# Introductions to JacobiNC

## Introduction to the Jacobi elliptic functions

### General

#### Historical remarks

Jacobi functions are named for the famous mathematician C. G. J. Jacobi. In 1827 he introduced the elliptic amplitude $\operatorname{am}(z \mid m)$ as the inverse function of the elliptic integral $F(z \mid m)$ by the variable $z$ and investigated the twelve functions $\operatorname{cd}(z \mid m)$, $\operatorname{cn}(z \mid m)$, $\operatorname{cs}(z \mid m)$, $\operatorname{dc}(z \mid m)$, $\operatorname{dn}(z \mid m)$, $\operatorname{ds}(z \mid m)$, $\operatorname{nc}(z \mid m)$, $\operatorname{nd}(z \mid m)$, $\operatorname{ns}(z \mid m)$, $\operatorname{sc}(z \mid m)$, $\operatorname{sd}(z \mid m)$, and $\operatorname{sn}(z \mid m)$. In the same year, N. H. Abel independently studied properties of these functions. But earlier K. F. Gauss (1799) gave some attention to one of these function, namely $\operatorname{sn}(z \mid m)$.

The modern notations for Jacobi functions were introduced later in the works of C. Gudermann (1838) (for functions cn, dn, and sn) and J. Glaisher (1882) (for functions cd, cs, dc, ds, nc, nd, ns, sc, and sd). V. A. Puiseux (1850) showed that amplitude am is a multivalued function.

#### Periodic functions

An analytic function $f(z)$ is called periodic if there exists a complex constant $\rho \neq 0$, such that $f(z+\rho) == f(z) /; z \in \mathbb{C}$. The number $\rho$ (with minimal possible value of $|\rho|$) is called the period of the function $f(z)$.

Examples of well-known singly periodic functions are the exponential functions and all the trigonometric and hyperbolic functions: $e^z$, sin(z), cos(z), csc(z), sec(z), tan(z), cot(z), sinh(z), cosh(z), csch(z), sech(z), tanh(z), and coth(z), which have periods $\rho = 2\pi i$, $\rho = 2\pi$, $\rho = \pi$, $\rho = 2\pi i$, and $\rho = \pi i$. The study of such functions can be restricted to any period-strip $\{z_0 + \alpha\rho /; 0 \leq \alpha < 1 \wedge z_0 \in \mathbb{C}\}$, because outside this strip, the values of these functions coincide with their corresponding values inside the strip.

Nonconstant analytic functions over the field of complex numbers cannot have more than two independent periods. So, generically, periodic functions can satisfy the following relations:

$$f(z + n\rho) == f(z) /; n \in \mathbb{Z}$$

$$f(z + m\rho_1 + n\rho_2) == f(z) /; \{m, n\} \in \mathbb{Z} \bigwedge \operatorname{Im}\left(\frac{\rho_1}{\rho_2}\right) \neq 0,$$

where $\rho$, $\rho_1$, and $\rho_2$ are periods (basic primitive periods). The condition $\operatorname{Im}\left(\frac{\rho_1}{\rho_2}\right) \neq 0$ for doubly periodic functions implies the existence of a period-parallelogram $\{z_0 + \alpha_1 \rho_1 + \alpha_2 \rho_2 /; 0 \leq \alpha_1 < 1 \bigwedge 0 \leq \alpha_1 < 1 \bigwedge z_0 \in \mathbb{C}\}$, which is the analog of the period-strip $\{z_0 + \alpha \rho /; 0 \leq \alpha < 1 \bigwedge z_0 \in \mathbb{C}\}$ for the singly periodic function with period $\rho$.

In the case $z_0 == 0 \bigwedge \operatorname{Im}\left(\frac{\rho_1}{\rho_2}\right) > 0$, this parallelogram is called the basic fundamental period-parallelogram: $\mathrm{P}_{0,0} == \{\alpha_1 \rho_1 + \alpha_2 \rho_2 /; 0 \leq \alpha_1 < 1 \bigwedge 0 \leq \alpha_2 < 1\}$. The two line segments $\{\alpha_i \rho_i /; 0 \leq \alpha_i < 1\} /; i == 1, 2$ lying on the boundary of the period-parallelogram and beginning from the origin 0 belong to $\mathrm{P}_{0,0}$. The region $\mathrm{P}_{0,0}$ includes only one corner point 0 from four points lying at the boundary of parallelogram with corners in $\{0, \rho_1, \rho_1 + \rho_2, \rho_2\}$. Sometimes the convention $\mathrm{P}_{0,0} == \{\alpha_1 \rho_1 + \alpha_2 \rho_2 /; -1/2 \leq \alpha_1 < 1/2 \bigwedge -1/2 \leq \alpha_2 < 1/2\}$ is used.

The set of all such period-parallelograms:

$$\mathrm{P}_{m,n} == \{m \rho_1 + n \rho_2 + \alpha_1 \rho_1 + \alpha_2 \rho_2 /; 0 \leq \alpha_1 < 1 \bigwedge 0 \leq \alpha_2 < 1 \bigwedge m \in \mathbb{Z} \bigwedge n \in \mathbb{Z}\}$$

covers all complex planes: $\mathbb{C} == \{\mathrm{P}_{m,n} \mid -\infty < m, n < \infty \bigwedge m \in \mathbb{Z} \bigwedge n \in \mathbb{Z}\}$.

Any doubly periodic function $\mathcal{E}(z)$ is called an elliptic function. The set of numbers $m \rho_1 + n \rho_2 /; \{m, n\} \in \mathbb{Z}$ is called the period-lattice for the elliptic function $\mathcal{E}(z)$.

An elliptic function $\mathcal{E}(z)$, which does not have poles in the period-parallelogram, is equal to the constant Liouville's theorem.

Nonconstant elliptic (doubly periodic) functions $\mathcal{E}(z)$ cannot be entire functions (this is not the case for singly periodic functions, for example, $\sin(z)$ is an entire function.

Any nonconstant elliptic function $\mathcal{E}(z)$ has at least two simple poles or at least one double pole in any period -parallelogram. The sum of all its residues at the poles inside a period-parallelogram is zero.

The number of zeros and poles of a nonconstant elliptic function $\mathcal{E}(z)$ in a fundamental period-parallelogram P are finite.

The number of zeros of $\mathcal{E}(z) - A$, where $A$ is any complex number, in a fundamental period-parallelogram $\mathrm{P}_{0,0}$ does not depend on the value $A$ and coincides with the number $s$ of the poles $b_1, b_2, \ldots, b_s$ counted according to their multiplicity ($s$ is called the order of the elliptic function $\mathcal{E}(z)$).

The simplest elliptic function has order two.

Let $a_1, a_2, \ldots, a_r$ (and $b_1, b_2, \ldots, b_s$) be the zeros (and poles) of a nonconstant elliptic function $\mathcal{E}(z)$ in a fundamental period-parallelogram $\mathrm{P}_{0,0}$, both listed one or more times according to their multiplicity. This results in the following:

$r == s$

$$\sum_{j=1}^{r} a_j - \sum_{k=1}^{s} b_k == \mu\ \rho_1 + \nu\,\rho_2\ /;\ a_j \in P_{0,0} \bigwedge b_k \in P_{0,0} \bigwedge \mathcal{E}(a_j) == 0 \bigwedge 1/\mathcal{E}(b_k) == 0 \bigwedge \mu \in \mathbb{Z} \bigwedge \nu \in \mathbb{Z}.$$

So, the number of zeros of a nonconstant elliptic function $\mathcal{E}(z)$ in the fundamental period-parallelogram $P_{0,0}$ is equal to the number of poles there and counted according to their
multiplicity. The sum of zeros of a nonconstant elliptic function $\mathcal{E}(z)$ in the fundamental period-parallelogram $P_{0,0}$ differs from the sum of its poles by a period $\mu\ \rho_1 + \nu\,\rho_2$, where $\mu \in \mathbb{Z} \bigwedge \nu \in \mathbb{Z}$ and the values of $\mu$, $\nu$ depend on the function $\mathcal{E}(z)$.

All elliptic functions $\mathcal{E}(z)$ satisfy a common fundamental property, which generalizes addition,
duplication, and multiple angle properties for the trigonometric and
hyperbolic functions such as $\sin(z_1 + z_2)$, $\sin(n\,z)\ /;\ n \in \mathbb{N}^+$. It can be formulated as:

$$\mathcal{E}\left(\textstyle\sum_{k=1}^{n} z_k\right),$$

which can be expressed as an algebraic function of $\mathcal{E}(z_k)\ /;\ 1 \le k \le n$. In other words, there exists an irreducible polynomial $C(t_1, t_2, \ldots, t_{n+1})$ in $n+1$ variables with constant coefficients, for which the following relation
holds:

$$C\left(\mathcal{E}(z_1), \mathcal{E}(z_2), \ldots, \mathcal{E}(z_n), \mathcal{E}\left(\sum_{k=1}^{n} z_k\right)\right) == 0.$$

Conversely, among all smooth functions, only elliptic functions and their
degenerations have algebraic addition theorems.

The simplest elliptic functions (of order two) can be divide into the
following two classes:

(1) Functions that at the period-parallelogram $P_{0,0}$ have only a double pole with a residue zero (e.g., the Weierstrass elliptic functions $\wp(z; g_2, g_3)$).

(2) Functions that at the period-parallelogram $P_{0,0}$ have only two simple poles with residues, which are equal in absolute
value but opposite in sign (e.g., the Jacobian elliptic functions $\mathrm{cd}(z \mid m)$, $\mathrm{cn}(z \mid m)$, $\mathrm{cs}(z \mid m)$, $\mathrm{dc}(z \mid m)$, $\mathrm{dn}(z \mid m)$, $\mathrm{ds}(z \mid m)$, $\mathrm{nc}(z \mid m)$, $\mathrm{nd}(z \mid m)$, $\mathrm{ns}(z \mid m)$, $\mathrm{sc}(z \mid m)$, $\mathrm{sd}(z \mid m)$, and $\mathrm{sn}(z \mid m)$).

Jacobian elliptic functions $\mathrm{cd}(z \mid m)$, $\mathrm{cn}(z \mid m)$, $\mathrm{cs}(z \mid m)$, $\mathrm{dc}(z \mid m)$, $\mathrm{dn}(z \mid m)$, $\mathrm{ds}(z \mid m)$, $\mathrm{nc}(z \mid m)$, $\mathrm{nd}(z \mid m)$, $\mathrm{ns}(z \mid m)$, $\mathrm{sc}(z \mid m)$, $\mathrm{sd}(z \mid m)$, and $\mathrm{sn}(z \mid m)$ arise as solutions to the differential equation:

$$w''(z) = \alpha\, w(z) + \beta\, w(z)^3,$$

with the following coefficients:

| | $\alpha$ | $\beta$ |
|---|---|---|
| **cd(*z* \| *m*)** | $-m-1$ | $2m$ |
| **cn(*z* \| *m*)** | $2m-1$ | $-2m$ |
| **cs(*z* \| *m*)** | $2-m$ | $2$ |
| **dc(*z* \| *m*)** | $-m-1$ | $2$ |
| **dn(*z* \| *m*)** | $2-m$ | $-2$ |
| **ds(*z* \| *m*)** | $2m-1$ | $2$ |
| **nc(*z* \| *m*)** | $2m-1$ | $2-2m$ |
| **nd(*z* \| *m*)** | $2-m$ | $2m-2$ |
| **ns(*z* \| *m*)** | $-m-1$ | $2$ |
| **sc(*z* \| *m*)** | $-m+2$ | $2-2m$ |
| **sd(*z* \| *m*)** | $2m-1$ | $2m(m-1)$ |
| **sn(*z* \| *m*)** | $-m-1$ | $2m$ |

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## Definitions of Jacobi functions

The Jacobi elliptic amplitude $\operatorname{am}(z \mid m)$ and the twelve Jacobi functions $\operatorname{cd}(z \mid m)$, $\operatorname{cn}(z \mid m)$, $\operatorname{cs}(z \mid m)$, $\operatorname{dc}(z \mid m)$, $\operatorname{dn}(z \mid m)$, $\operatorname{ds}(z \mid m)$, $\operatorname{nc}(z \mid m)$, $\operatorname{nd}(z \mid m)$, $\operatorname{ns}(z \mid m)$, $\operatorname{sc}(z \mid m)$, $\operatorname{sd}(z \mid m)$, and $\operatorname{sn}(z \mid m)$ are defined by the following formulas:

$$w = \operatorname{am}(z \mid m) \;/;\; z = F(w \mid m) \land 0 \le m \le 1$$

$$\operatorname{cn}(z \mid m) = \cos(\operatorname{am}(z \mid m))$$

$$\operatorname{dn}(z \mid m) = \frac{\partial \operatorname{am}(z \mid m)}{\partial z}$$

$$\operatorname{sn}(z \mid m) = \sin(\operatorname{am}(z \mid m))$$

$$\operatorname{cd}(z \mid m) = \frac{\operatorname{cn}(z \mid m)}{\operatorname{dn}(z \mid m)}$$

$$\operatorname{cs}(z \mid m) = \frac{\operatorname{cn}(z \mid m)}{\operatorname{sn}(z \mid m)}$$

$$\operatorname{ds}(z \mid m) = \frac{\operatorname{dn}(z \mid m)}{\operatorname{sn}(z \mid m)}$$

$$\operatorname{dc}(z \mid m) = \frac{1}{\operatorname{cd}(z \mid m)} = \frac{\operatorname{dn}(z \mid m)}{\operatorname{cn}(z \mid m)}$$

$$\operatorname{sc}(z \mid m) = \frac{1}{\operatorname{cs}(z \mid m)} = \frac{\operatorname{sn}(z \mid m)}{\operatorname{cn}(z \mid m)}$$

$$\operatorname{sd}(z \mid m) = \frac{1}{\operatorname{ds}(z \mid m)} = \frac{\operatorname{sn}(z \mid m)}{\operatorname{dn}(z \mid m)}$$

$$\operatorname{nc}(z \mid m) = \frac{1}{\operatorname{cn}(z \mid m)}$$

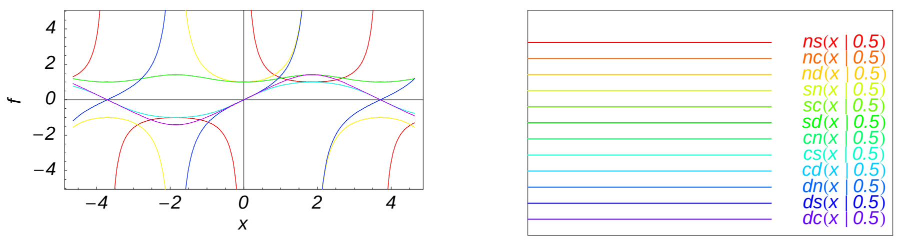
$$\operatorname{nd}(z \mid m) = \frac{1}{\operatorname{dn}(z \mid m)}$$

$$\operatorname{ns}(z \mid m) = \frac{1}{\operatorname{sn}(z \mid m)}.$$

It is apparent that the amplitude function $\operatorname{am}(z \mid m)$ is the inverse function to elliptic integral $F(w \mid m) = z$, and the functions $\operatorname{cn}(z \mid m)$, $\operatorname{sn}(z \mid m)$, and $\operatorname{dn}(z \mid m)$ are the basic Jacobi functions that are built as the cosine, sine, and derivative of the amplitude function $\operatorname{am}(z \mid m)$. The other nine Jacobi functions are the ratios of these three basic Jacobi functions or their reciprocal functions.

## A quick look at the Jacobi functions

Here is a quick look at the graphics for the twelve Jacobi elliptic functions along the real axis for $m = 1/2$.




## Connections within the group of Jacobi functions and with other elliptic functions

### Representations through related equivalent functions

The twelve Jacobi functions $\operatorname{cd}(z \mid m)$, $\operatorname{cn}(z \mid m)$, $\operatorname{cs}(z \mid m)$, $\operatorname{dc}(z \mid m)$, $\operatorname{dn}(z \mid m)$, $\operatorname{ds}(z \mid m)$, $\operatorname{nc}(z \mid m)$, $\operatorname{nd}(z \mid m)$, $\operatorname{ns}(z \mid m)$, $\operatorname{sc}(z \mid m)$, $\operatorname{sd}(z \mid m)$, and $\operatorname{sn}(z \mid m)$ can be represented through the Weierstrass sigma functions:

$$\operatorname{cd}(z \mid m) = \frac{\sigma_1\left(\frac{z}{\sqrt{e_1-e_3}}; g_2, g_3\right)}{\sigma_2\left(\frac{z}{\sqrt{e_1-e_3}}; g_2, g_3\right)} /;$$

$$\{\omega_1, \omega_2, \omega_3\} = \{\omega_1(g_2, g_3), -\omega_1(g_2, g_3) - \omega_3(g_2, g_3), \omega_3(g_2, g_3)\} \bigwedge m = \lambda\left(\frac{\omega_3}{\omega_1}\right) \bigwedge e_n = \wp(\omega_n; g_2, g_3) \bigwedge n \in \{1, 2, 3\}$$

$$\operatorname{cn}(z \mid m) = \frac{\sigma_1\left(\frac{z}{\sqrt{e_1-e_3}}; g_2, g_3\right)}{\sigma_3\left(\frac{z}{\sqrt{e_1-e_3}}; g_2, g_3\right)} /;$$

$$\{\omega_1, \omega_2, \omega_3\} = \{\omega_1(g_2, g_3), -\omega_1(g_2, g_3) - \omega_3(g_2, g_3), \omega_3(g_2, g_3)\} \bigwedge m = \lambda\left(\frac{\omega_3}{\omega_1}\right) \bigwedge e_n = \wp(\omega_n; g_2, g_3) \bigwedge n \in \{1, 2, 3\}$$

$$\operatorname{cs}(z \mid m) == \frac{1}{\sqrt{e_1 - e_3}} \frac{\sigma_1\left(\frac{z}{\sqrt{e_1-e_3}}; g_2, g_3\right)}{\sigma\left(\frac{z}{\sqrt{e_1-e_3}}; g_2, g_3\right)} /;$$

$$\{\omega_1, \omega_2, \omega_3\} == \{\omega_1(g_2, g_3), -\omega_1(g_2, g_3) - \omega_3(g_2, g_3), \omega_3(g_2, g_3)\} \bigwedge m == \lambda\left(\frac{\omega_3}{\omega_1}\right) \bigwedge e_n == \wp(\omega_n; g_2, g_3) \bigwedge n \in \{1, 2, 3\}$$

$$\operatorname{dc}(z \mid m) == \frac{\sigma_2\left(\frac{z}{\sqrt{e_1-e_3}}; g_2, g_3\right)}{\sigma_1\left(\frac{z}{\sqrt{e_1-e_3}}; g_2, g_3\right)} /;$$

$$\{\omega_1, \omega_2, \omega_3\} == \{\omega_1(g_2, g_3), -\omega_1(g_2, g_3) - \omega_3(g_2, g_3), \omega_3(g_2, g_3)\} \bigwedge m == \lambda\left(\frac{\omega_3}{\omega_1}\right) \bigwedge e_n == \wp(\omega_n; g_2, g_3) \bigwedge n \in \{1, 2, 3\}$$

$$\operatorname{dn}(z \mid m) == \frac{\sigma_2\left(\frac{z}{\sqrt{e_1-e_3}}; g_2, g_3\right)}{\sigma_3\left(\frac{z}{\sqrt{e_1-e_3}}; g_2, g_3\right)} /;$$

$$\{\omega_1, \omega_2, \omega_3\} == \{\omega_1(g_2, g_3), -\omega_1(g_2, g_3) - \omega_3(g_2, g_3), \omega_3(g_2, g_3)\} \bigwedge m == \lambda\left(\frac{\omega_3}{\omega_1}\right) \bigwedge e_n == \wp(\omega_n; g_2, g_3) \bigwedge n \in \{1, 2, 3\}$$

$$\operatorname{ds}(z \mid m) == \frac{1}{\sqrt{e_1 - e_3}} \frac{\sigma_2\left(\frac{z}{\sqrt{e_1-e_3}}; g_2, g_3\right)}{\sigma\left(\frac{z}{\sqrt{e_1-e_3}}; g_2, g_3\right)} /;$$

$$\{\omega_1, \omega_2, \omega_3\} == \{\omega_1(g_2, g_3), -\omega_1(g_2, g_3) - \omega_3(g_2, g_3), \omega_3(g_2, g_3)\} \bigwedge m == \lambda\left(\frac{\omega_3}{\omega_1}\right) \bigwedge e_n == \wp(\omega_n; g_2, g_3) \bigwedge n \in \{1, 2, 3\}$$

$$\operatorname{nc}(z \mid m) == \frac{\sigma_3\left(\frac{z}{\sqrt{e_1-e_3}}; g_2, g_3\right)}{\sigma_1\left(\frac{z}{\sqrt{e_1-e_3}}; g_2, g_3\right)} /;$$

$$\{\omega_1, \omega_2, \omega_3\} == \{\omega_1(g_2, g_3), -\omega_1(g_2, g_3) - \omega_3(g_2, g_3), \omega_3(g_2, g_3)\} \bigwedge m == \lambda\left(\frac{\omega_3}{\omega_1}\right) \bigwedge e_n == \wp(\omega_n; g_2, g_3) \bigwedge n \in \{1, 2, 3\}$$

$$\operatorname{nd}(z \mid m) == \frac{\sigma_3\left(\frac{z}{\sqrt{e_1-e_3}}; g_2, g_3\right)}{\sigma_2\left(\frac{z}{\sqrt{e_1-e_3}}; g_2, g_3\right)} /;$$

$$\{\omega_1, \omega_2, \omega_3\} == \{\omega_1(g_2, g_3), -\omega_1(g_2, g_3) - \omega_3(g_2, g_3), \omega_3(g_2, g_3)\} \bigwedge m == \lambda\left(\frac{\omega_3}{\omega_1}\right) \bigwedge e_n == \wp(\omega_n; g_2, g_3) \bigwedge n \in \{1, 2, 3\}$$

$$\operatorname{ns}(z \mid m) = \frac{1}{\sqrt{e_1 - e_3}} \frac{\sigma_3\left(\frac{z}{\sqrt{e_1-e_3}}; g_2, g_3\right)}{\sigma\left(\frac{z}{\sqrt{e_1-e_3}}; g_2, g_3\right)} /;$$

$$\{\omega_1, \omega_2, \omega_3\} = \{\omega_1(g_2, g_3), -\omega_1(g_2, g_3) - \omega_3(g_2, g_3), \omega_3(g_2, g_3)\} \bigwedge m = \lambda\left(\frac{\omega_3}{\omega_1}\right) \bigwedge e_n = \wp(\omega_n; g_2, g_3) \bigwedge n \in \{1, 2, 3\}$$

$$\operatorname{sc}(z \mid m) = \sqrt{e_1 - e_3} \frac{\sigma\left(\frac{z}{\sqrt{e_1-e_3}}; g_2, g_3\right)}{\sigma_1\left(\frac{z}{\sqrt{e_1-e_3}}; g_2, g_3\right)} /;$$

$$\{\omega_1, \omega_2, \omega_3\} = \{\omega_1(g_2, g_3), -\omega_1(g_2, g_3) - \omega_3(g_2, g_3), \omega_3(g_2, g_3)\} \bigwedge m = \lambda\left(\frac{\omega_3}{\omega_1}\right) \bigwedge e_n = \wp(\omega_n; g_2, g_3) \bigwedge n \in \{1, 2, 3\}$$

$$\operatorname{sd}(z \mid m) = \sqrt{e_1 - e_3} \frac{\sigma\left(\frac{z}{\sqrt{e_1-e_3}}; g_2, g_3\right)}{\sigma_2\left(\frac{z}{\sqrt{e_1-e_3}}; g_2, g_3\right)} /;$$

$$\{\omega_1, \omega_2, \omega_3\} = \{\omega_1(g_2, g_3), -\omega_1(g_2, g_3) - \omega_3(g_2, g_3), \omega_3(g_2, g_3)\} \bigwedge m = \lambda\left(\frac{\omega_3}{\omega_1}\right) \bigwedge e_n = \wp(\omega_n; g_2, g_3) \bigwedge n \in \{1, 2, 3\}$$

$$\operatorname{sn}(z \mid m) = \frac{\sqrt{e_1 - e_3}\, \sigma\left(\frac{z}{\sqrt{e_1-e_3}}; g_2, g_3\right)}{\sigma_3\left(\frac{z}{\sqrt{e_1-e_3}}; g_2, g_3\right)} /;$$

$$\{\omega_1, \omega_2, \omega_3\} = \{\omega_1(g_2, g_3), -\omega_1(g_2, g_3) - \omega_3(g_2, g_3), \omega_3(g_2, g_3)\} \bigwedge m = \lambda\left(\frac{\omega_3}{\omega_1}\right) \bigwedge e_n = \wp(\omega_n; g_2, g_3) \bigwedge n \in \{1, 2, 3\}.$$

