pi z}{2\,\omega_1}, q\right)}{\vartheta_1'(0, q)} \;/;\; q == \exp\left(\frac{\pi\, i\, \omega_3}{\omega_1}\right)$$

$$\sigma_1(z; g_2, g_3) == \frac{1}{2\,\sqrt[4]{q}} \left(\prod_{n=1}^{\infty} \frac{1}{\left(1 - q^{4n}\right)\left(1 + q^{2n}\right)}\right) \exp\left(\frac{\eta_1 z^2}{2\,\omega_1}\right) \vartheta_2\left(\frac{\pi z}{2\,\omega_1}, q\right) \;/;\; q == \exp\left(\frac{\pi\, i\, \omega_3}{\omega_1}\right)$$

$$\sigma_2(z; g_2, g_3) == \left(\prod_{n=1}^{\infty} \frac{1}{\left(1 - q^{2n}\right)\left(1 + q^{2n-1}\right)^2}\right) \exp\left(\frac{\eta_1 z^2}{2\,\omega_1}\right) \vartheta_3\left(\frac{\pi z}{2\,\omega_1}, q\right) \;/;\; q == \exp\left(\frac{\pi\, i\, \omega_3}{\omega_1}\right)$$

$$\sigma_3(z; g_2, g_3) == \left(\prod_{n=1}^{\infty} \frac{1}{\left(1 - q^{2n}\right)\left(1 - q^{2n-1}\right)^2}\right) \exp\left(\frac{\eta_1 z^2}{2\,\omega_1}\right) \vartheta_4\left(\frac{\pi z}{2\,\omega_1}, q\right) \;/;\; q == \exp\left(\frac{\pi\, i\, \omega_3}{\omega_1}\right)$$

$$\sigma_i(u; g_2, g_3) == \exp\left(\frac{\eta_1 z^2}{2\,\omega_1}\right) \frac{\vartheta_{i+1}\left(\frac{\pi z}{2\,\omega_1}, q\right)}{\vartheta_{i+1}(0, q)} \;/;\; i \in \{1, 2, 3\} \bigwedge q == \exp\left(\frac{\pi\, i\, \omega_3}{\omega_1}\right)$$

$$\zeta(z; g_2, g_3) == \frac{z\,\eta_1}{\omega_1} + \frac{\pi\, \vartheta_1'\left(\frac{\pi z}{2\,\omega_1}, q\right)}{2\,\omega_1\, \vartheta_1\left(\frac{\pi z}{2\,\omega_1}, q\right)} \;/;\; q == \exp\left(\frac{\pi\, i\, \omega_3}{\omega_1}\right)$$

$$\zeta(z+\omega_i;\,g_2,\,g_3) == \eta_i + \frac{z\,\eta_1}{\omega_1} + \frac{\pi\,\vartheta'_{i+1}\!\left(\frac{\pi z}{2\,\omega_1},\,q\right)}{2\,\omega_1\,\vartheta_{i+1}\!\left(\frac{\pi z}{2\,\omega_1},\,q\right)}\ /;\ i\in\{1,2,3\}\bigwedge q == \exp\!\left(\frac{\pi\,i\,\omega_3}{\omega_1}\right)$$

$$\zeta(\omega_1;\,g_2,\,g_3) == -\frac{\pi^2}{12\,\omega_1}\,\frac{\vartheta_1^{(3)}(0,\,q)}{\vartheta'_1(0,\,q)}\ /;\ q == \exp\!\left(\frac{\pi\,i\,\omega_3}{\omega_1}\right),$$

or through elliptic integrals and the inverse elliptic nome:

$$\{\zeta(\omega_1;\,g_2,\,g_3),\,\zeta(\omega_3;\,g_2,\,g_3)\} == \left\{\sqrt{e_1-e_3}\left(E(m)-\frac{e_1}{e_1-e_3}K(m)\right),\,-i\,\sqrt{e_1-e_3}\left(E(1-m)+\frac{e_3}{e_1-e_3}K(1-m)\right)\right\}\ /;$$
$$m == q^{-1}\!\left(\exp\!\left(\frac{\pi\,i\,\omega_3}{\omega_1}\right)\right)\bigwedge\{e_1,\,e_3\} == \{e_1(g_2,\,g_3),\,e_3(g_2,\,g_3)\}.$$

Relations to inverse functions

The Weierstrass function $\wp(z;\,g_2,\,g_3)$ and its derivative $\wp'(z;\,g_2,\,g_3)$ are interconnected with the inverse functions $\wp^{-1}(z;\,g_2,\,g_3)$ and $\wp^{-1}(z_1,\,z_2;\,g_2,\,g_3)$ by the following formulas:

$$\wp\!\left(\wp^{-1}(z;\,g_2,\,g_3);\,g_2,\,g_3\right) == z$$

$$\wp\!\left(\wp^{-1}(z_1,\,z_2;\,g_2,\,g_3);\,g_2,\,g_3\right) == z_1\ /;\ z_2 == \sqrt{4\,z_1^3-g_2\,z_1-g_3}$$

$$\wp'\!\left(\wp^{-1}(z_1,\,z_2;\,g_2,\,g_3);\,g_2,\,g_3\right) == z_2\ /;\ z_2 == \sqrt{4\,z_1^3-g_2\,z_1-g_3}\,.$$

Representations through other Weierstrass functions

Each of the Weierstrass functions $\wp(z;\,g_2,\,g_3)$, $\wp'(z;\,g_2,\,g_3)$, $\sigma(z;\,g_2,\,g_3)$, $\sigma_n(z,\,g_2,\,g_3)\ /;\ n\in\{1,2,3\}$, and $\zeta(z;\,g_2,\,g_3)$ can be expressed through the other Weierstrass functions using numerous formulas, for example:

$$\wp(z;\,g_2,\,g_3) == -\frac{\partial\zeta(z;\,g_2,\,g_3)}{\partial z}$$

$$48\,\wp(z;\,g_2,\,g_3)^4 - 24\,g_2\,\wp(z;\,g_2,\,g_3)^2 - 48\,g_3\,\wp(z;\,g_2,\,g_3) == g_2^2 + \frac{16\,\sigma(3\,z;\,g_2,\,g_3)}{\sigma(z;\,g_2,\,g_3)^9}$$

$$\wp(z;\,g_2,\,g_3) == e_i + \frac{\sigma_i(z;\,g_2,\,g_3)^2}{\sigma(z;\,g_2,\,g_3)^2}\ /;\ i\in\{1,2,3\}$$

$$\wp(z_1;\,g_2,\,g_3) - \wp(z_2;\,g_2,\,g_3) == -\frac{\sigma(z_1+z_2;\,g_2,\,g_3)\,\sigma(z_1-z_2;\,g_2,\,g_3)}{\sigma(z_1;\,g_2,\,g_3)^2\,\sigma(z_2;\,g_2,\,g_3)^2}$$

$$\wp(z;\,g_2,\,g_3) == -\frac{\sigma(z-z_0;\,g_2,\,g_3)\,\sigma(z+z_0;\,g_2,\,g_3)}{\sigma(z;\,g_2,\,g_3)^2\,\sigma(z_0;\,g_2,\,g_3)^2}\ /;\ z_0 == \wp^{-1}(0;\,g_2,\,g_3)$$

$$\frac{\wp'(z_1;\,g_2,\,g_3) - \wp'(z_2;\,g_2,\,g_3)}{\wp(z_1;\,g_2,\,g_3) - \wp(z_2;\,g_2,\,g_3)} == 2\,(\zeta(z_1+z_2;\,g_2,\,g_3) - \zeta(z_1;\,g_2,\,g_3) - \zeta(z_2;\,g_2,\,g_3))$$

$$\wp'(z; g_2, g_3) = \frac{\partial \wp(z; g_2, g_3)}{\partial z}$$

$$\wp'(z; g_2, g_3) = -\frac{\sigma(2z; g_2, g_3)}{\sigma(z; g_2, g_3)^4}$$

$$\wp'(z; g_2, g_3) = \frac{2\,\sigma(z-\omega_1; g_2, g_3)\,\sigma(z-\omega_2; g_2, g_3)\,\sigma(z-\omega_3; g_2, g_3)}{\sigma(z; g_2, g_3)^3\,\sigma(\omega_1; g_2, g_3)\,\sigma(\omega_2; g_2, g_3)\,\sigma(\omega_3; g_2, g_3)}$$

$$\wp'(z; g_2, g_3) = -\frac{2\,\sigma_1(z; g_2, g_3)\,\sigma_2(z; g_2, g_3)\,\sigma_3(z; g_2, g_3)}{\sigma(z; g_2, g_3)^3}$$

$$\sigma(z; g_2, g_3) = z\exp\left(\int_0^z \left(\zeta(t; g_2, g_3) - \frac{1}{t}\right) dt\right)$$

$$\sigma_i(z; g_2, g_3) = \sigma(z; g_2, g_3)\sqrt{\wp(z; g_2, g_3) - e_i} \;/;\; i \in \{1, 2, 3\}$$

$$\zeta(z; g_2, g_3) = \frac{1}{z} - \int_0^z \left(\wp(t; g_2, g_3) - \frac{1}{t^2}\right) dt$$

$$\zeta(z; g_2, g_3) = \frac{\partial \log(\sigma(z; g_2, g_3))}{\partial z}$$

$$\zeta(z; g_2, g_3) = \frac{\sigma'(z; g_2, g_3)}{\sigma(z; g_2, g_3)}.$$

Note that the Weierstrass functions $\sigma(z; g_2, g_3)$, $\sigma_n(z, g_2, g_3)$, $\zeta(z; g_2, g_3)$, $\wp(z; g_2, g_3)$, and $\wp'(z; g_2, g_3)$ form a chain with respect to differentiation:

$$\frac{\partial \sigma(z; g_2, g_3)}{\partial z} = \sigma(z; g_2, g_3)\,\zeta(z; g_2, g_3)$$

$$\frac{\partial \sigma_n(z, g_2, g_3)}{\partial z} = \sigma_n(z, g_2, g_3)\,(\zeta(z+\omega_n; g_2, g_3) - \eta_n)$$

$$\frac{\partial \zeta(z; g_2, g_3)}{\partial z} = -\wp(z; g_2, g_3)$$

$$\frac{\partial \wp(z; g_2, g_3)}{\partial z} = \wp'(z; g_2, g_3).$$

The best-known properties and formulas for Weierstrass functions and inverses

Simple values at zero

The Weierstrass functions $\wp(z; g_2, g_3)$, $\wp'(z; g_2, g_3)$, $\sigma(z; g_2, g_3)$, and $\zeta(z; g_2, g_3)$ have the following simple values at the origin point:

$\wp(0; 0, 0) = \tilde{\infty} \quad \wp'(0; 0, 0) = \tilde{\infty}$

$\sigma(0; 0, 0) = 0 \quad \zeta(0; 0, 0) = \tilde{\infty}.$

Specific values for specialized parameter

The Weierstrass functions $\wp(z; g_2, g_3)$, $\wp'(z; g_2, g_3)$, $\sigma(z; g_2, g_3)$, $\sigma_n(z, g_2, g_3) /; n \in \{1, 2, 3\}$, and $\zeta(z; g_2, g_3)$ can be represented through elementary functions, when $\{g_2, g_3\} = \{0, 0\}$ or $\{g_2, g_3\} = \{3, 1\}$:

$$\wp(z; 0, 0) = \frac{1}{z^2} \qquad \wp(z; 3, 1) = \frac{3}{2} \cot^2\left(\sqrt{\frac{3}{2}}\, z\right) + 1$$

$$\wp'(z; 0, 0) = -\frac{2}{z^3} \qquad \wp'(z; 3, 1) = -3\sqrt{\frac{3}{2}} \cot\left(\sqrt{\frac{3}{2}}\, z\right) \csc^2\left(\sqrt{\frac{3}{2}}\, z\right)$$

$$\sigma(z; 0, 0) = z \qquad \sigma(z; 3, 1) = \sqrt{\frac{2}{3}}\, e^{\frac{z^2}{4}} \sin\left(\sqrt{\frac{3}{2}}\, z\right)$$

$$\zeta(z; 0, 0) = \frac{1}{z} \qquad \zeta(z; 3, 1) = \frac{z}{2} + \sqrt{\frac{3}{2}} \cot\left(\sqrt{\frac{3}{2}}\, z\right)$$

$$\sigma_n(z; 0, 0) = 1 /; n \in \{1, 2, 3\}.$$

At points $z = m\,\omega_1 + n\,\omega_3 /; \{m, n\} \in \mathbb{Z}$, all Weierstrass functions $\wp(z; g_2, g_3)$, $\wp'(z; g_2, g_3)$, $\sigma(z; g_2, g_3)$, $\sigma_n(z, g_2, g_3) /; n \in \{1, 2, 3\}$, and $\zeta(z; g_2, g_3)$ can be equal to zero or can have poles and be equal to $\tilde{\infty}$:

$$\wp(2\,m\,\omega_1 + 2\,n\,\omega_3; g_2, g_3) = \tilde{\infty} /; \{m, n\} \in \mathbb{Z}$$

$$\wp'(2\,m\,\omega_1 + 2\,n\,\omega_3; g_2, g_3) = \tilde{\infty} /; \{m, n\} \in \mathbb{Z}$$

$$\wp'(m\,\omega_1 + n\,\omega_3; g_2, g_3) = 0 /; \left\{\frac{m-1}{2}, n\right\} \in \mathbb{Z} \bigvee \left\{m, \frac{n-1}{2}\right\} \in \mathbb{Z}$$

$$\sigma(2\,m\,\omega_1 + 2\,n\,\omega_3; g_2, g_3) = 0 /; \{m, n\} \in \mathbb{Z}$$

$$\sigma_j(2\,m\,\omega_1 + 2\,n\,\omega_3; g_2, g_3) = \tilde{\infty} /; \{m, n\} \in \mathbb{Z} \wedge j \in \{1, 2, 3\}$$

$$\sigma_i\left((2\,m+1)\,\omega_i + 2\,n\,\omega_j + 2\,r\,\omega_k; g_2, g_3\right) = 0 /; \{m, n, r\} \in \mathbb{Z} \wedge \{i, j, k\} \in \{1, 2, 3\} \wedge i \neq j \neq k$$

$$\zeta(2\,m\,\omega_1 + 2\,n\,\omega_3; g_2, g_3) = \tilde{\infty} /; \{m, n\} \in \mathbb{Z}.$$

The values of Weierstrass functions $\wp(z; g_2, g_3)$, $\wp'(z; g_2, g_3)$, $\sigma(z; g_2, g_3)$, $\sigma_n(z, g_2, g_3) /; n \in \{1, 2, 3\}$, and $\zeta(z; g_2, g_3)$ at the points $z = r\,\omega_j /; r \in \mathbb{Q}$ can sometimes be evaluated in closed form:

$$\wp(\omega_1; g_2, g_3) = \frac{3\,g_3}{g_2} /; \omega_3 = \tilde{\infty} \quad \wp(\omega_2; g_2, g_3) = -\frac{3\,g_3}{g_2} /; \omega_3 = \tilde{\infty} \quad \wp(\omega_3; g_2, g_3) = -\frac{3\,g_3}{g_2} /; \omega_3 = \tilde{\infty}$$

$$\wp\left(\frac{\omega_i}{2}; g_2, g_3\right) = e_i + \epsilon_{i,j}\,\epsilon_{i,k}\,\sqrt{e_i - e_j}\,\sqrt{e_i - e_k} /; \{i, j, k\} \in \{1, 2, 3\} \bigwedge i \neq j \neq k \bigwedge \epsilon_{\alpha,\beta} = \operatorname{sgn}\left(\frac{\pi}{2} - \left|\operatorname{Arg}\left(\frac{\sigma_\beta(\omega_\alpha; g_2, g_3)}{\sigma(\omega_\alpha; g_2, g_3)}\right)\right|\right)$$

$$\wp'\left(\omega_j; g_2, g_3\right) = 0 /; j \in \{1, 2, 3\}$$

$$\sigma(0; g_2, g_3) = 0 \quad \sigma\left(\frac{2\omega_i}{3}; g_2, g_3\right)^3 = -\frac{\exp\left(\frac{2\eta_i\omega_i}{3}\right)}{\wp'\left(\frac{2\omega_i}{3}; g_2, g_3\right)} /; i \in \{1, 2, 3\} \quad \wp(\omega_3; g_2, g_3) = -\frac{3 g_3}{g_2} /; \omega_3 = \tilde{\infty}$$

$$\sigma_j\left(\omega_j; g_2, g_3\right) = 0 /; j \in \{1, 2, 3\}$$

$$\sigma_i\left(\omega_j; g_2, g_3\right) = e^{-\eta_i\omega_j} \frac{\sigma(\omega_k; g_2, g_3)}{\sigma(\omega_i; g_2, g_3)} /; \{i, j, k\} \in \{1, 2, 3\} \bigwedge i \neq j \neq k$$

$$\sigma_j(0; g_2, g_3) = 1 /; j \in \{1, 2, 3\}$$

$$\frac{\sigma(\omega_k; g_2, g_3)^2}{\sigma(\omega_i; g_2, g_3)^2 \sigma\left(\omega_j; g_2, g_3\right)^2} = e^{2\eta_j\omega_i}\left(e_i - e_j\right) /; \{i, j, k\} \in \{1, 2, 3\} \bigwedge i \neq j \neq k$$

$$\frac{\sigma(\omega_k; g_2, g_3)}{\sigma\left(\omega_j; g_2, g_3\right)\sigma_j(\omega_i; g_2, g_3)} = e^{\eta_j\omega_i} /; \{i, j, k\} \in \{1, 2, 3\} \bigwedge i \neq j \neq k$$

$$\frac{\sigma_j(\omega_i; g_2, g_3)^2}{\sigma(\omega_i; g_2, g_3)^2} = e_i - e_j /; \{i, j\} \in \{1, 2, 3\} \bigwedge i \neq j$$

$$\left(\zeta\left(\frac{2\omega_i}{3}; g_2, g_3\right) - \frac{2\eta_i}{3}\right)^2 = \frac{1}{3}\wp\left(\frac{2\omega_i}{3}; g_2, g_3\right) /; i \in \{1, 2, 3\}.$$

