

# Introductions to Sin

## Introduction to the trigonometric functions

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### General

The six trigonometric functions sine  $\sin(z)$ , cosine  $\cos(z)$ , tangent  $\tan(z)$ , cotangent  $\cot(z)$ , cosecant  $\csc(z)$ , and secant  $\sec(z)$  are well known and among the most frequently used elementary functions. The most popular functions  $\sin(z)$ ,  $\cos(z)$ ,  $\tan(z)$ , and  $\cot(z)$  are taught worldwide in high school programs because of their natural appearance in problems involving angle measurement and their wide applications in the quantitative sciences.

The trigonometric functions share many common properties.

### Definitions of trigonometric functions

All trigonometric functions can be defined as simple rational functions of the exponential function of  $i z$ :

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$$

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}$$

$$\tan(z) = -\frac{i(e^{iz} - e^{-iz})}{e^{iz} + e^{-iz}}$$

$$\cot(z) = \frac{i(e^{iz} + e^{-iz})}{e^{iz} - e^{-iz}}$$

$$\csc(z) = \frac{2i}{e^{iz} - e^{-iz}}$$

$$\sec(z) = \frac{2}{e^{iz} + e^{-iz}}.$$

The functions  $\tan(z)$ ,  $\cot(z)$ ,  $\csc(z)$ , and  $\sec(z)$  can also be defined through the functions  $\sin(z)$  and  $\cos(z)$  using the following formulas:

$$\tan(z) = \frac{\sin(z)}{\cos(z)}$$

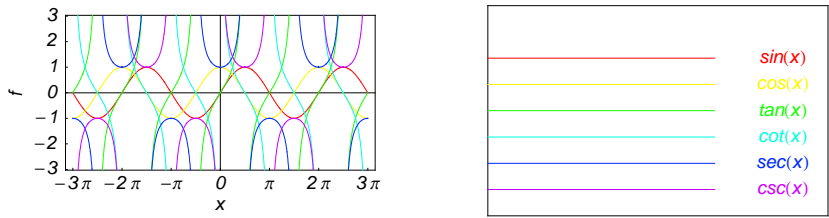
$$\cot(z) = \frac{\cos(z)}{\sin(z)}$$

$$\csc(z) = \frac{1}{\sin(z)}$$

$$\sec(z) = \frac{1}{\cos(z)}.$$

### A quick look at the trigonometric functions

Here is a quick look at the graphics for the six trigonometric functions along the real axis.



## Connections within the group of trigonometric functions and with other function groups

### Representations through more general functions

The trigonometric functions are particular cases of more general functions. Among these more general functions, four different classes of special functions are particularly relevant: Bessel, Jacobi, Mathieu, and hypergeometric functions.

For example,  $\sin(z)$  and  $\cos(z)$  have the following representations through Bessel, Mathieu, and hypergeometric functions:

$$\sin(z) = \sqrt{\frac{\pi z}{2}} J_{1/2}(z) \quad \sin(z) = -i \sqrt{\frac{\pi i z}{2}} I_{1/2}(i z) \quad \sin(z) = \sqrt{\frac{\pi z}{2}} Y_{-1/2}(z) \quad \sin(z) = \frac{i}{\sqrt{2\pi}} (\sqrt{i z} K_{1/2}(i z) - \sqrt{-i z} K_{1/2}(-i z))$$

$$\cos(z) = \sqrt{\frac{\pi z}{2}} J_{-1/2}(z) \quad \cos(z) = \sqrt{\frac{\pi i z}{2}} I_{-1/2}(i z) \quad \cos(z) = -\sqrt{\frac{\pi z}{2}} Y_{1/2}(z) \quad \cos(z) = \sqrt{\frac{i z}{2\pi}} K_{1/2}(i z) + \sqrt{-\frac{i z}{2\pi}} K_{1/2}(-i z)$$

$$\sin(z) = \text{Se}(1, 0, z) \quad \cos(z) = \text{Ce}(1, 0, z)$$

$$\sin(z) = z {}_0F_1\left(\frac{3}{2}; -\frac{z^2}{4}\right) \quad \cos(z) = {}_0F_1\left(\frac{1}{2}; -\frac{z^2}{4}\right).$$