The twelve Jacobi functions $\operatorname{cd}(z \mid m)$, $\operatorname{cn}(z \mid m)$, $\operatorname{cs}(z \mid m)$, $\operatorname{dc}(z \mid m)$, $\operatorname{dn}(z \mid m)$, $\operatorname{ds}(z \mid m)$, $\operatorname{nc}(z \mid m)$, $\operatorname{nd}(z \mid m)$, $\operatorname{ns}(z \mid m)$, $\operatorname{sc}(z \mid m)$, $\operatorname{sd}(z \mid m)$, and $\operatorname{sn}(z \mid m)$ can be represented through the Weierstrass $\wp$ function:

$$\operatorname{cd}(z \mid m)^2 = \frac{\wp\left(\frac{z}{\sqrt{e_1-e_3}}; g_2, g_3\right) - e_1}{\wp\left(\frac{z}{\sqrt{e_1-e_3}}; g_2, g_3\right) - e_2} /;$$

$$\{\omega_1, \omega_2, \omega_3\} = \{\omega_1(g_2, g_3), -\omega_1(g_2, g_3) - \omega_3(g_2, g_3), \omega_3(g_2, g_3)\} \bigwedge m = \lambda\left(\frac{\omega_3}{\omega_1}\right) \bigwedge e_n = \wp(\omega_n; g_2, g_3) \bigwedge n \in \{1, 2, 3\}$$

$$\operatorname{cn}(z \mid m)^2 = \frac{\wp\left(\frac{z}{\sqrt{e_1-e_3}}; g_2, g_3\right) - e_1}{\wp\left(\frac{z}{\sqrt{e_1-e_3}}; g_2, g_3\right) - e_3} /;$$

$$\{\omega_1, \omega_2, \omega_3\} = \{\omega_1(g_2, g_3), -\omega_1(g_2, g_3) - \omega_3(g_2, g_3), \omega_3(g_2, g_3)\} \bigwedge m = \lambda\left(\frac{\omega_3}{\omega_1}\right) \bigwedge e_n = \wp(\omega_n; g_2, g_3) \bigwedge n \in \{1, 2, 3\}$$

$$\operatorname{cs}(z \mid m)^2 = \frac{\wp\left(\frac{z}{\sqrt{e_1-e_3}}; g_2, g_3\right) - e_1}{e_1 - e_3} \;/;$$

$$\{\omega_1, \omega_2, \omega_3\} = \{\omega_1(g_2, g_3), -\omega_1(g_2, g_3) - \omega_3(g_2, g_3), \omega_3(g_2, g_3)\} \bigwedge m = \lambda\left(\frac{\omega_3}{\omega_1}\right) \bigwedge e_n = \wp(\omega_n; g_2, g_3) \bigwedge n \in \{1, 2, 3\}$$

$$\operatorname{dc}(z \mid m)^2 = \frac{\wp\left(\frac{z}{\sqrt{e_1-e_3}}; g_2, g_3\right) - e_2}{\wp\left(\frac{z}{\sqrt{e_1-e_3}}; g_2, g_3\right) - e_1} \;/;$$

$$\{\omega_1, \omega_2, \omega_3\} = \{\omega_1(g_2, g_3), -\omega_1(g_2, g_3) - \omega_3(g_2, g_3), \omega_3(g_2, g_3)\} \bigwedge m = \lambda\left(\frac{\omega_3}{\omega_1}\right) \bigwedge e_n = \wp(\omega_n; g_2, g_3) \bigwedge n \in \{1, 2, 3\}$$

$$\operatorname{dn}(z \mid m)^2 = \frac{\wp\left(\frac{z}{\sqrt{e_1-e_3}}; g_2, g_3\right) - e_2}{\wp\left(\frac{z}{\sqrt{e_1-e_3}}; g_2, g_3\right) - e_3} \;/;$$

$$\{\omega_1, \omega_2, \omega_3\} = \{\omega_1(g_2, g_3), -\omega_1(g_2, g_3) - \omega_3(g_2, g_3), \omega_3(g_2, g_3)\} \bigwedge m = \lambda\left(\frac{\omega_3}{\omega_1}\right) \bigwedge e_n = \wp(\omega_n; g_2, g_3) \bigwedge n \in \{1, 2, 3\}$$

$$\operatorname{ds}(z \mid m)^2 = \frac{\wp\left(\frac{z}{\sqrt{e_1-e_3}}; g_2, g_3\right) - e_2}{e_1 - e_3} \;/;$$

$$\{\omega_1, \omega_2, \omega_3\} = \{\omega_1(g_2, g_3), -\omega_1(g_2, g_3) - \omega_3(g_2, g_3), \omega_3(g_2, g_3)\} \bigwedge m = \lambda\left(\frac{\omega_3}{\omega_1}\right) \bigwedge e_n = \wp(\omega_n; g_2, g_3) \bigwedge n \in \{1, 2, 3\}$$

$$\operatorname{nc}(z \mid m)^2 = \frac{\wp\left(\frac{z}{\sqrt{e_1-e_3}}; g_2, g_3\right) - e_3}{\wp\left(\frac{z}{\sqrt{e_1-e_3}}; g_2, g_3\right) - e_1} \;/;$$

$$\{\omega_1, \omega_2, \omega_3\} = \{\omega_1(g_2, g_3), -\omega_1(g_2, g_3) - \omega_3(g_2, g_3), \omega_3(g_2, g_3)\} \bigwedge m = \lambda\left(\frac{\omega_3}{\omega_1}\right) \bigwedge e_n = \wp(\omega_n; g_2, g_3) \bigwedge n \in \{1, 2, 3\}$$

$$\operatorname{nd}(z \mid m)^2 = \frac{\wp\left(\frac{z}{\sqrt{e_1-e_3}}; g_2, g_3\right) - e_3}{\wp\left(\frac{z}{\sqrt{e_1-e_3}}; g_2, g_3\right) - e_2} \;/;$$

$$\{\omega_1, \omega_2, \omega_3\} = \{\omega_1(g_2, g_3), -\omega_1(g_2, g_3) - \omega_3(g_2, g_3), \omega_3(g_2, g_3)\} \bigwedge m = \lambda\left(\frac{\omega_3}{\omega_1}\right) \bigwedge e_n = \wp(\omega_n; g_2, g_3) \bigwedge n \in \{1, 2, 3\}$$

$$\mathrm{ns}(z \mid m)^2 = \frac{\wp\left(\frac{z}{\sqrt{e_1-e_3}}; g_2, g_3\right) - e_3}{e_1 - e_3} \;/;$$

$$\{\omega_1, \omega_2, \omega_3\} = \{\omega_1(g_2, g_3), -\omega_1(g_2, g_3) - \omega_3(g_2, g_3), \omega_3(g_2, g_3)\} \bigwedge m = \lambda\left(\frac{\omega_3}{\omega_1}\right) \bigwedge e_n = \wp(\omega_n; g_2, g_3) \bigwedge n \in \{1, 2, 3\}$$

$$\mathrm{sc}\left(z\sqrt{e_1 - e_3} \,\middle|\, m\right)^2 = \frac{e_1 - e_3}{\wp(z; g_2, g_3) - e_1} \;/;$$

$$\{\omega_1, \omega_2, \omega_3\} = \{\omega_1(g_2, g_3), -\omega_1(g_2, g_3) - \omega_3(g_2, g_3), \omega_3(g_2, g_3)\} \bigwedge m = \lambda\left(\frac{\omega_3}{\omega_1}\right) \bigwedge e_n = \wp(\omega_n; g_2, g_3) \bigwedge n \in \{1, 2, 3\}$$

$$\mathrm{sd}(z \mid m)^2 = \frac{e_1 - e_3}{\wp\left(\frac{z}{\sqrt{e_1-e_3}}; g_2, g_3\right) - e_2} \;/;$$

$$\{\omega_1, \omega_2, \omega_3\} = \{\omega_1(g_2, g_3), -\omega_1(g_2, g_3) - \omega_3(g_2, g_3), \omega_3(g_2, g_3)\} \bigwedge m = \lambda\left(\frac{\omega_3}{\omega_1}\right) \bigwedge e_n = \wp(\omega_n; g_2, g_3) \bigwedge n \in \{1, 2, 3\}$$

$$\mathrm{sn}(u \mid m)^2 = \frac{e_1 - e_3}{\wp\left(\frac{z}{\sqrt{e_1-e_3}}; g_2, g_3\right) - e_3} \;/;$$

$$\{\omega_1, \omega_2, \omega_3\} = \{\omega_1(g_2, g_3), -\omega_1(g_2, g_3) - \omega_3(g_2, g_3), \omega_3(g_2, g_3)\} \bigwedge m = \lambda\left(\frac{\omega_3}{\omega_1}\right) \bigwedge e_n = \wp(\omega_n; g_2, g_3) \bigwedge n \in \{1, 2, 3\}.$$

The twelve Jacobi functions $\mathrm{cd}(z \mid m)$, $\mathrm{cn}(z \mid m)$, $\mathrm{cs}(z \mid m)$, $\mathrm{dc}(z \mid m)$, $\mathrm{dn}(z \mid m)$, $\mathrm{ds}(z \mid m)$, $\mathrm{nc}(z \mid m)$, $\mathrm{nd}(z \mid m)$, $\mathrm{ns}(z \mid m)$, $\mathrm{sc}(z \mid m)$, $\mathrm{sd}(z \mid m)$, and $\mathrm{sn}(z \mid m)$ can be represented through the elliptic theta functions:

$$\mathrm{cd}(z \mid m) = m^{-1/4} \frac{\vartheta_2\left(\frac{z\pi}{2K(m)}, q(m)\right)}{\vartheta_3\left(\frac{z\pi}{2K(m)}, q(m)\right)}$$

$$\mathrm{cn}(z \mid m) = (1 - m)^{1/4} m^{-1/4} \frac{\vartheta_2\left(\frac{\pi z}{2K(m)}, q(m)\right)}{\vartheta_4\left(\frac{\pi z}{2K(m)}, q(m)\right)}$$

$$\mathrm{cs}(z \mid m) = \sqrt[4]{1 - m} \frac{\vartheta_2\left(\frac{\pi z}{2K(m)}, q(m)\right)}{\vartheta_1\left(\frac{\pi z}{2K(m)}, q(m)\right)}$$

$$\mathrm{dc}(z \mid m) = \sqrt[4]{m} \frac{\vartheta_3\left(\frac{\pi z}{2K(m)}, q(m)\right)}{\vartheta_2\left(\frac{\pi z}{2K(m)}, q(m)\right)}$$

$$\mathrm{dn}(z \mid m) = \sqrt[4]{1 - m} \frac{\vartheta_3\left(\frac{\pi z}{2K(m)}, q(m)\right)}{\vartheta_4\left(\frac{\pi z}{2K(m)}, q(m)\right)}$$

$$\text{ds}(z\mid m)=(m\,(1-m))^{1/4}\,\frac{\vartheta_3\left(\frac{\pi z}{2K(m)},\,q(m)\right)}{\vartheta_1\left(\frac{\pi z}{2K(m)},\,q(m)\right)}$$

$$\text{nc}(z\mid m)=\frac{\sqrt[4]{m}}{\sqrt[4]{1-m}}\,\frac{\vartheta_4\left(\frac{\pi z}{2K(m)},\,q(m)\right)}{\vartheta_2\left(\frac{\pi z}{2K(m)},\,q(m)\right)}$$

$$\text{nd}(z\mid m)=(1-m)^{-1/4}\,\frac{\vartheta_4\left(\frac{\pi z}{2K(m)},\,q(m)\right)}{\vartheta_3\left(\frac{\pi z}{2K(m)},\,q(m)\right)}$$

$$\text{ns}(z\mid m)=\sqrt[4]{m}\,\frac{\vartheta_4\left(\frac{\pi z}{2K(m)},\,q(m)\right)}{\vartheta_1\left(\frac{\pi z}{2K(m)},\,q(m)\right)}$$

$$\text{sc}(z\mid m)=(1-m)^{-1/4}\,\frac{\vartheta_1\left(\frac{\pi z}{2K(m)},\,q(m)\right)}{\vartheta_2\left(\frac{\pi z}{2K(m)},\,q(m)\right)}$$

$$\text{sd}(z\mid m)=\frac{1}{\sqrt[4]{m}\,\sqrt[4]{1-m}}\,\frac{1}{\vartheta_3\left(\frac{\pi z}{2K(m)},\,q(m)\right)}\,\vartheta_1\left(\frac{\pi z}{2K(m)},\,q(m)\right)$$

$$\text{sn}(z\mid m)=\frac{1}{\sqrt[4]{m}}\,\frac{\vartheta_1\left(\frac{\pi z}{2K(m)},\,q(m)\right)}{\vartheta_4\left(\frac{\pi z}{2K(m)},\,q(m)\right)}.$$

The twelve Jacobi functions $\text{cd}(z\mid m)$, $\text{cn}(z\mid m)$, $\text{cs}(z\mid m)$, $\text{dc}(z\mid m)$, $\text{dn}(z\mid m)$, $\text{ds}(z\mid m)$, $\text{nc}(z\mid m)$, $\text{nd}(z\mid m)$, $\text{ns}(z\mid m)$, $\text{sc}(z\mid m)$, $\text{sd}(z\mid m)$, and $\text{sn}(z\mid m)$ can be represented through the Neville theta functions:

$$\text{cd}(z\mid m)=\frac{\vartheta_c(z|m)}{\vartheta_d(z|m)}\quad \text{cn}(z\mid m)=\frac{\vartheta_c(z|m)}{\vartheta_n(z|m)}\quad \text{cs}(z\mid m)=\frac{\vartheta_c(z|m)}{\vartheta_s(z|m)}$$

$$\text{dc}(z\mid m)=\frac{\vartheta_d(z|m)}{\vartheta_c(z|m)}\quad \text{dn}(z\mid m)=\frac{\vartheta_d(z|m)}{\vartheta_n(z|m)}\quad \text{ds}(z\mid m)=\frac{\vartheta_d(z|m)}{\vartheta_s(z|m)}$$

$$\text{nc}(z\mid m)=\frac{\vartheta_n(z|m)}{\vartheta_c(z|m)}\quad \text{nd}(z\mid m)=\frac{\vartheta_n(z|m)}{\vartheta_d(z|m)}\quad \text{ns}(z\mid m)=\frac{\vartheta_n(z|m)}{\vartheta_s(z|m)}$$

$$\text{sc}(z\mid m)=\frac{\vartheta_s(z|m)}{\vartheta_c(z|m)}\quad \text{sd}(z\mid m)=\frac{\vartheta_s(z|m)}{\vartheta_d(z|m)}\quad \text{sn}(z\mid m)=\frac{\vartheta_s(z|m)}{\vartheta_n(z|m)}.$$

### Relations to inverse functions

The Jacobi functions $\text{am}(z\mid m)$, $\text{cd}(z\mid m)$, $\text{cn}(z\mid m)$, $\text{cs}(z\mid m)$, $\text{dc}(z\mid m)$, $\text{dn}(z\mid m)$, $\text{ds}(z\mid m)$, $\text{nc}(z\mid m)$, $\text{nd}(z\mid m)$, $\text{ns}(z\mid m)$, $\text{sc}(z\mid m)$, $\text{sd}(z\mid m)$, and $\text{sn}(z\mid m)$ are connected with the corresponding inverse functions by the following formulas:

$$\text{am}(F(z\mid m)\mid m)=z\;/;\;m<1\bigvee -\frac{3}{2}\leq z\leq\frac{3}{2}$$

$$F(\text{am}(z\mid m)\mid m)=z\;/;\;m\leq 1\land -2\leq z\leq 2$$

$\mathrm{cd}\left(\mathrm{cd}^{-1}(z \mid m) \mid m\right) = z \quad \mathrm{cn}\left(\mathrm{cn}^{-1}(z \mid m) \mid m\right) = z \quad \mathrm{cs}\left(\mathrm{cs}^{-1}(z \mid m) \mid m\right) = z$

$\mathrm{dc}\left(\mathrm{dc}^{-1}(z \mid m) \mid m\right) = z \quad \mathrm{dn}\left(\mathrm{dn}^{-1}(z \mid m) \mid m\right) = z \quad \mathrm{ds}\left(\mathrm{ds}^{-1}(z \mid m) \mid m\right) = z$

$\mathrm{nc}\left(\mathrm{nc}^{-1}(z \mid m) \mid m\right) = z \quad \mathrm{nd}\left(\mathrm{nd}^{-1}(z \mid m) \mid m\right) = z \quad \mathrm{ns}\left(\mathrm{ns}^{-1}(z \mid m) \mid m\right) = z$

$\mathrm{sc}\left(\mathrm{sc}^{-1}(z \mid m) \mid m\right) = z \quad \mathrm{sd}\left(\mathrm{sd}^{-1}(z \mid m) \mid m\right) = z \quad \mathrm{sn}\left(\mathrm{sn}^{-1}(z \mid m) \mid m\right) = z.$

### Representations through other Jacobi functions

By definition, the three basic Jacobi functions have the following
representations through the amplitude function am:

$$\mathrm{cn}(z \mid m) = \cos(\mathrm{am}(z \mid m))$$

$$\mathrm{dn}(z \mid m) = \sqrt{1 - m \sin^2(\mathrm{am}(z \mid m))} \;/;\; m < 1$$

$$\mathrm{sn}(z \mid m) = \sin(\mathrm{am}(z \mid m)).$$

The other nine Jacobi functions can be easily expressed through the three
basic Jacobi functions $\mathrm{cn}(z \mid m)$, $\mathrm{cn}(z \mid m)$, and $\mathrm{dn}(z \mid m)$ and, consequently, they can also be represented through the amplitude
function am.

The twelve Jacobi functions $\mathrm{cd}(z \mid m)$, $\mathrm{cn}(z \mid m)$, $\mathrm{cs}(z \mid m)$, $\mathrm{dc}(z \mid m)$, $\mathrm{dn}(z \mid m)$, $\mathrm{ds}(z \mid m)$, $\mathrm{nc}(z \mid m)$, $\mathrm{nd}(z \mid m)$, $\mathrm{ns}(z \mid m)$, $\mathrm{sc}(z \mid m)$, $\mathrm{sd}(z \mid m)$, and $\mathrm{sn}(z \mid m)$ are interconnected by formulas that include rational functions, simple
powers, and arithmetical operations from other Jacobi functions. These
formulas can be divided into the following eleven groups:

Representations of $\mathrm{cd}(z \mid m)$ through other Jacobi functions are:

$$\mathrm{cd}(z \mid m) = \frac{\mathrm{cn}(z|m)}{\mathrm{dn}(z|m)} \qquad \mathrm{cd}(z \mid m) = \mathrm{cn}(z \mid m)\, \mathrm{nd}(z \mid m)$$

$$\mathrm{cd}(z \mid m)^2 = \frac{\mathrm{cn}(z|m)^2}{m\, \mathrm{cn}(z|m)^2 - m + 1}$$

$$\mathrm{cd}(z \mid m) = \frac{\mathrm{cs}(z|m)}{\mathrm{ds}(z|m)} \qquad \mathrm{cd}(z \mid m) = \mathrm{cs}(z \mid m)\, \mathrm{sd}(z \mid m)$$

$$\mathrm{cd}(z \mid m)^2 = \frac{\mathrm{cs}(z|m)^2}{\mathrm{cs}(z|m)^2 - m + 1}$$

$$\mathrm{cd}(z \mid m) = \frac{1}{\mathrm{dc}(z|m)}$$

$$\mathrm{cd}(z \mid m) = \frac{1}{\mathrm{nc}(z|m)\, \mathrm{dn}(z|m)} \qquad \mathrm{cd}(z \mid m)^2 = \frac{\mathrm{dn}(z|m)^2 + m - 1}{m\, \mathrm{dn}(z|m)^2}$$

$$\mathrm{cd}(z \mid m) = \frac{1}{\mathrm{sc}(z|m)\, \mathrm{ds}(z|m)} \qquad \mathrm{cd}(z \mid m)^2 = 1 - \frac{1-m}{\mathrm{ds}(z|m)^2}$$

$$\mathrm{cd}(z \mid m) = \frac{\mathrm{nd}(z|m)}{\mathrm{nc}(z|m)} \qquad \mathrm{cd}(z \mid m)^2 = \frac{1}{m - (m-1)\, \mathrm{nc}(z|m)^2}$$

$$\mathrm{cd}(z \mid m) = \mathrm{nd}(i\, z \mid 1 - m) \qquad \mathrm{cd}(z \mid m)^2 = \frac{1 - (1-m)\, \mathrm{nd}(z|m)^2}{m}$$

$$\mathrm{cd}(z \mid m)^2 = \frac{\mathrm{ns}(z|m)^2 - 1}{\mathrm{ns}(z|m)^2 - m}$$

$\mathrm{cd}(z \mid m) = \frac{\mathrm{sd}(z|m)}{\mathrm{sc}(z|m)} \quad \mathrm{cd}(z \mid m)^2 = \frac{1}{1-(m-1)\,\mathrm{sc}(z|m)^2}$

$\mathrm{cd}(z \mid m)^2 = 1 - (1-m)\,\mathrm{sd}(z \mid m)^2$

$\mathrm{cd}(z \mid m)^2 = \frac{\mathrm{sn}(z|m)^2-1}{m\,\mathrm{sn}(z|m)^2-1}$.