The Weierstrass functions $\wp(z; g_2, g_3)$, $\sigma(z; g_2, g_3)$, and $\zeta(z; g_2, g_3)$ have rather simple values, when $z = \omega_j = \omega_j(g_2, g_3) /; j \in \{1, 2, 3\}$ and $\{g_2, g_3\} = \{0, 1\}$ or $\{g_2, g_3\} = \{1, 0\}$:

$$\wp(\omega_1; 0, 1) = \frac{1}{\sqrt[3]{4}} \qquad \wp(\omega_2; 0, 1) = 4^{-1/3} e^{4\pi i/3} \qquad \wp(\omega_3; 0, 1) = 4^{-1/3} e^{2\pi i/3}$$

$$\sigma(\omega_1; 0, 1) = e^{\frac{\pi}{4\sqrt{3}}} \frac{\sqrt[3]{2}}{\sqrt[4]{3}} \quad \sigma(\omega_2; 0, 1) = e^{\frac{\pi}{4\sqrt{3}}} \frac{\sqrt[3]{2}}{\sqrt[4]{3}} e^{\frac{4\pi i}{3}} \quad \sigma(\omega_3; 0, 1) = e^{\frac{\pi}{4\sqrt{3}}} \frac{\sqrt[3]{2}}{\sqrt[4]{3}} e^{\frac{2\pi i}{3}}$$

$$\zeta(\omega_1; 0, 1) = \frac{\pi}{2\omega_1\sqrt{3}} \qquad \zeta(\omega_2; 0, 1) = \frac{\pi}{2\omega_1\sqrt{3}} e^{2\pi i/3} \qquad \zeta(\omega_3; 0, 1) = \frac{\pi}{2\omega_1\sqrt{3}} e^{4\pi i/3}$$

$$\wp(\omega_1; 1, 0) = \frac{1}{2} \qquad \wp(\omega_2; 1, 0) = 0 \qquad \wp(\omega_3; 1, 0) = -\frac{1}{2}$$

$$\sigma(\omega_1; 1, 0) = e^{\pi/8}\sqrt[4]{2} \quad \sigma(\omega_2; 1, 0) = -e^{\pi/4}\sqrt{2}\, e^{\frac{i\pi}{4}} \quad \sigma(\omega_3; 1, 0) = e^{\pi/8}\sqrt[4]{2}\, i$$

$$\zeta(\omega_1; 1, 0) = \frac{\pi}{4\omega_1} \qquad \zeta(\omega_2; 1, 0) = \frac{\pi}{4\omega_1}(1 - i) \qquad \zeta(\omega_3; 1, 0) = -\frac{\pi i}{4\omega_1}.$$

The Weierstrass functions $\wp(z; g_2, g_3)$, $\wp'(z; g_2, g_3)$, $\sigma(z; g_2, g_3)$, and $\zeta(z; g_2, g_3)$ can be represented through elementary functions, when $\omega_3 = \tilde{\infty}$:

$$\wp(z; g_2(\omega_1, \tilde{\infty}), g_3(\omega_1, \tilde{\infty})) = \left(\frac{\pi}{2\omega_1}\right)^2 \left(\frac{1}{\sin^2\left(\frac{\pi z}{2\omega_1}\right)} - \frac{1}{3}\right) \quad \wp'(z; g_2(\omega_1, \tilde{\infty}), g_3(\omega_1, \tilde{\infty})) = -\frac{\pi^3}{4\omega_1^3}\cot\left(\frac{\pi z}{2\omega_1}\right)\csc^2\left(\frac{\pi z}{2\omega_1}\right)$$

$$\sigma(z; g_2(\omega_1, \tilde{\infty}), g_3(\omega_1, \tilde{\infty})) = \frac{2\omega_1}{\pi}\exp\left(\frac{1}{6}\left(\frac{\pi z}{2\omega_1}\right)^2\right)\sin\left(\frac{\pi z}{2\omega_1}\right) \quad \zeta(z; g_2(\omega_1, \tilde{\infty}), g_3(\omega_1, \tilde{\infty})) = \frac{1}{3}\left(\frac{\pi}{2\omega_1}\right)^2 z + \frac{\pi}{2\omega_1}\cot\left(\frac{\pi z}{2\omega_1}\right)$$

$\wp(z; g_2(\tilde{\infty}, \tilde{\infty}), g_3(\tilde{\infty}, \tilde{\infty})) == \frac{1}{z^2}$ $\wp'(z; g_2(\tilde{\infty}, \tilde{\infty}), g_3(\tilde{\infty}, \tilde{\infty})) == -\frac{2}{z^3}$ $\sigma(z; g_2(\tilde{\infty}, \tilde{\infty}), g_3(\tilde{\infty}, \tilde{\infty})) == z$ $\zeta(z; g_2(\tilde{\infty}, \tilde{\infty}), g_3(\tilde{\infty}, \tilde{\infty})) =$

$\wp(\tilde{\infty}; g_2(\tilde{\infty}, \tilde{\infty}), g_3(\tilde{\infty}, \tilde{\infty})) == 0$ $\wp'(\tilde{\infty}; g_2(\tilde{\infty}, \tilde{\infty}), g_3(\tilde{\infty}, \tilde{\infty})) == 0$ $\sigma(\tilde{\infty}; g_2(\tilde{\infty}, \tilde{\infty}), g_3(\tilde{\infty}, \tilde{\infty})) == \tilde{\infty}$ $\zeta(\tilde{\infty}; g_2(\tilde{\infty}, \tilde{\infty}), g_3(\tilde{\infty}, \tilde{\infty}))$

Analyticity

The Weierstrass functions $\wp(z; g_2, g_3)$, $\wp'(z; g_2, g_3)$, $\sigma(z; g_2, g_3)$, $\sigma_n(z, g_2, g_3) /; n \in \{1, 2, 3\}$, $\zeta(z; g_2, g_3)$, and $\wp^{-1}(z; g_2, g_3)$ are analytical functions of z, g_2, and g_3, which are defined in $\mathbb{C}^3$. The inverse Weierstrass function $\wp^{-1}(z_1, z_2; g_2, g_3)$ is an analytical function of z_1, z_2, g_2, g_3, which is also defined in $\mathbb{C}^3$, because z_2 is not an independent variable.

Poles and essential singularities

For fixed g_2, g_3, the Weierstrass functions $\wp(z; g_2, g_3)$, $\wp'(z; g_2, g_3)$, and $\zeta(z; g_2, g_3)$ have an infinite set of singular points:

(a) $z == 2\, m\, \omega_1(g_2, g_3) + 2\, n\, \omega_3(g_2, g_3)$, $\{m, n\} \in \mathbb{Z}$ are the poles of order 2 with residues 0 (for $\wp(z; g_2, g_3)$), of order 3 with residues 0 (for $\wp'(z; g_2, g_3)$) and simple poles with residues 1 (for $\zeta(z; g_2, g_3)$).

(b) $z == \tilde{\infty}$ is an essential singular point.

For fixed g_2, g_3, the Weierstrass functions $\sigma(z; g_2, g_3)$ and $\sigma_j(z; g_2, g_3)$, $j \in \{1, 2, 3\}$ have only one singular point at $z = \tilde{\infty}$. It is an essential singular point.

The Weierstrass functions $\wp^{-1}(z; g_2, g_3)$ and $\wp^{-1}(z_1, z_2; g_2, g_3)$ do not have poles and essential singularities with respect to their
variables.

Branch points and branch cuts

For fixed g_2, g_3, the Weierstrass functions $\wp(z; g_2, g_3)$, $\wp'(z; g_2, g_3)$, $\sigma(z; g_2, g_3)$, $\sigma_j(z; g_2, g_3)$, $j \in \{1, 2, 3\}$, and $\zeta(z; g_2, g_3)$ do not have branch points and branch cuts.

For fixed z, g_2, the inverse Weierstrass function $\wp^{-1}(z; g_2, g_3)$ has two branch points: $g_3 == 4\, z^3 - g_2\, z$, $g_3 = \tilde{\infty}$.

For fixed z, g_3, the inverse Weierstrass function $\wp^{-1}(z; g_2, g_3)$ has two branch points: $g_2 == 4\, z^2 - \frac{g_3}{z}$, $g_2 = \tilde{\infty}$.

For fixed g_2, g_3, the inverse Weierstrass function $\wp^{-1}(z; g_2, g_3)$ has four branch points: $z = \left(z; 4\, z^3 - g_2\, z - g_3\right)_k^{-1} /; k \in \{1, 2, 3\}$, $z = \tilde{\infty}$.

Periodicity

The Weierstrass functions $\wp(z; g_2, g_3)$ and $\wp'(z; g_2, g_3)$ are doubly periodic functions with respect to z with periods $2\, \omega_1$ and $2\, \omega_3$:

$$\wp(z + 2\, m\, \omega_1 + 2\, n\, \omega_3; g_2, g_3) == \wp(z; g_2, g_3) /; \{m, n\} \in \mathbb{Z}$$

$$\wp'(z + 2\, m\, \omega_1 + 2\, n\, \omega_3; g_2, g_3) == \wp'(z; g_2, g_3) /; \{m, n\} \in \mathbb{Z}.$$

The Weierstrass functions $\sigma(z; g_2, g_3)$, $\sigma_n(z, g_2, g_3)$, $n \in \{1, 2, 3\}$, and $\zeta(z; g_2, g_3)$ are quasi-periodic functions with respect to z:

$$\sigma(z + 2\,\omega_n; g_2, g_3) == -e^{2\,\eta_n (z+\omega_n)}\,\sigma(z; g_2, g_3)\ /;\ n \in \{1, 2, 3\}$$

$$\sigma(z + 2\,m\,\omega_1 + 2\,n\,\omega_2 + 2\,r\,\omega_3; g_2, g_3) == (-1)^{n\,r+r\,m+m\,n+m+n+r}\,e^{2\,(m\,\eta_1+n\,\eta_2+r\,\eta_3)(z+m\,\omega_1+n\,\omega_2+r\,\omega_3)}\,\sigma(z; g_2, g_3)\ /;\ \{m, n, r\} \in \mathbb{Z}$$

$$\sigma_n(z + 2\,\omega_n; g_2, g_3) == -e^{2\,\eta_n (z+\omega_n)}\,\sigma_n(z; g_2, g_3)\ /;\ n \in \{1, 2, 3\}$$

$$\sigma_n\left(z + 2\,\omega_j; g_2, g_3\right) == e^{2\,\eta_j (z+\omega_j)}\,\sigma_n(z; g_2, g_3)\ /;\ \{n, j\} \in \{1, 2, 3\} \bigwedge n \neq j$$

$$\sigma_i\left(z + 2\,m\,\omega_i + 2\,n\,\omega_j + 2\,r\,\omega_k; g_2, g_3\right) == (-1)^{n\,r+r\,m+m\,n+m}\,e^{2\,(m\,\eta_i+n\,\eta_j+r\,\eta_k)(z+m\,\omega_i+n\,\omega_j+r\,\omega_k)}\,\sigma_i(z; g_2, g_3)\ /;$$
$$\{i, j, k\} \in \{1, 2, 3\} \bigwedge i \neq j \neq k \bigwedge \{m, n, r\} \in \mathbb{Z}$$

$$\zeta(z + 2\,m\,\omega_1 + 2\,n\,\omega_2 + 2\,r\,\omega_3; g_2, g_3) == \zeta(z; g_2, g_3) + 2\,m\,\eta_1 + 2\,n\,\eta_2 + 2\,r\,\eta_3\ /;\ \{m, n, r\} \in \mathbb{Z}.$$

The inverse Weierstrass functions $\wp^{-1}(z; g_2, g_3)$ and $\wp^{-1}(z_1, z_2; g_2, g_3)$ do not have periodicity and symmetry.

Transformation of half-periods

The Weierstrass functions $\wp(z; g_2, g_3)$, $\wp'(z; g_2, g_3)$, $\sigma(z; g_2, g_3)$, $\sigma_j(z; g_2, g_3)$, $j \in \{1, 2, 3\}$, and $\zeta(z; g_2, g_3)$ are the invariant functions under the linear transformation of the half
-periods $\omega_1 \to a\,\omega_1 + b\,\omega_3,\ \omega_3 \to c\,\omega_1 + d\,\omega_3$ with integer coefficients a, b, c, and d, satisfying restrictions $a\,d - b\,c == \pm 1$ (modular transformations):

$$\wp(z; g_2(a\,\omega_1 + b\,\omega_3, c\,\omega_1 + d\,\omega_3), g_3(a\,\omega_1 + b\,\omega_3, c\,\omega_1 + d\,\omega_3)) == \wp(z; g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3))\ /;$$
$$\{a, b, c, d\} \in \mathbb{Z} \bigwedge a\,d - b\,c == \pm 1$$

$$\wp'(z; g_2(a\,\omega_1 + b\,\omega_3, c\,\omega_1 + d\,\omega_3), g_3(a\,\omega_1 + b\,\omega_3, c\,\omega_1 + d\,\omega_3)) == \wp'(z; g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3))\ /;$$
$$\{a, b, c, d\} \in \mathbb{Z} \bigwedge a\,d - b\,c == \pm 1$$

$$\sigma(z; g_2(a\,\omega_1 + b\,\omega_3, c\,\omega_1 + d\,\omega_3), g_3(a\,\omega_1 + b\,\omega_3, c\,\omega_1 + d\,\omega_3)) == \sigma(z; g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3))\ /;$$
$$\{a, b, c, d\} \in \mathbb{Z} \bigwedge a\,d - b\,c == \pm 1$$

$$\sigma_1(z; g_2(a\,\omega_1 + b\,\omega_3, c\,\omega_1 + d\,\omega_3), g_3(a\,\omega_1 + b\,\omega_3, c\,\omega_1 + d\,\omega_3))$$
$$\sigma_2(z; g_2(a\,\omega_1 + b\,\omega_3, c\,\omega_1 + d\,\omega_3), g_3(a\,\omega_1 + b\,\omega_3, c\,\omega_1 + d\,\omega_3))$$
$$\sigma_3(z; g_2(a\,\omega_1 + b\,\omega_3, c\,\omega_1 + d\,\omega_3), g_3(a\,\omega_1 + b\,\omega_3, c\,\omega_1 + d\,\omega_3)) == \sigma_1(z; g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3))$$
$$\sigma_2(z; g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3))\,\sigma_3(z; g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3))\ /;\ \{a, b, c, d\} \in \mathbb{Z} \bigwedge a\,d - b\,c == \pm 1$$

$$\zeta(z; g_2(a\,\omega_1 + b\,\omega_3, c\,\omega_1 + d\,\omega_3), g_3(a\,\omega_1 + b\,\omega_3, c\,\omega_1 + d\,\omega_3)) == \zeta(z; g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3))\ /;$$
$$\{a, b, c, d\} \in \mathbb{Z} \bigwedge a\,d - b\,c == \pm 1.$$