On the other hand, all trigonometric functions can be represented as degenerate cases of the corresponding doubly periodic Jacobi elliptic functions when their second parameter is equal to 0 or 1:

$$\sin(z) = \text{sd}(z | 0) = \text{sn}(z | 0) \quad \sin(z) = -i \text{sc}(i z | 1) = -i \text{sd}(i z | 1)$$

$$\cos(z) = \text{cd}(z | 0) = \text{cn}(z | 0) \quad \cos(z) = \text{nc}(i z | 1) = \text{nd}(i z | 1)$$

$$\tan(z) = \text{sc}(z | 0) \quad \tan(z) = -i \text{sn}(i z | 1)$$

$$\cot(z) = \text{cs}(z | 0) \quad \cot(z) = i \text{ns}(i z | 1)$$

$$\csc(z) = \text{ds}(z | 0) = \text{ns}(z | 0) \quad \csc(z) = i \text{cs}(i z | 1) = i \text{ds}(i z | 1)$$

$$\sec(z) = \text{dc}(z | 0) = \text{nc}(z | 0) \quad \sec(z) = \text{cn}(i z | 1) = \text{dn}(i z | 1).$$

### Representations through related equivalent functions

Each of the six trigonometric functions can be represented through the corresponding hyperbolic function:

$$\sin(z) = -i \sinh(i z) \quad \sin(i z) = i \sinh(z)$$

$$\cos(z) = \cosh(i z) \quad \cos(i z) = \cosh(z)$$

$$\tan(z) = -i \tanh(i z) \quad \tan(i z) = i \tanh(z)$$

$$\cot(z) = i \coth(i z) \quad \cot(i z) = -i \coth(z)$$

$$\csc(z) = i \text{csch}(i z) \quad \csc(i z) = -i \text{csch}(z)$$

$$\sec(z) = \text{sech}(i z) \quad \sec(i z) = \text{sech}(z).$$

### Relations to inverse functions

Each of the six trigonometric functions is connected with its corresponding inverse trigonometric function by two formulas. One is a simple formula, and the other is much more complicated because of the multivalued nature of the inverse function:

$$\begin{aligned} \sin(\sin^{-1}(z)) &= z & \sin^{-1}(\sin(z)) &= z /; -\frac{\pi}{2} < \operatorname{Re}(z) < \frac{\pi}{2} \vee \operatorname{Re}(z) = -\frac{\pi}{2} \wedge \operatorname{Im}(z) \geq 0 \vee \operatorname{Re}(z) = \frac{\pi}{2} \wedge \operatorname{Im}(z) \leq 0 \\ \cos(\cos^{-1}(z)) &= z & \cos^{-1}(\cos(z)) &= z /; 0 < \operatorname{Re}(z) < \pi \vee \operatorname{Re}(z) = 0 \wedge \operatorname{Im}(z) \geq 0 \vee \operatorname{Re}(z) = \pi \wedge \operatorname{Im}(z) \leq 0 \\ \tan(\tan^{-1}(z)) &= z & \tan^{-1}(\tan(z)) &= z /; |\operatorname{Re}(z)| < \frac{\pi}{2} \vee \operatorname{Re}(z) = -\frac{\pi}{2} \wedge \operatorname{Im}(z) < 0 \vee \operatorname{Re}(z) = \frac{\pi}{2} \wedge \operatorname{Im}(z) > 0 \\ \cot(\cot^{-1}(z)) &= z & \cot^{-1}(\cot(z)) &= z /; |\operatorname{Re}(z)| < \frac{\pi}{2} \vee \operatorname{Re}(z) = -\frac{\pi}{2} \wedge \operatorname{Im}(z) < 0 \vee \operatorname{Re}(z) = \frac{\pi}{2} \wedge \operatorname{Im}(z) \geq 0 \\ \csc(\csc^{-1}(z)) &= z & \csc^{-1}(\csc(z)) &= z /; |\operatorname{Re}(z)| < \frac{\pi}{2} \vee \operatorname{Re}(z) = -\frac{\pi}{2} \wedge \operatorname{Im}(z) \leq 0 \vee \operatorname{Re}(z) = \frac{\pi}{2} \wedge \operatorname{Im}(z) \geq 0 \\ \sec(\sec^{-1}(z)) &= z & \sec^{-1}(\sec(z)) &= z /; 0 < \operatorname{Re}(z) < \pi \vee \operatorname{Re}(z) = 0 \wedge \operatorname{Im}(z) \geq 0 \vee \operatorname{Re}(z) = \pi \wedge \operatorname{Im}(z) \leq 0. \end{aligned}$$