Representations of cn($z$ | $m$) through other Jacobi functions are:

$\mathrm{cn}(z \mid m) = \mathrm{cd}(z \mid m)\,\mathrm{dn}(z \mid m) \quad \mathrm{cn}(z \mid m) = \frac{\mathrm{cd}(z|m)}{\mathrm{nd}(z|m)} \quad \mathrm{cn}(z \mid m)^2 = \frac{(m-1)\,\mathrm{cd}(z|m)^2}{m\,\mathrm{cd}(z|m)^2-1}$

$\mathrm{cn}(z \mid m) = \frac{\mathrm{cs}(z|m)}{\mathrm{ns}(z|m)} \quad \mathrm{cn}(z \mid m) = \mathrm{cs}(z \mid m)\,\mathrm{sn}(z \mid m) \quad \mathrm{cn}(z \mid m)^2 = \frac{\mathrm{cs}(z|m)^2}{\mathrm{cs}(z|m)^2+1}$

$\mathrm{cn}(z \mid m) = \frac{\mathrm{dn}(z|m)}{\mathrm{dc}(z|m)} \quad \mathrm{cn}(z \mid m) = \frac{1}{\mathrm{dc}(z|m)\,\mathrm{nd}(z|m)} \quad \mathrm{cn}(z \mid m)^2 = \frac{1-m}{\mathrm{dc}(z|m)^2-m}$

$\mathrm{cn}(z \mid m)^2 = \frac{\mathrm{dn}(z|m)^2-1}{m} + 1$

$\mathrm{cn}(z \mid m)^2 = \frac{\mathrm{ds}(z|m)^2+m-1}{\mathrm{ds}(z|m)^2+m}$

$\mathrm{cn}(z \mid m) = \frac{1}{\mathrm{nc}(z|m)} \quad \mathrm{cn}(z \mid m) = \mathrm{nc}(i\,z \mid 1-m)$

$\mathrm{cn}(z \mid m)^2 = \frac{(m-1)\,\mathrm{nd}(z|m)^2+1}{m\,\mathrm{nd}(z|m)^2}$

$\mathrm{cn}(z \mid m) = \frac{1}{\mathrm{sc}(z|m)\,\mathrm{ns}(z|m)} \quad \mathrm{cn}(z \mid m)^2 = 1 - \frac{1}{\mathrm{ns}(z|m)^2}$

$\mathrm{cn}(z \mid m) = \frac{\mathrm{sn}(z|m)}{\mathrm{sc}(z|m)} \quad \mathrm{cn}(z \mid m)^2 = \frac{1}{\mathrm{sc}(z|m)^2+1}$

$\mathrm{cn}(z \mid m)^2 = \frac{(m-1)\,\mathrm{sd}(z|m)^2+1}{m\,\mathrm{sd}(z|m)^2+1}$

$\mathrm{cn}(z \mid m)^2 = 1 - \mathrm{sn}(z \mid m)^2$ .

Representations of cs($z$ | $m$) through other Jacobi functions are:

$\mathrm{cs}(z \mid m) = \mathrm{cd}(z \mid m)\,\mathrm{ds}(z \mid m) \quad \mathrm{cs}(z \mid m) = \frac{\mathrm{cd}(z|m)}{\mathrm{sd}(z|m)} \quad \mathrm{cs}(z \mid m)^2 = \frac{(m-1)\,\mathrm{cd}(z|m)^2}{\mathrm{cd}(z|m)^2-1}$

$\mathrm{cs}(z \mid m) = \mathrm{cn}(z \mid m)\,\mathrm{ns}(z \mid m) \quad \mathrm{cs}(z \mid m) = \frac{\mathrm{cn}(z|m)}{\mathrm{sn}(z|m)} \quad \mathrm{cs}(z \mid m)^2 = \frac{\mathrm{cn}(z|m)^2}{1-\mathrm{cn}(z|m)^2}$

$\mathrm{cs}(z \mid m) = \frac{\mathrm{ds}(z|m)}{\mathrm{dc}(z|m)} \quad \mathrm{cs}(z \mid m) = \frac{1}{\mathrm{dc}(z|m)\,\mathrm{sd}(z|m)} \quad \mathrm{cs}(z \mid m)^2 = \frac{m-1}{1-\mathrm{dc}(z|m)^2}$

$\mathrm{cs}(z \mid m)^2 = \frac{\mathrm{dn}(z|m)^2+m-1}{1-\mathrm{dn}(z|m)^2}$

$\mathrm{cs}(z \mid m)^2 = \mathrm{ds}(z \mid m)^2 + m - 1$

$\mathrm{cs}(z \mid m) = \frac{\mathrm{ns}(z|m)}{\mathrm{nc}(z|m)} \quad \mathrm{cs}(z \mid m) = \frac{1}{\mathrm{nc}(z|m)\,\mathrm{sn}(z|m)} \quad \mathrm{cs}(z \mid m)^2 = \frac{1}{\mathrm{nc}(z|m)^2-1}$

$$\text{cs}(z \mid m)^2 = \frac{(m-1)\,\text{nd}(z|m)^2+1}{\text{nd}(z|m)^2-1}$$

$$\text{cs}(z \mid m) = i\,\text{ns}(i\,z \mid 1-m) \quad \text{cs}(z \mid m)^2 = \text{ns}(z \mid m)^2 - 1$$

$$\text{cs}(z \mid m) = \frac{1}{\text{sc}(z|m)}$$

$$\text{cs}(z \mid m)^2 = \frac{(m-1)\,\text{sd}(z|m)^2+1}{\text{sd}(z|m)^2}$$

$$\text{cs}(z \mid m) = \frac{i}{\text{sn}(i\,z|1-m)} \quad \text{cs}(z \mid m)^2 = \frac{1-\text{sn}(z|m)^2}{\text{sn}(z|m)^2}\ .$$

Representations of $\text{dc}(z \mid m)$ through other Jacobi functions are:

$$\text{dc}(z \mid m) = \frac{1}{\text{cd}(z|m)}$$

$$\text{dc}(z \mid m) = \frac{\text{dn}(z|m)}{\text{cn}(z|m)} \quad \text{dc}(z \mid m) = \frac{1}{\text{nd}(z|m)\,\text{cn}(z|m)} \quad \text{dc}(z \mid m)^2 = \frac{m\,\text{cn}(z|m)^2-m+1}{\text{cn}(z|m)^2}$$

$$\text{dc}(z \mid m) = \frac{\text{ds}(z|m)}{\text{cs}(z|m)} \quad \text{dc}(z \mid m) = \frac{1}{\text{sd}(z|m)\,\text{cs}(z|m)} \quad \text{dc}(z \mid m)^2 = \frac{\text{cs}(z|m)^2-m+1}{\text{cs}(z|m)^2}$$

$$\text{dc}(z \mid m) = \text{dn}(z \mid m)\,\text{nc}(z \mid m) \quad \text{dc}(z \mid m) = \text{dn}(i\,z \mid 1-m) \quad \text{dc}(z \mid m)^2 = \frac{m\,\text{dn}(z|m)^2}{\text{dn}(z|m)^2+m-1}$$

$$\text{dc}(z \mid m) = \text{ds}(z \mid m)\,\text{sc}(z \mid m) \quad \text{dc}(z \mid m)^2 = \frac{\text{ds}(z|m)^2}{\text{ds}(z|m)^2+m-1}$$

$$\text{dc}(z \mid m) = \frac{\text{nc}(z|m)}{\text{nd}(z|m)} \quad \text{dc}(z \mid m)^2 = m - (m-1)\,\text{nc}(z \mid m)^2$$

$$\text{dc}(z \mid m) = \frac{1}{\text{nd}(i\,z|1-m)} \quad \text{dc}(z \mid m)^2 = \frac{m}{1-(1-m)\,\text{nd}(z|m)^2}$$

$$\text{dc}(z \mid m)^2 = \frac{\text{ns}(z|m)^2-m}{\text{ns}(z|m)^2-1}$$

$$\text{dc}(z \mid m) = \frac{\text{sc}(z|m)}{\text{sd}(z|m)} \quad \text{dc}(z \mid m)^2 = 1 - (m-1)\,\text{sc}(z \mid m)^2$$

$$\text{dc}(z \mid m)^2 = \frac{1}{1-(1-m)\,\text{sd}(z|m)^2}$$

$$\text{dc}(z \mid m)^2 = \frac{m\,\text{sn}(z|m)^2-1}{\text{sn}(z|m)^2-1}\ .$$

Representations of $\text{dn}(z \mid m)$ through other Jacobi functions are:

$$\text{dn}(z \mid m) = \frac{\text{cn}(z|m)}{\text{cd}(z|m)} \quad \text{dn}(z \mid m) = \frac{1}{\text{cd}(z|m)\,\text{nc}(z|m)} \quad \text{dn}(z \mid m)^2 = \frac{m-1}{m\,\text{cd}(z|m)^2-1}$$

$$\text{dn}(z \mid m) = \text{dc}(z \mid m)\,\text{cn}(z \mid m) \quad \text{dn}(z \mid m)^2 = 1 - m + m\,\text{cn}(z \mid m)^2$$

$$\text{dn}(z \mid m)^2 = \frac{\text{cs}(z|m)^2-m+1}{\text{cs}(z|m)^2+1}$$

$$\text{dn}(z \mid m) = \frac{\text{dc}(z|m)}{\text{nc}(z|m)} \quad \text{dn}(z \mid m) = \text{dc}(i\,z \mid 1-m) \quad \text{dn}(z \mid m)^2 = \frac{(m-1)\,\text{dc}(z|m)^2}{m-\text{dc}(z|m)^2}$$

$$\mathrm{dn}(z \mid m) = \mathrm{ds}(z \mid m)\,\mathrm{sn}(z \mid m) \quad \mathrm{dn}(z \mid m) = \frac{\mathrm{ds}(z|m)}{\mathrm{ns}(z|m)} \quad \mathrm{dn}(z \mid m)^2 = \frac{\mathrm{ds}(z|m)^2}{\mathrm{ds}(z|m)^2+m}$$

$$\mathrm{dn}(z \mid m)^2 = \frac{m}{\mathrm{nc}(z|m)^2} - m + 1$$

$$\mathrm{dn}(z \mid m) = \frac{1}{\mathrm{nd}(z|m)}$$

$$\mathrm{dn}(z \mid m) = \frac{1}{\mathrm{sd}(z|m)\,\mathrm{ns}(z|m)} \quad \mathrm{dn}(z \mid m)^2 = 1 - \frac{m}{\mathrm{ns}(z|m)^2}$$

$$\mathrm{dn}(z \mid m)^2 = \frac{(1-m)\,\mathrm{sc}(z|m)^2+1}{\mathrm{sc}(z|m)^2+1}$$

$$\mathrm{dn}(z \mid m) = \frac{\mathrm{sn}(z|m)}{\mathrm{sd}(z|m)} \quad \mathrm{dn}(z \mid m)^2 = \frac{1}{m\,\mathrm{sd}(z|m)^2+1}$$

$$\mathrm{dn}(z \mid m) = \sqrt{1-m}\;\mathrm{sn}(-i\,z + K(1-m) - i\,K(m) \mid 1-m) \quad \mathrm{dn}(z \mid m)^2 = 1 - m\,\mathrm{sn}(z \mid m)^2 \;.$$

Representations of ds$(z \mid m)$ through other Jacobi functions are:

$$\mathrm{ds}(z \mid m) = \frac{\mathrm{cs}(z|m)}{\mathrm{cd}(z|m)} \quad \mathrm{ds}(z \mid m) = \frac{1}{\mathrm{cd}(z|m)\,\mathrm{sc}(z|m)} \quad \mathrm{ds}(z \mid m)^2 = \frac{1-m}{1-\mathrm{cd}(z|m)^2}$$

$$\mathrm{ds}(z \mid m)^2 = \frac{m\,\mathrm{cn}(z|m)^2-m+1}{1-\mathrm{cn}(z|m)^2}$$

$$\mathrm{ds}(z \mid m) = \mathrm{dc}(z \mid m)\,\mathrm{cs}(z \mid m) \quad \mathrm{ds}(z \mid m)^2 = \mathrm{cs}(z \mid m)^2 - m + 1$$

$$\mathrm{ds}(z \mid m) = \frac{\mathrm{dc}(z|m)}{\mathrm{sc}(z|m)} \quad \mathrm{ds}(z \mid m)^2 = \frac{(1-m)\,\mathrm{dc}(z|m)^2}{\mathrm{dc}(z|m)^2-1}$$

$$\mathrm{ds}(z \mid m) = \mathrm{dn}(z \mid m)\,\mathrm{ns}(z \mid m) \quad \mathrm{ds}(z \mid m) = \frac{\mathrm{dn}(z|m)}{\mathrm{sn}(z|m)} \quad \mathrm{ds}(z \mid m)^2 = \frac{m\,\mathrm{dn}(z|m)^2}{1-\mathrm{dn}(z|m)^2}$$

$$\mathrm{ds}(z \mid m)^2 = \frac{(m-1)\,\mathrm{nc}(z|m)^2-m}{1-\mathrm{nc}(z|m)^2}$$

$$\mathrm{ds}(z \mid m) = \frac{\mathrm{ns}(z|m)}{\mathrm{nd}(z|m)} \quad \mathrm{ds}(z \mid m) = \frac{1}{\mathrm{nd}(z|m)\,\mathrm{sn}(z|m)} \quad \mathrm{ds}(z \mid m)^2 = \frac{m}{\mathrm{nd}(z|m)^2-1}$$

$$\mathrm{ds}(z \mid m)^2 = \mathrm{ns}(z \mid m)^2 - m$$

$$\mathrm{ds}(z \mid m)^2 = \frac{1-(m-1)\,\mathrm{sc}(z|m)^2}{\mathrm{sc}(z|m)^2}$$

$$\mathrm{ds}(z \mid m) = \frac{1}{\mathrm{sd}(z|m)}$$

$$\mathrm{ds}(z \mid m)^2 = \frac{1-m\,\mathrm{sn}(z|m)^2}{\mathrm{sn}(z|m)^2} \;.$$

Representations of nc$(z \mid m)$ through other Jacobi functions are:

$$\mathrm{nc}(z \mid m) = \frac{1}{\mathrm{dn}(z|m)\,\mathrm{cd}(z|m)} \quad \mathrm{nc}(z \mid m) = \frac{\mathrm{nd}(z|m)}{\mathrm{cd}(z|m)} \quad \mathrm{nc}(z \mid m)^2 = \frac{m\,\mathrm{cd}(z|m)^2-1}{(m-1)\,\mathrm{cd}(z|m)^2}$$

$$\mathrm{nc}(z \mid m) = \frac{1}{\mathrm{cn}(z|m)} \quad \mathrm{nc}(z \mid m) = \mathrm{cn}(i\,z \mid 1-m)$$

$\operatorname{nc}(z \mid m) = \frac{\operatorname{ns}(z|m)}{\operatorname{cs}(z|m)} \quad \operatorname{nc}(z \mid m) = \frac{1}{\operatorname{sn}(z|m)\operatorname{cs}(z|m)} \quad \operatorname{nc}(z \mid m)^2 = 1 + \frac{1}{\operatorname{cs}(z|m)^2}$

$\operatorname{nc}(z \mid m) = \frac{\operatorname{dc}(z|m)}{\operatorname{dn}(z|m)} \quad \operatorname{nc}(z \mid m) = \operatorname{nd}(z \mid m)\operatorname{dc}(z \mid m) \quad \operatorname{nc}(z \mid m)^2 = \frac{\operatorname{dc}(z|m)^2 - m}{1-m}$

$\operatorname{nc}(z \mid m)^2 = \frac{m}{\operatorname{dn}(z|m)^2 + m - 1}$

$\operatorname{nc}(z \mid m)^2 = \frac{\operatorname{ds}(z|m)^2 + m}{\operatorname{ds}(z|m)^2 + m - 1}$

$\operatorname{nc}(z \mid m)^2 = \frac{m\operatorname{nd}(z|m)^2}{1-(1-m)\operatorname{nd}(z|m)^2}$

$\operatorname{nc}(z \mid m) = \operatorname{ns}(z \mid m)\operatorname{sc}(z \mid m) \quad \operatorname{nc}(z \mid m)^2 = \frac{\operatorname{ns}(z|m)^2}{\operatorname{ns}(z|m)^2 - 1}$

$\operatorname{nc}(z \mid m) = \frac{\operatorname{sc}(z|m)}{\operatorname{sn}(z|m)} \quad \operatorname{nc}(z \mid m)^2 = \operatorname{sc}(z \mid m)^2 + 1$

$\operatorname{nc}(z \mid m)^2 = \frac{m\operatorname{sd}(z|m)^2 + 1}{(m-1)\operatorname{sd}(z|m)^2 + 1}$

$\operatorname{nc}(z \mid m)^2 = \frac{1}{1-\operatorname{sn}(z|m)^2}.$

Representations of $\operatorname{nd}(z \mid m)$ through other Jacobi functions are:

$\operatorname{nd}(z \mid m) = \frac{\operatorname{cd}(z|m)}{\operatorname{cn}(z|m)} \quad \operatorname{nd}(z \mid m) = \operatorname{nc}(z \mid m)\operatorname{cd}(z \mid m)$

$\operatorname{nd}(z \mid m) = \operatorname{cd}(i\, z \mid 1-m) \quad \operatorname{nd}(z \mid m)^2 = \frac{m\operatorname{cd}(z|m)^2 - 1}{m-1}$

$\operatorname{nd}(z \mid m) = \frac{1}{\operatorname{cn}(z|m)\operatorname{dc}(z|m)} \quad \operatorname{nd}(z \mid m)^2 = \frac{1}{m\operatorname{cn}(z|m)^2 - m + 1}$

$\operatorname{nd}(z \mid m)^2 = \frac{\operatorname{cs}(z|m)^2 + 1}{\operatorname{cs}(z|m)^2 - m + 1}$

$\operatorname{nd}(z \mid m) = \frac{\operatorname{nc}(z|m)}{\operatorname{dc}(z|m)} \quad \operatorname{nd}(z \mid m)^2 = \frac{m - \operatorname{dc}(z|m)^2}{(m-1)\operatorname{dc}(z|m)^2}$

$\operatorname{nd}(z \mid m) = \frac{1}{\operatorname{dn}(z|m)}$

$\operatorname{nd}(z \mid m) = \frac{\operatorname{ns}(z|m)}{\operatorname{ds}(z|m)} \quad \operatorname{nd}(z \mid m) = \frac{1}{\operatorname{sn}(z|m)\operatorname{ds}(z|m)} \quad \operatorname{nd}(z \mid m)^2 = \frac{\operatorname{ds}(z|m)^2 + m}{\operatorname{ds}(z|m)^2}$

$\operatorname{nd}(z \mid m)^2 = \frac{\operatorname{nc}(z|m)^2}{(1-m)\operatorname{nc}(z|m)^2 + m}$

$\operatorname{nd}(z \mid m) = \operatorname{ns}(z \mid m)\operatorname{sd}(z \mid m) \quad \operatorname{nd}(z \mid m)^2 = \frac{\operatorname{ns}(z|m)^2}{\operatorname{ns}(z|m)^2 - m}$

$\operatorname{nd}(z \mid m)^2 = \frac{\operatorname{sc}(z|m)^2 + 1}{(1-m)\operatorname{sc}(z|m)^2 + 1}$

$\operatorname{nd}(z \mid m) = \frac{\operatorname{sd}(z|m)}{\operatorname{sn}(z|m)} \quad \operatorname{nd}(z \mid m)^2 = m\operatorname{sd}(z \mid m)^2 + 1 \quad \operatorname{nd}(z \mid m) = \frac{1}{\operatorname{dn}(z|m)}$

$\operatorname{nd}(z \mid m)^2 = \frac{1}{1 - m\operatorname{sn}(z|m)^2} \quad \operatorname{nd}(z \mid m) = \frac{1}{\sqrt{1-m}\ \operatorname{sn}(K(1-m) - i\, K(m) - i\, z|1-m)}.$

Representations of $\text{ns}(z \mid m)$ through other Jacobi functions are:

$$\text{ns}(z \mid m)^2 = \frac{m\,\text{cd}(z|m)^2-1}{\text{cd}(z|m)^2-1}$$

$$\text{ns}(z \mid m) = \frac{\text{cs}(z|m)}{\text{cn}(z|m)} \quad \text{ns}(z \mid m) = \frac{1}{\text{cn}(z|m)\,\text{sc}(z|m)} \quad \text{ns}(z \mid m)^2 = \frac{1}{1-\text{cn}(z|m)^2}$$

$$\text{ns}(z \mid m) = \text{nc}(z \mid m)\,\text{cs}(z \mid m) \quad \text{ns}(z \mid m) = i\,\text{cs}(i\,z \mid 1-m)$$

$$\text{ns}(z \mid m)^2 = \frac{m-\text{dc}(z|m)^2}{1-\text{dc}(z|m)^2}$$

$$\text{ns}(z \mid m) = \frac{\text{ds}(z|m)}{\text{dn}(z|m)} \quad \text{ns}(z \mid m) = \frac{1}{\text{dn}(z|m)\,\text{sd}(z|m)} \quad \text{ns}(z \mid m)^2 = \frac{m}{1-\text{dn}(z|m)^2}$$

$$\text{ns}(z \mid m) = \text{nd}(z \mid m)\,\text{ds}(z \mid m) \quad \text{ns}(z \mid m)^2 = \text{ds}(z \mid m)^2 + m$$

$$\text{ns}(z \mid m) = \frac{\text{nc}(z|m)}{\text{sc}(z|m)} \quad \text{ns}(z \mid m)^2 = \frac{\text{nc}(z|m)^2}{\text{nc}(z|m)^2-1}$$

$$\text{ns}(z \mid m) = \frac{\text{nd}(z|m)}{\text{sd}(z|m)} \quad \text{ns}(z \mid m)^2 = \frac{m\,\text{nd}(z|m)^2}{\text{nd}(z|m)^2-1}$$

$$\text{ns}(z \mid m) = \frac{i}{\text{sc}(i\,z|1-m)} \quad \text{ns}(z \mid m)^2 = \frac{\text{sc}(z|m)^2+1}{\text{sc}(z|m)^2}$$

$$\text{ns}(z \mid m)^2 = \frac{m\,\text{sd}(z|m)^2+1}{\text{sd}(z|m)^2}$$

$$\text{ns}(z \mid m) = \frac{1}{\text{sn}(z|m)}\,.$$

Representations of $\text{sc}(z \mid m)$ through other Jacobi functions are:

$$\text{sc}(z \mid m) = \frac{1}{\text{ds}(z|m)\,\text{cd}(z|m)} \quad \text{sc}(z \mid m) = \frac{\text{sd}(z|m)}{\text{cd}(z|m)} \quad \text{sc}(z \mid m)^2 = \frac{\text{cd}(z|m)^2-1}{(m-1)\,\text{cd}(z|m)^2}$$

$$\text{sc}(z \mid m) = \frac{1}{\text{ns}(z|m)\,\text{cn}(z|m)} \quad \text{sc}(z \mid m) = \frac{\text{sn}(z|m)}{\text{cn}(z|m)} \quad \text{sc}(z \mid m)^2 = \frac{1-\text{cn}(z|m)^2}{\text{cn}(z|m)^2}$$

$$\text{sc}(z \mid m) = \frac{1}{\text{cs}(z|m)}$$

$$\text{sc}(z \mid m) = \frac{\text{dc}(z|m)}{\text{ds}(z|m)} \quad \text{sc}(z \mid m) = \text{sd}(z \mid m)\,\text{dc}(z \mid m) \quad \text{sc}(z \mid m)^2 = \frac{1-\text{dc}(z|m)^2}{m-1}$$

$$\text{sc}(z \mid m)^2 = \frac{1-\text{dn}(z|m)^2}{\text{dn}(z|m)^2+m-1}$$

$$\text{sc}(z \mid m)^2 = \frac{1}{\text{ds}(z|m)^2+m-1}$$

$$\text{sc}(z \mid m) = \frac{\text{nc}(z|m)}{\text{ns}(z|m)} \quad \text{sc}(z \mid m) = \text{sn}(z \mid m)\,\text{nc}(z \mid m) \quad \text{sc}(z \mid m)^2 = \text{nc}(z \mid m)^2 - 1$$

$$\text{sc}(z \mid m)^2 = \frac{\text{nd}(z|m)^2-1}{(m-1)\,\text{nd}(z|m)^2+1}$$

$$\text{sc}(z \mid m) = -\frac{i}{\text{ns}(i\,z|1-m)} \quad \text{sc}(z \mid m)^2 = \frac{1}{\text{ns}(z|m)^2-1}$$

$\operatorname{sc}(z \mid m)^2 == \frac{\operatorname{sd}(z|m)^2}{(m-1)\operatorname{sd}(z|m)^2+1}$

$\operatorname{sc}(z \mid m) == -i\operatorname{sn}(i\,z \mid 1-m) \quad \operatorname{sc}(z \mid m)^2 == \frac{\operatorname{sn}(z|m)^2}{1-\operatorname{sn}(z|m)^2}$ .