Homogeneity

The Weierstrass functions $\wp(z; g_2, g_3)$, $\wp'(z; g_2, g_3)$, $\sigma(z; g_2, g_3)$, $\sigma_j(z; g_2, g_3)$, $j \in \{1, 2, 3\}$, and $\zeta(z; g_2, g_3)$ satisfy the following homogeneity type relations:

$$\wp(z\,t; g_2, g_3) == \frac{1}{t^2}\,\wp\left(z; g_2\,t^4, g_3\,t^6\right)\ /;\ t \in \mathbb{R}$$

$$\wp(\lambda\,z; g_2(\lambda\,\omega_1, \lambda\,\omega_3), g_3(\lambda\,\omega_1, \lambda\,\omega_3)) == \wp\left(\lambda\,z; \frac{g_2(\omega_1, \omega_3)}{\lambda^4}, \frac{g_3(\omega_1, \omega_3)}{\lambda^6}\right)$$

$$\wp(\lambda z; g_2(\lambda \omega_1, \lambda \omega_3), g_3(\lambda \omega_1, \lambda \omega_3)) = \frac{1}{\lambda^2} \wp(z; g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3))$$

$$\wp'(z t; g_2, g_3) = \frac{1}{t^3} \wp'\left(z; g_2 t^4, g_3 t^6\right) /; t \in \mathbb{R}$$

$$\wp'(\lambda z; g_2(\lambda \omega_1, \lambda \omega_3), g_3(\lambda \omega_1, \lambda \omega_3)) = \wp'\left(\lambda z; \frac{g_2(\omega_1, \omega_3)}{\lambda^4}, \frac{g_3(\omega_1, \omega_3)}{\lambda^6}\right)$$

$$\wp'(\lambda z; g_2(\lambda \omega_1, \lambda \omega_3), g_3(\lambda \omega_1, \lambda \omega_3)) = \frac{1}{\lambda^3} \wp'(z; g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3))$$

$$\sigma(z t; g_2, g_3) = t \sigma\left(z; g_2 t^4, g_3 t^6\right) /; t \in \mathbb{R}$$

$$\sigma(\lambda z; g_2(\lambda \omega_1, \lambda \omega_3), g_3(\lambda \omega_1, \lambda \omega_3)) = \lambda \sigma(z; g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3))$$

$$\sigma(\lambda z; g_2(\lambda \omega_1, \lambda \omega_3), g_3(\lambda \omega_1, \lambda \omega_3)) = \sigma\left(\lambda z; \frac{g_2(\omega_1, \omega_3)}{\lambda^4}, \frac{g_3(\omega_1, \omega_3)}{\lambda^6}\right)$$

$$\sigma_n(\lambda z; g_2(\lambda \omega_1, \lambda \omega_3), g_3(\lambda \omega_1, \lambda \omega_3)) = \sigma_n(z; g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)) /; n \in \{1, 2, 3\}$$

$$\sigma_n(\lambda z; g_2(\lambda \omega_1, \lambda \omega_3), g_3(\lambda \omega_1, \lambda \omega_3)) = \sigma_n\left(\lambda z; \frac{g_2(\omega_1, \omega_3)}{\lambda^4}, \frac{g_3(\omega_1, \omega_3)}{\lambda^6}\right) /; n \in \{1, 2, 3\}$$

$$\zeta(z t; g_2, g_3) = \frac{1}{t} \zeta\left(z; g_2 t^4, g_3 t^6\right) /; t \in \mathbb{R}$$

$$\zeta(\lambda z; g_2(\lambda \omega_1, \lambda \omega_3), g_3(\lambda \omega_1, \lambda \omega_3)) = \zeta\left(\lambda z; \frac{g_2(\omega_1, \omega_3)}{\lambda^4}, \frac{g_3(\omega_1, \omega_3)}{\lambda^6}\right)$$

$$\zeta(\lambda z; g_2(\lambda \omega_1, \lambda \omega_3), g_3(\lambda \omega_1, \lambda \omega_3)) = \frac{1}{\lambda} \zeta(z; g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)).$$

Parity and symmetry

The Weierstrass functions $\wp(z; g_2, g_3)$, $\wp'(z; g_2, g_3)$, $\sigma(z; g_2, g_3)$, $\sigma_j(z; g_2, g_3)$, $j \in \{1, 2, 3\}$, $\zeta(z; g_2, g_3)$, and $\wp^{-1}(z; g_2, g_3)$ have mirror symmetry:

$$\wp(\bar{z}; \overline{g_2}, \overline{g_3}) = \overline{\wp(z; g_2, g_3)} \quad \wp'(\bar{z}; \overline{g_2}, \overline{g_3}) = \overline{\wp'(z; g_2, g_3)}$$

$$\sigma(\bar{z}; \overline{g_2}, \overline{g_3}) = \overline{\sigma(z; g_2, g_3)} \quad \sigma_n(\bar{z}; \overline{g_2}, \overline{g_3}) = \overline{\sigma_n(z; g_2, g_3)} /; n \in \{1, 2, 3\}$$

$$\zeta(\bar{z}; \overline{g_2}, \overline{g_3}) = \overline{\zeta(z; g_2, g_3)} \quad \wp^{-1}(\bar{z}; \overline{g_2}, \overline{g_3}) = \overline{\wp^{-1}(z; g_2, g_3)}\ .$$

The Weierstrass functions $\wp(z; g_2, g_3)$ and $\sigma_n(z, g_2, g_3)$, $n \in \{1, 2, 3\}$ are even functions with respect to z:

$$\wp(-z; g_2, g_3) = \wp(z; g_2, g_3)$$

$$\sigma_n(-z; g_2, g_3) = \sigma_n(z; g_2, g_3) /; n \in \{1, 2, 3\}.$$

The Weierstrass functions $\wp'(z; g_2, g_3)$, $\sigma(z; g_2, g_3)$, and $\zeta(z; g_2, g_3)$ are odd functions with respect to z:

$$\wp'(-z; g_2, g_3) == -\wp'(z; g_2, g_3)$$

$$\sigma(-z; g_2, g_3) == -\sigma(z; g_2, g_3)$$

$$\zeta(-z; g_2, g_3) == -\zeta(z; g_2, g_3).$$

Series representations

The Weierstrass functions $\wp(z; g_2, g_3)$, $\wp'(z; g_2, g_3)$, $\sigma(z; g_2, g_3)$, and $\zeta(z; g_2, g_3)$ have the following series expansions at the point $z == 0$:

$$\wp(z; g_2, g_3) \propto \frac{1}{z^2} + \frac{g_2}{20} z^2 + \frac{g_3}{28} z^4 + \ldots /; (z \to 0)$$

$$\wp(z; g_2, g_3) == \frac{1}{z^2} + \sum_{k=2}^{\infty} a_k z^{2k-2} /; a_2 == \frac{g_2}{20} \bigwedge a_3 == \frac{g_3}{28} \bigwedge a_k == \frac{3}{(2k+1)(k-3)} \sum_{l=2}^{k-2} a_l a_{k-l}$$

$$\wp'(z; g_2, g_3) \propto -\frac{2}{z^3} + \frac{g_2}{10} z + \frac{g_3}{7} z^3 + \ldots /; (z \to 0)$$

$$\wp'(z; g_2, g_3) == -\frac{2}{z^3} + \sum_{k=2}^{\infty} (2k-2) a_k z^{2k-3} /; a_2 == \frac{g_2}{20} \bigwedge a_3 == \frac{g_3}{28} \bigwedge a_k == \frac{3}{(2k+1)(k-3)} \sum_{l=2}^{k-2} a_l a_{k-l}$$

$$\wp'(z; g_2, g_3) == -\frac{2}{z^3} + 2 \sum_{k=1}^{\infty} k (2k+1) \sum_{\substack{m, n=-\infty \\ \{m,n\} \neq \{0,0\}}}^{\infty} \frac{1}{(2 m \omega_1 + 2 n \omega_3)^{2k+2}} z^{2k-1}$$

$$\sigma(z; g_2, g_3) \propto z - \frac{g_2}{240} z^5 - \frac{g_3}{840} z^7 - \frac{g_2^2}{161\,280} z^9 + \ldots /; (z \to 0)$$

$$\sigma(z; g_2, g_3) == \sum_{k=0}^{\infty} d_k z^{2k+1} /;$$

$$d_0 == 1 \bigwedge d_1 == 0 \bigwedge d_2 == -\frac{g_2}{240} \bigwedge d_3 == -\frac{g_3}{840} \bigwedge d_4 == -\frac{g_2^2}{161\,280} \bigwedge d_n == \sum_{k_1=0}^{n} \sum_{k_2=0}^{n} \cdots \sum_{k_n=0}^{n} \delta_{n-\sum_{j=1}^{n} j k_j, 0} \prod_{j=1}^{n} \frac{c_j^{k_j}}{k_j!} \bigwedge$$

$$c_j == -\frac{a_j}{2 j (2j-1)} \bigwedge a_1 == 0 \bigwedge a_2 == \frac{g_2}{20} \bigwedge a_3 == \frac{g_3}{28} \bigwedge a_k == \frac{3}{(2k+1)(k-3)} \sum_{l=2}^{k-2} a_l a_{k-l}$$

$$\sigma(z; g_2, g_3) == \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{a_{m,n} \left(\frac{g_2}{2}\right)^m (2 g_3)^n}{(4m+6n+1)!} z^{4m+6n+1} /; a_{0,0} == 1 \land a_{m,n} == 0 /;$$

$$m < 0 \bigvee n < 0 \bigwedge a_{m,n} == \frac{16}{3} (n+1) a_{m-2,n+1} - \frac{1}{3} (2m + 3n - 1)(4m + 6n - 1) a_{m-1,n} + 3 (m+1) a_{m+1,n-1}$$

$$\zeta(z; g_2, g_3) \propto \frac{1}{z} - \frac{g_2}{60} z^3 - \frac{g_3}{140} z^5 - \ldots /; (z \to 0)$$

$$\zeta(z; g_2, g_3) == \frac{1}{z} - \sum_{k=2}^{\infty} \frac{a_k z^{2k-1}}{2k-1} /; a_2 == \frac{g_2}{20} \bigwedge a_3 == \frac{g_3}{28} \bigwedge a_k == \frac{3}{(2k+1)(k-3)} \sum_{l=2}^{k-2} a_l a_{k-l}$$

$$\zeta(z; g_2, g_3) = \frac{1}{z} - \sum_{k=1}^{\infty} \sum_{\substack{m,n=-\infty \\ \{m,n\}\neq\{0,0\}}}^{\infty} \frac{1}{(2 m \omega_1 + 2 n \omega_3)^{2k+2}} z^{2k+1}.$$

The inverse Weierstrass function $\wp^{-1}(z; g_2, g_3)$ has the following series expansion at the point $z = \tilde{\infty}$:

$$\wp^{-1}(z; g_2, g_3) \propto \sqrt{\frac{1}{z}} + \frac{1}{40} g_2 \left(\frac{1}{z}\right)^{5/2} + \frac{1}{56} g_3 \left(\frac{1}{z}\right)^{7/2} + \frac{1}{384} g_2^2 \left(\frac{1}{z}\right)^{9/2} + \frac{3}{704} g_2 g_3 \left(\frac{1}{z}\right)^{11/2} + \frac{5 g_2^3 + 24 g_3^2}{13312} \left(\frac{1}{z}\right)^{13/2} + \frac{g_2^2 g_3}{1024} \left(\frac{1}{z}\right)^{15/2} + \frac{5\left(7 g_2^4 + 96 g_3^2 g_2\right)}{557056} \left(\frac{1}{z}\right)^{17/2} + \frac{5\left(7 g_3 g_2^3 + 8 g_3^3\right)}{155648} \left(\frac{1}{z}\right)^{19/2} + \frac{3 g_2^5 + 80 g_3^2 g_2^2}{262144} \left(\frac{1}{z}\right)^{21/2} + O\left(\left(\frac{1}{z}\right)^{23/2}\right).$$

q-series representations

The Weierstrass functions $\wp(z; g_2, g_3)$, $\wp'(z; g_2, g_3)$, $\log(\sigma(z; g_2, g_3))$, $\log\left(\sigma_j(z; g_2, g_3)\right)$, $j \in \{1, 2, 3\}$, and $\zeta(z; g_2, g_3)$ have the following so-called q-series representations:

$$\wp(z; g_2, g_3) = -\frac{\eta_1}{\omega_1} + \left(\frac{\pi}{2\omega_1}\right)^2 \csc^2\left(\frac{\pi z}{2\omega_1}\right) - \frac{2\pi^2}{\omega_1^2} \sum_{k=1}^{\infty} \frac{k q^{2k}}{1-q^{2k}} \cos\left(\frac{k \pi z}{\omega_1}\right) /; q = \exp\left(\frac{\pi i \omega_3}{\omega_1}\right)$$

$$\wp(z+\omega_1; g_2, g_3) = -\frac{\eta_1}{\omega_1} + \left(\frac{\pi}{2\omega_1}\right)^2 \sec^2\left(\frac{\pi z}{2\omega_1}\right) - \frac{2\pi^2}{\omega_1^2} \sum_{k=1}^{\infty} (-1)^k \frac{k q^{2k}}{1-q^{2k}} \cos\left(\frac{k \pi z}{\omega_1}\right) /; q = \exp\left(\frac{\pi i \omega_3}{\omega_1}\right)$$

$$\wp(z+\omega_2; g_2, g_3) = -\frac{\eta_1}{\omega_1} - \frac{2\pi^2}{\omega_1^2} \sum_{k=1}^{\infty} (-1)^k \frac{k q^{k}}{1-q^{2k}} \cos\left(\frac{k \pi z}{\omega_1}\right) /; q = \exp\left(\frac{\pi i \omega_3}{\omega_1}\right)$$

$$\wp(z+\omega_3; g_2, g_3) = -\frac{\eta_1}{\omega_1} - \frac{2\pi^2}{\omega_1^2} \sum_{k=1}^{\infty} \frac{k q^{k}}{1-q^{2k}} \cos\left(\frac{k \pi z}{\omega_1}\right) /; q = \exp\left(\frac{\pi i \omega_3}{\omega_1}\right)$$

$$\wp'(z; g_2, g_3) = -\frac{\pi^3}{4\omega_1^3} \cot\left(\frac{\pi z}{2\omega_1}\right) \csc^2\left(\frac{\pi z}{2\omega_1}\right) + \frac{2\pi^3}{\omega_1^3} \sum_{k=1}^{\infty} \frac{k^2 q^{2k}}{1-q^{2k}} \sin\left(\frac{k \pi z}{\omega_1}\right) /; q = \exp\left(\frac{\pi i \omega_3}{\omega_1}\right)$$

$$\wp'(z+\omega_1; g_2, g_3) = \frac{\pi^3}{4\omega_1^3} \tan\left(\frac{\pi z}{2\omega_1}\right) \sec^2\left(\frac{\pi z}{2\omega_1}\right) + \frac{2\pi^3}{\omega_1^3} \sum_{k=1}^{\infty} (-1)^k \frac{k^2 q^{2k}}{1-q^{2k}} \sin\left(\frac{k \pi z}{\omega_1}\right) /; q = \exp\left(\frac{\pi i \omega_3}{\omega_1}\right)$$

$$\wp'(z+\omega_2; g_2, g_3) = \frac{2\pi^3}{\omega_1^3} \sum_{k=1}^{\infty} (-1)^k \frac{k^2 q^{k}}{1-q^{2k}} \sin\left(\frac{k \pi z}{\omega_1}\right) /; q = \exp\left(\frac{\pi i \omega_3}{\omega_1}\right)$$

$$\wp'(z+\omega_3; g_2, g_3) = \frac{2\pi^3}{\omega_1^3} \sum_{k=1}^{\infty} \frac{k^2 q^{k}}{1-q^{2k}} \sin\left(\frac{k \pi z}{\omega_1}\right) /; q = \exp\left(\frac{\pi i \omega_3}{\omega_1}\right)$$

$$\log(\sigma(z; g_2, g_3)) = \log\left(\frac{2\omega_1}{\pi}\right) + \frac{\eta_1 z^2}{2\omega_1} + \log\left(\sin\left(\frac{\pi z}{2\omega_1}\right)\right) + 4 \sum_{k=1}^{\infty} \frac{q^{2k}}{k\left(1-q^{2k}\right)} \sin^2\left(\frac{k \pi z}{2\omega_1}\right) /; q = \exp\left(\frac{\pi i \omega_3}{\omega_1}\right)$$

$$\log(\sigma_1(z; g_2, g_3)) = \frac{\eta_1 z^2}{2\omega_1} + \log\left(\cos\left(\frac{\pi z}{2\omega_1}\right)\right) + 4 \sum_{k=1}^{\infty} (-1)^k \frac{q^{2k}}{k\left(1-q^{2k}\right)} \sin^2\left(\frac{k \pi z}{2\omega_1}\right) /; q = \exp\left(\frac{\pi i \omega_3}{\omega_1}\right)$$

$$\log(\sigma_2(z; g_2, g_3)) == \frac{\eta_1 z^2}{2\omega_1} + 4\sum_{k=1}^{\infty}(-1)^k \frac{q^k}{k(1-q^{2k})}\sin^2\left(\frac{k\pi z}{2\omega_1}\right) /; q == \exp\left(\frac{\pi i \omega_3}{\omega_1}\right)$$

$$\log(\sigma_3(z; g_2, g_3)) == \frac{\eta_1 z^2}{2\omega_1} + 4\sum_{k=1}^{\infty}\frac{q^k}{k(1-q^{2k})}\sin^2\left(\frac{k\pi z}{2\omega_1}\right) /; q == \exp\left(\frac{\pi i \omega_3}{\omega_1}\right)$$

$$\zeta(z; g_2, g_3) == \frac{\eta_1 z}{\omega_1} + \frac{\pi}{2\omega_1}\cot\left(\frac{\pi z}{2\omega_1}\right) + \frac{2\pi}{\omega_1}\sum_{k=1}^{\infty}\frac{q^{2k}}{1-q^{2k}}\sin\left(\frac{k\pi z}{\omega_1}\right) /; q == \exp\left(\frac{\pi i \omega_3}{\omega_1}\right)$$

$$\zeta(z; g_2, g_3) == \frac{\eta_1 z}{\omega_1} + \frac{\pi}{2\omega_1}\cot\left(\frac{\pi z}{2\omega_1}\right) + \frac{2\pi}{\omega_1}\sum_{k=1}^{\infty}\frac{q^{2k}\sin\left(\frac{\pi z}{\omega_1}\right)}{1-2\cos\left(\frac{\pi z}{\omega_1}\right)q^{2k}+q^{4k}} /; q == \exp\left(\frac{\pi i \omega_3}{\omega_1}\right)$$

$$\zeta(\omega_1; g_2, g_3) == == \frac{\pi^2}{12\omega_1} - \frac{2\pi^2}{\omega_1}\sum_{k=1}^{\infty}\frac{k q^{2k}}{1-q^{2k}} /; q == \exp\left(\frac{\pi i \omega_3}{\omega_1}\right).$$

