Introductions to KroneckerDelta2

Introduction to the tensor functions

General

The tensor functions discrete delta and Kronecker delta first appeared in the works L. Kronecker (1866, 1903) and T. Levi–Civita (1896).

Definitions of the tensor functions

For all possible values of their arguments, the discrete delta functions $\delta(n)$ and $\delta(n_1, n_2, \ldots)$, Kronecker delta functions δ_n and $\delta_{n_1,n_2,\ldots}$, and signature (Levi–Civita symbol) $\varepsilon_{n_1,n_2,\ldots,n_d}$ are defined by the formulas:

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & \text{True} \end{cases}$$

$$\delta(n_1, n_2, \ldots) = 1 \;/;\; n_1 = n_2 = \ldots = 0$$

$$\delta(n_1, n_2, \ldots) = 0 \;/;\; \neg\, n_1 = n_2 = \ldots = 0$$

$$\delta_n = \begin{cases} 1 & n = 0 \\ 0 & \text{True} \end{cases}$$

$$\delta_{n_1,n_2,\ldots} = 1 \;/;\; n_1 = n_2 = \ldots \land n_1 \in \mathbb{Q} \land n_2 \in \mathbb{Q} \land \ldots$$

$$\delta_{n_1,n_2,\ldots} = 0 \;/;\; \neg\, n_1 = n_2 = \ldots.$$

In other words, the Kronecker delta function is equal to 1 if all its arguments are equal.

In the case of one variable, the discrete delta function $\delta(n)$ coincides with the Kronecker delta function δ_n. In the case of several variables, the discrete delta function $\delta(n_1, n_2, \ldots, n_m)$ coincides with Kronecker delta function $\delta_{n_1,n_2,\ldots,n_m,0}$:

$$\delta(n) = \delta_n$$

$$\delta(n_1, n_2, \ldots, n_m) = \delta_{n_1,n_2,\ldots,n_m,0}$$

$$\varepsilon_{n_1,n_2,\ldots,n_d} = (-1)^t$$

$$\varepsilon_{n_1,n_2,\ldots,n_d} = 0 \;/;\; n_i = n_j \land 1 \le i \le d \land 1 \le j \le d,$$

where t is the number of permutations needed to go from the sorted version of $\{n_1, n_2, \ldots, n_d\}$ to $\{n_1, n_2, \ldots, n_d\}$.

Connections within the group of tensor functions and with other function groups

Representations through equivalent functions

The tensor functions $\delta(n)$, $\delta(n_1, n_2, \ldots)$, δ_n, and $\delta_{n_1,n_2,\ldots}$ have the following representations through equivalent functions:

$\delta(n)$ and $\delta(n_1, n_2, \ldots, n_m)$	δ_n and $\delta_{n_1,n_2,\ldots,n_m}$
$\delta(n) = \delta(n_1, n_2, \ldots, n_m)$ /; $n_1 = n \wedge m = 1$	$\delta_n = \delta(n)$
$\delta(n) = \delta_{n,0}$	$\delta_{n_1,n_2} = \delta_{n_1-n_2}$
$\delta(n) = \delta_n$	$\delta_{n_1,n_2,\ldots,n_m} = \delta_{n_1-n_m,n_2-n_m,\ldots,n_{m-1}-n_m}$
$\delta(n_1, n_2, \ldots, n_m) = \delta_{n_1,n_2,\ldots,n_m,0}$	$\delta_{n_1,n_2,\ldots,n_m,0} = \delta(n_1, n_2, \ldots, n_m)$

The best-known properties and formulas of the tensor functions

Simple values at zero and infinity

The tensor functions $\delta(n)$, $\delta(n_1, n_2, \ldots)$, δ_n, and $\delta_{n_1,n_2,\ldots}$ can have unit values at infinity:

$\delta(n)$ and $\delta(n_1, n_2)$	δ_n and δ_{n_1,n_2}
$\delta(\infty) = 0$	$\delta_\infty = 0$
$\delta(-\infty) = 0$	$\delta_{-\infty} = 0$
$\delta(\infty, -\infty) = 0$	$\delta_{\infty,-\infty} = 0$
$\delta(-\infty, \infty) = 0$	$\delta_{-\infty,\infty} = 0$

Specific values for specialized variables

The tensor functions $\delta(n)$, $\delta(n_1, n_2, \ldots)$, δ_n, $\delta_{n_1,n_2,\ldots}$, and $\varepsilon_{n_1, n_2, \ldots, n_d}$ have the following values for some specialized variables:

$\delta(n)$	δ_n	ε_n
$\delta(0) = 1$	$\delta_0 = 1$	$\varepsilon_0 = 1$
$\delta(1) = 0$	$\delta_1 = 0$	$\varepsilon_1 = 1$
$\delta(-1) = 0$	$\delta_{-1} = 0$	$\varepsilon_{-1} = 1$
$\delta(2) = 0$	$\delta_2 = 0$	$\varepsilon_2 = 1$
…	…	…
$\delta(n) = 0$ /; $n \neq 0$	$\delta_n = 0$ /; $n \neq 0$	$\varepsilon_n = 1$

$\delta(n_1, n_2)$	δ_{n_1,n_2}	ε_{n_1, n_2}
$\delta(0, 0) = 1$	$\delta_{1,1} = 1$	$\varepsilon_{1, 1} = 0$
$\delta(1, 2) = 0$	$\delta_{1,2} = 0$	$\varepsilon_{1, 2} = 1$
$\delta(2, 1) = 0$	$\delta_{2,1} = 0$	$\varepsilon_{2, 1} = -1$
$\delta(2, 2) = 0$	$\delta_{2,2} = 1$	$\varepsilon_{2, 2} = 0$
…	…	…
$\delta(n, 0) = \delta_n$	$\delta_{n,0} = \delta_n$	$\varepsilon_{z, a} = -1$
$\delta(0, n) = \delta_n$	$\delta_{0,n} = \delta_n$	$\varepsilon_{a, z} = 1$

$\delta(n_1, n_2, n_3)$	δ_{n_1,n_2,n_3}	$\varepsilon_{n_1,n_2,n_3}$
$\delta(0, 0, 0) == 1$	$\delta_{1,1,1} == 1$	$\varepsilon_{1,1,2} == 0$
$\delta(0, 0, 1) == 0$	$\delta_{1,1,2} == 0$	$\varepsilon_{1,2,3} == 1$
$\delta(0, 1, 0) == 0$	$\delta_{1,2,1} == 0$	$\varepsilon_{1,3,2} == -1$
$\delta(1, 0, 0) == 0$	$\delta_{2,1,1} == 0$	$\varepsilon_{2,3,1} == 1$
$\delta(1, 1, 1) == 0$	$\delta_{1,1,1} == 1$	$\varepsilon_{2,1,3} == -1$
$\delta(2, 2, 2) == 0$	$\delta_{2,2,2} == 1$	$\varepsilon_{3,1,2} == 1$
$\delta(-1, -1, -1) == 0$	$\delta_{-1,-1,-1} == 1$	$\varepsilon_{3,2,1} == -1$

Analyticity

$\delta(n)$ and δ_n are nonanalytical functions defined over $\mathbb{C}$. Their possible values are 0 and 1.

$\delta(n_1, n_2, \ldots, n_m)$ and $\delta_{n_1,n_2,\ldots,n_m}$ are nonanalytical functions defined over $\mathbb{C}^m$. Their possible values are 0 and 1.

$\varepsilon_{n_1,n_2,\ldots,n_d}$ is a nonanalytical function, defined over the set of tuples of complex numbers with possible values 0 and ± 1.

Periodicity

The tensor functions $\delta(n)$, $\delta(n_1, n_2, \ldots)$, δ_n, $\delta_{n_1,n_2,\ldots}$, and $\varepsilon_{n_1,n_2,\ldots,n_d}$ do not have periodicity.

Parity and symmetry quasi-permutation symmetry

The tensor functions $\delta(n)$, $\delta(n_1, n_2, \ldots)$, δ_n, and $\delta_{n_1,n_2,\ldots}$ are even functions:

$$\delta(-n) == \delta(n)$$

$$\delta(-n_1, -n_2, \ldots, -n_m) == \delta(n_1, n_2, \ldots, n_m)$$

$$\delta_{-n} = \delta_n$$

$$\delta_{-n_1,-n_2,\ldots,-n_m} == \delta_{n_1,n_2,\ldots,n_m}.$$

The tensor functions $\delta(n_1, n_2, \ldots)$, $\delta_{n_1,n_2,\ldots}$, and $\varepsilon_{n_1,n_2,\ldots,n_d}$ have permutation symmetry, for example:

$\delta(n_1, n_2, \ldots)$	$\delta_{n_1,n_2,\ldots}$	$\varepsilon_{n_1,n_2,\ldots}$
$\delta(m, n) == \delta(n, m)$	$\delta_{m,n} == \delta_{n,m}$	$\varepsilon_{n_1,n_2,\ldots,n_d} == -\varepsilon_{n_2,n_1,\ldots,n_d}$
$\delta(n_1, n_2, \ldots, n_k, \ldots, n_j, \ldots, n_m) == \delta(n_1, n_2, \ldots, n_j, \ldots, n_k, \ldots, n_m) /; n_k \neq n_j \land k \neq j$	$\delta_{n_1,n_2,\ldots,n_k,\ldots,n_j,\ldots,n_m} == \delta_{n_1,n_2,\ldots,n_j,\ldots,n_k,\ldots,n_m} /; n_k \neq n_j \land k \neq j$	$\varepsilon_{n_1,n_2,\ldots,n_k,\ldots,n_j,\ldots,n_d} == -\varepsilon_{n_1,n_2,\ldots,n_j,\ldots,n_k,\ldots,n_d}$
$\delta(n_1, n_2, \ldots, n_m) == \delta(n_2, n_3, \ldots, n_d, n_1)$	$\delta_{n_1,n_2,\ldots,n_d} == \delta_{n_2,n_3,\ldots,n_d,n_1}$	$\varepsilon_{n_1,n_2,\ldots,n_d} == (-1)^{d+1}\, \varepsilon_{n_2,n_3,\ldots,n_d,n_1}$

Integral representations

The discrete delta function $\delta(n)$ and Kronecker delta function $\delta_{n,m}$ have the following integral representations along the interval $(0, 2\pi)$ and unit circle $|z| == 1$:

$$\delta(n) = \frac{1}{2\pi}\int_0^{2\pi} e^{i\,t\,n}\,dt$$

$$\delta_{n,m} = \frac{1}{2\pi}\int_0^{2\pi} e^{i\,t\,(n-m)}\,dt$$

$$\delta_{n,m} == \frac{1}{2\pi i}\int_{|z|==1} z^{n-m-1}\,dz.$$

Transformations

The tensor functions $\delta(n)$, $\delta(n_1, n_2, \ldots)$, δ_n, $\delta_{n_1,n_2,\ldots}$, and $\varepsilon_{n_1,n_2,\ldots,n_d}$ satisfy various identities, for example:

$$\delta(-n) == \delta(n)$$

$$\delta(-n_1, -n_2, \ldots, -n_m) == \delta(n_1, n_2, \ldots, n_m)$$

$$\delta_{-n} = \delta_n$$

$$\delta_{-n_1,-n_2,\ldots,-n_m} == \delta_{n_1,n_2,\ldots,n_m}$$

$$\delta_{-n,m} == \delta_{n,m} - \theta(|n| - |m|)\,\text{sgn}(n\,m) \;/;\; n \in \mathbb{Z} \land m \in \mathbb{Z}$$

$$\delta(n_1)\,\delta(n_2) == \delta(n_1, n_2)$$

$$\delta(n_1, n_2, \ldots, n_m)\,\delta(n_{m+1}, n_{m+2}, \ldots, n_{m+r}) == \delta(n_1, n_2, \ldots, n_m, n_{m+1}, n_{m+2}, \ldots, n_{m+r})$$

$$\delta_{n_1,n_2,\ldots,n_m}\,\delta_{n_{m+1},n_{m+2},\ldots,n_{m+r}} == \delta_{n_1,n_2,\ldots,n_m,n_{m+1},n_{m+2},\ldots,n_{m+r}} \;/;\; m \geq 2 \land r \geq 2$$

$$\varepsilon_{n_1,n_2,\ldots,n_d}\,\varepsilon_{m_1,m_2,\ldots,m_d} == -\sum_{\text{permutations}(m_1,m_2,\ldots,m_d)} \varepsilon_{m_1,m_2,\ldots,m_d} \prod_{k=1}^{d} \delta_{n_k,m_k}$$

$$\varepsilon_{n_1,n_2,\ldots,n_{r-1},n_r,n_{r+1},\ldots,n_d}\,\varepsilon_{n_1,n_2,\ldots,n_{r-1},m_r,m_{r+1},\ldots,m_d} == -\frac{(d-r)!\,r!}{d!} \sum_{\text{permutations}(m_r,m_{r+1},\ldots,m_d)} \varepsilon_{m_r,m_{r+1},\ldots,m_d} \prod_{k=r}^{d} \delta_{n_k,m_k}.$$

Complex characteristics

The tensor functions $\delta(n)$, $\delta(n_1, n_2, \ldots)$, δ_n, $\delta_{n_1,n_2,\ldots}$, and $\varepsilon_{n_1,n_2,\ldots,n_d}$ have the following complex characteristics:

	$\delta(n_1, n_2, \ldots, n_m)$	$\delta_{n_1,n_2,\ldots,n_m}$	$\varepsilon_{n_1,n_2,\ldots,n_d}$
Abs	$\lvert\delta(n_1, n_2, \ldots, n_m)\rvert == \delta(n_1, n_2, \ldots, n_m)$	$\lvert\delta_{n_1,n_2,\ldots,n_m}\rvert == \delta_{n_1,n_2,\ldots,n_m}$	$\lvert\varepsilon_{n_1,n_2,\ldots,n_d}\rvert == \sqrt{\varepsilon_{n_1,n}}$
Arg	$\text{Arg}(\delta(n_1, n_2, \ldots, n_m)) == \tan^{-1}(\delta(n_1, n_2, \ldots, n_m), 0)$	$\text{Arg}(\delta_{n_1,n_2,\ldots,n_m}) == \tan^{-1}(\delta_{n_1,n_2,\ldots,n_m}, 0)$	$\text{Arg}(\varepsilon_{n_1,n_2,\ldots,n_d}) ==$ tan
Re	$\text{Re}(\delta(n_1, n_2, \ldots, n_m)) == \delta(n_1, n_2, \ldots, n_m)$	$\text{Re}(\delta_{n_1,n_2,\ldots,n_m}) == \delta_{n_1,n_2,\ldots,n_m}$	$\text{Re}(\varepsilon_{n_1,n_2,\ldots,n_d}) == \varepsilon_{n_1,}$
Im	$\text{Im}(\delta(n_1, n_2, \ldots, n_m)) == 0$	$\text{Im}(\delta_{n_1,n_2,\ldots,n_m}) == 0$	$\text{Im}(\varepsilon_{n_1,n_2,\ldots,n_d}) == 0$
Conjugate	$\overline{\delta(n_1, n_2, \ldots, n_m)} == \delta(n_1, n_2, \ldots, n_m)$	$\overline{\delta_{n_1,n_2,\ldots,n_m}} == \delta_{n_1,n_2,\ldots,n_m}$	$\overline{\varepsilon_{n_1,n_2,\ldots,n_d}} == \varepsilon_{n_1,n_2,\ldots,}$

Differentiation

Differentiation of the tensor functions $\delta(n)$ and δ_n can be provided by the following formulas:

$$\frac{\partial \delta(n)}{\partial n} == 0$$

$$\frac{\partial \delta_n}{\partial n} = 0.$$

Fractional integro-differentiation of the tensor functions $\delta(n)$ and δ_n can be provided by the following formulas:

$$\frac{\partial^\alpha \delta(n)}{\partial n^\alpha} == \frac{n^{-\alpha}\, \delta(n)}{\Gamma(1-\alpha)}$$

$$\frac{\partial^\alpha \delta_n}{\partial n^\alpha} = \frac{n^{-\alpha}\, \delta_n}{\Gamma(1-\alpha)}.$$

Indefinite integration

Indefinite integration of the tensor functions $\delta(n)$ and δ_n can be provided by the following formulas:

$$\int \delta(z)\, dz == \delta(z)\, z$$

$$\int \delta_z\, dz = \delta_z\, z.$$

Summation

The following relations represent the sifting properties of the Kronecker and discrete delta functions:

$$\sum_{k=-\infty}^{\infty} \delta_{k,n}\, a_k = a_n$$

$$\sum_{k=-\infty}^{\infty} \delta(k,\, n)\, a_k = a_0.$$

There exist various formulas including finite summation of signature $\varepsilon_{n_1, n_2, \ldots, n_d}$, for example:

$$\sum_{\tau_1=1}^{n} \sum_{\tau_2=1}^{n} \ldots \sum_{\tau_r=1}^{n} \varepsilon_{\tau_1,\tau_2,\ldots,\tau_r,\nu_{r+1},\ldots,\nu_n}\, \varepsilon_{\tau_1,\tau_2,\ldots,\tau_r,\mu_{r+1},\ldots,\mu_n} == r! \sum_{\mu_{r+1}=1}^{n} \sum_{\mu_{r+2}=1}^{n} \ldots \sum_{\mu_n=1}^{n} \varepsilon_{\mu_{r+1},\ldots,\mu_n} \prod_{k=r+1}^{n} \delta_{\nu_k,\mu_k}.$$

Applications of the tensor functions

The tensor functions have numerous applications throughout mathematics, number theory, analysis, and other fields.

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