Representations of $\operatorname{sd}(z \mid m)$ through other Jacobi functions are:

$\operatorname{sd}(z \mid m) == \frac{\operatorname{cd}(z|m)}{\operatorname{cs}(z|m)} \quad \operatorname{sd}(z \mid m) == \operatorname{sc}(z \mid m)\operatorname{cd}(z \mid m) \quad \operatorname{sd}(z \mid m)^2 == \frac{1-\operatorname{cd}(z|m)^2}{1-m}$

$\operatorname{sd}(z \mid m)^2 == \frac{1-\operatorname{cn}(z|m)^2}{m\operatorname{cn}(z|m)^2-m+1}$

$\operatorname{sd}(z \mid m) == \frac{1}{\operatorname{cs}(z|m)\operatorname{dc}(z|m)} \quad \operatorname{sd}(z \mid m)^2 == \frac{1}{\operatorname{cs}(z|m)^2-m+1}$

$\operatorname{sd}(z \mid m) == \frac{\operatorname{sc}(z|m)}{\operatorname{dc}(z|m)} \quad \operatorname{sd}(z \mid m)^2 == \frac{1-\operatorname{dc}(z|m)^2}{(m-1)\operatorname{dc}(z|m)^2}$

$\operatorname{sd}(z \mid m) == \frac{1}{\operatorname{ns}(z|m)\operatorname{dn}(z|m)} \quad \operatorname{sd}(z \mid m) == \frac{\operatorname{sn}(z|m)}{\operatorname{dn}(z|m)} \quad \operatorname{sd}(z \mid m)^2 == \frac{1-\operatorname{dn}(z|m)^2}{m\operatorname{dn}(z|m)^2}$

$\operatorname{sd}(z \mid m) == \frac{1}{\operatorname{ds}(z|m)}$

$\operatorname{sd}(z \mid m)^2 == \frac{1-\operatorname{nc}(z|m)^2}{(m-1)\operatorname{nc}(z|m)^2-m}$

$\operatorname{sd}(z \mid m) == \frac{\operatorname{nd}(z|m)}{\operatorname{ns}(z|m)} \quad \operatorname{sd}(z \mid m) == \operatorname{sn}(z \mid m)\operatorname{nd}(z \mid m) \quad \operatorname{sd}(z \mid m)^2 == \frac{\operatorname{nd}(z|m)^2-1}{m}$

$\operatorname{sd}(z \mid m)^2 == \frac{1}{\operatorname{ns}(z|m)^2-m}$

$\operatorname{sd}(z \mid m)^2 == \frac{\operatorname{sc}(z|m)^2}{1-(m-1)\operatorname{sc}(z|m)^2}$

$\operatorname{sd}(z \mid m)^2 == \frac{\operatorname{sn}(z|m)^2}{1-m\operatorname{sn}(z|m)^2}$ .

Representations of $\operatorname{sn}(z \mid m)$ through other Jacobi functions are:

$\operatorname{sn}(z \mid m)^2 == \frac{\operatorname{cd}(z|m)^2-1}{m\operatorname{cd}(z|m)^2-1}$

$\operatorname{sn}(z \mid m) == \frac{\operatorname{cn}(z|m)}{\operatorname{cs}(z|m)} \quad \operatorname{sn}(z \mid m) == \operatorname{sc}(z \mid m)\operatorname{cn}(z \mid m) \quad \operatorname{sn}(z \mid m)^2 == 1-\operatorname{cn}(z \mid m)^2$

$\operatorname{sn}(z \mid m) == \frac{1}{\operatorname{cs}(z|m)\operatorname{nc}(z|m)} \quad \operatorname{sn}(z \mid m) == -\frac{i}{\operatorname{cs}(i\,z|1-m)}$

$\operatorname{sn}(z \mid m)^2 == \frac{1-\operatorname{dc}(z|m)^2}{m-\operatorname{dc}(z|m)^2}$

$\operatorname{sn}(z \mid m) == \frac{\operatorname{dn}(z|m)}{\operatorname{ds}(z|m)} \quad \operatorname{sn}(z \mid m) == \operatorname{sd}(z \mid m)\operatorname{dn}(z \mid m) \quad \operatorname{sn}(z \mid m)^2 == \frac{1-\operatorname{dn}(z|m)^2}{m}$

$\operatorname{sn}(z \mid m) == \frac{1}{\operatorname{ds}(z|m)\operatorname{nd}(z|m)} \quad \operatorname{sn}(z \mid m)^2 == \frac{1}{\operatorname{ds}(z|m)^2+m}$

$\operatorname{sn}(z \mid m) == \frac{\operatorname{sc}(z|m)}{\operatorname{nc}(z|m)} \quad \operatorname{sn}(z \mid m)^2 == 1-\frac{1}{\operatorname{nc}(z|m)^2}$

$\mathrm{sn}(z \mid m) = \frac{\mathrm{sd}(z|m)}{\mathrm{nd}(z|m)} \quad \mathrm{sn}(z \mid m)^2 = \frac{\mathrm{nd}(z|m)^2 - 1}{m\,\mathrm{nd}(z|m)^2}$

$\mathrm{sn}(z \mid m) = \frac{1}{\mathrm{ns}(z|m)}$

$\mathrm{sn}(z \mid m) = -i\,\mathrm{sc}(i\,z \mid 1 - m) \quad \mathrm{sn}(z \mid m)^2 = \frac{\mathrm{sc}(z|m)^2}{\mathrm{sc}(z|m)^2 + 1}$

$\mathrm{sn}(z \mid m)^2 = \frac{\mathrm{sd}(z|m)^2}{m\,\mathrm{sd}(z|m)^2 + 1}$ .

## The best-known properties and formulas for Jacobi functions

### Real values for real arguments

For real values of arguments $z$ and $m$, the values of all Jacobi functions $\mathrm{am}(z \mid m)$, $\mathrm{cd}(z \mid m)$, $\mathrm{cn}(z \mid m)$, $\mathrm{cs}(z \mid m)$, $\mathrm{dc}(z \mid m)$, $\mathrm{dn}(z \mid m)$, $\mathrm{ds}(z \mid m)$, $\mathrm{nc}(z \mid m)$, $\mathrm{nd}(z \mid m)$, $\mathrm{ns}(z \mid m)$, $\mathrm{sc}(z \mid m)$, $\mathrm{sd}(z \mid m)$, and $\mathrm{sn}(z \mid m)$ are real (or infinity).

### Simple values at zero

All thirteen Jacobi functions $\mathrm{am}(z \mid m)$, $\mathrm{cd}(z \mid m)$, $\mathrm{cn}(z \mid m)$, $\mathrm{cs}(z \mid m)$, $\mathrm{dc}(z \mid m)$, $\mathrm{dn}(z \mid m)$, $\mathrm{ds}(z \mid m)$, $\mathrm{nc}(z \mid m)$, $\mathrm{nd}(z \mid m)$, $\mathrm{ns}(z \mid m)$, $\mathrm{sc}(z \mid m)$, $\mathrm{sd}(z \mid m)$, and $\mathrm{sn}(z \mid m)$ have the following simple values at the origin:

$\mathrm{am}(0 \mid 0) = 0$

$\mathrm{cd}(0 \mid 0) = 1 \quad \mathrm{cn}(0 \mid 0) = 1 \quad \mathrm{cs}(0 \mid 0) = \tilde{\infty}$
$\mathrm{dc}(0 \mid 0) = 1 \quad \mathrm{dn}(0 \mid 0) = 1 \quad \mathrm{ds}(0 \mid 0) = \tilde{\infty}$
$\mathrm{nc}(0 \mid 0) = 1 \quad \mathrm{nd}(0 \mid 0) = 1 \quad \mathrm{ns}(0 \mid 0) = \tilde{\infty}$
$\mathrm{sc}(0 \mid 0) = 0 \quad \mathrm{sd}(0 \mid 0) = 0 \quad \mathrm{sn}(0 \mid 0) = 0.$

### Specific values for specialized parameter values

All Jacobi functions $\mathrm{am}(z \mid m)$, $\mathrm{cd}(z \mid m)$, $\mathrm{cn}(z \mid m)$, $\mathrm{cs}(z \mid m)$, $\mathrm{dc}(z \mid m)$, $\mathrm{dn}(z \mid m)$, $\mathrm{ds}(z \mid m)$, $\mathrm{nc}(z \mid m)$, $\mathrm{nd}(z \mid m)$, $\mathrm{ns}(z \mid m)$, $\mathrm{sc}(z \mid m)$, $\mathrm{sd}(z \mid m)$, and $\mathrm{sn}(z \mid m)$ can be represented through elementary functions when $m = 0$ or $m = 1$. The twelve elliptic functions degenerate into trigonometric and
hyperbolic functions:

$\mathrm{am}(z \mid 0) = z \quad \mathrm{am}(z \mid 1) = 2\tan^{-1}(e^z) - \frac{\pi}{2} \quad \mathrm{am}(K(m) \mid m) = \frac{\pi}{2}$

$\operatorname{cd}(z \mid 0) = \cos(z)$ $\operatorname{cd}\left(z + \frac{\pi}{2} \,\middle|\, 0\right) = -\sin(z)$ $\operatorname{cd}(z \mid 1) = 1$

$\operatorname{cn}(z \mid 0) = \cos(z)$ $\operatorname{cn}\left(z + \frac{\pi}{2} \,\middle|\, 0\right) = -\sin(z)$ $\operatorname{cn}(z \mid 1) = \operatorname{sech}(z)$

$\operatorname{cs}(z \mid 0) = \cot(z)$ $\operatorname{cs}\left(z + \frac{\pi}{2} \,\middle|\, 0\right) = -\tan(z)$ $\operatorname{cs}(z \mid 1) = \operatorname{csch}(z)$

$\operatorname{dc}(z \mid 0) = \sec(z)$ $\operatorname{dc}\left(z + \frac{\pi}{2} \,\middle|\, 0\right) = -\csc(z)$ $\operatorname{dc}(z \mid 1) = 1$

$\operatorname{dn}(z \mid 0) = 1$ $\operatorname{dn}(z \mid 1) = \operatorname{sech}(z)$ $\operatorname{dn}\left(z + \frac{\pi i}{2} \,\middle|\, 1\right) = -i \operatorname{csch}(z)$

$\operatorname{ds}(z \mid 0) = \csc(z)$ $\operatorname{ds}\left(z + \frac{\pi}{2} \,\middle|\, 0\right) = \sec(z)$ $\operatorname{ds}\left(z + \frac{\pi i}{2} \,\middle|\, 1\right) = -i \operatorname{sech}(z)$

$\operatorname{nc}(z \mid 0) = \sec(z)$ $\operatorname{nc}\left(z + \frac{\pi}{2} \,\middle|\, 0\right) = -\csc(z)$ $\operatorname{nc}(z \mid 1) = \cosh(z)$

$\operatorname{nd}(z \mid 0) = 1$ $\operatorname{nd}(z \mid 1) = \cosh(z)$ $\operatorname{nd}\left(z + \frac{\pi i}{2} \,\middle|\, 1\right) = i \sinh(z)$

$\operatorname{ns}(z \mid 0) = \csc(z)$ $\operatorname{ns}\left(z + \frac{\pi}{2} \,\middle|\, 0\right) = \sec(z)$ $\operatorname{ns}(z \mid 1) = \coth(z)$

$\operatorname{sc}(z \mid 0) = \tan(z)$ $\operatorname{sc}\left(z + \frac{\pi}{2} \,\middle|\, 0\right) = -\cot(z)$ $\operatorname{sc}(z \mid 1) = \sinh(z)$

$\operatorname{sd}(z \mid 0) = \sin(z)$ $\operatorname{sd}\left(z + \frac{\pi}{2} \,\middle|\, 0\right) = \cos(z)$ $\operatorname{sd}(z \mid 1) = \sinh(z)$

$\operatorname{sn}(z \mid 0) = \sin(z)$ $\operatorname{sn}\left(z + \frac{\pi}{2} \,\middle|\, 0\right) = \cos(z)$ $\operatorname{sn}(z \mid 1) = \tanh(z).$

All Jacobi functions $\operatorname{am}(z \mid m)$, $\operatorname{cd}(z \mid m)$, $\operatorname{cn}(z \mid m)$, $\operatorname{cs}(z \mid m)$, $\operatorname{dc}(z \mid m)$, $\operatorname{dn}(z \mid m)$, $\operatorname{ds}(z \mid m)$, $\operatorname{nc}(z \mid m)$, $\operatorname{nd}(z \mid m)$, $\operatorname{ns}(z \mid m)$, $\operatorname{sc}(z \mid m)$, $\operatorname{sd}(z \mid m)$, and $\operatorname{sn}(z \mid m)$ have very simple values at $z = 0$:

$\operatorname{am}(0 \mid m) = 0$

$\operatorname{cd}(0 \mid m) = 1$ $\operatorname{cn}(0 \mid m) = 1$ $\operatorname{cs}(0 \mid m) = \tilde{\infty}$

$\operatorname{dc}(0 \mid m) = 1$ $\operatorname{dn}(0 \mid m) = 1$ $\operatorname{ds}(0 \mid m) = \tilde{\infty}$

$\operatorname{nc}(0 \mid m) = 1$ $\operatorname{nd}(0 \mid m) = 1$ $\operatorname{ns}(0 \mid m) = \tilde{\infty}$

$\operatorname{sc}(0 \mid m) = 0$ $\operatorname{sd}(0 \mid m) = 0$ $\operatorname{sn}(0 \mid m) = 0.$

The twelve Jacobi functions $\operatorname{cd}(z \mid m)$, $\operatorname{cn}(z \mid m)$, $\operatorname{cs}(z \mid m)$, $\operatorname{dc}(z \mid m)$, $\operatorname{dn}(z \mid m)$, $\operatorname{ds}(z \mid m)$, $\operatorname{nc}(z \mid m)$, $\operatorname{nd}(z \mid m)$, $\operatorname{ns}(z \mid m)$, $\operatorname{sc}(z \mid m)$, $\operatorname{sd}(z \mid m)$, and $\operatorname{sn}(z \mid m)$ have the following values at the half-quarter-period points:

$$\operatorname{cd}\left(\frac{K(m)}{2} \,\middle|\, m\right) = \frac{1}{\sqrt{1+\sqrt{1-m}}} \qquad \operatorname{cn}\left(\frac{K(m)}{2} \,\middle|\, m\right) = \frac{\sqrt[4]{1-m}}{\sqrt{1+\sqrt{1-m}}}$$

$$\operatorname{cs}\left(\frac{K(m)}{2} \,\middle|\, m\right) = \sqrt[4]{1-m} \qquad \operatorname{dc}\left(\frac{K(m)}{2} \,\middle|\, m\right) = \sqrt{1+\sqrt{1-m}}$$

$$\operatorname{dn}\left(\frac{K(m)}{2} \,\middle|\, m\right) = \sqrt[4]{1-m} \qquad \operatorname{ds}\left(\frac{K(m)}{2} \,\middle|\, m\right) = \sqrt[4]{1-m}\,\sqrt{1+\sqrt{1-m}}$$

$$\operatorname{nc}\left(\frac{K(m)}{2} \,\middle|\, m\right) = \frac{\sqrt{1+\sqrt{1-m}}}{\sqrt[4]{1-m}} \qquad \operatorname{nd}\left(\frac{K(m)}{2} \,\middle|\, m\right) = \frac{1}{\sqrt[4]{1-m}}$$

$$\operatorname{ns}\left(\frac{K(m)}{2} \,\middle|\, m\right) = \sqrt{1+\sqrt{1-m}} \qquad \operatorname{sc}\left(\frac{K(m)}{2} \,\middle|\, m\right) = \frac{1}{\sqrt[4]{1-m}}$$

$$\operatorname{sd}\left(\frac{K(m)}{2} \,\middle|\, m\right) = \frac{1}{\sqrt[4]{1-m}\,\sqrt{1+\sqrt{1-m}}} \qquad \operatorname{sn}\left(\frac{K(m)}{2} \,\middle|\, m\right) = \frac{1}{\sqrt{1+\sqrt{1-m}}}.$$

The partial derivatives of all Jacobi functions $\operatorname{am}(z \mid m)$, $\operatorname{cd}(z \mid m)$, $\operatorname{cn}(z \mid m)$, $\operatorname{cs}(z \mid m)$, $\operatorname{dc}(z \mid m)$, $\operatorname{dn}(z \mid m)$, $\operatorname{ds}(z \mid m)$, $\operatorname{nc}(z \mid m)$, $\operatorname{nd}(z \mid m)$, $\operatorname{ns}(z \mid m)$, $\operatorname{sc}(z \mid m)$, $\operatorname{sd}(z \mid m)$, and $\operatorname{sn}(z \mid m)$ at the points $z = 0$, $m = 0$, or $m = 1$ can be represented through trigonometric functions, for example:

$\text{am}^{(1,0)}(0, m) == 1$ $\text{am}^{(3,0)}(0, m) == -m$

$\text{am}^{(0,1)}(z, 0) == \frac{1}{8}(\sin(2z) - 2z)$ $\text{am}^{(0,2)}(z, 0) == \frac{1}{128}(-16\cos(2z)z - 20z + 16\sin(2z) + \sin(4z))$

$\text{am}^{(0,1)}(z, 1) == \frac{1}{4}(z\,\text{sech}(z) - \sinh(z))$ $\text{am}^{(0,2)}(z, 1) == \frac{1}{32}(-4z\cosh(z) + 9\sinh(z) - z\,\text{sech}(z)(2z\tanh(z) + 5))$.

### Analyticity

All Jacobi functions $\text{am}(z \mid m)$, $\text{cd}(z \mid m)$, $\text{cn}(z \mid m)$, $\text{cs}(z \mid m)$, $\text{dc}(z \mid m)$, $\text{dn}(z \mid m)$, $\text{ds}(z \mid m)$, $\text{nc}(z \mid m)$, $\text{nd}(z \mid m)$, $\text{ns}(z \mid m)$, $\text{sc}(z \mid m)$, $\text{sd}(z \mid m)$, and $\text{sn}(z \mid m)$ are analytical meromorphic functions of $z$ and $m$ that are defined over $\mathbb{C}^2$.

### Poles and essential singularities

The amplitude function $\text{am}(z \mid m)$ does not have poles and essential singularities with respect to $m$ and $z$.

For fixed $m$, all Jacobi functions $\text{cd}(z \mid m)$, $\text{cn}(z \mid m)$, $\text{cs}(z \mid m)$, $\text{dc}(z \mid m)$, $\text{dn}(z \mid m)$, $\text{ds}(z \mid m)$, $\text{nc}(z \mid m)$, $\text{nd}(z \mid m)$, $\text{ns}(z \mid m)$, $\text{sc}(z \mid m)$, $\text{sd}(z \mid m)$, and $\text{sn}(z \mid m)$ have an infinite set of singular points, including simple poles in finite points and an essential singular point $z == \tilde{\infty}$.

The following formulas describe the sets of the simple poles for the corresponding Jacobi functions:

$\text{cd}((2r+1)K(m) + (2s+1)iK(1-m) \mid m) == \tilde{\infty} /; \{r, s\} \in \mathbb{Z}$

$\text{cn}(2rK(m) + (2s+1)iK(1-m) \mid m) == \tilde{\infty} /; \{r, s\} \in \mathbb{Z}$

$\text{cs}(2rK(m) + 2siK(1-m) \mid m) == \tilde{\infty} /; \{r, s\} \in \mathbb{Z}$

$\text{dc}((2r+1)K(m) + 2isK(1-m) \mid m) == \tilde{\infty} /; \{r, s\} \in \mathbb{Z}$

$\text{dn}(2rK(m) + i(2s+1)K(1-m) \mid m) == \tilde{\infty} /; \{r, s\} \in \mathbb{Z}$

$\text{ds}(2rK(m) + 2siK(1-m) \mid m) == \tilde{\infty} /; \{r, s\} \in \mathbb{Z}$

$\text{nc}((2r+1)K(m) + 2isK(1-m) \mid m) == \tilde{\infty} /; \{r, s\} \in \mathbb{Z}$

$\text{nd}((2r+1)K(m) + i(2s+1)K(1-m) \mid m) == \tilde{\infty} /; \{r, s\} \in \mathbb{Z}$

$\text{ns}(2rK(m) + 2isK(1-m) \mid m) == \tilde{\infty} /; \{r, s\} \in \mathbb{Z}$

$\text{sc}((2r+1)K(m) + 2siK(1-m) \mid m) == \tilde{\infty} /; \{r, s\} \in \mathbb{Z}$

$\text{sd}((2r+1)K(m) + (2s+1)iK(1-m) \mid m) == \tilde{\infty} /; \{r, s\} \in \mathbb{Z}$

$\text{sn}(2rK(m) + (2s+1)iK(1-m) \mid m) == \tilde{\infty} /; \{r, s\} \in \mathbb{Z}$.