Other series representations

The Weierstrass functions $\wp(z; g_2, g_3)$, $\wp'(z; g_2, g_3)$, $\sigma(z; g_2, g_3)$, $\sigma_j(z; g_2, g_3)$, $j \in \{1, 2, 3\}$, and $\zeta(z; g_2, g_3)$ with $\{g_2, g_3\} == \{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\}$ can be represented through series of different forms, for example:

$$\wp(z; g_2, g_3) == \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\frac{1}{(z+2m\omega_1+2n\omega_3)^2} - \frac{1}{2}\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}\frac{1}{(m\omega_1+n\omega_3)^2} - \frac{\pi^2}{12\omega_1^2}$$

$$\wp(z; g_2, g_3) == -\frac{\eta_i}{\omega_i} + \frac{\pi^2}{4\omega_i^2}\sum_{n=-\infty}^{\infty}\csc^2\left(\frac{\pi(z-2n\omega_j)}{2\omega_i}\right) /; \{i, j\} \in \{1, 2, 3\} \bigwedge i \neq j$$

$$\wp'(z; g_2, g_3) == -2\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\frac{1}{(z-2m\omega_1-2n\omega_3)^3}$$

$$\wp'(z; g_2, g_3) == -\frac{\pi^3}{4\omega_i^3}\sum_{n=-\infty}^{\infty}\csc^2\left(\frac{\pi(z-2n\omega_j)}{2\omega_i}\right)\cot\left(\frac{\pi(z-2n\omega_j)}{2\omega_i}\right) /; \{i, j\} \in \{1, 2, 3\} \bigwedge i \neq j$$

$$\wp'(z; g_2, g_3) == -\frac{\pi^3}{4\omega_1^3}\sum_{n=-\infty}^{\infty}\csc^2\left(\frac{\pi(z-2n\omega_3)}{2\omega_1}\right)\cot\left(\frac{\pi(z-2n\omega_3)}{2\omega_1}\right)$$

$$\sigma(z; g_2, g_3) == z\exp\left(-\sum_{j=2}^{\infty}\frac{z^{2j}}{2j}\sum_{\substack{m,n=-\infty \\ \{m,n\}\neq\{0,0\}}}^{\infty}\frac{1}{(2m\omega_1+2n\omega_3)^{2j}}\right)$$

$$\sigma_i(z; g_2, g_3) == z\exp\left(-\sum_{j=2}^{\infty}\frac{z^{2j}}{2j}\sum_{\substack{m,n=-\infty \\ \{m,n\}\neq\{0,0\}}}^{\infty}\frac{1}{(2m\omega_1+2n\omega_3)^{2j}}\right)\left(-e_i + \frac{1}{z^2} + \sum_{j=1}^{\infty}(2j+1)z^{2j}\sum_{\substack{m,n=-\infty \\ \{m,n\}\neq\{0,0\}}}^{\infty}\frac{1}{(2m\omega_1+2n\omega_3)^{2j+2}}\right)^{1/2} /;$$

$i \in \{1, 2, 3\}$

$$\zeta(z;\,g_2,\,g_3)=\frac{1}{z}+\sum_{\substack{m,n=-\infty\\ \{m,n\}\neq\{0,0\}}}^{\infty}\frac{z}{(2\,m\,\omega_1+2\,n\,\omega_3)^2}+\frac{1}{2\,m\,\omega_1+2\,n\,\omega_3}+\frac{1}{z-2\,m\,\omega_1-2\,n\,\omega_3}$$

$$\zeta(z;\,g_2,\,g_3)=\frac{z\,\eta_1}{\omega_1}+\frac{\pi}{2\,\omega_1}\sum_{n=-\infty}^{\infty}\cot\left(\frac{(z-2\,n\,\omega_3)\,\pi}{2\,\omega_1}\right)$$

$$\zeta(z;\,g_2,\,g_3)=\frac{\eta_i\,z}{\omega_i}+\frac{\pi}{2\,\omega_i}\cot\left(\frac{\pi\,z}{2\,\omega_i}\right)+\frac{\pi}{2\,\omega_i}\sum_{k=-\infty}^{\infty}\cot\left(\frac{\pi\,k\,\omega_j}{\omega_i}\right)+\cot\left(\pi\,\frac{z-2\,k\,\omega_j}{2\,\omega_i}\right)\;/;\;\{i,\,j\}\in\{1,\,2,\,3\}\bigwedge i\neq j.$$

Integral representations

The Weierstrass functions and their inverses $\wp(z;\,g_2,\,g_3)$, $\wp'(z;\,g_2,\,g_3)$, $\sigma(z;\,g_2,\,g_3)$, $\zeta(z;\,g_2,\,g_3)$, $\wp^{-1}(z;\,g_2,\,g_3)$, and $\wp^{-1}(z_1,\,z_2;\,g_2,\,g_3)$ can be represented through the following integrals from elementary or Weierstrass functions:

$$\wp(z;\,g_2,\,g_3)=\frac{1}{z^2}+\frac{1}{4}\int_0^{\infty}t\left(\frac{\left(\cosh(t\,\omega_3)+e^{-\frac{t}{2}\omega_3}\sinh\left(\frac{t\,\omega_3}{2}\right)\right)\left(\cosh\left(\frac{t\,z}{2}\right)-1\right)}{\sinh\left(\frac{1}{2}\,t\,(\omega_1-\omega_3)\right)\sinh\left(\frac{1}{2}\,t\,(\omega_1+\omega_3)\right)}+\frac{e^{\frac{i\,t}{2}\omega_3}\left(1-\cos\left(\frac{t\,z}{2}\right)\right)\cos\left(\frac{t\,\omega_3}{2}\right)}{\sin\left(\frac{1}{2}\,t\,(\omega_1-\omega_3)\right)\sin\left(\frac{1}{2}\,t\,(\omega_1+\omega_3)\right)}\right)dt$$

$$\wp(z;\,g_2,\,g_3)=w\;/;\;z=\int_{\infty}^{w}\frac{1}{\sqrt{4\,t^3-g_2\,t-g_3}}\,dt\bigwedge w\in\mathbb{R}$$

$$z=\int_{\wp(z;g_2,g_3)}^{\infty}\frac{dt}{\sqrt{4\,t^3-g_2\,t-g_3}}$$

$$\wp'(z;\,g_2,\,g_3)=-\frac{2}{z^3}+\frac{1}{8}\int_0^{\infty}t^2\left(\frac{e^{\frac{i\,t}{2}\omega_2}\cos\left(\frac{t\,\omega_2}{2}\right)\sin\left(\frac{t\,z}{2}\right)}{\sin\left(\frac{1}{2}\,t\,(\omega_1-\omega_2)\right)\sin\left(\frac{1}{2}\,t\,(\omega_1+\omega_2)\right)}+\frac{\sinh\left(\frac{t\,z}{2}\right)\left(\cosh(t\,\omega_2)+e^{-\frac{t}{2}\omega_2}\sinh\left(\frac{t\,\omega_2}{2}\right)\right)}{\sinh\left(\frac{1}{2}\,t\,(\omega_1-\omega_2)\right)\sinh\left(\frac{1}{2}\,t\,(\omega_1+\omega_2)\right)}\right)dt$$

$$\wp'(z;\,g_2,\,g_3)=\sqrt{4\,w^3-g_2\,w-g_3}\;/;\;z=\int_{\infty}^{w}\frac{1}{\sqrt{4\,t^3-g_2\,t-g_3}}\,dt\bigwedge w\in\mathbb{R}$$

$$\sigma(z;\,g_2,\,g_3)=z\exp\left(\int_0^z\left(\zeta(t;\,g_2,\,g_3)-\frac{1}{t}\right)dt\right)$$

$$\zeta(z;\,g_2,\,g_3)=\frac{1}{z}-\frac{1}{4}\int_0^{\infty}\left(\frac{\left(t\,z-2\sin\left(\frac{t\,z}{2}\right)\right)e^{\frac{i\,t}{2}\omega_2}\cos\left(\frac{t\,\omega_2}{2}\right)}{\sin\left(\frac{1}{2}\,t\,(\omega_1-\omega_2)\right)\sin\left(\frac{1}{2}\,t\,(\omega_1+\omega_2)\right)}-\frac{\left(t\,z-2\sinh\left(\frac{t\,z}{2}\right)\right)\left(\cosh(t\,\omega_2)+e^{-\frac{t}{2}\omega_2}\sinh\left(\frac{t\,\omega_2}{2}\right)\right)}{\sinh\left(\frac{1}{2}\,t\,(\omega_1-\omega_2)\right)\sinh\left(\frac{1}{2}\,t\,(\omega_1+\omega_2)\right)}\right)dt$$

$$\zeta(z;\,g_2,\,g_3)=\frac{1}{z}-\int_0^z\left(\wp(t;\,g_2,\,g_3)-\frac{1}{t^2}\right)dt$$

$$\wp^{-1}(z;\,g_2,\,g_3)=\int_{\infty}^{z}\frac{1}{\sqrt{4\,t^3-g_2\,t-g_3}}\,dt\;/;\;z\in\mathbb{R}\bigwedge\mathrm{Re}\left(4\,z^3-g_2\,z-g_3\right)>0$$

$$\wp^{-1}(z_1, z_2; g_2, g_3) = \int_{\infty}^{z_1} \frac{1}{\sqrt{4 t^3 - g_2 t - g_3}} \, dt \;/;\; z_2 = \sqrt{4 z_1^3 - g_2 z_1 - g_3}$$

Product representations

The Weierstrass functions $\wp(z; g_2, g_3)$, $\sigma(z; g_2, g_3)$, and $\sigma_j(z; g_2, g_3)$, $j \in \{1, 2, 3\}$ have the following product representations:

$$\wp(z; g_2, g_3) = e_i + \frac{\pi^2}{4 \omega_i^2} \cot^2\left(\frac{\pi z}{2 \omega_i}\right) \prod_{k=1}^{\infty} \tan^4\left(\frac{k \pi \omega_j}{\omega_i}\right) \left(\frac{\cos^2\left(\frac{k \pi \omega_j}{\omega_i}\right) - \sin^2\left(\frac{\pi z}{2 \omega_i}\right)}{\sin^2\left(\frac{k \pi \omega_j}{\omega_i}\right) - \sin^2\left(\frac{\pi z}{2 \omega_i}\right)}\right)^2 \;/;\; \{i, j\} \in \{1, 2, 3\} \bigwedge i \neq j$$

$$\sigma(z; g_2, g_3) = \frac{2 \omega_i}{\pi} \exp\left(\frac{\eta_i z^2}{2 \omega_i}\right) \sin\left(\frac{\pi z}{2 \omega_i}\right) \prod_{n=1}^{\infty} \left(1 - \frac{\sin^2\left(\frac{\pi z}{2 \omega_i}\right)}{\sin^2\left(\frac{n \pi \omega_j}{\omega_i}\right)}\right) \;/;\; \{i, j\} \in \{1, 2, 3\} \bigwedge i \neq j$$

$$\sigma(z; g_2, g_3) = z \prod_{k=1}^{\infty} \exp\left(-\frac{G_{2k+2}(\omega_1, \omega_3) z^{2k+2}}{2k+2}\right) \;/;\; G_p(\omega_1, \omega_3) = \sum_{\substack{m, n=-\infty \\ \{m,n\} \neq \{0,0\}}}^{\infty} \frac{1}{(2 m \omega_1 + 2 n \omega_3)^p}$$

$$\sigma_i(z; g_2, g_3) = \exp\left(\frac{\eta_i z^2}{2 \omega_i}\right) \cos\left(\frac{\pi z}{2 \omega_i}\right) \prod_{n=1}^{\infty} \left(1 - \frac{\sin^2\left(\frac{\pi z}{2 \omega_i}\right)}{\cos^2\left(\frac{n \pi \omega_j}{\omega_i}\right)}\right) \;/;\; \{i, j\} \in \{1, 2, 3\} \bigwedge i \neq j$$

$$\sigma_i(z; g_2, g_3) = \exp\left(-\frac{e_i z^2}{2}\right) \prod_{\substack{m, n=-\infty \\ \{m,n\} \neq \{0,0\}}}^{\infty} \left(\left(1 - \frac{z}{2 m \omega_1 + 2 n \omega_2 - \omega_i}\right) \exp\left(\frac{z}{2 m \omega_1 + 2 n \omega_2 - \omega_i} + \frac{z^2}{2 (2 m \omega_1 + 2 n \omega_2 - \omega_i)^2}\right)\right) \;/;$$

$i \in \{1, 2, 3\}$.

q-product representations

The Weierstrass functions $\wp(z; g_2, g_3)$, $\sigma(z; g_2, g_3)$, and $\sigma_j(z; g_2, g_3)$, $j \in \{1, 2, 3\}$ can be represented as so-called q-products by the following formulas:

$$\wp(z; g_2, g_3) = e_1 + \frac{\pi^2}{4 \omega_1^2} \cot^2\left(\frac{\pi z}{2 \omega_1}\right) \prod_{n=1}^{\infty} \left(\frac{1 - q^{2n}}{1 + q^{2n}}\right)^4 \left(\frac{1 + 2 q^{2n} \cos\left(\frac{\pi z}{\omega_1}\right) + q^{4n}}{1 - 2 q^{2n} \cos\left(\frac{\pi z}{\omega_1}\right) + q^{4n}}\right)^2 \;/;\; q = \exp\left(\frac{\pi i \omega_3}{\omega_1}\right).$$

$$\wp(z; g_2, g_3) = e_2 + \frac{\pi^2}{4 \omega_1^2} \csc^2\left(\frac{\pi z}{2 \omega_1}\right) \prod_{n=1}^{\infty} \left(\frac{1 - q^{2n}}{1 + q^{2n-1}}\right)^4 \left(\frac{1 + 2 q^{2n-1} \cos\left(\frac{\pi z}{\omega_1}\right) + q^{4n-2}}{1 - 2 q^{2n} \cos\left(\frac{\pi z}{\omega_1}\right) + q^{4n}}\right)^2 \;/;\; q = \exp\left(\frac{\pi i \omega_3}{\omega_1}\right).$$

$$\wp(z; g_2, g_3) = e_3 + \frac{\pi^2}{4 \omega_1^2} \csc^2\left(\frac{\pi z}{2 \omega_1}\right) \prod_{n=1}^{\infty} \left(\frac{1 - q^{2n}}{1 - q^{2n-1}}\right)^4 \left(\frac{1 - 2 q^{2n-1} \cos\left(\frac{\pi z}{\omega_1}\right) + q^{4n-2}}{1 - 2 q^{2n} \cos\left(\frac{\pi z}{\omega_1}\right) + q^{4n}}\right)^2 \;/;\; q = \exp\left(\frac{\pi i \omega_3}{\omega_1}\right).$$

$$\sigma(z;\,g_2,\,g_3) = \frac{2\,\omega_1}{\pi}\sin\left(\frac{\pi\,z}{2\,\omega_1}\right)\exp\left(\frac{\eta_1\,z^2}{2\,\omega_1}\right)\prod_{n=1}^{\infty}\frac{1-2\,q^{2n}\cos\left(\frac{\pi z}{\omega_1}\right)+q^{4n}}{\left(1-q^{2n}\right)^2}\;/;\;q = \exp\left(\frac{\pi\,i\,\omega_3}{\omega_1}\right).$$

$$\sigma(z;\,g_2,\,g_3) = \frac{2\,\omega_1}{\pi}\sin\left(\frac{\pi\,z}{2\,\omega_1}\right)\exp\left(\frac{\eta_1\,z^2}{2\,\omega_1}\right)\left(\prod_{n=1}^{\infty}\frac{1-q^{2n}\exp\left(-\frac{i\pi z}{\omega_1}\right)}{1-q^{2n}}\right)\left(\prod_{n=1}^{\infty}\frac{1-q^{2n}\exp\left(\frac{i\pi z}{\omega_1}\right)}{1-q^{2n}}\right)\;/;\;q = \exp\left(\frac{\pi\,i\,\omega_3}{\omega_1}\right).$$