### Representations through other trigonometric functions

Each of the six trigonometric functions can be represented by any other trigonometric function as a rational function of that function with linear arguments. For example, the sine function can be representative as a group-defining function because the other five functions can be expressed as follows:

$$\begin{aligned} \cos(z) &= \sin\left(\frac{\pi}{2} - z\right) & \cos^2(z) &= 1 - \sin^2(z) \\ \tan(z) &= \frac{\sin(z)}{\cos(z)} = \frac{\sin(z)}{\sin\left(\frac{\pi}{2} - z\right)} & \tan^2(z) &= \frac{\sin^2(z)}{1 - \sin^2(z)} \\ \cot(z) &= \frac{\cos(z)}{\sin(z)} = \frac{\sin\left(\frac{\pi}{2} - z\right)}{\sin(z)} & \cot^2(z) &= \frac{1 - \sin^2(z)}{\sin^2(z)} \\ \csc(z) &= \frac{1}{\sin(z)} & \csc^2(z) &= \frac{1}{\sin^2(z)} \\ \sec(z) &= \frac{1}{\cos(z)} = \frac{1}{\sin\left(\frac{\pi}{2} - z\right)} & \sec^2(z) &= \frac{1}{1 - \sin^2(z)}. \end{aligned}$$

All six trigonometric functions can be transformed into any other trigonometric function of this group if the argument  $z$  is replaced by  $p\pi/2 + qz$  with  $q^2 = 1 \wedge p \in \mathbb{Z}$ :

$$\begin{aligned} \sin(-z - 2\pi) &= -\sin(z) & \sin(z - 2\pi) &= \sin(z) \\ \sin\left(-z - \frac{3\pi}{2}\right) &= \cos(z) & \sin\left(z - \frac{3\pi}{2}\right) &= \cos(z) \\ \sin(-z - \pi) &= \sin(z) & \sin(z - \pi) &= -\sin(z) \\ \sin\left(-z - \frac{\pi}{2}\right) &= -\cos(z) & \sin\left(z - \frac{\pi}{2}\right) &= -\cos(z) \\ \sin\left(z + \frac{\pi}{2}\right) &= \cos(z) & \sin\left(\frac{\pi}{2} - z\right) &= \cos(z) \\ \sin(z + \pi) &= -\sin(z) & \sin(\pi - z) &= \sin(z) \\ \sin\left(z + \frac{3\pi}{2}\right) &= -\cos(z) & \sin\left(\frac{3\pi}{2} - z\right) &= -\cos(z) \\ \sin(z + 2\pi) &= \sin(z) & \sin(2\pi - z) &= -\sin(z) \end{aligned}$$

$$\begin{aligned} \cos(-z - 2\pi) &= \cos(z) & \cos(z - 2\pi) &= \cos(z) \\ \cos\left(-z - \frac{3\pi}{2}\right) &= \sin(z) & \cos\left(z - \frac{3\pi}{2}\right) &= -\sin(z) \\ \cos(-z - \pi) &= -\cos(z) & \cos(z - \pi) &= -\cos(z) \\ \cos\left(-z - \frac{\pi}{2}\right) &= -\sin(z) & \cos\left(z - \frac{\pi}{2}\right) &= \sin(z) \\ \cos\left(z + \frac{\pi}{2}\right) &= -\sin(z) & \cos\left(\frac{\pi}{2} - z\right) &= \sin(z) \\ \cos(z + \pi) &= -\cos(z) & \cos(\pi - z) &= -\cos(z) \\ \cos\left(z + \frac{3\pi}{2}\right) &= \sin(z) & \cos\left(\frac{3\pi}{2} - z\right) &= -\sin(z) \\ \cos(z + 2\pi) &= \cos(z) & \cos(2\pi - z) &= \cos(z) \end{aligned}$$

$$\begin{aligned} \tan(-z - \pi) &= -\tan(z) & \tan(z - \pi) &= \tan(z) \\ \tan\left(-z - \frac{\pi}{2}\right) &= \cot(z) & \tan\left(z - \frac{\pi}{2}\right) &= -\cot(z) \\ \tan\left(z + \frac{\pi}{2}\right) &= -\cot(z) & \tan\left(\frac{\pi}{2} - z\right) &= \cot(z) \\ \tan(z + \pi) &= \tan(z) & \tan(\pi - z) &= -\tan(z) \end{aligned}$$