The values of the residues of the Jacobi functions at the simple poles are given by the following formulas:

$$\text{res}_z(\text{cd}(z \mid m))\,((2s+1)\,i\,K(1-m)+(2r+1)\,K(m)) = \frac{(-1)^{r-1}}{\sqrt{m}} \;/;\; \{r, s\} \in \mathbb{Z}$$

$$\text{res}_z(\text{cn}(z \mid m))\,((2s+1)\,i\,K(1-m)+2\,r\,K(m)) = \frac{(-1)^{r+s-1}\,i}{\sqrt{m}} \;/;\; \{r, s\} \in \mathbb{Z}$$

$$\text{res}_z(\text{cs}(z \mid m))\,(2\,s\,i\,K(1-m)+2\,r\,K(m)) = (-1)^{s} \;/;\; \{r, s\} \in \mathbb{Z}$$

$$\text{res}_z(\text{dc}(z \mid m))\,(2\,s\,i\,K(1-m)+(2r+1)\,K(m)) = (-1)^{r-1} \;/;\; \{r, s\} \in \mathbb{Z}$$

$$\text{res}_z(\text{dn}(z \mid m))\,((2s+1)\,i\,K(1-m)+2\,r\,K(m)) = (-1)^{s-1}\,i \;/;\; \{r, s\} \in \mathbb{Z}$$

$$\text{res}_z(\text{ds}(z \mid m))\,(2\,s\,i\,K(1-m)+2\,r\,K(m)) = (-1)^{r+s} \;/;\; \{r, s\} \in \mathbb{Z}$$

$$\text{res}_z(\text{nc}(z \mid m))\,(2\,s\,i\,K(1-m)+(2r+1)\,K(m)) = \frac{(-1)^{r+s-1}}{\sqrt{1-m}} \;/;\; \{r, s\} \in \mathbb{Z}$$

$$\text{res}_z(\text{nd}(z \mid m))\,((2s+1)\,i\,K(1-m)+(2r+1)\,K(m)) = \frac{(-1)^{s-1}\,i}{\sqrt{1-m}} \;/;\; \{r, s\} \in \mathbb{Z}$$

$$\text{res}_z(\text{ns}(z \mid m))\,(2\,s\,i\,K(1-m)+2\,r\,K(m)) = (-1)^{r} \;/;\; \{r, s\} \in \mathbb{Z}$$

$$\text{res}_z(\text{sc}(z \mid m))\,(2\,s\,i\,K(1-m)+(2r+1)\,K(m)) = \frac{(-1)^{s-1}}{\sqrt{1-m}} \;/;\; \{r, s\} \in \mathbb{Z}$$

$$\text{res}_z(\text{sd}(z \mid m))\,((2s+1)\,i\,K(1-m)+(2r+1)\,K(m)) = \frac{(-1)^{r+s-1}\,i}{\sqrt{m}\,\sqrt{1-m}} \;/;\; \{r, s\} \in \mathbb{Z}$$

$$\text{res}_z(\text{sn}(z \mid m))\,((2s+1)\,i\,K(1-m)+2\,r\,K(m)) = \frac{(-1)^{r}}{\sqrt{m}} \;/;\; \{r, s\} \in \mathbb{Z}.$$

### Branch points and branch cuts

For fixed $z$, all Jacobi functions $\text{cd}(z \mid m)$, $\text{cn}(z \mid m)$, $\text{cs}(z \mid m)$, $\text{dc}(z \mid m)$, $\text{dn}(z \mid m)$, $\text{ds}(z \mid m)$, $\text{nc}(z \mid m)$, $\text{nd}(z \mid m)$, $\text{ns}(z \mid m)$, $\text{sc}(z \mid m)$, $\text{sd}(z \mid m)$, and $\text{sn}(z \mid m)$ are meromorphic functions in $m$ that have no branch points and branch cuts.

For fixed $m$, all Jacobi functions $\text{cd}(z \mid m)$, $\text{cn}(z \mid m)$, $\text{cs}(z \mid m)$, $\text{dc}(z \mid m)$, $\text{dn}(z \mid m)$, $\text{ds}(z \mid m)$, $\text{nc}(z \mid m)$, $\text{nd}(z \mid m)$, $\text{ns}(z \mid m)$, $\text{sc}(z \mid m)$, $\text{sd}(z \mid m)$, and $\text{sn}(z \mid m)$ do not have branch points and branch cuts.

### Periodicity

The Jacobi amplitude $\text{am}(z \mid m)$ is a pseudo-periodic function with respect to $z$ with period $2\,i\,K(1-m)$ and pseudo-period $2\,K(m)$:

$$\text{am}(z+2\,r\,K(m)+2\,i\,s\,K(1-m) \mid m) = \text{am}(z \mid m)+r\,\pi \;/;\; \{r, s\} \in \mathbb{Z} \land 0 \le m < 1.$$

The Jacobi functions $\mathrm{cd}(z \mid m)$, $\mathrm{dc}(z \mid m)$, $\mathrm{ns}(z \mid m)$, and $\mathrm{sn}(z \mid m)$ are doubly periodic functions with respect to $z$ with periods $2\,i\,K(1-m)$ and $4\,K(m)$. The Jacobi functions $\mathrm{cs}(z \mid m)$, $\mathrm{dn}(z \mid m)$, $\mathrm{nd}(z \mid m)$, and $\mathrm{sc}(z \mid m)$ are doubly periodic functions with respect to $z$ with periods $4\,i\,K(1-m)$ and $2\,K(m)$. The Jacobi functions $\mathrm{cn}(z \mid m)$, $\mathrm{ds}(z \mid m)$, $\mathrm{nc}(z \mid m)$, and $\mathrm{sd}(z \mid m)$ are doubly periodic functions with respect to $z$ with periods $4\,i\,K(1-m)$ and $4\,K(m)$. That periodicity can be described by the following formulas:

$$\mathrm{cd}(z+4\,K(m) \mid m) = \mathrm{cd}(z \mid m) \quad \mathrm{cd}(z+2\,i\,K(1-m) \mid m) = \mathrm{cd}(z \mid m)$$

$$\mathrm{cn}(z+4\,K(m) \mid m) = \mathrm{cn}(z \mid m) \quad \mathrm{cn}(z+4\,i\,K(1-m) \mid m) = \mathrm{cn}(z \mid m)$$

$$\mathrm{cs}(z+2\,K(m) \mid m) = \mathrm{cs}(z \mid m) \quad \mathrm{cs}(z+4\,i\,K(1-m) \mid m) = \mathrm{cs}(z \mid m)$$

$$\mathrm{dc}(z+4\,K(m) \mid m) = \mathrm{dc}(z \mid m) \quad \mathrm{dc}(z+2\,i\,K(1-m) \mid m) = \mathrm{dc}(z \mid m)$$

$$\mathrm{dn}(z+2\,K(m) \mid m) = \mathrm{dn}(z \mid m) \quad \mathrm{dn}(z+4\,i\,K(1-m) \mid m) = \mathrm{dn}(z \mid m)$$

$$\mathrm{ds}(z+4\,K(m) \mid m) = \mathrm{ds}(z \mid m) \quad \mathrm{ds}(z+4\,i\,K(1-m) \mid m) = \mathrm{ds}(z \mid m)$$

$$\mathrm{nc}(z+4\,K(m) \mid m) = \mathrm{nc}(z \mid m) \quad \mathrm{nc}(z+4\,i\,K(1-m) \mid m) = \mathrm{nc}(z \mid m)$$

$$\mathrm{nd}(z+2\,K(m) \mid m) = \mathrm{nd}(z \mid m) \quad \mathrm{nd}(z+4\,i\,K(1-m) \mid m) = \mathrm{nd}(z \mid m)$$

$$\mathrm{ns}(z+4\,K(m) \mid m) = \mathrm{ns}(z \mid m) \quad \mathrm{ns}(z+2\,i\,K(1-m) \mid m) = \mathrm{ns}(z \mid m)$$

$$\mathrm{sc}(z+2\,K(m) \mid m) = \mathrm{sc}(z \mid m) \quad \mathrm{sc}(z+4\,i\,K(1-m) \mid m) = \mathrm{sc}(z \mid m)$$

$$\mathrm{sd}(z+4\,K(m) \mid m) = \mathrm{sd}(z \mid m) \quad \mathrm{sd}(z+4\,i\,K(1-m) \mid m) = \mathrm{sd}(z \mid m)$$

$$\mathrm{sn}(z+4\,K(m) \mid m) = \mathrm{sn}(z \mid m) \quad \mathrm{sn}(z+2\,i\,K(1-m) \mid m) = \mathrm{sn}(z \mid m).$$

The periodicity of Jacobi functions follow from more general formulas that also describe quasi-periodicity situations such as $\mathrm{sn}(z+2\,K(m) \mid m) = -\mathrm{sn}(z \mid m)$:

$$\mathrm{cd}(z+2\,r\,K(m)+2\,s\,i\,K(1-m) \mid m) = (-1)^r\,\mathrm{cd}(z \mid m) \;/;\; \{r,\,s\} \in \mathbb{Z}$$

$$\mathrm{cn}(z+2\,i\,s\,K(1-m)+2\,r\,K(m) \mid m) = (-1)^{r+s}\,\mathrm{cn}(z \mid m) \;/;\; \{r,\,s\} \in \mathbb{Z}$$

$$\mathrm{cs}(z+2\,i\,s\,K(1-m)+2\,r\,K(m) \mid m) = (-1)^{s}\,\mathrm{cs}(z \mid m) \;/;\; \{r,\,s\} \in \mathbb{Z}$$

$$\mathrm{dc}(z+2\,i\,s\,K(1-m)+2\,r\,K(m) \mid m) = (-1)^{r}\,\mathrm{dc}(z \mid m) \;/;\; \{r,\,s\} \in \mathbb{Z}$$

$$\mathrm{dn}(z+2\,i\,s\,K(1-m)+2\,r\,K(m) \mid m) = (-1)^{s}\,\mathrm{dn}(z \mid m) \;/;\; \{r,\,s\} \in \mathbb{Z}$$

$$\mathrm{ds}(z+2\,i\,s\,K(1-m)+2\,r\,K(m) \mid m) = (-1)^{r+s}\,\mathrm{ds}(z \mid m) \;/;\; \{r,\,s\} \in \mathbb{Z}$$

$$\mathrm{nc}(z+2\,i\,s\,K(1-m)+2\,r\,K(m) \mid m) = (-1)^{r+s}\,\mathrm{nc}(z \mid m) \;/;\; \{r,\,s\} \in \mathbb{Z}$$

$$\mathrm{nd}(z+2\,i\,s\,K(1-m)+2\,r\,K(m) \mid m) = (-1)^{s}\,\mathrm{nd}(z \mid m) \;/;\; \{r,\,s\} \in \mathbb{Z}$$

$$\mathrm{ns}(z+2\,i\,s\,K(1-m)+2\,r\,K(m) \mid m) = (-1)^{r}\,\mathrm{ns}(z \mid m) \;/;\; \{r,\,s\} \in \mathbb{Z}$$

$$\mathrm{sc}(z+2\,i\,s\,K(1-m)+2\,r\,K(m) \mid m) = (-1)^{s}\,\mathrm{sc}(z \mid m) \;/;\; \{r,\,s\} \in \mathbb{Z}$$

$$\mathrm{sd}(z+2\,i\,s\,K(1-m)+2\,r\,K(m) \mid m) = (-1)^{r+s}\,\mathrm{sd}(z \mid m) \;/;\; \{r,\,s\} \in \mathbb{Z}$$

$\text{sn}(z + 2\,i\,s\,K(1-m) + 2\,r\,K(m) \mid m) = (-1)^r\,\text{sn}(z \mid m) \;/;\; \{r, s\} \in \mathbb{Z}.$

### Parity and symmetry

All Jacobi functions $\text{am}(z \mid m)$, $\text{cd}(z \mid m)$, $\text{cn}(z \mid m)$, $\text{cs}(z \mid m)$, $\text{dc}(z \mid m)$, $\text{dn}(z \mid m)$, $\text{ds}(z \mid m)$, $\text{nc}(z \mid m)$, $\text{nd}(z \mid m)$, $\text{ns}(z \mid m)$, $\text{sc}(z \mid m)$, $\text{sd}(z \mid m)$, and $\text{sn}(z \mid m)$ have mirror symmetry:

$$\text{am}(\bar{z} \mid \bar{m}) = \overline{\text{am}(z \mid m)}$$

$$\text{cd}(\bar{z} \mid \bar{m}) = \overline{\text{cd}(z \mid m)} \quad \text{cn}(\bar{z} \mid \bar{m}) = \overline{\text{cn}(z \mid m)} \quad \text{cs}(\bar{z} \mid \bar{m}) = \overline{\text{cs}(z \mid m)}$$
$$\text{dc}(\bar{z} \mid \bar{m}) = \overline{\text{dc}(z \mid m)} \quad \text{dn}(\bar{z} \mid \bar{m}) = \overline{\text{dn}(z \mid m)} \quad \text{ds}(\bar{z} \mid \bar{m}) = \overline{\text{ds}(z \mid m)}$$
$$\text{nc}(\bar{z} \mid \bar{m}) = \overline{\text{nc}(z \mid m)} \quad \text{nd}(\bar{z} \mid \bar{m}) = \overline{\text{nd}(z \mid m)} \quad \text{ns}(\bar{z} \mid \bar{m}) = \overline{\text{ns}(z \mid m)}$$
$$\text{sc}(\bar{z} \mid \bar{m}) = \overline{\text{sc}(z \mid m)} \quad \text{sd}(\bar{z} \mid \bar{m}) = \overline{\text{sd}(z \mid m)} \quad \text{sn}(\bar{z} \mid \bar{m}) = \overline{\text{sn}(z \mid m)}.$$

The Jacobi functions $\text{cd}(z \mid m)$, $\text{cn}(z \mid m)$, $\text{dc}(z \mid m)$, $\text{dn}(z \mid m)$, $\text{nc}(z \mid m)$, and $\text{nd}(-z \mid m)$ are even functions with respect to $z$:

$$\text{cd}(-z \mid m) = \text{cd}(z \mid m) \quad \text{cn}(-z \mid m) = \text{cn}(z \mid m)$$

$$\text{dc}(-z \mid m) = \text{dc}(z \mid m) \quad \text{dn}(-z \mid m) = \text{dn}(z \mid m)$$

$$\text{nc}(-z \mid m) = \text{nc}(z \mid m) \quad \text{nd}(-z \mid m) = \text{nd}(z \mid m).$$

The Jacobi functions $\text{am}(z \mid m)$, $\text{cs}(z \mid m)$, $\text{ds}(z \mid m)$, $\text{ns}(z \mid m)$, $\text{sc}(z \mid m)$, $\text{sd}(z \mid m)$, and $\text{sn}(z \mid m)$ are odd functions with respect to $z$:

$$\text{am}(-z \mid m) = -\text{am}(z \mid m)$$

$$\text{cs}(-z \mid m) = -\text{cs}(z \mid m) \quad \text{ds}(-z \mid m) = -\text{ds}\,(z \mid m)$$

$$\text{ns}(-z \mid m) = -\text{ns}\,(z \mid m) \quad \text{sc}(-z \mid m) = -\text{sc}\,(z \mid m)$$

$$\text{sd}(-z \mid m) = -\text{sd}\,(z \mid m) \quad \text{sn}(-z \mid m) = -\text{sn}\,(z \mid m).$$

### Series representations

The Jacobi functions $\text{am}(z \mid m)$, $\text{cd}(z \mid m)$, $\text{cn}(z \mid m)$, $\text{cs}(z \mid m)$, $\text{dc}(z \mid m)$, $\text{dn}(z \mid m)$, $\text{ds}(z \mid m)$, $\text{nc}(z \mid m)$, $\text{nd}(z \mid m)$, $\text{ns}(z \mid m)$, $\text{sc}(z \mid m)$, $\text{sd}(z \mid m)$, and $\text{sn}(z \mid m)$ have the following series expansions at the point $z = 0$:

$$\text{am}(z \mid m) \propto z - \frac{m}{6} z^3 + \frac{1}{120}\left(4\,m + m^2\right) z^5 + \ldots \;/;\; (z \to 0)$$

$$\text{cd}(z \mid m) \propto 1 + \frac{1}{2}(m-1)\,z^2 + \frac{1}{24}\left(1 - 6\,m + 5\,m^2\right) z^4 + \ldots \;/;\; (z \to 0)$$

$$\text{cn}(z \mid m) \propto 1 - \frac{z^2}{2} + \frac{1}{24}(1 + 4\,m)\,z^4 + \ldots \;/;\; (z \to 0)$$

$$\text{cs}(z \mid m) \propto \frac{1}{z} + \frac{1}{6}(m-2)\,z + \frac{1}{360}\left(7\,m^2 + 8\,m - 8\right) z^3 + \ldots \;/;\; (z \to 0)$$

$$\text{dc}(z \mid m) \propto 1 + \frac{1}{2}(1-m)\,z^2 + \frac{1}{24}\left(5 - 6\,m + m^2\right) z^4 + \ldots \;/;\; (z \to 0)$$

$$\operatorname{dn}(z \mid m) \propto 1 - \frac{m z^2}{2} + \frac{m(m+4)}{24} z^4 + \ldots /; (z \to 0)$$

$$\operatorname{ds}(z \mid m) \propto \frac{1}{z} + \frac{1}{6}(1 - 2m) z + \frac{1}{360}\left(7 + 8m - 8m^2\right) z^3 + \ldots /; (z \to 0)$$

$$\operatorname{nc}(z \mid m) \propto 1 + \frac{z^2}{2} + \frac{1}{24}(5 - 4m) z^4 + \ldots /; (z \to 0)$$

$$\operatorname{nd}(z \mid m) \propto 1 + \frac{m z^2}{2} + \frac{1}{24}\left(-4m + 5m^2\right) z^4 + \ldots /; (z \to 0)$$

$$\operatorname{ns}(z \mid m) \propto \frac{1}{z} + \frac{1}{6}(1 + m) z + \frac{1}{360}\left(7 - 22m + 7m^2\right) z^3 + \ldots /; (z \to 0)$$

$$\operatorname{sc}(z \mid m) \propto z + \frac{1}{6}(2 - m) z^3 + \frac{1}{120}\left(16 - 16m + m^2\right) z^5 + \ldots /; (z \to 0)$$

$$\operatorname{sd}(z \mid m) \propto z + \frac{1}{6}(2m - 1) z^3 + \frac{1}{120}\left(1 - 16m + 16m^2\right) z^5 + \ldots /; (z \to 0)$$

$$\operatorname{sn}(z \mid m) \propto z - \frac{1+m}{6} z^3 + \frac{\left(1 + 14m + m^2\right) z^5}{120} + \ldots /; (z \to 0).$$

The Jacobi functions $\operatorname{am}(z \mid m)$, $\operatorname{cd}(z \mid m)$, $\operatorname{cn}(z \mid m)$, $\operatorname{cs}(z \mid m)$, $\operatorname{dc}(z \mid m)$, $\operatorname{dn}(z \mid m)$, $\operatorname{ds}(z \mid m)$, $\operatorname{nc}(z \mid m)$, $\operatorname{nd}(z \mid m)$, $\operatorname{ns}(z \mid m)$, $\operatorname{sc}(z \mid m)$, $\operatorname{sd}(z \mid m)$, and $\operatorname{sn}(z \mid m)$ have the following series expansions at the point $m == 0$:

$$\operatorname{am}(z \mid m) \propto z + \frac{1}{8}(\sin(2z) - 2z) m + \frac{1}{256}(-16\cos(2z) z - 20z + 16\sin(2z) + \sin(4z)) m^2 + \ldots /; (m \to 0)$$

$$\operatorname{cd}(z \mid m) \propto \cos(z) + \frac{1}{4}\sin(z)(z + \cos(z)\sin(z)) m + \frac{1}{256}\left(\left(7 - 8z^2\right)\cos(z) - 8\cos(3z) + \cos(5z) + 24z\sin(z) - 12z\sin(3z)\right) m^2 + \ldots /; (m \to 0)$$

$$\operatorname{cn}(z \mid m) \propto \cos(z) + \frac{1}{8}\sin(z)(2z - \sin(2z)) m + \frac{1}{256}\left(-\left(8z^2 + 9\right)\cos(z) + 8\cos(3z) + \cos(5z) + 16z\sin(z) + 12z\sin(3z)\right) m^2 + \ldots /; (m \to 0)$$

$$\operatorname{cs}(z \mid m) \propto \cot(z) + \frac{1}{4}\left(z\csc^2(z) - \cot(z)\right) m + \frac{1}{512}\left(4\left(8z^2 - 3\right)\cos(z) + 13\cos(3z) - \cos(5z) + 8z\sin(z)\right)\csc^3(z) m^2 + \ldots /; (m \to 0)$$

$$\operatorname{dc}(z \mid m) \propto \sec(z) - \frac{1}{4}(z + \cos(z)\sin(z))\tan(z)\sec(z) m + \frac{1}{512}\left(-8\cos(2z) z^2 + 24z^2 + 4\sin(2z) z + 4\sin(4z) z + 5\cos(4z) - 5\right)\sec^3(z) m^2 + \ldots /; (m \to 0)$$

$$\operatorname{dn}(z \mid m) \propto 1 - \frac{1}{2}\sin^2(z) m - \frac{1}{32}\sin(z)(-8z\cos(z) + 5\sin(z) + \sin(3z)) m^2 + \ldots /; (m \to 0)$$

$$\text{ds}(z \mid m) \propto \csc(z) + \frac{1}{16}(4 z \cos(z) - 7 \sin(z) + \sin(3 z)) \csc^2(z)\, m + \frac{1}{512}\left(24 z^2 + 12 \sin(2 z)\, z - 4 \sin(4 z)\, z + 8\left(z^2 + 4\right) \cos(2 z) - 3 \cos(4 z) - 29\right) \csc^3(z)\, m^2 + \ldots /; (m \to 0)$$

$$\text{nc}(z \mid m) \propto \sec(z) + \frac{1}{4}(\sin(z) - z \sec(z)) \tan(z)\, m - \frac{1}{512}\left(8 \cos(2 z)\, z^2 - 24 z^2 + 44 \sin(2 z)\, z + 4 \sin(4 z)\, z + 11 \cos(4 z) - 11\right) \sec^3(z)\, m^2 + \ldots /; (m \to 1)$$

$$\text{nd}(z \mid m) \propto 1 + \frac{1}{2} \sin^2(z)\, m - \frac{1}{32} \sin(z)\, (8 z \cos(z) - 11 \sin(z) + \sin(3 z))\, m^2 + \ldots /; (m \to 0)$$

$$\text{ns}(z \mid m) \propto \csc(z) + \frac{1}{4} \cot(z)\, (z - \cos(z) \sin(z)) \csc(z)\, m + \frac{1}{512}\left(8 \cos(2 z)\, z^2 + 24 z^2 - 4 \sin(2 z)\, z + 4 \sin(4 z)\, z + 5 \cos(4 z) - 5\right) \csc^3(z)\, m^2 + \ldots /; (m \to 0)$$

$$\text{sc}(z \mid m) \propto \tan(z) + \frac{1}{4}\left(\tan(z) - z \sec^2(z)\right) m - \frac{1}{512}\left(72 z \cos(z) + 2\left(-16 z^2 - 18 \cos(2 z) + \cos(4 z) - 19\right) \sin(z)\right) \sec^3(z)\, m^2 + \ldots /; (m \to 0)$$

$$\text{sd}(z \mid m) \propto \sin(z) - \frac{1}{16}(4 z \cos(z) - 7 \sin(z) + \sin(3 z))\, m + \frac{1}{256}\left(-48 z \cos(z) + 12 z \cos(3 z) - \left(8 z^2 - 79\right) \sin(z) - 16 \sin(3 z) + \sin(5 z)\right) m^2 + \ldots /; (m \to 0)$$

$$\text{sn}(z \mid m) \propto \sin(z) + \frac{1}{8} \cos(z)\, (\sin(2 z) - 2 z)\, m + \frac{1}{256}\left(-24 z \cos(z) - 12 z \cos(3 z) + 2\left(-4 z^2 + 9 \cos(2 z) + \cos(4 z) + 8\right) \sin(z)\right) m^2 + \ldots /; (m \to 0).$$