$$\sigma(\omega_1;\,g_2,\,g_3) = \frac{2\,\omega_1}{\pi}\exp\left(\frac{\eta_1\,\omega_1}{2}\right)\left(\prod_{n=1}^{\infty}\frac{1+q^{2n}}{1-q^{2n}}\right)^2\;/;\;q = \exp\left(\frac{\pi\,i\,\omega_3}{\omega_1}\right).$$

$$\sigma(\omega_2;\,g_2,\,g_3) = -\sqrt{i}\;\frac{\omega_1}{\pi}\exp\left(\frac{\eta_2\,\omega_2}{2}\right)\frac{1}{\sqrt[4]{q}}\left(\prod_{n=1}^{\infty}\frac{1+q^{2n-1}}{1-q^{2n}}\right)^2\;/;\;q = \exp\left(\frac{\pi\,i\,\omega_3}{\omega_1}\right).$$

$$\sigma(\omega_3;\,g_2,\,g_3) = i\;\frac{\omega_1}{\pi}\exp\left(\frac{\eta_3\,\omega_3}{2}\right)\frac{1}{\sqrt[4]{q}}\left(\prod_{n=1}^{\infty}\frac{1-q^{2n-1}}{1-q^{2n}}\right)^2\;/;\;q = \exp\left(\frac{\pi\,i\,\omega_3}{\omega_1}\right).$$

$$\sigma_1(z;\,g_2,\,g_3) = \cos\left(\frac{\pi\,z}{2\,\omega_1}\right)\exp\left(\frac{\eta_1\,z^2}{2\,\omega_1}\right)\prod_{n=1}^{\infty}\frac{1+2\,q^{2n}\cos\left(\frac{\pi z}{\omega_1}\right)+q^{4n}}{\left(1+q^{2n}\right)^2}\;/;\;q = \exp\left(\frac{\pi\,i\,\omega_3}{\omega_1}\right).$$

$$\sigma_2(z;\,g_2,\,g_3) = \exp\left(\frac{\eta_1\,z^2}{2\,\omega_1}\right)\prod_{n=1}^{\infty}\frac{1+2\,q^{2n-1}\cos\left(\frac{\pi z}{\omega_1}\right)+q^{4n-2}}{\left(1+q^{2n-1}\right)^2}\;/;\;q = \exp\left(\frac{\pi\,i\,\omega_3}{\omega_1}\right).$$

$$\sigma_3(z;\,g_2,\,g_3) = \exp\left(\frac{\eta_1\,z^2}{2\,\omega_1}\right)\prod_{n=1}^{\infty}\frac{1-2\,q^{2n-1}\cos\left(\frac{\pi z}{\omega_1}\right)+q^{4n-2}}{\left(1-q^{2n-1}\right)^2}\;/;\;q = \exp\left(\frac{\pi\,i\,\omega_3}{\omega_1}\right).$$

$$\sigma_1(\omega_2;\,g_2,\,g_3) = -\frac{i\sqrt{i}}{2\sqrt[4]{q}}\exp\left(\frac{\eta_2\,\omega_2}{2}\right)\left(\prod_{n=1}^{\infty}\frac{1-q^{2n-1}}{1+q^{2n}}\right)^2\;/;\;q = \exp\left(\frac{\pi\,i\,\omega_3}{\omega_1}\right).$$

$$\sigma_1(\omega_3;\,g_2,\,g_3) = \frac{1}{2\sqrt[4]{q}}\exp\left(\frac{\eta_3\,\omega_3}{2}\right)\left(\prod_{n=1}^{\infty}\frac{1+q^{2n-1}}{1+q^{2n}}\right)^2\;/;\;q = \exp\left(\frac{\pi\,i\,\omega_3}{\omega_1}\right).$$

$$\sigma_2(\omega_1;\,g_2,\,g_3) = \exp\left(\frac{\eta_1\,\omega_1}{2}\right)\left(\prod_{n=1}^{\infty}\frac{1-q^{2n-1}}{1+q^{2n-1}}\right)^2\;/;\;q = \exp\left(\frac{\pi\,i\,\omega_3}{\omega_1}\right).$$

$$\sigma_2(\omega_3;\,g_2,\,g_3) = 2\sqrt[4]{q}\;\exp\left(\frac{\eta_3\,\omega_3}{2}\right)\left(\prod_{n=1}^{\infty}\frac{1+q^{2n}}{1+q^{2n-1}}\right)^2\;/;\;q = \exp\left(\frac{\pi\,i\,\omega_3}{\omega_1}\right).$$

$$\sigma_3(\omega_1;\,g_2,\,g_3) = \exp\left(\frac{\eta_1\,\omega_1}{2}\right)\left(\prod_{n=1}^{\infty}\frac{1+q^{2n-1}}{1-q^{2n-1}}\right)^2\;/;\;q = \exp\left(\frac{\pi\,i\,\omega_3}{\omega_1}\right).$$

$$\sigma_3(\omega_2;\,g_2,\,g_3) = 2\sqrt{i}\;\sqrt[4]{q}\;\exp\left(\frac{\eta_2\,\omega_2}{2}\right)\left(\prod_{n=1}^{\infty}\frac{1+q^{2n}}{1-q^{2n-1}}\right)^2\;/;\;q = \exp\left(\frac{\pi\,i\,\omega_3}{\omega_1}\right).$$

Transformations

The Weierstrass functions $\wp(z; g_2, g_3)$, $\wp'(z; g_2, g_3)$, $\sigma(z; g_2, g_3)$, $\sigma_n(z, g_2, g_3) /; n \in \{1, 2, 3\}$, and $\zeta(z; g_2, g_3)$ satisfy numerous relations that can provide transformations of its
arguments. One of these transformations simplifies argument $i z$ to z, for example:

$$\wp(i z; g_2, g_3) = -\wp(z; g_2, -g_3)$$

$$\wp'(i z; g_2, g_3) = i \wp'(z; g_2, -g_3)$$

$$\zeta(i z; g_2, g_3) = -i \zeta(z; g_2, -g_3).$$

Other transformations are described by so-called addition formulas:

$$\wp(z_1 + z_2; g_2, g_3) = \frac{1}{4} \left(\frac{\wp'(z_1; g_2, g_3) - \wp'(z_2; g_2, g_3)}{\wp(z_1; g_2, g_3) - \wp(z_2; g_2, g_3)} \right)^2 - \wp(z_1; g_2, g_3) - \wp(z_2; g_2, g_3)$$

$$\wp(z_1 + z_2; g_2, g_3) - \wp(z_1 - z_2; g_2, g_3) = -\frac{\wp'(z_1; g_2, g_3) \wp'(z_2; g_2, g_3)}{(\wp(z_1; g_2, g_3) - \wp(z_2; g_2, g_3))^2}$$

$$\wp\left(z \pm \omega_j; g_2, g_3\right) = e_j + \frac{\left(e_j - e_k\right)\left(e_j - e_l\right)}{\wp(z; g_2, g_3) - e_j} /; \{i, j, k\} \in \{1, 2, 3\} \land i \neq j \neq k$$

$$\wp'(z_1 + z_2; g_2, g_3) = (\wp(z_1 + z_2; g_2, g_3) (\wp'(z_1; g_2, g_3) - \wp'(z_2; g_2, g_3)) + \wp(z_1; g_2, g_3) \wp'(z_2; g_2, g_3) - \wp'(z_1; g_2, g_3) \wp(z_2; g_2, g_3)) / (\wp(z_2; g_2, g_3) - \wp(z_1; g_2, g_3))$$

$$\sigma(z_2 + z_1; g_2, g_3) = \exp\left(z_1 \zeta(z_2; g_2, g_3) - \frac{z_1^2}{2} \wp(z_2; g_2, g_3)\right) \sigma(z_2; g_2, g_3)$$
$$\prod_{m,n=-\infty}^{\infty} \left(1 - \frac{z_1}{2 m \omega_1 + 2 n \omega_3 - z_2}\right) \exp\left(\frac{z_1}{2 m \omega_1 + 2 n \omega_3 - z_2} + \frac{z_1^2}{2 (2 m \omega_1 + 2 n \omega_3 - z_2)^2}\right)$$

$$\sigma(z_1 + z_2; g_2, g_3) \sigma(z_1 - z_2; g_2, g_3) = -\sigma(z_1; g_2, g_3)^2 \sigma(z_2; g_2, g_3)^2 (\wp(z_1; g_2, g_3) - \wp(z_2; g_2, g_3))$$

$$\sigma(z_1 + z_2; g_2, g_3) \sigma(z_1 - z_2; g_2, g_3) = \sigma(z_1; g_2, g_3)^2 \sigma_i(z_2; g_2, g_3)^2 - \sigma(z_2; g_2, g_3)^2 \sigma_i(z_1; g_2, g_3)^2 /; i \in \{1, 2, 3\}$$

$$\sigma(z \pm \omega_i; g_2, g_3) = \pm e^{\pm \eta_i z} \sigma(\omega_i; g_2, g_3) \sigma_i(\omega_i; g_2, g_3) /; i \in \{1, 2, 3\}$$

$$\sigma_i(z_1 + z_2; g_2, g_3) \sigma_i(z_1 - z_2; g_2, g_3) = \sigma_i(z_1; g_2, g_3)^2 \sigma_i(z_2; g_2, g_3)^2 - \left(e_i - e_j\right) (e_i - e_k) \sigma(z_1; g_2, g_3)^2 \sigma(z_2; g_2, g_3)^2 /; \{i, j, k\} \in \{1, 2, 3\} \land i \neq j \neq k$$

$$\sigma_i(z_1 + z_2; g_2, g_3) \sigma_i(z_1 - z_2; g_2, g_3) = \sigma_i(z_1; g_2, g_3)^2 \sigma_j(z_2; g_2, g_3)^2 - \left(e_i - e_j\right) \sigma(z_2; g_2, g_3)^2 \sigma_k(z_1; g_2, g_3)^2 /; \{i, j, k\} \in \{1, 2, 3\} \land i \neq j \neq k$$

$$\sigma_j(z \pm \omega_i; g_2, g_3) = e^{\pm \eta_i z} \sigma_j(\omega_i; g_2, g_3) \sigma_k(z; g_2, g_3) /; \{i, j, k\} \in \{1, 2, 3\} \land i \neq j \neq k$$

$$\zeta(z_1 \pm z_2; g_2, g_3) = \zeta(z_1; g_2, g_3) \pm \zeta(z_2; g_2, g_3) + \frac{1}{2} \frac{\wp'(z_1; g_2, g_3) \mp \wp'(z_2; g_2, g_3)}{\wp(z_1; g_2, g_3) - \wp(z_2; g_2, g_3)}$$

$$\zeta(z_1+z_2; g_2, g_3)+\zeta(z_1-z_2; g_2, g_3) == 2\,\zeta(z_1; g_2, g_3)+\frac{\wp'(z_1; g_2, g_3)}{\wp(z_1; g_2, g_3)-\wp(z_2; g_2, g_3)}$$

$$\zeta(z\pm\omega_i; g_2, g_3) == \zeta(z; g_2, g_3)\pm\eta_i+\frac{1}{2}\,\frac{\wp'(z; g_2, g_3)}{\wp(z; g_2, g_3)-\eta_i}\ /;\ i\in\{1, 2, 3\}.$$

Half-angle formulas provide one more type of transformation, for example:

$$\wp\left(\frac{z}{2}; g_2, g_3\right) == \wp(z; g_2, g_3)+\epsilon_2\,\epsilon_3\sqrt{\wp(z; g_2, g_3)-e_2}\sqrt{\wp(z; g_2, g_3)-e_3}+\epsilon_3\,\epsilon_1\sqrt{\wp(z; g_2, g_3)-e_3}\sqrt{\wp(z; g_2, g_3)-e_1}+ \\ \epsilon_1\,\epsilon_2\sqrt{\wp(z; g_2, g_3)-e_1}\sqrt{\wp(z; g_2, g_3)-e_2}\ /;\ \epsilon_n == \operatorname{sgn}\left(\frac{\pi}{2}-\left|\operatorname{Arg}\left(\frac{\sigma_n(z; g_2, g_3)}{\sigma(z; g_2, g_3)}\right)\right|\right)\bigwedge n\in\{1, 2, 3\}.$$

The Weierstrass functions $\wp(z; g_2, g_3)$, $\wp'(z; g_2, g_3)$, $\sigma(z; g_2, g_3)$, and $\zeta(z; g_2, g_3)$ satisfy the following double-angle formulas:

$$\wp(2\,z; g_2, g_3) == \frac{\left(\wp(z; g_2, g_3)^2+\frac{g_2}{4}\right)^2+2\,g_3\,\wp(z; g_2, g_3)}{4\,\wp(z; g_2, g_3)^3-g_2\,\wp(z; g_2, g_3)-g_3}$$

$$\wp(2\,z; g_2, g_3) == -2\,\wp(z; g_2, g_3)+\frac{1}{4}\left(\frac{\wp''(z; g_2, g_3)}{\wp'(z; g_2, g_3)}\right)^2$$

$$\wp(2\,z; g_2, g_3) == \frac{\left(\wp(z; g_2, g_3)^2+\frac{g_2}{4}\right)^2+2\,g_3\,\wp(z; g_2, g_3)}{\wp'(z; g_2, g_3)^2}$$

$$\wp'(2\,z; g_2, g_3) == \frac{1}{4\,\wp'(z; g_2, g_3)^3}\left(-4\,\wp'(z; g_2, g_3)^4+12\,\wp(z; g_2, g_3)\,\wp'(z; g_2, g_3)^2\,\wp''(z; g_2, g_3)-\wp''(z; g_2, g_3)^3\right)$$

$$\sigma(2\,z; g_2, g_3) == \frac{2\,\sigma(z; g_2, g_3)\,\sigma(\omega_1-z; g_2, g_3)\,\sigma(\omega_2-z; g_2, g_3)\,\sigma(\omega_3-z; g_2, g_3)}{\sigma(\omega_1; g_2, g_3)\,\sigma(\omega_2; g_2, g_3)\,\sigma(\omega_3; g_2, g_3)}$$

$$\sigma(2\,z; g_2, g_3) == 2\,\sigma(z; g_2, g_3)\,\sigma_1(z; g_2, g_3)\,\sigma_2(z; g_2, g_3)\,\sigma_3(z; g_2, g_3)$$

$$\sigma(2\,z; g_2, g_3) == \sigma(z; g_2, g_3)\left(2\left(\frac{\partial\sigma(z; g_2, g_3)}{\partial z}\right)^3-3\,\sigma(z; g_2, g_3)\,\frac{\partial^2\sigma(z; g_2, g_3)}{\partial z^2}\,\frac{\partial\sigma(z; g_2, g_3)}{\partial z}+\frac{\partial^3\sigma(z; g_2, g_3)}{\partial z^3}\,\sigma(z; g_2, g_3)^2\right)$$

$$\sigma(2\,z; g_2, g_3) == -\wp'(z; g_2, g_3)\,\sigma(z; g_2, g_3)^4$$

$$\zeta(2\,z; g_2, g_3) == \frac{1}{2}\left(\zeta(z; g_2, g_3)+\zeta(z-\omega_1; g_2, g_3)+\zeta(z-\omega_2; g_2, g_3)+\zeta(z-\omega_3; g_2, g_3)\right)$$

$$\zeta(2\,z; g_2, g_3) == 2\,\zeta(z; g_2, g_3)+\frac{\wp''(z; g_2, g_3)}{2\,\wp'(z; g_2, g_3)}.$$

These formulas can be expanded on triple angle formulas, for example:

$$\sigma(3\,z;\,g_2,\,g_3) = -\wp'(z;\,g_2,\,g_3)^2\,\sigma(z;\,g_2,\,g_3)^9\,(\wp(2\,z;\,g_2,\,g_3) - \wp(z;\,g_2,\,g_3))$$

$$\sigma(3\,z;\,g_2,\,g_3) = \sigma(z;\,g_2,\,g_3)^9\left(3\,\wp(z;\,g_2,\,g_3)^4 - \frac{3}{2}\,g_2\,\wp(z;\,g_2,\,g_3)^2 - 3\,g_3\,\wp(z;\,g_2,\,g_3) - \frac{g_2^2}{16}\right)$$

$$\zeta(3\,z;\,g_2,\,g_3) = 3\,\zeta(z;\,g_2,\,g_3) + \frac{4\,\wp'(z;\,g_2,\,g_3)^3}{\wp'(z;\,g_2,\,g_3)\,\wp^{(3)}(z;\,g_2,\,g_3) - \wp''(z;\,g_2,\,g_3)^2}.$$