$$\begin{aligned} \cot(-z - \pi) &= -\cot(z) & \cot(z - \pi) &= \cot(z) \\ \cot\left(-z - \frac{\pi}{2}\right) &= \tan(z) & \cot\left(z - \frac{\pi}{2}\right) &= -\tan(z) \\ \cot\left(z + \frac{\pi}{2}\right) &= -\tan(z) & \cot\left(\frac{\pi}{2} - z\right) &= \tan(z) \\ \cot(z + \pi) &= \cot(z) & \cot(\pi - z) &= -\cot(z) \end{aligned}$$

$$\begin{aligned} \csc(-z - 2\pi) &= -\csc(z) & \csc(z - 2\pi) &= \csc(z) \\ \csc\left(-z - \frac{3\pi}{2}\right) &= \sec(z) & \csc\left(z - \frac{3\pi}{2}\right) &= \sec(z) \\ \csc(-z - \pi) &= \csc(z) & \csc(z - \pi) &= -\csc(z) \\ \csc\left(-z - \frac{\pi}{2}\right) &= -\sec(z) & \csc\left(z - \frac{\pi}{2}\right) &= -\sec(z) \\ \csc\left(z + \frac{\pi}{2}\right) &= \sec(z) & \csc\left(\frac{\pi}{2} - z\right) &= \sec(z) \\ \csc(z + \pi) &= -\csc(z) & \csc(\pi - z) &= \csc(z) \\ \csc\left(z + \frac{3\pi}{2}\right) &= -\sec(z) & \csc\left(\frac{3\pi}{2} - z\right) &= -\sec(z) \\ \csc(z + 2\pi) &= \csc(z) & \csc(2\pi - z) &= -\csc(z) \end{aligned}$$

$$\begin{aligned} \sec(-z - 2\pi) &= \sec(z) & \sec(z - 2\pi) &= \sec(z) \\ \sec\left(-z - \frac{3\pi}{2}\right) &= \csc(z) & \sec\left(z - \frac{3\pi}{2}\right) &= -\csc(z) \\ \sec(-z - \pi) &= -\sec(z) & \sec(z - \pi) &= -\sec(z) \\ \sec\left(-z - \frac{\pi}{2}\right) &= -\csc(z) & \sec\left(z - \frac{\pi}{2}\right) &= \csc(z) \\ \sec\left(z + \frac{\pi}{2}\right) &= -\csc(z) & \sec\left(\frac{\pi}{2} - z\right) &= \csc(z) \\ \sec(z + \pi) &= -\sec(z) & \sec(\pi - z) &= -\sec(z) \\ \sec\left(z + \frac{3\pi}{2}\right) &= \csc(z) & \sec\left(\frac{3\pi}{2} - z\right) &= -\csc(z) \\ \sec(z + 2\pi) &= \sec(z) & \sec(2\pi - z) &= \sec(z). \end{aligned}$$

## The best-known properties and formulas for trigonometric functions

### Real values for real arguments

For real values of argument  $z$ , the values of all the trigonometric functions are real (or infinity).

In the points  $z = 2\pi n/m$ ;  $n \in \mathbb{Z} \wedge m \in \mathbb{Z}$ , the values of trigonometric functions are algebraic. In several cases they can even be rational numbers or integers (like  $\sin(\pi/2) = 1$  or  $\sin(\pi/6) = 1/2$ ). The values of trigonometric functions can be expressed using only square roots if  $n \in \mathbb{Z}$  and  $m$  is a product of a power of 2 and distinct Fermat primes  $\{3, 5, 17, 257, \dots\}$ .

### Simple values at zero

All trigonometric functions have rather simple values for arguments  $z = 0$  and  $z = \pi/2$ :

$$\begin{aligned}\sin(0) &= 0 & \sin\left(\frac{\pi}{2}\right) &= 1 \\ \cos(0) &= 1 & \cos\left(\frac{\pi}{2}\right) &= 0 \\ \tan(0) &= 0 & \tan\left(\frac{\pi}{2}\right) &= \infty \\ \cot(0) &= \infty & \cot\left(\frac{\pi}{2}\right) &= 0 \\ \csc(0) &= \infty & \csc\left(\frac{\pi}{2}\right) &= 1 \\ \sec(0) &= 1 & \sec\left(\frac{\pi}{2}\right) &= \infty.\end{aligned}$$