The Jacobi functions am$(z \mid m)$, cd$(z \mid m)$, cn$(z \mid m)$, cs$(z \mid m)$, dc$(z \mid m)$, dn$(z \mid m)$, ds$(z \mid m)$, nc$(z \mid m)$, nd$(z \mid m)$, ns$(z \mid m)$, sc$(z \mid m)$, sd$(z \mid m)$, and sn$(z \mid m)$ have the following series expansions at the point $m == 1$:

$$\text{am}(z \mid m) \propto 2 \tan^{-1}(e^z) - \frac{\pi}{2} + \frac{1}{4}(z\, \text{sech}(z) - \sinh(z))\, (m - 1) + \frac{1}{64}(-4 z \cosh(z) + 9 \sinh(z) - z\, \text{sech}(z)\, (2 z \tanh(z) + 5))\, (m - 1)^2 + \ldots /; (m \to 1)$$

$$\text{cd}(z \mid m) \propto 1 + \frac{1}{2} \sinh^2(z)\, (m - 1) + \frac{1}{32} \sinh(z)\, (8 z \cosh(z) - 11 \sinh(z) + \sinh(3 z))\, (m - 1)^2 + \ldots /; (m \to 1)$$

$$\text{cn}(z \mid m) \propto \text{sech}(z) + \frac{1}{4}(\sinh(z) - z\, \text{sech}(z)) \tanh(z)\, (m - 1) + \frac{1}{512}\left(8 \cosh(2 z)\, z^2 - 24 z^2 + 44 \sinh(2 z)\, z + 4 \sinh(4 z)\, z - 11 \cosh(4 z) + 11\right) \text{sech}^3(z)\, (m - 1)^2 + \ldots /; (m \to 1)$$

$$\operatorname{cs}(z \mid m) \propto \operatorname{csch}(z) + \frac{1}{4} \coth(z) (\cosh(z) - z \operatorname{csch}(z)) (m-1) + \frac{1}{512} \left(8 \cosh(2z) z^2 + 24 z^2 - 4 \sinh(2z) z + 4 \sinh(4z) z - 5 \cosh(4z) + 5\right) \operatorname{csch}^3(z) (m-1)^2 - \ldots /; (m \to 1)$$

$$\operatorname{dc}(z \mid m) \propto 1 - \frac{1}{2} \sinh^2(z) (m-1) + \frac{1}{32} \sinh(z) (-8 z \cosh(z) + 5 \sinh(z) + \sinh(3z)) (m-1)^2 + \ldots /; (m \to 1)$$

$$\operatorname{dn}(z \mid m) \propto \operatorname{sech}(z) - \frac{1}{4} \tanh(z) (z + \cosh(z) \sinh(z)) \operatorname{sech}(z) (m-1) + \frac{1}{512} \left(8 \cosh(2z) z^2 - 24 z^2 - 4 \sinh(2z) z - 4 \sinh(4z) z + 5 \cosh(4z) - 5\right) \operatorname{sech}^3(z) (m-1)^2 + \ldots /; (m \to 1)$$

$$\operatorname{ds}(z \mid m) \propto \operatorname{csch}(z) - \frac{1}{16} (4 z \cosh(z) - 7 \sinh(z) + \sinh(3z)) \operatorname{csch}^2(z) (m-1) + \frac{1}{512} \left(24 z^2 + 12 \sinh(2z) z - 4 \sinh(4z) z + 8 \left(z^2 - 4\right) \cosh(2z) + 3 \cosh(4z) + 29\right) \operatorname{csch}^3(z) (m-1)^2 + \ldots /; (m \to 1)$$

$$\operatorname{nc}(z \mid m) \propto \cosh(z) - \frac{1}{8} \sinh(z) (\sinh(2z) - 2z) (m-1) + \frac{1}{256} \left(\left(8 z^2 - 9\right) \cosh(z) + 8 \cosh(3z) + \cosh(5z) - 16 z \sinh(z) - 12 z \sinh(3z)\right) (m-1)^2 + \ldots /; (m \to 1)$$

$$\operatorname{nd}(z \mid m) \propto \cosh(z) + \frac{1}{4} \sinh(z) (z + \cosh(z) \sinh(z)) (m-1) + \frac{1}{256} \left(\left(8 z^2 + 7\right) \cosh(z) - 8 \cosh(3z) + \cosh(5z) - 24 z \sinh(z) + 12 z \sinh(3z)\right) (m-1)^2 + \ldots /; (m \to 1)$$

$$\operatorname{ns}(z \mid m) \propto \coth(z) + \frac{1}{4} \left(\coth(z) - z \operatorname{csch}^2(z)\right) (m-1) + \frac{1}{512} \left(4 \left(8 z^2 + 3\right) \cosh(z) - 13 \cosh(3z) + \cosh(5z) + 8 z \sinh(z)\right) \operatorname{csch}^3(z) (m-1)^2 + \ldots /; (m \to 1)$$

$$\operatorname{sc}(z \mid m) \propto \sinh(z) - \frac{1}{8} \cosh(z) (\sinh(2z) - 2z) (m-1) + \frac{1}{256} \left(-24 z \cosh(z) - 12 z \cosh(3z) + \left(8 z^2 + 7\right) \sinh(z) + 8 \sinh(3z) + \sinh(5z)\right) (m-1)^2 + \ldots /; (m \to 1)$$

$$\operatorname{sd}(z \mid m) \propto \sinh(z) + \frac{1}{16} (4 z \cosh(z) - 7 \sinh(z) + \sinh(3z)) (m-1) + \frac{1}{256} \left(-48 z \cosh(z) + 12 z \cosh(3z) + 2 \left(4 z^2 - 15 \cosh(2z) + \cosh(4z) + 32\right) \sinh(z)\right) (m-1)^2 + \ldots /; (m \to 1)$$

$$\operatorname{sn}(z \mid m) \propto \tanh(z) + \frac{1}{4} \left(z \operatorname{sech}^2(z) - \tanh(z)\right) (m-1) - \frac{1}{512} \left(72 z \cosh(z) + 4 \left(8 z^2 - 5\right) \sinh(z) - 19 \sinh(3z) + \sinh(5z)\right) \operatorname{sech}^3(z) (m-1)^2 + \ldots /; (m \to 1).$$

## *q*-series representations

The Jacobi functions $\operatorname{am}(z \mid m)$, $\operatorname{cd}(z \mid m)$, $\operatorname{cn}(z \mid m)$, $\operatorname{cs}(z \mid m)$, $\operatorname{dc}(z \mid m)$, $\operatorname{dn}(z \mid m)$, $\operatorname{ds}(z \mid m)$, $\operatorname{nc}(z \mid m)$, $\operatorname{nd}(z \mid m)$, $\operatorname{ns}(z \mid m)$, $\operatorname{sc}(z \mid m)$, $\operatorname{sd}(z \mid m)$, and $\operatorname{sn}(z \mid m)$ have the following so-called $q$-series representations:

$$\operatorname{am}(z \mid m) = \frac{\pi z}{2 K(m)} + 2 \sum_{k=1}^{\infty} \frac{q(m)^k}{k \left(q(m)^{2k} + 1\right)} \sin\left(\frac{k \pi z}{K(m)}\right)$$

$$\operatorname{cd}(z \mid m) = \frac{2\pi}{\sqrt{m}\, K(m)} \sum_{k=0}^{\infty} \frac{(-1)^k q(m)^{k+1/2}}{1 - q(m)^{2k+1}} \cos\left(\frac{(2k+1)\pi z}{2 K(m)}\right)$$

$$\operatorname{cn}(z \mid m) = \frac{2\pi}{\sqrt{m}\, K(m)} \sum_{n=0}^{\infty} \frac{q(m)^{n+\frac{1}{2}}}{1 + q(m)^{2n+1}} \cos\left((2n+1) \frac{\pi z}{2 K(m)}\right)$$

$$\operatorname{cs}(z \mid m) = \frac{\pi}{2 K(m)} \cot\left(\frac{\pi z}{2 K(m)}\right) - \frac{2\pi}{K(m)} \sum_{k=1}^{\infty} \frac{q(m)^{2k}}{q(m)^{2k} + 1} \sin\left(\frac{k \pi z}{K(m)}\right)$$

$$\operatorname{dc}(z \mid m) = \frac{\pi}{2 K(m)} \sec\left(\frac{\pi z}{2 K(m)}\right) + \frac{2\pi}{K(m)} \sum_{k=0}^{\infty} \frac{(-1)^k q(m)^{2k+1}}{1 - q(m)^{2k+1}} \cos\left(\frac{(2k+1)\pi z}{2 K(m)}\right)$$

$$\operatorname{dn}(z \mid m) = \frac{\pi}{2 K(m)} + \frac{2\pi}{K(m)} \sum_{n=1}^{\infty} \frac{q(m)^n}{1 + q(m)^{2n}} \cos\left(\frac{n \pi z}{K(m)}\right)$$

$$\operatorname{ds}(z \mid m) = \frac{\pi}{2 K(m)} \csc\left(\frac{\pi z}{2 K(m)}\right) - \frac{2\pi}{K(m)} \sum_{k=0}^{\infty} \frac{q(m)^{2k+1}}{q(m)^{2k+1} + 1} \sin\left(\frac{(2k+1)\pi z}{2 K(m)}\right)$$

$$\operatorname{nc}(z \mid m) = \frac{\pi}{2 \sqrt{1-m}\, K(m)} \sec\left(\frac{\pi z}{2 K(m)}\right) - \frac{2\pi}{\sqrt{1-m}\, K(m)} \sum_{k=0}^{\infty} \frac{(-1)^k q(m)^{2k+1}}{q(m)^{2k+1} + 1} \cos\left(\frac{(2k+1)\pi z}{2 K(m)}\right)$$

$$\operatorname{nd}(z \mid m) = \frac{\pi}{2 \sqrt{1-m}\, K(m)} + \frac{2\pi}{\sqrt{1-m}\, K(m)} \sum_{k=1}^{\infty} \frac{(-1)^k q(m)^k}{q(m)^{2k} + 1} \cos\left(\frac{k \pi z}{K(m)}\right)$$

$$\operatorname{ns}(z \mid m) = \frac{\pi}{2 K(m)} \csc\left(\frac{\pi z}{2 K(m)}\right) + \frac{2\pi}{K(m)} \sum_{k=0}^{\infty} \frac{q(m)^{2k+1}}{1 - q(m)^{2k+1}} \sin\left(\frac{(2k+1)\pi z}{2 K(m)}\right)$$

$$\operatorname{sc}(z \mid m) = \frac{\pi}{2 \sqrt{1-m}\, K(m)} \tan\left(\frac{\pi z}{2 K(m)}\right) + \frac{2\pi}{\sqrt{1-m}\, K(m)} \sum_{k=1}^{\infty} \frac{(-1)^k q(m)^{2k}}{q(m)^{2k} + 1} \sin\left(\frac{k \pi z}{K(m)}\right)$$

$$\operatorname{sd}(z \mid m) = \frac{2\pi}{\sqrt{m}\, \sqrt{1-m}\, K(m)} \sum_{k=0}^{\infty} \frac{(-1)^k q(m)^{k+1/2}}{q(m)^{2k+1} + 1} \sin\left(\frac{(2k+1)\pi z}{2 K(m)}\right)$$

$$\operatorname{sn}(z \mid m) = \frac{2\pi}{\sqrt{m}\, K(m)} \sum_{n=0}^{\infty} \frac{q(m)^{n+\frac{1}{2}}}{1 - q(m)^{2n+1}} \sin\left((2n+1) \frac{\pi z}{2 K(m)}\right),$$

where $q(m)$ is the elliptic nome and $K(m)$ is the complete elliptic integral.

## Product representations

The twelve Jacobi functions $\operatorname{cd}(z \mid m)$, $\operatorname{cn}(z \mid m)$, $\operatorname{cs}(z \mid m)$, $\operatorname{dc}(z \mid m)$, $\operatorname{dn}(z \mid m)$, $\operatorname{ds}(z \mid m)$, $\operatorname{nc}(z \mid m)$, $\operatorname{nd}(z \mid m)$, $\operatorname{ns}(z \mid m)$, $\operatorname{sc}(z \mid m)$, $\operatorname{sd}(z \mid m)$, and $\operatorname{sn}(z \mid m)$ have the following product representations:

$$\operatorname{cd}(z \mid m) = \frac{2\sqrt[4]{q(m)}}{\sqrt[4]{m}} \cos\left(\frac{\pi z}{2K(m)}\right) \prod_{k=1}^{\infty} \frac{1 + 2q(m)^{2k}\cos\left(\frac{\pi z}{K(m)}\right) + q(m)^{4k}}{1 + 2q(m)^{2k-1}\cos\left(\frac{\pi z}{K(m)}\right) + q(m)^{4k-2}}$$

$$\operatorname{cn}(z \mid m) = 2\sqrt[4]{q(m)}\, \frac{\sqrt[4]{1-m}}{\sqrt[4]{m}} \cos\left(\frac{\pi z}{2K(m)}\right) \prod_{n=1}^{\infty} \frac{1 + 2q(m)^{2n}\cos\left(\frac{\pi z}{K(m)}\right) + q(m)^{4n}}{1 - 2q(m)^{2n-1}\cos\left(\frac{\pi z}{K(m)}\right) + q(m)^{4n-2}}$$

$$\operatorname{cs}(z \mid m) = \sqrt[4]{1-m}\, \cot\left(\frac{\pi z}{2K(m)}\right) \prod_{k=1}^{\infty} \frac{1 + 2q(m)^{2k}\cos\left(\frac{\pi z}{K(m)}\right) + q(m)^{4k}}{1 - 2q(m)^{2k}\cos\left(\frac{\pi z}{K(m)}\right) + q(m)^{4k}}$$

$$\operatorname{dc}(z \mid m) = \frac{\sqrt[4]{m}}{2\sqrt[4]{q(m)}} \sec\left(\frac{\pi z}{2K(m)}\right) \prod_{k=1}^{\infty} \frac{1 + 2q(m)^{2k-1}\cos\left(\frac{\pi z}{K(m)}\right) + q(m)^{4k-2}}{1 + 2q(m)^{2k}\cos\left(\frac{\pi z}{K(m)}\right) + q(m)^{4k}}$$

$$\operatorname{dn}(z \mid m) = \sqrt[4]{1-m} \prod_{n=1}^{\infty} \frac{1 + 2q(m)^{2n-1}\cos\left(\frac{\pi z}{K(m)}\right) + q\,(m)^{4n-2}}{1 - 2q(m)^{2n-1}\cos\left(\frac{\pi z}{K(m)}\right) + q\,(m)^{4n-2}}$$

$$\operatorname{ds}(z \mid m) = \sqrt[4]{1-m}\, \frac{\sqrt[4]{m}}{2\sqrt[4]{q(m)}} \csc\left(\frac{\pi z}{2K(m)}\right) \prod_{k=1}^{\infty} \frac{1 + 2q(m)^{2k-1}\cos\left(\frac{\pi z}{K(m)}\right) + q(m)^{4k-2}}{1 - 2q(m)^{2k}\cos\left(\frac{\pi z}{K(m)}\right) + q(m)^{4k}}$$

$$\operatorname{nc}(z \mid m) = \frac{1}{2}\, \frac{1}{\sqrt[4]{q(m)}}\, \frac{\sqrt[4]{m}}{\sqrt[4]{1-m}} \sec\left(\frac{\pi z}{2K(m)}\right) \prod_{n=1}^{\infty} \frac{1 - 2q(m)^{2n-1}\cos\left(\frac{\pi z}{K(m)}\right) + q(m)^{4n-2}}{1 + 2q(m)^{2n}\cos\left(\frac{\pi z}{K(m)}\right) + q(m)^{4n}}$$

$$\operatorname{nd}(z \mid m) = \frac{1}{\sqrt[4]{1-m}} \prod_{n=1}^{\infty} \frac{1 - 2q(m)^{2n-1}\cos\left(\frac{\pi z}{K(m)}\right) + q\,(m)^{4n-2}}{1 + 2q(m)^{2n-1}\cos\left(\frac{\pi z}{K(m)}\right) + q\,(m)^{4n-2}}$$

$$\operatorname{ns}(z \mid m) = \frac{1}{2}\, \frac{\sqrt[4]{m}}{\sqrt[4]{q(m)}} \csc\left(\frac{\pi z}{2K(m)}\right) \prod_{k=1}^{\infty} \frac{1 - 2q(m)^{2k-1}\cos\left(\frac{\pi z}{K(m)}\right) + q(m)^{4k-2}}{1 - 2q(m)^{2k}\cos\left(\frac{\pi z}{K(m)}\right) + q(m)^{4k}}$$

$$\operatorname{sc}(z \mid m) = \frac{1}{\sqrt[4]{1-m}} \tan\left(\frac{\pi z}{2K(m)}\right) \prod_{k=1}^{\infty} \frac{1 - 2q(m)^{2k}\cos\left(\frac{\pi z}{K(m)}\right) + q(m)^{4k}}{1 + 2q(m)^{2k}\cos\left(\frac{\pi z}{K(m)}\right) + q(m)^{4k}}$$

$$\operatorname{sd}(z \mid m) = \frac{2\sqrt[4]{q(m)}}{\sqrt[4]{m}\,\sqrt[4]{1-m}} \sin\left(\frac{\pi z}{2K(m)}\right) \prod_{k=1}^{\infty} \frac{1 - 2q(m)^{2k}\cos\left(\frac{\pi z}{K(m)}\right) + q(m)^{4k}}{1 + 2q(m)^{2k-1}\cos\left(\frac{\pi z}{K(m)}\right) + q(m)^{4k-2}}$$

$$\operatorname{sn}(z \mid m) = 2\frac{\sqrt[4]{q(m)}}{\sqrt[4]{m}} \sin\left(\frac{\pi z}{2K(m)}\right) \prod_{k=1}^{\infty} \frac{1 - 2q(m)^{2k}\cos\left(\frac{\pi z}{K(m)}\right) + q(m)^{4k}}{1 - 2q(m)^{2k-1}\cos\left(\frac{\pi z}{K(m)}\right) + q(m)^{4k-2}},$$

where $q(m)$ is the elliptic nome and $K(m)$ is the complete elliptic integral.

### Transformations

The amplitude function $\operatorname{am}(z \mid m)$ satisfies numerous relations that allow for transformations of its arguments, for example:

$$\operatorname{am}\left(\sqrt{1-m}\, z \,\middle|\, \frac{m}{m-1}\right) = \frac{\pi}{2} - \operatorname{am}(K(m) - z \mid m) \;/;\; m \notin (1, \infty)$$

$$\operatorname{am}\left(\sqrt{1-m}\, z \,\middle|\, \frac{m}{m-1}\right) = -\frac{\pi}{2} + \operatorname{am}(K(m) - z \mid m) \;/;\; m > 1.$$

The twelve Jacobi functions $\operatorname{cd}(z \mid m)$, $\operatorname{cn}(z \mid m)$, $\operatorname{cs}(z \mid m)$, $\operatorname{dc}(z \mid m)$, $\operatorname{dn}(z \mid m)$, $\operatorname{ds}(z \mid m)$, $\operatorname{nc}(z \mid m)$, $\operatorname{nd}(z \mid m)$, $\operatorname{ns}(z \mid m)$, $\operatorname{sc}(z \mid m)$, $\operatorname{sd}(z \mid m)$, and $\operatorname{sn}(z \mid m)$ with specific arguments can sometimes be represented through elliptic functions with other mostly simpler arguments, for example:

$$\operatorname{cd}(i z \mid m) = \operatorname{nd}(z \mid 1-m) \quad \operatorname{cn}(i z \mid m) = \frac{1}{\operatorname{cn}(z|1-m)} \quad \operatorname{cs}(i z, m) = -i \operatorname{ns}(z, 1-m)$$

$$\operatorname{dc}(i z \mid m) = \operatorname{dn}(z \mid 1-m) \quad \operatorname{dn}(i z \mid m) = \frac{\operatorname{dn}(z|1-m)}{\operatorname{cn}(z|1-m)} \quad \operatorname{ds}(i z \mid m) = -i \operatorname{ds}(z \mid 1-m)$$

$$\operatorname{nc}(i z \mid m) = \operatorname{cn}(z \mid 1-m) \quad \operatorname{nd}(i z \mid m) = \frac{\operatorname{nd}(z|1-m)}{\operatorname{nc}(z|1-m)} \quad \operatorname{ns}(i z \mid m) = -i \operatorname{cs}(z \mid 1-m)$$

$$\operatorname{sc}(i z, m) = i \operatorname{sn}(z, 1-m) \quad \operatorname{sd}(i z \mid m) = i \operatorname{sd}(z \mid 1-m) \quad \operatorname{sn}(i z \mid m) = i \frac{\operatorname{sn}(z|1-m)}{\operatorname{cn}(z|1-m)}$$

$$\operatorname{cd}(z \mid 1-m) = \operatorname{nd}(i z \mid m) \quad \operatorname{cn}(z \mid 1-m) = \frac{1}{\operatorname{cn}(i z|m)} \quad \operatorname{cs}(z \mid 1-m) = i \operatorname{ns}(i z \mid m)$$

$$\operatorname{dc}(z \mid 1-m) = \operatorname{dn}(i z \mid m) \quad \operatorname{dn}(z \mid 1-m) = \frac{\operatorname{dn}(i z|m)}{\operatorname{cn}(i z|m)} \quad \operatorname{ds}(z \mid 1-m) = i \operatorname{ds}(i z \mid m)$$

$$\operatorname{nc}(z \mid 1-m) = \operatorname{cn}(i z \mid m) \quad \operatorname{nd}(z \mid 1-m) = \frac{\operatorname{nd}(i z|m)}{\operatorname{nc}(i z|m)} \quad \operatorname{ns}(z \mid 1-m) = i \operatorname{cs}(i z \mid m)$$

$$\operatorname{sc}(z \mid 1-m) = -i \operatorname{sn}(i z \mid m) \quad \operatorname{sd}(z \mid 1-m) = -i \operatorname{sd}(i z \mid m) \quad \operatorname{sn}(z \mid 1-m) = -i \frac{\operatorname{sn}(i z|m)}{\operatorname{cn}(i z|m)}$$

$$\operatorname{cd}\left(\sqrt{m}\, z \,\middle|\, \tfrac{1}{m}\right) = \operatorname{dc}(z \mid m) \quad \operatorname{cn}\left(\sqrt{m}\, z \,\middle|\, \tfrac{1}{m}\right) = \operatorname{dn}(z \mid m) \quad \operatorname{cs}\left(\sqrt{m}\, z \,\middle|\, \tfrac{1}{m}\right) = \frac{1}{\sqrt{m}} \operatorname{ds}(z \mid m)$$