Generally the following multiple angle formulas take place:

$$\wp(n\,z;\,g_2,\,g_3) = \frac{1}{n^2}\sum_{j=0}^{n-1}\sum_{k=0}^{n-1}\wp\left(z - \frac{2\,j\,\omega_1 + 2\,k\,\omega_3}{n};\,g_2,\,g_3\right)\,/;\,n \in \mathbb{N}^+$$

$$\wp'(n\,z;\,g_2,\,g_3) = \frac{1}{n^3}\sum_{j=0}^{n-1}\sum_{k=0}^{n-1}\wp'\left(z - \frac{2\,j\,\omega_1 + 2\,k\,\omega_3}{n};\,g_2,\,g_3\right)\,/;\,n \in \mathbb{N}^+$$

$$\sigma(n\,z;\,g_2,\,g_3) = n\,(-1)^{n^2-1}\,e^{-n\,(n-1)\,\eta_2\,z}\,\sigma(z;\,g_2,\,g_3)\prod_{\substack{j,\,k=0\\ \{j,k\}\neq\{0,0\}}}^{n-1}\frac{\sigma\left(z - \frac{2\,j\,\omega_1+2\,k\,\omega_3}{n};\,g_2,\,g_3\right)}{\sigma\left(\frac{2\,j\,\omega_1+2\,k\,\omega_3}{n};\,g_2,\,g_3\right)}\,/;\,n-1 \in \mathbb{N}^+$$

$$\zeta(n\,z;\,g_2,\,g_3) = -(n-1)\,\eta_2 + \frac{1}{n}\sum_{j=0}^{n-1}\sum_{k=0}^{n-1}\zeta\left(z - \frac{2\,j\,\omega_1 + 2\,k\,\omega_3}{n};\,g_2,\,g_3\right)\,/;\,n \in \mathbb{N}^+.$$

Sometimes transformations have a symmetrical character, which includes operations like determinate, for example:

$$\begin{vmatrix} \wp'(z_1;\,g_2,\,g_3) & \wp(z_1;\,g_2,\,g_3) & 1 \\ \wp'(z_2;\,g_2,\,g_3) & \wp(z_2;\,g_2,\,g_3) & 1 \\ -\wp'\,(z_1+z_2;\,g_2,\,g_3) & \wp(z_1+z_2;\,g_2,\,g_3) & 1 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & \wp(z_0;\,g_2,\,g_3) & \wp'(z_0;\,g_2,\,g_3) & \ldots & \wp^{(n-1)}(z_0;\,g_2,\,g_3) \\ 1 & \wp(z_1;\,g_2,\,g_3) & \wp'(z_1;\,g_2,\,g_3) & \ldots & \wp^{(n-1)}(z_1;\,g_2,\,g_3) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \wp(z_n;\,g_2,\,g_3) & \wp'(z_n;\,g_2,\,g_3) & \ldots & \wp^{(n-1)}(z_n;\,g_2,\,g_3) \end{vmatrix} =$$

$$(-1)^n\,\sigma\left(\sum_{i=0}^{n} z_i;\,g_2,\,g_3\right)\left(\prod_{k=0}^{n}\frac{k!}{\sigma(z_k;\,g_2,\,g_3)^{n+1}}\right)\prod_{i=1}^{n-1}\prod_{j=i+1}^{n}\sigma\left(z_j - z_i;\,g_2,\,g_3\right)\,/;\,n \in \mathbb{N}^+.$$

A special class of transformation includes the simplification of Weierstrass functions $\wp(z;\,g_2,\,g_3)$, $\wp'(z;\,g_2,\,g_3)$, $\sigma(z;\,g_2,\,g_3)$, $\sigma_n(z,\,g_2,\,g_3)\,/;\,n \in \{1,\,2,\,3\}$, and $\zeta(z;\,g_2,\,g_3)$ with invariants $g_2\left(\frac{\omega_1}{n},\,\omega_3\right)$ and $g_3\left(\frac{\omega_1}{n},\,\omega_3\right)$, where $n \in \mathbb{N}^+$, for example:

$$\wp\left(z; g_2\left(\frac{\omega_1}{2}, \omega_2\right), g_3\left(\frac{\omega_1}{2}, \omega_2\right)\right) = \left(\wp(z; g_2, g_3)^2 - e_1\, \wp(z; g_2, g_3) + (e_1 - e_2)(e_1 - e_3)\right) / (\wp(z; g_2, g_3) - e_1)$$

$$\wp\left(z; g_2\left(\frac{\omega_1}{2}, \omega_3\right), g_3\left(\frac{\omega_1}{2}, \omega_3\right)\right) = \wp(z; g_2, g_3) + \wp(z + \omega_1; g_2, g_3) - e_1$$

$$\wp\left(z; g_2\left(\frac{\omega_1}{3}, \omega_3\right), g_3\left(\frac{\omega_1}{3}, \omega_3\right)\right) = \wp\left(z + \frac{2\,\omega_1}{3}; g_2, g_3\right) - 2\,\wp\left(\frac{2\,\omega_1}{3}; g_2, g_3\right) + \wp\left(z + \frac{4\,\omega_1}{3}; g_2, g_3\right) + \wp(z; g_2, g_3)$$

$$\wp\left(z; g_2\left(\frac{\omega_1}{n}, \omega_3\right), g_3\left(\frac{\omega_1}{n}, \omega_3\right)\right) = \sum_{k=1}^{n-1}\left(\wp\left(z + \frac{2\,k\,\omega_1}{n}; g_2, g_3\right) - \wp\left(\frac{2\,k\,\omega_1}{n}; g_2, g_3\right)\right) + \wp(z; g_2, g_3) \;/;\; n \in \mathbb{N}^+$$

$$\wp'\left(z; g_2\left(\frac{\omega_1}{2}, \omega_3\right), g_3\left(\frac{\omega_1}{2}, \omega_3\right)\right) = \wp'(z; g_2, g_3) + \wp'(z + \omega_1; g_2, g_3)$$

$$\wp'\left(z; g_2\left(\frac{\omega_1}{3}, \omega_3\right), g_3\left(\frac{\omega_1}{3}, \omega_3\right)\right) = \wp'(z; g_2, g_3) + \wp'\left(z + \frac{2\,\omega_1}{3}; g_2, g_3\right) + \wp'\left(z + \frac{4\,\omega_1}{3}; g_2, g_3\right)$$

$$\wp'\left(z; g_2\left(\frac{\omega_1}{n}, \omega_3\right), g_3\left(\frac{\omega_1}{n}, \omega_3\right)\right) = \wp'(z; g_2, g_3) + \sum_{k=1}^{n-1}\wp'\left(z + \frac{2\,k\,\omega_1}{n}; g_2, g_3\right) \;/;\; n \in \mathbb{N}^+$$

$$\sigma\left(z; g_2\left(\frac{\omega_1}{2}, \omega_3\right), g_3\left(\frac{\omega_1}{2}, \omega_3\right)\right) = \exp\left(\frac{e_1\, z^2}{2}\right)\sigma(z; g_2, g_3)\,\sigma_1(z; g_2, g_3)$$

$$\sigma\left(z; g_2\left(\frac{\omega_1}{3}, \omega_3\right), g_3\left(\frac{\omega_1}{3}, \omega_3\right)\right) = \exp\left(z^2\,\wp\left(\frac{2\,\omega_1}{3}; g_2, g_3\right) - 2\,z\,\eta_1\right) \frac{\sigma(z; g_2, g_3)\,\sigma\left(z + \frac{2\,\omega_1}{3}; g_2, g_3\right)\sigma\left(z + \frac{4\,\omega_1}{3}; g_2, g_3\right)}{\sigma\left(\frac{2\,\omega_1}{3}; g_2, g_3\right)\sigma\left(\frac{4\,\omega_1}{3}; g_2, g_3\right)}$$

$$\sigma\left(z; g_2\left(\frac{\omega_1}{n}, \omega_3\right), g_3\left(\frac{\omega_1}{n}, \omega_3\right)\right) = \exp\left(\frac{z^2}{2}\sum_{k=1}^{n-1}\wp\left(\frac{2\,k\,\omega_1}{n}; g_2, g_3\right) - z\sum_{k=1}^{n-1}\frac{2\,k\,\eta_1}{n}\right)\sigma(z; g_2, g_3)\prod_{k=1}^{n-1}\frac{\sigma\left(z + \frac{2\,k\,\omega_1}{n}; g_2, g_3\right)}{\sigma\left(\frac{2\,k\,\omega_1}{n}; g_2, g_3\right)} \;/;\; n \in \mathbb{N}^+$$

$$\sigma\left(z; g_2\left(\frac{\omega_1}{n}, \omega_3\right), g_3\left(\frac{\omega_1}{n}, \omega_3\right)\right) =$$
$$\exp\left(\frac{z^2}{2}\sum_{j=1}^{n-1}\wp\left(\frac{j\,2\,\omega_1}{n}; g_2, g_3\right) - z\sum_{j=1}^{n-1}\zeta\left(\frac{j\,2\,\omega_1}{n}; g_2, g_3\right)\right)\sigma(z; g_2, g_3)\prod_{j=1}^{n-1}\frac{\sigma\left(z + \frac{2\,j\,\omega_1}{n}; g_2, g_3\right)}{\sigma\left(\frac{j\,2\,\omega_1}{n}; g_2, g_3\right)} \;/;\; n \in \mathbb{N}^+$$

$$\sigma_1\left(z; g_2\left(\frac{\omega_1}{2}, \omega_3\right), g_3\left(\frac{\omega_1}{2}, \omega_3\right)\right) = \exp\left(\frac{e_1\, z^2}{2}\right)\left(\sigma_1(z; g_2, g_3)^2 - e^{\eta_1\,\omega_1}\,\frac{\sigma(z; g_2, g_3)^2}{\sigma(\omega_1; g_2, g_3)^2}\right)$$

$$\sigma_2\left(z; g_2\left(\frac{\omega_1}{2}, \omega_3\right), g_3\left(\frac{\omega_1}{2}, \omega_3\right)\right) = \exp\left(\frac{e_1\, z^2}{2}\right)\left(\sigma_1(z; g_2, g_3)^2 + e^{\eta_1\,\omega_1}\,\frac{\sigma(z; g_2, g_3)^2}{\sigma(\omega_1; g_2, g_3)^2}\right)$$

$$\sigma_3\left(z; g_2\left(\frac{\omega_1}{2}, \omega_3\right), g_3\left(\frac{\omega_1}{2}, \omega_3\right)\right) = \exp\left(\frac{e_1\, z^2}{2}\right)\sigma_2(z; g_2, g_3)\,\sigma_3(z; g_2, g_3)$$

$$\sigma_i\left(z; g_2\left(\frac{\omega_1}{3}, \omega_3\right), g_3\left(\frac{\omega_1}{3}, \omega_3\right)\right) = \exp\left(z^2\,\wp\left(\frac{2\,\omega_1}{3}; g_2, g_3\right) - 2\,z\,\eta_1\right)$$
$$\left(\sigma_i(z; g_2, g_3)\,\sigma_i\left(z + \frac{2\,\omega_1}{3}; g_2, g_3\right)\sigma_i\left(z + \frac{4\,\omega_1}{3}; g_2, g_3\right)\right) \Big/ \left(\sigma_i\left(\frac{2\,\omega_1}{3}; g_2, g_3\right)\sigma_i\left(\frac{4\,\omega_1}{3}; g_2, g_3\right)\right) /; i \in \{1, 2, 3\}$$

$$\sigma_1\left(z; g_2\left(\frac{\omega_1}{n}, \omega_3\right), g_3\left(\frac{\omega_1}{n}, \omega_3\right)\right) =$$
$$\exp\left(\frac{(n-1)\,z\,\eta_1}{n} + \sum_{k=1}^{n-1}\left(\frac{z^2}{2}\,\wp\left(\frac{2\,k\,\omega_1}{n}; g_2, g_3\right) - \frac{(2\,k-n+1)\,\eta_1\,z}{n}\right)\right)\prod_{k=0}^{n-1}\frac{\sigma_1\left(z + \frac{(2\,k-n+1)\,\omega_1}{n}; g_2, g_3\right)}{\sigma_1\left(\frac{(2\,k-n+1)\,\omega_1}{n}; g_2, g_3\right)} /; n \in \mathbb{N}^+$$

$$\sigma_2\left(z; g_2\left(\frac{\omega_1}{n}, \omega_3\right), g_3\left(\frac{\omega_1}{n}, \omega_3\right)\right) =$$
$$\exp\left(\frac{(n-1)\,z\,\eta_1}{n} + \sum_{k=1}^{n-1}\left(\frac{z^2}{2}\,\wp\left(\frac{2\,k\,\omega_1}{n}; g_2, g_3\right) - \frac{(2\,k-n+1)\,\eta_1\,z}{n}\right)\right)\prod_{k=0}^{n-1}\frac{\sigma_2\left(z + \frac{(2\,k-n+1)\,\omega_1}{n}; g_2, g_3\right)}{\sigma_2\left(\frac{(2\,k-n+1)\,\omega_1}{n}; g_2, g_3\right)} /; n \in \mathbb{N}^+$$

$$\sigma_3\left(z; g_2\left(\frac{\omega_1}{n}, \omega_3\right), g_3\left(\frac{\omega_1}{n}, \omega_3\right)\right) = \exp\left(\sum_{k=1}^{n-1}\left(\frac{z^2}{2}\,\wp\left(\frac{2\,k\,\omega_1}{n}; g_2, g_3\right) - \frac{2\,k\,\eta_1\,z}{n}\right)\right)\prod_{k=0}^{n-1}\frac{\sigma_3\left(z + \frac{2\,k\,\omega_1}{n}; g_2, g_3\right)}{\sigma_3\left(\frac{2\,k\,\omega_1}{n}; g_2, g_3\right)} /; n \in \mathbb{N}^+$$

$$\zeta\left(z; g_2\left(\frac{\omega_1}{2}, \omega_3\right), g_3\left(\frac{\omega_1}{2}, \omega_3\right)\right) = e_1\,z + \zeta(z; g_2, g_3) + \zeta(z + \omega_1; g_2, g_3) - \eta_1$$

$$\zeta\left(z; g_2\left(\frac{\omega_1}{3}, \omega_3\right), g_3\left(\frac{\omega_1}{3}, \omega_3\right)\right) = \zeta\,(z; g_2, g_3) + \zeta\left(z + \frac{2\,\omega_1}{3}; g_2, g_3\right) + \zeta\left(z + \frac{4\,\omega_1}{3}; g_2, g_3\right) + 2\,z\,\wp\left(\frac{2\,\omega_1}{3}; g_2, g_3\right) - 2\,\eta_1$$

$$\zeta\left(z; g_2\left(\frac{\omega_1}{n}, \omega_3\right), g_3\left(\frac{\omega_1}{n}, \omega_3\right)\right) = \zeta(z; g_2, g_3) + \sum_{k=1}^{n-1}\left(\zeta\left(z + \frac{2\,k\,\omega_1}{n}; g_2, g_3\right) + z\,\wp\left(\frac{2\,k\,\omega_1}{n}; g_2, g_3\right) - \frac{2\,k\,\eta_1}{n}\right) /; n \in \mathbb{N}^+.$$

Identities

The Weierstrass functions satisfy numerous functional identities, for example:

$$\frac{\wp'(a; g_2, g_3) - \wp'(b; g_2, g_3)}{\wp(a; g_2, g_3) - \wp(b; g_2, g_3)} = \frac{\wp'(b; g_2, g_3) - \wp'(c; g_2, g_3)}{\wp(b; g_2, g_3) - \wp(c; g_2, g_3)} /; a + b + c = 2\,m\,\omega_1 + 2\,n\,\omega_3 \bigwedge \{m, n\} \in \mathbb{Z}$$

$$\frac{\wp'(a; g_2, g_3) - \wp'(b; g_2, g_3)}{\wp(a; g_2, g_3) - \wp(b; g_2, g_3)} = \frac{\wp'(c; g_2, g_3) - \wp'(a; g_2, g_3)}{\wp(c; g_2, g_3) - \wp(a; g_2, g_3)} /; a + b + c = 2\,m\,\omega_1 + 2\,n\,\omega_3 \bigwedge \{m, n\} \in \mathbb{Z}$$

$$\frac{\wp(b; g_2, g_3)\,\wp'(c; g_2, g_3) - \wp(c; g_2, g_3)\,\wp'(b; g_2, g_3)}{\wp(b; g_2, g_3) - \wp(c; g_2, g_3)} = \frac{\wp(c; g_2, g_3)\,\wp'(a; g_2, g_3) - \wp(a; g_2, g_3)\,\wp'(c; g_2, g_3)}{\wp(c; g_2, g_3) - \wp(a; g_2, g_3)} /;$$
$$a + b + c = 2\,m\,\omega_1 + 2\,n\,\omega_3 \bigwedge \{m, n\} \in \mathbb{Z}$$

$$\frac{\wp(b;g_2,g_3)\,\wp'(c;g_2,g_3)-\wp(c;g_2,g_3)\,\wp'(b;g_2,g_3)}{\wp(b;g_2,g_3)-\wp(c;g_2,g_3)} = \frac{\wp(a;g_2,g_3)\,\wp'(b;g_2,g_3)-\wp(b;g_2,g_3)\,\wp'(a;g_2,g_3)}{\wp(a;g_2,g_3)-\wp(b;g_2,g_3)} \;/;\; a+b+c = 2\,m\,\omega_1 + 2\,n\,\omega_3 \bigwedge \{m,n\}\in\mathbb{Z}$$

$$\sigma_i(z;g_2,g_3)^2 - \sigma_j(z;g_2,g_3)^2 = (e_j - e_i)\,\sigma(z;g_2,g_3)^2 \;/;\; \{i,j\}\in\{1,2,3\} \bigwedge i\neq j$$

$$(e_k - e_j)\,\sigma_i(z;g_2,g_3)^2 + (e_i - e_k)\,\sigma_j(z;g_2,g_3)^2 + (e_j - e_i)\,\sigma_k(z;g_2,g_3)^2 = 0 \;/;\; \{i,j\}\in\{1,2,3\} \bigwedge i\neq j$$

$$\wp^{-1}(z_1;g_2,g_3) + \wp^{-1}(z_2;g_2,g_3) = \wp^{-1}(z_3;g_2,g_3) \;/;\; \sqrt{4\,z_3^3 - g_2\,z_3 - g_3}\,(z_2 - z_1) + \sqrt{4\,z_1^3 - g_2\,z_1 - g_3}\,(z_3 - z_2) + \sqrt{4\,z_2^3 - g_2\,z_2 - g_3}\,(z_1 - z_3) = 0.$$