### Analyticity

All trigonometric functions are defined for all complex values of  $z$ , and they are analytical functions of  $z$  over the whole complex  $z$ -plane and do not have branch cuts or branch points. The two functions  $\sin(z)$  and  $\cos(z)$  are entire functions with an essential singular point at  $z = \infty$ . All other trigonometric functions are meromorphic functions with simple poles at points  $z = \pi k$ ;  $k \in \mathbb{Z}$  for  $\csc(z)$  and  $\cot(z)$ , and at points  $z = \pi/2 + \pi k$ ;  $k \in \mathbb{Z}$  for  $\sec(z)$  and  $\tan(z)$ .

### Periodicity

All trigonometric functions are periodic functions with a real period ( $2\pi$  or  $\pi$ ):

$$\begin{aligned}\sin(z) &= \sin(z + 2\pi) & \sin(z + 2\pi k) &= \sin(z) /; k \in \mathbb{Z} \\ \cos(z) &= \cos(z + 2\pi) & \cos(z + 2\pi k) &= \cos(z) /; k \in \mathbb{Z} \\ \tan(z) &= \tan(z + \pi) & \tan(z + \pi k) &= \tan(z) /; k \in \mathbb{Z} \\ \cot(z) &= \cot(z + \pi) & \cot(z + \pi k) &= \cot(z) /; k \in \mathbb{Z} \\ \csc(z) &= \csc(z + 2\pi) & \csc(z + 2\pi k) &= \csc(z) /; k \in \mathbb{Z} \\ \sec(z) &= \sec(z + 2\pi) & \sec(z + 2\pi k) &= \sec(z) /; k \in \mathbb{Z}.\end{aligned}$$

### Parity and symmetry

All trigonometric functions have parity (either odd or even) and mirror symmetry:

$$\begin{aligned}\sin(-z) &= -\sin(z) & \sin(\bar{z}) &= \overline{\sin(z)} \\ \cos(-z) &= \cos(z) & \cos(\bar{z}) &= \overline{\cos(z)} \\ \tan(-z) &= -\tan(z) & \tan(\bar{z}) &= \overline{\tan(z)} \\ \cot(-z) &= -\cot(z) & \cot(\bar{z}) &= \overline{\cot(z)} \\ \csc(-z) &= -\csc(z) & \csc(\bar{z}) &= \overline{\csc(z)} \\ \sec(-z) &= \sec(z) & \sec(\bar{z}) &= \overline{\sec(z)}.\end{aligned}$$

### Simple representations of derivatives

The derivatives of all trigonometric functions have simple representations that can be expressed through other trigonometric functions:

$$\begin{aligned} \frac{\partial \sin(z)}{\partial z} &= \cos(z) & \frac{\partial \cos(z)}{\partial z} &= -\sin(z) & \frac{\partial \tan(z)}{\partial z} &= \sec^2(z) \\ \frac{\partial \cot(z)}{\partial z} &= -\csc^2(z) & \frac{\partial \csc(z)}{\partial z} &= -\cot(z) \csc(z) & \frac{\partial \sec(z)}{\partial z} &= \sec(z) \tan(z). \end{aligned}$$

### Simple differential equations

The solutions of the simplest second-order linear ordinary differential equation with constant coefficients can be represented through  $\sin(z)$  and  $\cos(z)$ :

$$\begin{aligned} w''(z) + w(z) &= 0 /; w(z) = \cos(z) \wedge w(0) = 1 \wedge w'(0) = 0 \\ w''(z) + w(z) &= 0 /; w(z) = \sin(z) \wedge w(0) = 0 \wedge w'(0) = 1 \\ w''(z) + w(z) &= 0 /; w(z) = c_1 \cos(z) + c_2 \sin(z). \end{aligned}$$

All six trigonometric functions satisfy first-order nonlinear differential equations:

$$\begin{aligned} w'(z) - \sqrt{1 - (w(z))^2} &= 0 /; w(z) = \sin(z) \wedge w(0) = 0 \wedge |\operatorname{Re}(z)| < \frac{\pi}{2} \\ w'(z) - \sqrt{1 - (w(z))^2} &= 0 /; w(z) = \cos(z) \wedge w(0) = 1 \wedge |\operatorname{Re}(z)| < \frac{\pi}{2} \\ w'(z) - w(z)^2 - 1 &= 0 /; w(z) = \tan(z) \wedge w(0) = 0 \\ w'(z) + w(z)^2 + 1 &= 0 /; w(z) = \cot(z) \wedge w\left(\frac{\pi}{2}\right) = 0 \\ w'(z)^2 - w(z)^4 + w(z)^2 &= 0 /; w(z) = \csc(z) \\ w'(z)^2 - w(z)^4 + w(z)^2 &= 0 /; w(z) = \sec(z). \end{aligned}$$