$$\operatorname{dc}\left(\sqrt{m}\, z \,\middle|\, \tfrac{1}{m}\right) = \operatorname{cd}(z \mid m) \quad \operatorname{dn}\left(\sqrt{m}\, z \,\middle|\, \tfrac{1}{m}\right) = \operatorname{cn}(z \mid m) \quad \operatorname{ds}\left(\sqrt{m}\, z \,\middle|\, \tfrac{1}{m}\right) = \frac{\operatorname{cs}(z|m)}{\sqrt{m}}$$

$$\operatorname{nc}\left(\sqrt{m}\, z \,\middle|\, \tfrac{1}{m}\right) = \operatorname{nd}(z \mid m) \quad \operatorname{nd}\left(\sqrt{m}\, z \,\middle|\, \tfrac{1}{m}\right) = \operatorname{nc}(z \mid m) \quad \operatorname{ns}\left(\sqrt{m}\, z \,\middle|\, \tfrac{1}{m}\right) = \frac{1}{\sqrt{m}} \operatorname{ns}(z \mid m)$$

$$\operatorname{sc}\left(\sqrt{m}\, z \,\middle|\, \tfrac{1}{m}\right) = \sqrt{m}\, \operatorname{sd}(z \mid m) \quad \operatorname{sd}\left(\sqrt{m}\, z \,\middle|\, \tfrac{1}{m}\right) = \sqrt{m}\, \operatorname{sc}(z \mid m) \quad \operatorname{sn}\left(\sqrt{m}\, z \,\middle|\, \tfrac{1}{m}\right) = \sqrt{m}\, \operatorname{sn}(z \mid m)$$

$$\operatorname{cd}\left(\sqrt{1-m}\, z \,\middle|\, \tfrac{m}{m-1}\right) = \operatorname{cn}(z \mid m) \quad \operatorname{cn}\left(\sqrt{1-m}\, z \,\middle|\, \tfrac{m}{m-1}\right) = \frac{\operatorname{cn}(z|m)}{\operatorname{dn}(z|m)} \quad \operatorname{cs}\left(\sqrt{1-m}\, z \,\middle|\, \tfrac{m}{m-1}\right) = \frac{1}{\sqrt{1-m}} \operatorname{cs}(z \mid m)$$

$$\operatorname{dc}\left(\sqrt{1-m}\, z \,\middle|\, \tfrac{m}{m-1}\right) = \operatorname{nc}(z \mid m) \quad \operatorname{dn}\left(\sqrt{1-m}\, z \,\middle|\, \tfrac{m}{m-1}\right) = \frac{1}{\operatorname{dn}(z|m)} \quad \operatorname{ds}\left(\sqrt{1-m}\, z \,\middle|\, \tfrac{m}{m-1}\right) = \frac{\operatorname{ns}(z|m)}{\sqrt{1-m}}$$

$$\operatorname{nc}\left(\sqrt{1-m}\, z \,\middle|\, \tfrac{m}{m-1}\right) = \frac{\operatorname{nc}(z|m)}{\operatorname{nd}(z|m)} \quad \operatorname{nd}\left(\sqrt{1-m}\, z \,\middle|\, \tfrac{m}{m-1}\right) = \operatorname{dn}(z \mid m) \quad \operatorname{ns}\left(\sqrt{1-m}\, z \,\middle|\, \tfrac{m}{m-1}\right) = \frac{1}{\sqrt{1-m}} \operatorname{ds}(z \mid m)$$

$$\operatorname{sc}\left(\sqrt{1-m}\, z \,\middle|\, \frac{m}{m-1}\right) = \sqrt{1-m}\, \operatorname{sc}(z \mid m)$$

$$\operatorname{sd}\left(\sqrt{1-m}\, z \,\middle|\, \frac{m}{m-1}\right) = \sqrt{1-m}\, \operatorname{sn}(z \mid m)$$

$$\text{sn}\left(\sqrt{1-m}\; z \,\middle|\, \frac{m}{m-1}\right) == \sqrt{1-m}\; \frac{\text{sn}(z \mid m)}{\text{dn}(z \mid m)}.$$

The twelve Jacobi functions $\text{cd}(z \mid m)$, $\text{cn}(z \mid m)$, $\text{cs}(z \mid m)$, $\text{dc}(z \mid m)$, $\text{dn}(z \mid m)$, $\text{ds}(z \mid m)$, $\text{nc}(z \mid m)$, $\text{nd}(z \mid m)$, $\text{ns}(z \mid m)$, $\text{sc}(z \mid m)$, $\text{sd}(z \mid m)$, and $\text{sn}(z \mid m)$ with the argument complex $z == u + v$ can be represented through elliptic functions with arguments $z == u$ and $z == v$, for example:

$$\text{cd}(u+v \mid m) == \frac{\text{cn}(u \mid m)\,\text{cn}(v \mid m) - \text{sn}(u \mid m)\,\text{dn}(u \mid m)\,\text{sn}(v \mid m)\,\text{dn}(v \mid m)}{\text{dn}(u \mid m)\,\text{dn}(v \mid m) - m\,\text{sn}(u \mid m)\,\text{cn}(u \mid m)\,\text{sn}(v \mid m)\,\text{cn}(v \mid m)}$$

$$\text{cn}(u+v \mid m) == \frac{\text{cn}(u \mid m)\,\text{cn}(v \mid m) - \text{sn}(u \mid m)\,\text{dn}(u \mid m)\,\text{sn}(v \mid m)\,\text{dn}(v \mid m)}{1 - m\,\text{sn}(u \mid m)^2\,\text{sn}(v \mid m)^2}$$

$$\text{cs}(u+v \mid m) == \frac{\text{cn}(u \mid m)\,\text{cn}(v \mid m) - \text{sn}(u \mid m)\,\text{dn}(u \mid m)\,\text{sn}(v \mid m)\,\text{dn}(v \mid m)}{\text{cn}(v \mid m)\,\text{dn}(v \mid m)\,\text{sn}(u \mid m) + \text{cn}(u \mid m)\,\text{dn}(u \mid m)\,\text{sn}(v \mid m)}$$

$$\text{dc}(u+v \mid m) == \frac{\text{dn}(u \mid m)\,\text{dn}(v \mid m) - m\,\text{sn}(u \mid m)\,\text{cn}(u \mid m)\,\text{sn}(v \mid m)\,\text{cn}(v \mid m)}{\text{cn}(u \mid m)\,\text{cn}(v \mid m) - \text{sn}(u \mid m)\,\text{dn}(u \mid m)\,\text{sn}(v \mid m)\,\text{dn}(v \mid m)}$$

$$\text{dn}(a+v \mid m) == \frac{\text{dn}(u \mid m)\,\text{dn}(v \mid m) - m\,\text{sn}(u \mid m)\,\text{cn}(u \mid m)\,\text{sn}(v \mid m)\,\text{cn}(v \mid m)}{1 - m\,\text{sn}(u \mid m)^2\,\text{sn}(v \mid m)^2}$$

$$\text{ds}(u+v \mid m) == \frac{\text{dn}(u \mid m)\,\text{dn}(v \mid m) - m\,\text{sn}(u \mid m)\,\text{cn}(u \mid m)\,\text{sn}(v \mid m)\,\text{cn}(v \mid m)}{\text{cn}(v \mid m)\,\text{dn}(v \mid m)\,\text{sn}(u \mid m) + \text{cn}(u \mid m)\,\text{dn}(u \mid m)\,\text{sn}(v \mid m)}$$

$$\text{nc}(u+v \mid m) == \frac{1 - m\,\text{sn}(u \mid m)^2\,\text{sn}(v \mid m)^2}{\text{cn}(u \mid m)\,\text{cn}(v \mid m) - \text{sn}(u \mid m)\,\text{dn}(u \mid m)\,\text{sn}(v \mid m)\,\text{dn}(v \mid m)}$$

$$\text{nd}(u+v \mid m) == \frac{1 - m\,\text{sn}(u \mid m)^2\,\text{sn}(v \mid m)^2}{\text{dn}(u \mid m)\,\text{dn}(v \mid m) - m\,\text{sn}(u \mid m)\,\text{cn}(u \mid m)\,\text{sn}(v \mid m)\,\text{cn}(v \mid m)}$$

$$\text{ns}(u+v \mid m) == \frac{1 - m\,\text{sn}(u \mid m)^2\,\text{sn}(v \mid m)^2}{\text{cn}(v \mid m)\,\text{dn}(v \mid m)\,\text{sn}(u \mid m) + \text{cn}(u \mid m)\,\text{dn}(u \mid m)\,\text{sn}(v \mid m)}$$

$$\text{sc}(u+v \mid m) == \frac{\text{cn}(v \mid m)\,\text{dn}(v \mid m)\,\text{sn}(u \mid m) + \text{cn}(u \mid m)\,\text{dn}(u \mid m)\,\text{sn}(v \mid m)}{\text{cn}(u \mid m)\,\text{cn}(v \mid m) - \text{sn}(u \mid m)\,\text{dn}(u \mid m)\,\text{sn}(v \mid m)\,\text{dn}(v \mid m)}$$

$$\text{sd}(u+v \mid m) == \frac{\text{cn}(v \mid m)\,\text{dn}(v \mid m)\,\text{sn}(u \mid m) + \text{cn}(u \mid m)\,\text{dn}(u \mid m)\,\text{sn}(v \mid m)}{\text{dn}(u \mid m)\,\text{dn}(v \mid m) - m\,\text{sn}(u \mid m)\,\text{cn}(u \mid m)\,\text{sn}(v \mid m)\,\text{cn}(v \mid m)}$$

$$\text{sn}(u+v \mid m) == \frac{\text{cn}(u \mid m)\,\text{dn}(u \mid m)\,\text{sn}(v \mid m) + \text{cn}(v \mid m)\,\text{dn}(v \mid m)\,\text{sn}(u \mid m)}{1 - m\,\text{sn}(u \mid m)^2\,\text{sn}(v \mid m)^2}.$$

The twelve Jacobi functions $\text{cd}(z \mid m)$, $\text{cn}(z \mid m)$, $\text{cs}(z \mid m)$, $\text{dc}(z \mid m)$, $\text{dn}(z \mid m)$, $\text{ds}(z \mid m)$, $\text{nc}(z \mid m)$, $\text{nd}(z \mid m)$, $\text{ns}(z \mid m)$, $\text{sc}(z \mid m)$, $\text{sd}(z \mid m)$, and $\text{sn}(z \mid m)$ satisfy the following half-angle formulas:

$$\operatorname{cd}\left(\frac{z}{2}\,\middle|\,m\right)^2 = \frac{\operatorname{cn}(z|m)+\operatorname{dn}(z|m)}{1-m+\operatorname{dn}(z|m)+m\operatorname{cn}(z|m)} \quad \operatorname{cn}\left(\frac{z}{2}\,\middle|\,m\right)^2 = \frac{\operatorname{cn}(z|m)+\operatorname{dn}(z|m)}{1+\operatorname{dn}(z|m)} \quad \operatorname{cs}\left(\frac{z}{2}\,\middle|\,m\right)^2 = \frac{\operatorname{cn}(z|m)+\operatorname{dn}(z|m)}{1-\operatorname{cn}(z|m)}$$

$$\operatorname{dc}\left(\frac{z}{2}\,\middle|\,m\right)^2 = \frac{1-m+\operatorname{dn}(z|m)+m\operatorname{cn}(z|m)}{\operatorname{cn}(z|m)+\operatorname{dn}(z|m)} \quad \operatorname{dn}\left(\frac{z}{2}\,\middle|\,m\right)^2 = \frac{1-m+\operatorname{dn}(z|m)+m\operatorname{cn}(z|m)}{1+\operatorname{dn}(z|m)} \quad \operatorname{ds}\left(\frac{z}{2}\,\middle|\,m\right)^2 = \frac{1-m+\operatorname{dn}(z|m)+m\operatorname{cn}(z|m)}{1-\operatorname{cn}(z|m)}$$

$$\operatorname{nc}\left(\frac{z}{2}\,\middle|\,m\right)^2 = \frac{1+\operatorname{dn}(z|m)}{\operatorname{cn}(z|m)+\operatorname{dn}(z|m)} \quad \operatorname{nd}\left(\frac{z}{2}\,\middle|\,m\right)^2 = \frac{1+\operatorname{dn}(z|m)}{1-m+\operatorname{dn}(z|m)+m\operatorname{cn}(z|m)} \quad \operatorname{ns}\left(\frac{z}{2}\,\middle|\,m\right)^2 = \frac{1+\operatorname{dn}(z|m)}{1-\operatorname{cn}(z|m)}$$

$$\operatorname{sc}\left(\frac{z}{2}\,\middle|\,m\right)^2 = \frac{1-\operatorname{cn}(z|m)}{\operatorname{cn}(z|m)+\operatorname{dn}(z|m)} \quad \operatorname{sd}\left(\frac{z}{2}\,\middle|\,m\right)^2 = \frac{1-\operatorname{cn}(z|m)}{1-m+\operatorname{dn}(z|m)+m\operatorname{cn}(z|m)} \quad \operatorname{sn}\left(\frac{z}{2}\,\middle|\,m\right)^2 = \frac{1-\operatorname{cn}(z|m)}{1+\operatorname{dn}(z|m)}.$$

The twelve Jacobi functions $\operatorname{cd}(z \mid m)$, $\operatorname{cn}(z \mid m)$, $\operatorname{cs}(z \mid m)$, $\operatorname{dc}(z \mid m)$, $\operatorname{dn}(z \mid m)$, $\operatorname{ds}(z \mid m)$, $\operatorname{nc}(z \mid m)$, $\operatorname{nd}(z \mid m)$, $\operatorname{ns}(z \mid m)$, $\operatorname{sc}(z \mid m)$, $\operatorname{sd}(z \mid m)$, and $\operatorname{sn}(z \mid m)$ satisfy the following double-angle (or multiplication) formulas:

$$\operatorname{cd}(2z \mid m) = \frac{\operatorname{cn}(z \mid m)^2 - \operatorname{sn}(z \mid m)^2 \operatorname{dn}(z \mid m)^2}{\operatorname{dn}(z \mid m)^2 - m \operatorname{sn}(z \mid m)^2 \operatorname{cn}(z \mid m)^2}$$

$$\operatorname{cn}(2z \mid m) = \frac{\operatorname{cn}(z \mid m)^2 - \operatorname{sn}(z \mid m)^2 \operatorname{dn}(z \mid m)^2}{1 - m \operatorname{sn}(z \mid m)^4}$$

$$\operatorname{cn}(n z \mid m) = \left(\frac{m}{1-m}\right)^{\frac{n^2-1}{4}} \prod_{\mu,\nu=0}^{n-1} \operatorname{cn}\left(z + \frac{4K(m)(\mu + \nu\tau)}{n} \,\middle|\, m\right) /; \frac{n+1}{2} \in \mathbb{N}^+$$

$$\operatorname{cs}(2z \mid m) = \frac{\operatorname{cn}(z \mid m)^2 - \operatorname{sn}(z \mid m)^2 \operatorname{dn}(z \mid m)^2}{2 \operatorname{sn}(z \mid m) \operatorname{cn}(z \mid m) \operatorname{dn}(z \mid m)}$$

$$\operatorname{dc}(2z \mid m) = \frac{\operatorname{dn}(z \mid m)^2 - m \operatorname{sn}(z \mid m)^2 \operatorname{cn}(z \mid m)^2}{\operatorname{cn}(z \mid m)^2 - \operatorname{sn}(z \mid m)^2 \operatorname{dn}(z \mid m)^2}$$

$$\operatorname{dn}(2z \mid m) = \frac{\operatorname{dn}(z \mid m)^2 - m \operatorname{sn}(z \mid m)^2 \operatorname{cn}(z \mid m)^2}{1 - m \operatorname{sn}(z \mid m)^4}$$

$$\operatorname{dn}(n z \mid m) = \left(\frac{1}{1-m}\right)^{\frac{n^2-1}{4}} \prod_{\mu,\nu=0}^{n-1} \operatorname{dn}\left(z + \frac{4K(m)(\mu + \nu\tau)}{n} \,\middle|\, m\right) /; \frac{n+1}{2} \in \mathbb{N}^+$$

$$\operatorname{ds}(2z \mid m) = \frac{\operatorname{dn}(z \mid m)^2 - m \operatorname{sn}(z \mid m)^2 \operatorname{cn}(z \mid m)^2}{2 \operatorname{sn}(z \mid m) \operatorname{cn}(z \mid m) \operatorname{dn}(z \mid m)}$$

$$\operatorname{nc}(2z \mid m) = \frac{1 - m \operatorname{sn}(z \mid m)^4}{\operatorname{cn}(z \mid m)^2 - \operatorname{sn}(z \mid m)^2 \operatorname{dn}(z \mid m)^2}$$

$$\operatorname{nc}(n z \mid m) = \left(\frac{1-m}{m}\right)^{\frac{n^2-1}{4}} \prod_{\mu,\nu=0}^{n-1} \operatorname{nc}\left(z + \frac{4K(m)(\mu + \nu\tau)}{n} \,\middle|\, m\right) /; \frac{n+1}{2} \in \mathbb{N}^+$$

$$\operatorname{nd}(2z \mid m) = \frac{1 - m \operatorname{sn}(z \mid m)^4}{\operatorname{dn}(z \mid m)^2 - m \operatorname{sn}(z \mid m)^2 \operatorname{cn}(z \mid m)^2}$$

$$\operatorname{nd}(n z \mid m) = (1-m)^{\frac{n^2-1}{4}} \prod_{\mu,\nu=0}^{n-1} \operatorname{nd}\left(z + \frac{4K(m)(\mu + \nu\tau)}{n} \,\middle|\, m\right) /; \frac{n+1}{2} \in \mathbb{N}^+$$

$$\operatorname{ns}(2z \mid m) = \frac{1 - m\,\operatorname{sn}(z \mid m)^4}{2\,\operatorname{sn}(z \mid m)\,\operatorname{cn}(z \mid m)\,\operatorname{dn}(z \mid m)}$$

$$\operatorname{ns}(n\,z, m) = (-1)^{\frac{n-1}{2}}\, m^{\frac{1-n^2}{4}} \prod_{\mu=-\frac{n-1}{2}}^{\frac{n-1}{2}} \prod_{\nu=-\frac{n-1}{2}}^{\frac{n-1}{2}} \operatorname{ns}\left(z + 2\,\frac{\mu\,K(m) + i\,\nu\,K(1-m)}{n} \,\middle|\, m\right) /;\ \frac{n+1}{2} \in \mathbb{N}^+$$

$$\operatorname{sc}(2z \mid m) = \frac{2\,\operatorname{sn}(z \mid m)\,\operatorname{cn}(z \mid m)\,\operatorname{dn}(z \mid m)}{\operatorname{cn}(z \mid m)^2 - \operatorname{sn}(z \mid m)^2\,\operatorname{dn}(z \mid m)^2}$$

$$\operatorname{sd}(2z \mid m) = \frac{2\,\operatorname{sn}(z \mid m)\,\operatorname{cn}(z \mid m)\,\operatorname{dn}(z \mid m)}{\operatorname{dn}(z \mid m)^2 - m\,\operatorname{sn}(z \mid m)^2\,\operatorname{cn}(z \mid m)^2}$$

$$\operatorname{sn}(2z \mid m) = \frac{2\,\operatorname{sn}(z \mid m)\,\operatorname{cn}(z \mid m)\,\operatorname{dn}(z \mid m)}{1 - m\,\operatorname{sn}(z \mid m)^4}$$

$$\operatorname{sn}(n\,z, m) = (-1)^{\frac{n-1}{2}}\, m^{\frac{n^2-1}{4}} \prod_{\mu=-\frac{n-1}{2}}^{\frac{n-1}{2}} \prod_{\nu=-\frac{n-1}{2}}^{\frac{n-1}{2}} \operatorname{sn}\left(z + 2\,\frac{\mu\,K(m) + i\,\nu\,K(1-m)}{n} \,\middle|\, m\right) /;\ \frac{n+1}{2} \in \mathbb{N}^+.$$

## Identities

The twelve Jacobi functions $\operatorname{cd}(z \mid m)$, $\operatorname{cn}(z \mid m)$, $\operatorname{cs}(z \mid m)$, $\operatorname{dc}(z \mid m)$, $\operatorname{dn}(z \mid m)$, $\operatorname{ds}(z \mid m)$, $\operatorname{nc}(z \mid m)$, $\operatorname{nd}(z \mid m)$, $\operatorname{ns}(z \mid m)$, $\operatorname{sc}(z \mid m)$, $\operatorname{sd}(z \mid m)$, and $\operatorname{sn}(z \mid m)$ satisfy the following nonlinear functional equations:

$$m\,w(z)^4 - 2\,w(z)^2 + \left(m\,w(z)^4 - 2\,m\,w(z)^2 + 1\right) w(2z) + 1 = 0 /;\ w(z) = \operatorname{cd}(z \mid m)$$

$$m\,w(z)^4 - 2\,(m-1)\,w(z)^2 + m + \left(m\,w(z)^4 - 2\,m\,w(z)^2 + m - 1\right) w(2z) - 1 = 0 /;\ w(z) = \operatorname{cn}(z \mid m)$$

$$4\,w(z)^2 \left(w(z)^2 + 1\right)\left(w(z)^2 - m + 1\right) w(2z)^2 - \left(w(z)^4 + m - 1\right)^2 = 0 /;\ w(z) = \operatorname{cs}(z \mid m)$$

$$w(z)^4 - 2\,m\,w(z)^2 + m + \left(w(z)^4 - 2\,w(z)^2 + m\right) w(2z) = 0 /;\ w(z) = \operatorname{dc}(z \mid m)$$

$$w(z)^4 + 2\,(m-1)\,w(z)^2 - m + \left(w(z)^4 - 2\,w(z)^2 - m + 1\right) w(2z) + 1 = 0 /;\ w(z) = \operatorname{dn}(z \mid m)$$

$$4\,w(z)^2 \left(w(z)^2 + m - 1\right)\left(w(z)^2 + m\right) w(2z)^2 - \left(w(z)^4 - m^2 + m\right)^2 = 0 /;\ w(z) = \operatorname{ds}(z \mid m)$$

$$(m-1)\,w(z)^4 - 2\,m\,w(z)^2 + m + \left((m-1)\,w(z)^4 + 2\,(1-m)\,w(z)^2 + m\right) w(2z) = 0 /;\ w(z) = \operatorname{nc}(z \mid m)$$

$$(m-1)\,w(z)^4 + 2\,w(z)^2 + \left((m-1)\,w(z)^4 + 2\,(1-m)\,w(z)^2 - 1\right) w(2z) - 1 = 0 /;\ w(z) = \operatorname{nd}(z \mid m)$$

$$4\,w(z)^2 \left(w(z)^2 - 1\right)\left(w(z)^2 - m\right) w(2z)^2 - \left(m - w(z)^4\right)^2 = 0 /;\ w(z) = \operatorname{ns}(z \mid m)$$

$$\left((m-1)\,w(z)^4 + 1\right)^2 w(2z)^2 - 4\left(w(z)^2 + 1\right)\left((1-m)\,w(z)^2 + 1\right) w(z)^2 = 0 /;\ w(z) = \operatorname{sc}(z \mid m)$$

$$\left((m-1)\,m\,w(z)^4 - 1\right)^2 w(2z)^2 - 4\,w(z)^2 \left((m-1)\,m\,w(z)^4 + (2m-1)\,w(z)^2 + 1\right) = 0 /;\ w(z) = \operatorname{sd}(z \mid m)$$

$$\left(m\,w(z)^4 - 1\right)^2 w(2z)^2 - 4\,w(z)^2 \left(w(z)^2 - 1\right)\left(m\,w(z)^2 - 1\right) = 0 /;\ w(z) = \operatorname{sn}(z \mid m).$$