Representations of derivatives

The first two derivatives of all Weierstrass functions $\wp(z;g_2,g_3)$, $\wp'(z;g_2,g_3)$, $\sigma(z;g_2,g_3)$, $\sigma_n(z,g_2,g_3)$ /; $n\in\{1,2,3\}$, and $\zeta(z;g_2,g_3)$, and their inverses $\wp^{-1}(z;g_2,g_3)$ and $\wp^{-1}(z_1,z_2;g_2,g_3)$ with respect to variable z can also be expressed through Weierstrass functions:

$$\frac{\partial\wp(z;g_2,g_3)}{\partial z} = \wp'(z;g_2,g_3) \qquad \frac{\partial^2\wp(z;g_2,g_3)}{\partial z^2} = 6\,\wp(z;g_2,g_3)^2 - \frac{g_2}{2}$$

$$\frac{\partial\wp'(z;g_2,g_3)}{\partial z} = 6\,\wp(z;g_2,g_3)^2 - \frac{g_2}{2} \qquad \frac{\partial^2\wp'(z;g_2,g_3)}{\partial z^2} = 12\,\wp(z;g_2,g_3)\,\wp'(z;g_2,g_3)$$

$$\frac{\partial\sigma(z;g_2,g_3)}{\partial z} = \sigma(z;g_2,g_3)\,\zeta(z;g_2,g_3) \qquad \frac{\partial^2\sigma(z;g_2,g_3)}{\partial z^2} = \sigma(z;g_2,g_3)\left(\zeta(z;g_2,g_3)^2 - \wp(z;g_2,g_3)\right)$$

$$\frac{\partial\sigma_n(z,g_2,g_3)}{\partial z} = \sigma_n(z,g_2,g_3)\,(\zeta(z+\omega_n;g_2,g_3) - \eta_n) \qquad \frac{\partial^2\sigma_n(z,g_2,g_3)}{\partial z^2} = \sigma_n(z,g_2,g_3)\left((\eta_n - \zeta(z+\omega_n;g_2,g_3))^2 - \wp(z+\omega_n;g_2,g_3)\right)$$

$$\frac{\partial\zeta(z;g_2,g_3)}{\partial z} = -\wp(z;g_2,g_3) \qquad \frac{\partial^2\zeta(z;g_2,g_3)}{\partial z^2} = -\wp'(z;g_2,g_3)$$

$$\frac{\partial\wp^{-1}(z;g_2,g_3)}{\partial z} = \frac{1}{\wp'(\wp^{-1}(z;g_2,g_3);g_2,g_3)} \qquad \frac{\partial\wp^{-1}(z;g_2,g_3)}{\partial z} = \frac{1}{\sqrt{4\,z^3 - g_2\,z - g_3}}$$

$$\frac{\partial^2\wp^{-1}(z;g_2,g_3)}{\partial z^2} = \frac{g_2 - 12\,z^2}{2\,\wp'\left(\wp^{-1}(z;g_2,g_3);g_2,g_3\right)^3}$$

$$\frac{\partial\wp^{-1}(z_1,z_2;g_2,g_3)}{\partial z_1} = \frac{1}{\wp'\left(\wp^{-1}(z_1;g_2,g_3);g_2,g_3\right)} \;/;\; z_2 = \sqrt{4\,z_1^3 - g_2\,z_1 - g_3}$$

$$\frac{\partial\wp^{-1}(z_1,z_2;g_2,g_3)}{\partial z_1} = \frac{1}{\sqrt{4\,z_1^3 - g_2\,z_1 - g_3}} \;/;\; z_2 = \sqrt{4\,z_1^3 - g_2\,z_1 - g_3}$$

$$\frac{\partial^2\wp^{-1}(z_1,z_2;g_2,g_3)}{\partial z_1^2} = \frac{g_2 - 12\,\wp\left(\wp^{-1}(z_1;g_2,g_3);g_2,g_3\right)^2}{2\,\wp'\left(\wp^{-1}(z_1;g_2,g_3);g_2,g_3\right)^3} \;/;\; z_2 = \sqrt{4\,z_1^3 - g_2\,z_1 - g_3}\,.$$

The first derivatives of Weierstrass functions $\wp(z; g_2, g_3)$, $\wp'(z; g_2, g_3)$, $\sigma(z; g_2, g_3)$, and $\zeta(z; g_2, g_3)$ with respect to parameter g_2 can also be expressed through Weierstrass functions by the following
formulas:

$$\frac{\partial \wp(z; g_2, g_3)}{\partial g_2} = \frac{1}{4\left(g_2^3 - 27 g_3^2\right)} \left(2 \wp(z; g_2, g_3) g_2^2 + 6 g_3 g_2 - 36 g_3 \wp(z; g_2, g_3)^2 + \wp'(z; g_2, g_3) \left(g_2^2 z - 18 g_3 \zeta(z; g_2, g_3)\right)\right)$$

$$\frac{\partial \wp'(z; g_2, g_3)}{\partial g_2} = \frac{1}{8\left(g_2^3 - 27 g_3^2\right)} \left(-z g_2^3 + 12 z \wp(z; g_2, g_3)^2 g_2^2 + 6\left(g_2^2 - 18 g_3 \wp(z; g_2, g_3)\right) \wp'(z; g_2, g_3) + 18 g_3 \left(g_2 - 12 \wp(z; g_2, g_3)^2\right) \zeta(z; g_2, g_3)\right)$$

$$\frac{\partial \sigma(z; g_2, g_3)}{\partial g_2} = \frac{1}{16\left(g_2^3 - 27 g_3^2\right)} \sigma(z; g_2, g_3) \left(4 g_2^2 (z \zeta(z; g_2, g_3) - 1) + 36 g_3 \left(\wp(z; g_2, g_3) - \zeta(z; g_2, g_3)^2\right) - 3 g_2 g_3 z^2\right)$$

$$\frac{\partial \zeta(z; g_2, g_3)}{\partial g_2} = \frac{1}{8\left(g_2^3 - 27 g_3^2\right)} \left(2\left(g_2^2 + 18 g_3 \wp(z; g_2, g_3)\right) \zeta(z; g_2, g_3) - z g_2 (3 g_3 + 2 g_2 \wp(z; g_2, g_3)) + 18 g_3 \wp'(z; g_2, g_3)\right).$$

The first derivatives of Weierstrass functions $\wp(z; g_2, g_3)$, $\wp'(z; g_2, g_3)$, $\sigma(z; g_2, g_3)$, and $\zeta(z; g_2, g_3)$ with respect to parameter g_3 can also be expressed through Weierstrass functions by the following
formulas:

$$\frac{\partial \wp(z; g_2, g_3)}{\partial g_3} = \frac{1}{2\left(g_2^3 - 27 g_3^2\right)} \left(12 \wp(z; g_2, g_3)^2 g_2 - 18 g_3 \wp(z; g_2, g_3) - 2 g_2^2 + (6 g_2 \zeta(z; g_2, g_3) - g_3 z) \wp'(z; g_2, g_3)\right)$$

$$\frac{\partial \wp'(z; g_2, g_3)}{\partial g_3} = \frac{1}{4\left(g_2^3 - 27 g_3^2\right)} \left(9 z g_2 g_3 - 108 z g_3 \wp(z; g_2, g_3)^2 + (36 g_2 \wp(z; g_2, g_3) - 54 g_3) \wp'(z; g_2, g_3) + \left(72 g_2 \wp(z; g_2, g_3)^2 - 6 g_2^2\right) \zeta(z; g_2, g_3)\right)$$

$$\frac{\partial \sigma(z; g_2, g_3)}{\partial g_3} = \frac{1}{8\left(g_2^3 - 27 g_3^2\right)} \sigma(z; g_2, g_3) \left(z^2 g_2^2 + 12 \zeta(z; g_2, g_3)^2 g_2 - 12 \wp(z; g_2, g_3) g_2 - 36 g_3 (z \zeta(z; g_2, g_3) - 1)\right)$$

$$\frac{\partial \zeta(z; g_2, g_3)}{\partial g_3} = \frac{1}{4\left(g_2^3 - 27 g_3^2\right)} \left(z\left(g_2^2 + 18 g_3 \wp(z; g_2, g_3)\right) - 6 g_2 \wp'(z; g_2, g_3) - 6 (3 g_3 + 2 g_2 \wp(z; g_2, g_3)) \zeta(z; g_2, g_3)\right).$$

Weierstrass invariants g_2 and g_3 can be expressed as functions of half-periods ω_1 and ω_3. This property allows obtaining the following formulas for the first
derivatives of Weierstrass functions $\wp(z; g_2, g_3)$, $\wp'(z; g_2, g_3)$, $\sigma(z; g_2, g_3)$, and $\zeta(z; g_2, g_3)$ with respect to half-period ω_1:

$$\frac{\partial \wp(z; g_2, g_3)}{\partial \omega_1} = -\frac{2 \omega_1}{\pi \omega_3} \sqrt{-\frac{\omega_3^2}{\omega_1^2}} \left(\omega_3 \left(2 \wp(z; g_2, g_3)^2 + \zeta(z; g_2, g_3) \wp'(z; g_2, g_3) - \frac{g_2}{3}\right) - \eta_3 (2 \wp(z; g_2, g_3) + z \wp'(z; g_2, g_3))\right) /; \operatorname{Im}\left(\frac{\omega_3}{\omega_1}\right) \neq 0$$

$$\frac{\partial \wp'(z; g_2, g_3)}{\partial \omega_1} = -\frac{\omega_1}{\pi\,\omega_3}\sqrt{-\frac{\omega_3^2}{\omega_1^2}}\left(\omega_3\left(6\,\wp(z; g_2, g_3)\,\wp'(z; g_2, g_3) + 12\,\zeta(z; g_2, g_3)\,\wp(z; g_2, g_3)^2 - g_2\,\zeta(z; g_2, g_3)\right) - \eta_3\left(6\,\wp'(z; g_2, g_3) + 12\,z\,\wp(z; g_2, g_3)^2 - g_2\,z\right)\right) /; \operatorname{Im}\left(\frac{\omega_3}{\omega_1}\right) \neq 0$$

$$\frac{\partial \sigma(z; g_2, g_3)}{\partial \omega_1} = \frac{\omega_1}{\pi\,\omega_3}\sqrt{-\frac{\omega_3^2}{\omega_1^2}}\left(\omega_3\left(\wp(z; g_2, g_3) - \zeta(z; g_2, g_3)^2 - \frac{1}{12}\,g_2\,z^2\right) + 2\,\eta_3\,(z\,\zeta(z; g_2, g_3) - 1)\right)\sigma(z; g_2, g_3) /; \operatorname{Im}\left(\frac{\omega_3}{\omega_1}\right) \neq 0$$

$$\frac{\partial \zeta(z; g_2, g_3)}{\partial \omega_1} = \frac{\omega_1}{\pi\,\omega_3}\sqrt{-\frac{\omega_3^2}{\omega_1^2}}\left(\omega_3\left(\wp'(z; g_2, g_3) + 2\,\zeta(z; g_2, g_3)\,\wp(z; g_2, g_3) - \frac{1}{6}\,g_2\,z\right) + 2\,\eta_3\,(\zeta(z; g_2, g_3) - z\,\wp(z; g_2, g_3))\right) /; \operatorname{Im}\left(\frac{\omega_3}{\omega_1}\right) \neq 0.$$

Similar formulas take place for the first derivatives of Weierstrass functions $\wp(z; g_2, g_3)$, $\wp'(z; g_2, g_3)$, $\sigma(z; g_2, g_3)$, and $\zeta(z; g_2, g_3)$ with respect to half-period ω_3:

$$\frac{\partial \wp(z; g_2, g_3)}{\partial \omega_3} = \frac{2\,\omega_1}{\pi\,\omega_3}\sqrt{-\frac{\omega_3^2}{\omega_1^2}}\left(\omega_1\left(2\,\wp(z; g_2, g_3)^2 + \zeta(z; g_2, g_3)\,\wp'(z; g_2, g_3) - \frac{g_2}{3}\right) - \eta_1\,(2\,\wp(z; g_2, g_3) + z\,\wp'(z; g_2, g_3))\right) /; \operatorname{Im}\left(\frac{\omega_3}{\omega_1}\right) \neq 0$$

$$\frac{\partial \wp'(z; g_2, g_3)}{\partial \omega_3} = \frac{\omega_1}{\pi\,\omega_3}\sqrt{-\frac{\omega_3^2}{\omega_1^2}}\left(\omega_1\left(6\,\wp(z; g_2, g_3)\,\wp'(z; g_2, g_3) + 12\,\zeta(z; g_2, g_3)\,\wp(z; g_2, g_3)^2 - g_2\,\zeta(z; g_2, g_3)\right) - \eta_1\left(6\,\wp'(z; g_2, g_3) + 12\,z\,\wp(z; g_2, g_3)^2 - g_2\,z\right)\right) /; \operatorname{Im}\left(\frac{\omega_3}{\omega_1}\right) \neq 0$$

$$\frac{\partial \sigma(z; g_2, g_3)}{\partial \omega_3} = -\frac{\omega_1}{\pi\,\omega_3}\sqrt{-\frac{\omega_3^2}{\omega_1^2}}\left(\omega_1\left(\wp(z; g_2, g_3) - \zeta(z; g_2, g_3)^2 - \frac{1}{12}\,g_2\,z^2\right) + 2\,\eta_1\,(z\,\zeta(z; g_2, g_3) - 1)\right)\sigma(z; g_2, g_3) /; \operatorname{Im}\left(\frac{\omega_3}{\omega_1}\right) \neq 0$$

$$\frac{\partial \zeta(z; g_2, g_3)}{\partial \omega_3} = -\frac{\omega_1}{\pi\,\omega_3}\sqrt{-\frac{\omega_3^2}{\omega_1^2}}\left(\omega_1\left(\wp'(z; g_2, g_3) + 2\,\zeta(z; g_2, g_3)\,\wp(z; g_2, g_3) - \frac{1}{6}\,g_2\,z\right) + 2\,\eta_1\,(\zeta(z; g_2, g_3) - z\,\wp(z; g_2, g_3))\right) /; \operatorname{Im}\left(\frac{\omega_3}{\omega_1}\right) \neq 0.$$