## Applications of trigonometric functions

### Triangle theorems

The prime application of the trigonometric functions are triangle theorems. In a triangle,  $a$ ,  $b$ , and  $c$  represent the lengths of the sides opposite to the angles,  $\Delta$  the area,  $R$  the circumradius, and  $r$  the inradius. Then the following identities hold:

$$\begin{aligned} \alpha + \beta + \gamma &= \pi \\ \frac{\sin(\alpha)}{a} &= \frac{\sin(\beta)}{b} = \frac{\sin(\gamma)}{c} \\ \sin(\alpha) \sin(\beta) \sin(\gamma) &= \frac{\Delta}{2R^2} & \sin(\alpha) &= \frac{2\Delta}{bc} \\ \cos(\alpha) &= \frac{b^2 + c^2 - a^2}{2bc} & \cot(\alpha) &= \frac{b^2 + c^2 - a^2}{4\Delta} \\ \sin\left(\frac{\alpha}{2}\right) \sin\left(\frac{\beta}{2}\right) \sin\left(\frac{\gamma}{2}\right) &= \frac{r}{4R} & \cos(\alpha) + \cos(\beta) + \cos(\gamma) &= 1 + \frac{r}{R} \end{aligned}$$

$$\cot(\alpha) + \cot(\beta) + \cot(\gamma) = \frac{a^2 + b^2 + c^2}{4\Delta}$$

$$\tan(\alpha) + \tan(\beta) + \tan(\gamma) = \tan(\alpha)\tan(\beta)\tan(\gamma)$$

$$\cot(\alpha)\cot(\beta) + \cot(\alpha)\cot(\gamma) + \cot(\beta)\cot(\gamma) = 1$$

$$\cos^2(\alpha) + \cos^2(\beta) + \cos^2(\gamma) = 1 - 2\cos(\alpha)\cos(\beta)\cos(\gamma)$$

$$\frac{\tan\left(\frac{\alpha}{2}\right)\tan\left(\frac{\beta}{2}\right)}{\tan\left(\frac{\alpha}{2}\right) + \tan\left(\frac{\beta}{2}\right)} = \frac{r}{c}$$

For a right-angle triangle the following relations hold:

$$\sin(\alpha) = \frac{a}{c}; \gamma = \frac{\pi}{2} \quad \cos(\alpha) = \frac{b}{c}; \gamma = \frac{\pi}{2}$$

$$\tan(\alpha) = \frac{a}{b}; \gamma = \frac{\pi}{2} \quad \cot(\alpha) = \frac{b}{a}; \gamma = \frac{\pi}{2}$$

$$\csc(\alpha) = \frac{c}{a}; \gamma = \frac{\pi}{2} \quad \sec(\alpha) = \frac{c}{b}; \gamma = \frac{\pi}{2}$$

### Other applications

Because the trigonometric functions appear virtually everywhere in quantitative sciences, it is impossible to list their numerous applications in teaching, science, engineering, and art.

## Introduction to the Sine Function

### Defining the sine function

The sine function is one of the oldest mathematical functions. It was used in ancient Greece and India, and in 1140, R. de Chesters translated Abu Ja'far Muhammed ibn Musa al-Khwarizme's works and used the word "sine" (in Latin, "sinus").

The classical definition of the sine function for real arguments is: "the sine of an angle  $\alpha$  in a right-angle triangle is the ratio of the length of the opposite leg to the length of the hypotenuse." This description of  $\sin(\alpha)$  is valid for  $0 < \alpha < \pi/2$  when the triangle is nondegenerate. This approach to sine can be expanded to arbitrary real values of  $\alpha$  if consideration is given to the arbitrary point  $\{x, y\}$  in the  $x,y$ -Cartesian plane and  $\sin(\alpha)$  is defined as the ratio  $y / (x^2 + y^2)^{1/2}$ , assuming that  $\alpha$  is the value of the angle between the positive direction of the  $x$ -axis and the direction from the origin to the point  $\{x, y\}$ .