### Simple representations of derivatives

The derivatives of all Jacobi functions $\text{am}(z \mid m)$, $\text{cd}(z \mid m)$, $\text{cn}(z \mid m)$, $\text{cs}(z \mid m)$, $\text{dc}(z \mid m)$, $\text{dn}(z \mid m)$, $\text{ds}(z \mid m)$, $\text{nc}(z \mid m)$, $\text{nd}(z \mid m)$, $\text{ns}(z \mid m)$, $\text{sc}(z \mid m)$, $\text{sd}(z \mid m)$, and $\text{sn}(z \mid m)$ with respect to variable $z$ have rather simple and symmetrical representations that can be expressed
through other Jacobi functions:

$$\frac{\partial \text{am}(z \mid m)}{\partial z} == \text{dn}(z \mid m)$$

$$\frac{\partial \text{cd}(z \mid m)}{\partial z} == (m-1)\, \text{nd}(z \mid m)\, \text{sd}(z \mid m)$$

$$\frac{\partial \text{cn}(z \mid m)}{\partial z} == -\text{sn}(z \mid m)\, \text{dn}(z \mid m)$$

$$\frac{\partial \text{cs}(z \mid m)}{\partial z} == -\text{ds}(z \mid m)\, \text{ns}(z \mid m)$$

$$\frac{\partial \text{dc}(z \mid m)}{\partial z} == (1-m)\, \text{nc}(z \mid m)\, \text{sc}(z \mid m)$$

$$\frac{\partial \text{dn}(z \mid m)}{\partial z} == -m\, \text{sn}(z \mid m)\, \text{cn}(z \mid m)$$

$$\frac{\partial \text{ds}(z \mid m)}{\partial z} == -\text{cs}(z \mid m)\, \text{ns}(z \mid m)$$

$$\frac{\partial \text{nc}(z \mid m)}{\partial z} == \text{dc}(z \mid m)\, \text{sc}(z \mid m)$$

$$\frac{\partial \text{nd}(z \mid m)}{\partial z} == m\, \text{cd}(z \mid m)\, \text{sd}(z \mid m)$$

$$\frac{\partial \text{ns}(z, m)}{\partial z} == -\text{cs}(z \mid m)\, \text{ds}(z \mid m)$$

$$\frac{\partial \text{sc}(z \mid m)}{\partial z} == \text{dc}(z \mid m)\, \text{nc}(z \mid m)$$

$$\frac{\partial \text{sd}(z \mid m)}{\partial z} == \text{cd}(z \mid m)\, \text{nd}(z \mid m)$$

$$\frac{\partial \text{sn}(z \mid m)}{\partial z} == \text{cn}(z \mid m)\, \text{dn}(z \mid m).$$

The derivatives of all Jacobi functions $\text{am}(z \mid m)$, $\text{cd}(z \mid m)$, $\text{cn}(z \mid m)$, $\text{cs}(z \mid m)$, $\text{dc}(z \mid m)$, $\text{dn}(z \mid m)$, $\text{ds}(z \mid m)$, $\text{nc}(z \mid m)$, $\text{nd}(z \mid m)$, $\text{ns}(z \mid m)$, $\text{sc}(z \mid m)$, $\text{sd}(z \mid m)$, and $\text{sn}(z \mid m)$ with respect to variable $m$ have more complicated representations that include other Jacobi functions
and the elliptic integral $E(\text{am}(z \mid m) \mid m)$:

$$\frac{\partial \operatorname{am}(z \mid m)}{\partial m} == \frac{((m-1)\,z + E(\operatorname{am}(z \mid m) \mid m))\operatorname{dn}(z \mid m) - m\operatorname{cn}(z \mid m)\operatorname{sn}(z \mid m)}{2\,(m-1)\,m} \;/;\; m < 1 \bigvee -\frac{3}{2} \leq z \leq \frac{3}{2}$$

$$\frac{\partial \operatorname{cd}(z \mid m)}{\partial m} == \frac{((m-1)\,z + E(\operatorname{am}(z \mid m) \mid m))\operatorname{nd}(z \mid m)\operatorname{sd}(z \mid m)}{2\,m}$$

$$\frac{\partial \operatorname{cn}(z \mid m)}{\partial m} == \frac{1}{2\,m\,(1-m)}\left(\operatorname{sn}(z \mid m)\operatorname{dn}(z \mid m)\left((m-1)\,z + E(\operatorname{am}(z \mid m) \mid m) - m\operatorname{sn}(z \mid m)\operatorname{cd}(z \mid m)\right)\right)$$

$$\frac{\partial \operatorname{cs}(z \mid m)}{\partial m} == \frac{1}{2\,m\,(1-m)}\left(\operatorname{ns}(z \mid m)\operatorname{ds}(z \mid m)\left((m-1)\,z + E(\operatorname{am}(z \mid m) \mid m) - m\operatorname{sn}(z \mid m)\operatorname{cd}(z \mid m)\right)\right)$$

$$\frac{\partial \operatorname{dc}(z \mid m)}{\partial m} == \frac{\operatorname{sc}(z \mid m)\operatorname{nc}(z \mid m)\left((1-m)\,z - E(\operatorname{am}(z \mid m) \mid m)\right)}{2\,m}$$

$$\frac{\partial \operatorname{dn}(z \mid m)}{\partial m} == \frac{1}{2\,(1-m)}\left(\operatorname{sn}(z \mid m)\operatorname{cn}(z \mid m)\left((m-1)\,z + E(\operatorname{am}(z \mid m) \mid m) - \operatorname{dn}(z \mid m)\operatorname{sc}(z \mid m)\right)\right)$$

$$\frac{\partial \operatorname{ds}(z \mid m)}{\partial m} == \frac{1}{2\,m\,(1-m)}\left(\operatorname{cs}(z \mid m)\operatorname{ns}(z \mid m)\left((m-1)\,z + E(\operatorname{am}(z \mid m) \mid m) - m\operatorname{dn}(z \mid m)\operatorname{sc}(z \mid m)\right)\right)$$

$$\frac{\partial \operatorname{nc}(z \mid m)}{\partial m} == \frac{1}{2\,m\,(1-m)}\left(\operatorname{sc}(z \mid m)\operatorname{dc}(z \mid m)\left((1-m)\,z - E(\operatorname{am}(z \mid m) \mid m) + m\operatorname{cd}(z \mid m)\operatorname{sn}(z \mid m)\right)\right)$$

$$\frac{\partial \operatorname{nd}(z \mid m)}{\partial m} == \frac{\operatorname{sd}(z \mid m)\operatorname{cd}(z \mid m)\left((1-m)\,z - E(\operatorname{am}(z \mid m) \mid m) + \operatorname{dn}(z \mid m)\operatorname{sc}(z \mid m)\right)}{2\,(1-m)}$$

$$\frac{\partial \operatorname{ns}(z \mid m)}{\partial m} == \frac{1}{2\,m\,(1-m)}\left(\operatorname{ds}(z \mid m)\operatorname{cs}(z \mid m)\left(-(1-m)\,z + E(\operatorname{am}(z \mid m) \mid m) - m\operatorname{sn}(z \mid m)\operatorname{cd}(z \mid m)\right)\right)$$

$$\frac{\partial \operatorname{sc}(z \mid m)}{\partial m} == \frac{\operatorname{nc}(z \mid m)\operatorname{dc}(z \mid m)\left((1-m)\,z - E(\operatorname{am}(z \mid m) \mid m) + m\operatorname{cd}(z \mid m)\operatorname{sn}(z \mid m)\right)}{2\,m\,(1-m)}$$

$$\frac{\partial \operatorname{sd}(z \mid m)}{\partial m} == \frac{\operatorname{cd}(z \mid m)\operatorname{nd}(z \mid m)\left((1-m)\,z - E(\operatorname{am}(z \mid m) \mid m) + m\operatorname{dn}(z \mid m)\operatorname{sc}(z \mid m)\right)}{2\,m\,(1-m)}$$

$$\frac{\partial \operatorname{sn}(z \mid m)}{\partial m} == \frac{1}{2\,m\,(1-m)}\left(\operatorname{dn}(z \mid m)\operatorname{cn}(z \mid m)\left((1-m)\,z - E(\operatorname{am}(z \mid m) \mid m) + m\operatorname{cd}(z \mid m)\operatorname{sn}(z \mid m)\right)\right).$$

### Integration

The indefinite integrals of the twelve Jacobi functions $\operatorname{cd}(z \mid m)$, $\operatorname{cn}(z \mid m)$, $\operatorname{cs}(z \mid m)$, $\operatorname{dc}(z \mid m)$, $\operatorname{dn}(z \mid m)$, $\operatorname{ds}(z \mid m)$, $\operatorname{nc}(z \mid m)$, $\operatorname{nd}(z \mid m)$, $\operatorname{ns}(z \mid m)$, $\operatorname{sc}(z \mid m)$, $\operatorname{sd}(z \mid m)$, and $\operatorname{sn}(z \mid m)$ with respect to variable $z$ can be expressed through Jacobi and elementary functions by the following
formulas:

$$\int \operatorname{cd}(z \mid m)\, dz == \frac{\log\left(\operatorname{nd}(z \mid m) + \sqrt{m}\,\operatorname{sd}(z \mid m)\right)}{\sqrt{m}}$$

$$\int \operatorname{cn}(z \mid m)\, dz = \frac{\cos^{-1}(\operatorname{dn}(z \mid m))\, \operatorname{sn}(z \mid m)}{\sqrt{1 - \operatorname{dn}(z \mid m)^2}}$$

$$\int \operatorname{cs}(z \mid m)\, dz = \log(\operatorname{ns}(z \mid m) - \operatorname{ds}(z \mid m))$$

$$\int \operatorname{dc}(z \mid m)\, dz = \log(\operatorname{nc}(z \mid m) + \operatorname{sc}(z \mid m))$$

$$\int \operatorname{dn}(z \mid m)\, dz = \operatorname{am}(z \mid m)$$

$$\int \operatorname{ds}(z \mid m)\, dz = \log\left(\frac{1 - \operatorname{cn}(z \mid m)}{\operatorname{sn}(z \mid m)}\right)$$

$$\int \operatorname{nc}(z \mid m)\, dz = \frac{\log\left(\operatorname{dc}(z \mid m) + \sqrt{1-m}\, \operatorname{sc}(z \mid m)\right)}{\sqrt{1-m}}$$

$$\int \operatorname{nd}(z \mid m)\, dz = \frac{\sqrt{1 - \operatorname{cd}(z \mid m)^2}\, \cos^{-1}(\operatorname{cd}(z \mid m))}{(1-m)\, \operatorname{sd}(z \mid m)}$$

$$\int \operatorname{ns}(z \mid m)\, dz = \log(\operatorname{ds}(z \mid m) - \operatorname{cs}(z \mid m))$$

$$\int \operatorname{sc}(z \mid m)\, dz = \frac{\log\left(\operatorname{dc}(z \mid m) + \sqrt{1-m}\, \operatorname{nc}(z \mid m)\right)}{\sqrt{1-m}}$$

$$\int \operatorname{sd}(z \mid m)\, dz = -\frac{\sin^{-1}\left(\sqrt{m}\, \operatorname{cd}(z \mid m)\right) \sqrt{1 - m\, \operatorname{cd}(z \mid m)^2}\, \operatorname{dn}(z \mid m)}{(1-m)\sqrt{m}}$$

$$\int \operatorname{sn}(z \mid m)\, dz = \frac{1}{\sqrt{m}} \log\left(\operatorname{dn}(z \mid m) - \sqrt{m}\, \operatorname{cn}(z \mid m)\right).$$

### Differential equations

The Jacobi amplitude $\operatorname{am}(z \mid m)$ satisfies the following differential equations:

$$\frac{\partial w(z)}{\partial z} - \operatorname{dn}(z \mid m) = 0 /;\ w(z) = \operatorname{am}(z \mid m)$$

$$w^{(3)}(z)\left(1 - w'(z)^2\right) - w'(z)^5 + 2\, w'(z)^3 + w''(z)^2\, w'(z) - w'(z) = 0 /;\ w(z) = \operatorname{am}(z \mid m).$$

All Jacobi functions $\operatorname{am}(z \mid m)$, $\operatorname{cd}(z \mid m)$, $\operatorname{cn}(z \mid m)$, $\operatorname{cs}(z \mid m)$, $\operatorname{dc}(z \mid m)$, $\operatorname{dn}(z \mid m)$, $\operatorname{ds}(z \mid m)$, $\operatorname{nc}(z \mid m)$, $\operatorname{nd}(z \mid m)$, $\operatorname{ns}(z \mid m)$, $\operatorname{sc}(z \mid m)$, $\operatorname{sd}(z \mid m)$, and $\operatorname{sn}(z \mid m)$ are special solutions of ordinary second-order nonlinear differential equations:

$$w''(z)^2 + \left(w'(z)^2 + m - 1\right)\left(w'(z)^2 - 1\right) = 0 /;\ w(z) = \operatorname{am}(z \mid m)$$

$$w''(z) - w(z)\left(2\, m\, w(z)^2 - m - 1\right) = 0 /;\ w(z) = \operatorname{cd}(z \mid m)$$

$$w''(z) + w(z)\left(2\, m\, w(z)^2 - 2\, m + 1\right) = 0 \,/;\, w(z) = \mathrm{cn}(z \mid m)$$

$$w''(z) - w(z)\left(2\, w(z)^2 - m + 2\right) = 0 \,/;\, w(z) = \mathrm{cs}(z \mid m)$$

$$w''(z) - w(z)\left(2\, w(z)^2 - m - 1\right) = 0 \,/;\, w(z) = \mathrm{dc}(z \mid m)$$

$$w''(z) + w(z)\left(2\, w(z)^2 + m - 2\right) = 0 \,/;\, w(z) = \mathrm{dn}(z \mid m)$$

$$w''(z) - w(z)\left(2\, w(z)^2 + 2\, m - 1\right) = 0 \,/;\, w(z) = \mathrm{ds}(z \mid m)$$

$$w''(z) - w(z)\left(2\,(1 - m)\, w(z)^2 + 2\, m - 1\right) = 0 \,/;\, w(z) = \mathrm{nc}(z \mid m)$$

$$w''(z) + w(z)\left(2\,(1 - m)\, w(z)^2 + m - 2\right) = 0 \,/;\, w(z) = \mathrm{nd}(z \mid m)$$

$$w''(z) - w(z)\left(2\, w(z)^2 - m - 1\right) = 0 \,/;\, w(z) = \mathrm{ns}(z \mid m)$$

$$w''(z) - w(z)\left(2\,(1 - m)\, w(z)^2 - m + 2\right) = 0 \,/;\, w(z) = \mathrm{sc}(z \mid m)$$

$$w''(z) + w(z)\left(2\, m\,(1 - m)\, w(z)^2 - 2\, m + 1\right) = 0 \,/;\, w(z) = \mathrm{sd}(z \mid m)$$

$$w''(z) - w(z)\left(2\, m\, w(z)^2 - m - 1\right) = 0 \,/;\, w(z) = \mathrm{sn}(z \mid m).$$

The twelve Jacobi functions $\mathrm{cd}(z \mid m)$, $\mathrm{cn}(z \mid m)$, $\mathrm{cs}(z \mid m)$, $\mathrm{dc}(z \mid m)$, $\mathrm{dn}(z \mid m)$, $\mathrm{ds}(z \mid m)$, $\mathrm{nc}(z \mid m)$, $\mathrm{nd}(z \mid m)$, $\mathrm{ns}(z \mid m)$, $\mathrm{sc}(z \mid m)$, $\mathrm{sd}(z \mid m)$, and $\mathrm{sn}(z \mid m)$ satisfy very complicated ordinary differential equations with respect to variable $m$, for example:

$$\begin{aligned}
&m^3 z^2 w(m)^{12} + \left(-3 z^2 m^3 - 3 z^2 m^2 - 1\right) w(m)^{10} + \left(3 z^2 m^3 + 9 z^2 m^2 + 3 z^2 m + 4\right) w(m)^8 + \\
&\quad \left(-z^2 m^3 - 9 z^2 m^2 - 9 z^2 m - z^2 - 6\right) w(m)^6 + \left(3 m^2 z^2 + 9 m z^2 + 3 z^2 + 4\right) w(m)^4 + \left(-3 m z^2 - 3 z^2 - 1\right) w(m)^2 + \\
&\quad \left(-64 (m-1)^2 m^4 w(m)^6 + 64 (m-1)^2 m^3 (m+1) w(m)^4 - 16 (m-1)^2 m^2 (m+1)^2 w(m)^2\right) w'(m)^4 + \\
&\quad \Big(64 (m-1) m^4 w(m)^7 - 32 (m-1) m^2 (m+2)(3 m-1) w(m)^5 + \\
&\qquad 32 (m-1) m \left(m^3 + 7 m^2 - m - 1\right) w(m)^3 - 32 (m-1) m (m+1)(2 m-1) w(m)\Big) w'(m)^3 + z^2 + \\
&\quad \Big(-16 m^2 \left(m^2 - m + 1\right) w(m)^8 + 8 m (m+1)\left(4 m^2 - m + 1\right) w(m)^6 - 16 (m+1)\left(m^3 + 5 m^2 - 4 m + 1\right) w(m)^4 + \\
&\qquad 8\left(7 m^3 + 12 m^2 - 15 m + 4\right) w(m)^2 - 16 \left(2 m - 1\right)^2\Big) w'(m)^2 + \Big(-16 (m-1)^2 m^4 w(m)^8 + 32 (m-1)^2 m^3 (m+1) w(m)^6 - \\
&\qquad 16 (m-1)^2 m^2 \left(m^2 + 4 m + 1\right) w(m)^4 + 32 (m-1)^2 m^2 (m+1) w(m)^2 - 16 (m-1)^2 m^2\Big) w''(m)^2 + \\
&\quad \left(-8 m^2 w(m)^9 + 8 (m+1)(3 m-1) w(m)^7 - 24\left(m^2 + 2 m - 1\right) w(m)^5 + 8\left(m^2 + 6 m - 3\right) w(m)^3 - 8 (2 m-1) w(m)\right) w'(m) + \\
&\quad \Big(-8 (m-1) m^2 w(m)^9 + 8 (m-1) m (3 m+1) w(m)^7 - 24 (m-1) m (m+1) w(m)^5 + \\
&\qquad 8 (m-1) m (m+3) w(m)^3 - 8 (m-1) m\, w(m) + \Big(64 (m-1)^2 m^4 w(m)^7 - 96 (m-1)^2 m^3 (m+1) w(m)^5 + \\
&\qquad\quad 32 (m-1)^2 m^2 \left(m^2 + 4 m + 1\right) w(m)^3 - 32 (m-1)^2 m^2 (m+1) w(m)\Big) w'(m)^2 + \\
&\qquad \Big(-32 (m-1) m^4 w(m)^8 + 32 (m-1) m^2 \left(2 m^2 + 3 m - 1\right) w(m)^6 - 32 (m-1) m \left(m^3 + 6 m^2 - 1\right) w(m)^4 + \\
&\qquad\quad 32 (m-1) m \left(3 m^2 + 3 m - 2\right) w(m)^2 - 32 (m-1) m (2 m-1)\Big) w'(m)\Big) w''(m) \,/;\, w(m) = \mathrm{sn}(z \mid m).
\end{aligned}$$

## Zeros

All Jacobi functions $\mathrm{am}(z \mid m)$, $\mathrm{cd}(z \mid m)$, $\mathrm{cn}(z \mid m)$, $\mathrm{cs}(z \mid m)$, $\mathrm{dc}(z \mid m)$, $\mathrm{dn}(z \mid m)$, $\mathrm{ds}(z \mid m)$, $\mathrm{nc}(z \mid m)$, $\mathrm{nd}(z \mid m)$, $\mathrm{ns}(z \mid m)$, $\mathrm{sc}(z \mid m)$, $\mathrm{sd}(z \mid m)$, and $\mathrm{sn}(z \mid m)$ are equal to zero in the points $z = k\, K(m) + n\, i\, K(1 - m)$, where $K(m)$ is the complete elliptic integral of the first kind and $k$, $n$ are even or odd integers:

$$\mathrm{am}(2\, s\, i\, K(1 - m) \mid m) = 0 \,/;\, s \in \mathbb{Z}$$

$\text{cd}((2r+1)K(m)+2si K(1-m) \mid m) == 0 \,/;\, \{r, s\} \in \mathbb{Z}$

$\text{cn}((2r+2s+1)K(m)+2si K(1-m) \mid m) == 0 \,/;\, \{r, s\} \in \mathbb{Z}$

$\text{cs}((2r+1)K(m)+2si K(1-m) \mid m) == 0 \,/;\, \{r, s\} \in \mathbb{Z}$

$\text{dc}((2r+1)K(m)+(2s+1)i K(1-m) \mid m) == 0 \,/;\, \{r, s\} \in \mathbb{Z}$

$\text{dn}((2r+1)K(m)+(2s+1)i K(1-m) \mid m) == 0 \,/;\, \{r, s\} \in \mathbb{Z}$

$\text{ds}((2r+1)K(m)+(2s+1)i K(1-m) \mid m) == 0 \,/;\, \{r, s\} \in \mathbb{Z}$

$\text{nc}(2rK(m)+(2s+1)i K(1-m) \mid m) == 0 \,/;\, \{r, s\} \in \mathbb{Z}$

$\text{nd}(2rK(m)+(2s+1)i K(1-m) \mid m) == 0 \,/;\, \{r, s\} \in \mathbb{Z}$

$\text{ns}(2rK(m)+(2s+1)i K(1-m) \mid m) == 0 \,/;\, \{r, s\} \in \mathbb{Z}$

$\text{sc}(2rK(m)+2si K(1-m) \mid m) == 0 \,/;\, \{r, s\} \in \mathbb{Z}$

$\text{sd}(2rK(m)+2si K(1-m) \mid m) == 0 \,/;\, \{r, s\} \in \mathbb{Z}$

$\text{sn}(2rK(m)+2si K(1-m) \mid m) == 0 \,/;\, \{r, s\} \in \mathbb{Z}.$

## Applications of Jacobi elliptic functions

Applications of Jacobi elliptic functions include conformal mappings, electrostatics and magnetostatics, fluid dynamics, mechanics of tops, nonlinear integrable equations, the Ising model of statistical mechanics, celestial mechanics, closed-form solutions for nonlinear Schr ödinger equations, analysis of exactly solvable chaos-exhibiting sequences, solution of equations of motion for quartic potentials, and elliptic function theory.

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