The n^{th} derivatives of all Weierstrass functions $\wp(z; g_2, g_3)$, $\wp'(z; g_2, g_3)$, $\sigma(z; g_2, g_3)$, $\sigma_n(z, g_2, g_3) /; n \in \{1, 2, 3\}$, $\zeta(z; g_2, g_3)$, and their inverses $\wp^{-1}(z; g_2, g_3)$ and $\wp^{-1}(z_1, z_2; g_2, g_3)$ with respect to variable z can be represented by the following formulas:

$$\frac{\partial^n \wp(z; g_2, g_3)}{\partial z^n} = \frac{\partial}{\partial z} \frac{\partial^{n-1} \wp(z; g_2, g_3)}{\partial z^{n-1}} \;/;\; n \in \mathbb{N}^+$$

$$\frac{\partial^k \wp'(z; g_2, g_3)}{\partial z^k} = (-1)^{k-1} (k+2)! \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{(z - 2 m \omega_1 - 2 n \omega_3)^{k+3}} \;/;\; k \in \mathbb{N}^+$$

$$\frac{\partial^n \sigma(z; g_2, g_3)}{\partial z^n} = (2 \omega_1)^{1-n} \pi^{n-\frac{1}{2}} \left(\prod_{m=1}^{\infty} \frac{1}{1 - q^{2m}}\right)^3 \sum_{j=0}^{n} {}_2\tilde{F}_2\left(\frac{1}{2}, 1; \frac{1-j}{2}, \frac{2-j}{2}; \frac{z^2 \zeta(\omega_1; g_2, g_3)}{2 \omega_1}\right) \left(\frac{4 \omega_1}{\pi z}\right)^j \binom{n}{j} \sum_{k=0}^{\infty} (-1)^k q^{k(k+1)} (2k+1)^{n-j} \sin\left(\frac{\pi((2k+1) z + (n-j) \omega_1)}{2 \omega_1}\right) \;/;\; n \in \mathbb{N}^+$$

$$\frac{\partial^n \sigma_j(z, g_2, g_3)}{\partial z^n} = \frac{(2 \omega_1)^{1-n} \pi^{n-\frac{1}{2}} e^{-\eta_j z}}{\sigma_j(z, g_2, g_3)} \left(\prod_{k=1}^{\infty} \frac{1}{1 - q^{2k}}\right)^3 \sum_{j=0}^{n} {}_2\tilde{F}_2\left(\frac{1}{2}, 1; \frac{1-j}{2}, \frac{2-j}{2}; \frac{(z+\omega_j)^2 \zeta(\omega_1; g_2, g_3)}{2 \omega_1}\right) \binom{n}{j} \left(\frac{4 \omega_1}{\pi (z + \omega_j)}\right)^j \sum_{k=0}^{\infty} (-1)^k q^{k(k+1)} (2k+1)^{n-j} \sin\left(\frac{1}{2 \omega_1} \left(\pi \left((n-j) \omega_1 + (2k+1)(z + \omega_j)\right)\right)\right) - \eta_j \sigma_j(z, g_2, g_3) \;/;\; n \in \mathbb{N}^+ \bigwedge j \in \{1, 2, 3\}$$

$$\frac{\partial^n \zeta(z; g_2, g_3)}{\partial z^n} = -\frac{n}{2} \left(\frac{\pi}{\omega_1}\right)^{n+1} \left(\sum_{k=0}^{n-1} \sum_{j=0}^{k-1} \frac{(-1)^j 2^{-2k}}{k+1} \binom{n-1}{k} \sin^{-2k-2}\left(\frac{\pi z}{2 \omega_1}\right) \binom{2k}{j} (k-j)^{n-1} \sin\left(\frac{\pi z (k-j)}{\omega_1} + \frac{n \pi}{2}\right)\right) - \frac{\pi^2 \delta_{n-1}}{4 \omega_1^2} \csc^2\left(\frac{\pi z}{2 \omega_1}\right) + \frac{z^{1-n} \eta_1}{\omega_1 \Gamma(2-n)} + \frac{2 \pi^{n+1}}{\omega_1^{n+1}} \sum_{k=1}^{\infty} \frac{q^{2k} k^n}{1 - q^{2k}} \sin\left(\frac{\pi n}{2} + \frac{k \pi z}{\omega_1}\right) \;/;\; n \in \mathbb{N}^+$$

$$\frac{\partial^n \wp^{-1}(z; g_2, g_3)}{\partial z^n} = \frac{\delta_{n-1}}{\sqrt{4 z^3 - g_2 z - g_3}} + \wp^{-1}(z; g_2, g_3) \delta_n + \sum_{m=1}^{n-1} \frac{1}{m!} \left(\frac{1}{2} - m\right)_m \sum_{j=0}^{m-1} (-1)^j \binom{m}{j} \left(4 z^3 - g_2 z - g_3\right)^{j-m-\frac{1}{2}} \sum_{k_1=0}^{m-j} \sum_{k_2=0}^{m-j} \sum_{k_3=0}^{m-j} (-1)^{n+k_2+k_3-1} \delta_{m-j, k_1+k_2+k_3} (k_1 + k_2 + k_3; k_1, k_2, k_3) 4^{k_1} g_2^{k_2} g_3^{k_3} (-3 k_1 - k_2)_{n-1} z^{-n+3 k_1 + k_2 + 1} \;/;\; n \in \mathbb{N}$$

$$\frac{\partial^n \wp^{-1}(z_1, z_2; g_2, g_3)}{\partial z_1^n} = \frac{\delta_{n-1}}{\sqrt{4 z_1^3 - g_2 z_1 - g_3}} + \wp^{-1}(z_1, z_2; g_2, g_3) \delta_n + \sum_{m=1}^{n-1} \frac{1}{m!} \left(\frac{1}{2} - m\right)_m \sum_{j=0}^{m-1} (-1)^j \binom{m}{j} \left(4 z_1^3 - g_2 z_1 - g_3\right)^{j-m-\frac{1}{2}} \sum_{k_1=0}^{m-j} \sum_{k_2=0}^{m-j} \sum_{k_3=0}^{m-j} (-1)^{n+k_2+k_3-1} \delta_{m-j, k_1+k_2+k_3} (k_1 + k_2 + k_3; k_1, k_2, k_3) 4^{k_1} g_2^{k_2} g_3^{k_3} (-3 k_1 - k_2)_{n-1} z_1^{-n+3 k_1 + k_2 + 1} \;/;\; n \in \mathbb{N} \bigwedge z_2 = \sqrt{4 z_1^3 - g_2 z_1 - g_3}.$$

Integration

The indefinite integrals of Weierstrass functions $\wp(z; g_2, g_3)$, $\wp'(z; g_2, g_3)$, $\sigma(z; g_2, g_3)$, and $\zeta(z; g_2, g_3)$ with respect to variable z can be expressed by the following formulas:

$$\int \wp(z; g_2, g_3) \, dz = -\zeta(z; g_2, g_3)$$

$$\int \wp'(z; g_2, g_3)\, dz = \wp(z; g_2, g_3)$$

$$\int \sigma(z; g_2, g_3)\, dz = \frac{\omega_1^{3/2}}{\sqrt{2\pi\eta_1}} \left(\prod_{m=1}^{\infty} \frac{1}{1-q^{2m}} \right)^3$$
$$\sum_{k=0}^{\infty} (-1)^k q^{k(k+1)} \exp\left(\frac{\pi^2 (2k+1)^2}{8\eta_1 \omega_1} \right) \left(\operatorname{erf}\left(\frac{\pi (2k+1) + 2 i z \eta_1}{2\sqrt{2\eta_1 \omega_1}} \right) + \operatorname{erf}\left(\frac{\pi (2k+1) - 2 i z \eta_1}{2\sqrt{2\eta_1 \omega_1}} \right) \right) /;\ q = \exp\left(\frac{\pi i \omega_3}{\omega_1} \right)$$

$$\int \zeta(z; g_2, g_3)\, dz = \log(\sigma(z; g_2, g_3)).$$

Summation

Finite and infinite sums including Weierstrass functions can sometimes be evaluated in closed forms, for example:

$$\sum_{\substack{j,k=0 \\ \{j,k\}\neq\{0,0\}}}^{n-1} \wp\left(\frac{2j\omega_1 + 2k\omega_3}{n}; g_2, g_3 \right) = 0 /;\ n-1 \in \mathbb{N}^+$$

$$\sum_{k=1}^{n-1} \wp\left(\frac{2k\omega_1}{n}; g_2, g_3 \right) = \frac{n}{\omega_1} \left(\zeta\left(\frac{\omega_1}{n}; g_2\left(\frac{\omega_1}{n}, \omega_3 \right), g_3\left(\frac{\omega_1}{n}, \omega_3 \right) \right) - \eta_1 \right)$$

$$\sum_{\substack{j,k=0 \\ \{j,k\}\neq\{0,0\}}}^{n-1} \zeta\left(\frac{2j\omega_1 + 2k\omega_3}{n}; g_2, g_3 \right) = -n(n-1)\eta_2 /;\ n-1 \in \mathbb{N}^+.$$

Differential equations

The Weierstrass functions $\wp(z; g_2, g_3)$, $\wp'(z; g_2, g_3)$, $\sigma(z; g_2, g_3)$, $\zeta(z; g_2, g_3)$, and their inverses $\wp^{-1}(z; g_2, g_3)$ and $\wp^{-1}(z_1, z_2; g_2, g_3)$ satisfy the following nonlinear differential equations:

$$w'(z)^2 - 4w(z)^3 + g_2 w(z) + g_3 /;\ w(z) = \wp(z+a; g_2, g_3)$$

$$w''(z) - 6w(z)^2 + \frac{g_2}{2} = 0 /;\ w(z) = \wp(a+z; g_2, g_3)$$

$$w^{(3)}(z) - 12 w(z) w'(z) = 0 /;\ w(z) = \wp(a+z; g_2, g_3)$$

$$\wp'(z; g_2, g_3)^2 = 4(\wp(z; g_2, g_3) - \wp(\omega_1; g_2, g_3))(\wp(z; g_2, g_3) - \wp(\omega_2; g_2, g_3))(\wp(z; g_2, g_3) - \wp(\omega_3; g_2, g_3))$$

$$2w'(z)^3 - 3g_2 w'(z)^2 - 27w(z)^4 - 54g_3 w(z)^2 + g_2^3 - 27g_3^2 = 0 /;\ w(z) = \wp'(z; g_2, g_3)$$

$$w'(z)^3 - w(z)^2 (w(z)-a)^2 = 0 /;\ w(z) = \frac{a}{2} + \frac{27}{16} \wp'\left(\frac{z}{2}; 0, -\frac{64}{729} a^2 \right)$$

$$w'(z)^3 - \left(w(z)^3 - 3a\, w(z)^2 + 3w(z) \right)^2 = 0 /;\ w(z) = \frac{2}{a - 3\wp'\left(z; 0, \frac{1}{27}\left(4-3a^2\right)\right)}$$

$$\sigma'(z; g_2, g_3) == \zeta(z; g_2, g_3)\, \sigma(z; g_2, g_3)$$

$$\frac{\partial \log(\sigma(z; g_2, g_3))}{\partial z} == \zeta(z; g_2, g_3)$$

$$2\, \sigma(z; g_2, g_3)\, \frac{\partial^4 \sigma(z; g_2, g_3)}{\partial z^4} - 8\, \frac{\partial \sigma(z; g_2, g_3)}{\partial z}\, \frac{\partial^3 \sigma(z; g_2, g_3)}{\partial z^3} + 6 \left(\frac{\partial^2 \sigma(z; g_2, g_3)}{\partial z^2}\right)^2 - g_2\, \sigma(z; g_2, g_3)^2 == 0$$

$$\left(\frac{\partial^2 \zeta(z; g_2, g_3)}{\partial z^2}\right)^2 + 4 \left(\frac{\partial \zeta(z; g_2, g_3)}{\partial z}\right)^3 - g_2\, \frac{\partial \zeta(z; g_2, g_3)}{\partial z} + g_3 == 0$$

$$\left(4\, z^3 - g_2\, z - g_3\right) w'(z)^2 - 1 == 0 \;/;\; w(z) == \wp^{-1}(z; g_2, g_3)$$

$$\left(4\, z_1^3 - g_2\, z_1 - g_3\right) w'(z_1)^2 - 1 == 0 \;/;\; w(z_1) == \wp^{-1}(z_1, z_2; g_2, g_3).$$

The Weierstrass functions $\wp(z; g_2, g_3)$, $\wp'(z; g_2, g_3)$, $\sigma(z; g_2, g_3)$, and $\zeta(z; g_2, g_3)$ are the special solutions of the corresponding partial differential
equations:

$$z\, \frac{\partial \wp(z; g_2, g_3)}{\partial z} - 4\, g_2\, \frac{\partial \wp(z; g_2, g_3)}{\partial g_2} - 6\, g_3\, \frac{\partial \wp(z; g_2, g_3)}{\partial g_3} + 2\, \wp(z; g_2, g_3) == 0$$

$$12\, g_3\, \frac{\partial \wp(z; g_2, g_3)}{\partial g_2} + \frac{2}{3}\, g_2^2\, \frac{\partial \wp(z; g_2, g_3)}{\partial g_3} - 2\, \zeta(z; g_2, g_3)\, \frac{\partial \wp(z; g_2, g_3)}{\partial z} == \frac{2}{3}\, \frac{\partial^2 \wp(z; g_2, g_3)}{\partial z^2} - \frac{g_2}{3}$$

$$\omega_1\, \frac{\partial \wp(z; g_2, g_3)}{\partial \omega_1} + \omega_3\, \frac{\partial \wp(z; g_2, g_3)}{\partial \omega_3} + z\, \wp(z; g_2, g_3) == -2\, \wp(z; g_2, g_3)$$

$$\eta_1\, \frac{\partial \wp(z; g_2, g_3)}{\partial \omega_1} + \eta_3\, \frac{\partial \wp(z; g_2, g_3)}{\partial \omega_3} + \zeta(z; g_2, g_3)\, \frac{\partial \wp(z; g_2, g_3)}{\partial z} == -\frac{1}{3}\, \frac{\partial^2 \wp(z; g_2, g_3)}{\partial x^2} + \frac{g_2}{6}$$

$$z\, \frac{\partial^2 \wp'(z; g_2, g_3)}{\partial z^2} + \frac{\partial \wp'(z; g_2, g_3)}{\partial z} - 4\, g_2\, \frac{\partial \wp'(z; g_2, g_3)}{\partial g_2} - 6\, g_2\, \frac{\partial \wp'(z; g_2, g_3)}{\partial g_3} - 2\, \wp'(z; g_2, g_3) == 0$$

$$4\, g_2\, \frac{\partial \wp'(z; g_2, g_3)}{\partial g_2} + 6\, g_2\, \frac{\partial \wp'(z; g_2, g_3)}{\partial g_3} - 6\, \wp(z; g_2, g_3)^2 - 12\, z\, \wp'(z; g_2, g_3)\, \wp(z; g_2, g_3) + 2\, \wp'(z; g_2, g_3) + \frac{g_2}{2} == 0$$

$$\left(g_2^3 - 27\, g_3^2\right) \frac{\partial \sigma(z; g_2, g_3)}{\partial g_2} == -\frac{9}{4}\, g_3\, \frac{\partial^2 \sigma(z; g_2, g_3)}{\partial z^2} - \frac{1}{4}\, g_2^2\, \sigma(z; g_2, g_3) - \frac{3}{16}\, g_2\, g_3\, z^2\, \sigma(z; g_2, g_3) + \frac{1}{4}\, g_2^2\, z\, \frac{\partial \sigma(z; g_2, g_3)}{\partial z}$$

$$\left(g_2^3 - 27\, g_3^2\right) \frac{\partial \sigma(z; g_2, g_3)}{\partial g_3} == \frac{3}{2}\, g_2\, \frac{\partial^2 \sigma(z; g_2, g_3)}{\partial z^2} + \frac{9}{2}\, g_3\, \sigma(z; g_2, g_3) + \frac{1}{8}\, g_2^2\, z^2\, \sigma(z; g_2, g_3) - \frac{9}{2}\, g_2\, z\, \frac{\partial \sigma(z; g_2, g_3)}{\partial z}$$

$$z\, \frac{\partial \sigma(z; g_2, g_3)}{\partial z} - 4\, g_2\, \frac{\partial \sigma(z; g_2, g_3)}{\partial g_2} - 6\, g_3\, \frac{\partial \sigma(z; g_2, g_3)}{\partial g_3} - \sigma(z; g_2, g_3) == 0$$

$$\frac{\partial^2 \sigma(z; g_2, g_3)}{\partial z^2} - 12\, g_3\, \frac{\partial \sigma(z; g_2, g_3)}{\partial g_2} - \frac{2}{3}\, g_2\, \frac{\partial \sigma(z; g_2, g_3)}{\partial g_3} + \frac{1}{12}\, \sigma(z; g_2, g_3)\, z^2 == 0$$

$$72\,g_3\,\frac{\partial\zeta(\omega_1;\,g_2,\,g_3)}{\partial g_2}+4\,g_2^2\,\frac{\partial\zeta(\omega_1;\,g_2,\,g_3)}{\partial g_3}-\omega_1\,g_2==0\;/;\;\{\omega_1,\,\omega_3\}=\{\omega_1(g_2,\,g_3),\,\omega_3\,(g_2,\,g_3)\}$$

$$72\,g_3\,\frac{\partial\zeta(\omega_3;\,g_2,\,g_3)}{\partial g_2}+4\,g_2^2\,\frac{\partial\zeta(\omega_3;\,g_2,\,g_3)}{\partial g_3}-\omega_3\,g_2==0\;/;\;\{\omega_1,\,\omega_3\}=\{\omega_1(g_2,\,g_3),\,\omega_3\,(g_2,\,g_3)\}.$$

Applications of Weierstrass functions and inverses

Applications of Weierstrass functions include integrable nonlinear differential equations, motion in cubic and quartic potentials, description of the movement of a spherical pendulum, and construction of minimal surfaces.

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