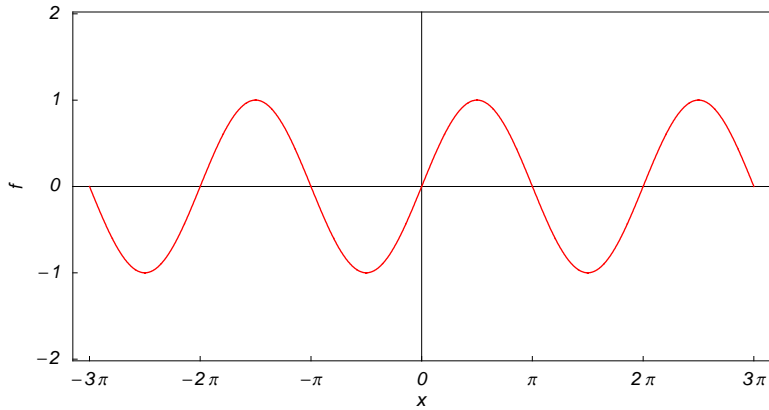
The following formula can also be used as a definition of the sine function:

$$\sin(z) = z - \frac{z^3}{6} + \frac{z^5}{120} - \dots = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)!}$$

This series converges for all finite numbers  $z$ .

### A quick look at the sine function

Here is a graphic of the sine function  $f(x) = \sin(x)$  for real values of its argument  $x$ .



### Representation through more general functions

The function  $\sin(z)$  is a particular case of more complicated mathematical functions. For example, it is a special case of the generalized hypergeometric function  ${}_0F_1(; a; w)$  with the parameter  $a = \frac{3}{2}$  at  $w = -\frac{z^2}{4}$ , multiplied by  $z$ :

$$\sin(z) = z {}_0F_1\left(\frac{3}{2}; -\frac{z^2}{4}\right).$$

It is also a particular case of the Bessel function  $J_\nu(z)$  with the parameter  $\nu = \frac{1}{2}$ , multiplied by  $\sqrt{\frac{\pi z}{2}}$ :

$$\sin(z) = \sqrt{\frac{\pi z}{2}} J_{\frac{1}{2}}(z).$$

Other Bessel functions can also be expressed through sine functions for similar values of the parameter:

$$\sin(z) = -i \sqrt{\frac{\pi iz}{2}} I_{\frac{1}{2}}(iz) \quad \sin(z) = \sqrt{\frac{\pi z}{2}} Y_{-\frac{1}{2}}(z) \quad \sin(z) = \frac{i}{\sqrt{2\pi}} \left( \sqrt{iz} K_{\frac{1}{2}}(iz) - \sqrt{-iz} K_{\frac{1}{2}}(-iz) \right).$$

Struve functions can also degenerate into the sine function for similar values of the parameter:

$$\sin(z) = \sqrt{z} \sqrt{\frac{\pi}{2}} \mathbf{H}_{-\frac{1}{2}}(z) \quad \sin(z) = -i \sqrt{iz} \sqrt{\frac{\pi}{2}} \mathbf{L}_{-\frac{1}{2}}(iz).$$

But the function  $\sin(z)$  is also a degenerate case of the doubly periodic Jacobi elliptic functions when their second parameter is equal to 0 or 1:

$$\begin{aligned} \sin(z) &= \operatorname{sd}(z | 0) = \operatorname{sn}(z | 0) \\ \sin(z) &= -i \operatorname{sc}(iz | 1) = -i \operatorname{sd}(iz | 1). \end{aligned}$$

Finally, the function  $\sin(z)$  is the particular case of another class of functions—the Mathieu functions:

$$\sin(z) = \operatorname{Se}(1, 0, z).$$

### Definition of the sine function for a complex argument

In the complex  $z$ -plane, the function  $\sin(z)$  is defined using the exponential function  $e^w$  in the points  $w = iz$  and  $w = -iz$  through the formula:



$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}.$$

The key role in this definition of  $\sin(z)$  belongs to the famous Euler formula connecting the exponential, the sine, and the cosine functions:

$$e^{iz} = \cos(z) + i \sin(z).$$

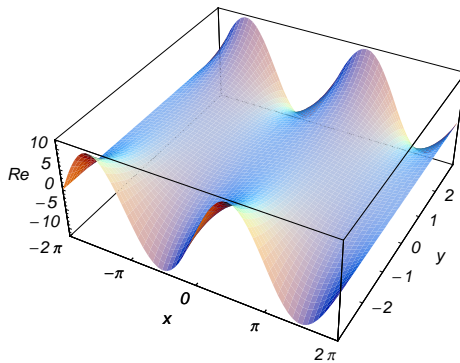
Changing  $z$  to  $-z$ , the Euler formula becomes:

$$e^{-iz} = \cos(z) - i \sin(z).$$

Taking the difference of the preceding formulas and dividing by  $2i$  gives the following result:

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}.$$

Here are two graphics showing the real and imaginary parts of the sine function over the complex plane.



## The best-known properties and formulas for the sine function

### Values in points

Students usually learn the following basic table of sine function values for special points of the circle:

$$\begin{aligned} \sin(0) &= 0 & \sin\left(\frac{\pi}{6}\right) &= \frac{1}{2} & \sin\left(\frac{\pi}{4}\right) &= \frac{1}{\sqrt{2}} & \sin\left(\frac{\pi}{3}\right) &= \frac{\sqrt{3}}{2} \\ \sin\left(\frac{\pi}{2}\right) &= 1 & \sin\left(\frac{2\pi}{3}\right) &= \frac{\sqrt{3}}{2} & \sin\left(\frac{3\pi}{4}\right) &= \frac{1}{\sqrt{2}} & \sin\left(\frac{5\pi}{6}\right) &= \frac{1}{2} \\ \sin(\pi) &= 0 & \sin\left(\frac{7\pi}{6}\right) &= -\frac{1}{2} & \sin\left(\frac{5\pi}{4}\right) &= -\frac{1}{\sqrt{2}} & \sin\left(\frac{4\pi}{3}\right) &= -\frac{\sqrt{3}}{2} \\ \sin\left(\frac{3\pi}{2}\right) &= -1 & \sin\left(\frac{5\pi}{3}\right) &= -\frac{\sqrt{3}}{2} & \sin\left(\frac{7\pi}{4}\right) &= -\frac{1}{\sqrt{2}} & \sin\left(\frac{11\pi}{6}\right) &= -\frac{1}{2} \end{aligned}$$

$$\sin(2\pi) = 0 \quad \sin(\pi m) = 0 \ ; \ m \in \mathbb{Z} \quad \sin\left(\pi\left(\frac{1}{2} + m\right)\right) = (-1)^m \ ; \ m \in \mathbb{Z} .$$

### General characteristics

For real values of argument  $z$ , the values of  $\sin(z)$  are real.

In the points  $z = 2\pi n/m$ ;  $n \in \mathbb{Z} \wedge m \in \mathbb{Z}$ , the values of  $\sin(z)$  are algebraic. In several cases they can even be rational numbers, 0, or 1. Here are some examples:

$$\sin(0) = 0 \quad \sin\left(\frac{\pi}{6}\right) = \frac{1}{2} \quad \sin\left(\frac{\pi}{2}\right) = 1.$$

The values of  $\sin\left(\frac{n\pi}{m}\right)$  can be expressed using only square roots if  $n \in \mathbb{Z}$  and  $m$  is a product of a power of 2 and distinct Fermat primes  $\{3, 5, 17, 257, \dots\}$ .

The function  $\sin(z)$  is an entire analytical function of  $z$  that is defined over the whole complex  $z$ -plane and does not have branch cuts and branch points. It has an essential singular point at  $z = \infty$ . It is a periodic function with the real period  $2\pi$ :

$$\sin(z + 2\pi) = \sin(z)$$

$$\sin(z) = \sin(z + 2\pi k) \quad ; \quad k \in \mathbb{Z}$$

$$\sin(z) = (-1)^k \sin(z + \pi k) \quad ; \quad k \in \mathbb{Z}.$$

The function  $\sin(z)$  is an odd function with mirror symmetry:

$$\sin(-z) = -\sin(z) \quad \sin(\bar{z}) = \overline{\sin(z)}.$$

### Differentiation

The derivatives of  $\sin(z)$  have simple representations using either the  $\sin(z)$  function or the  $\cos(z)$  function:

$$\frac{\partial \sin(z)}{\partial z} = \cos(z) \quad \frac{\partial^n \sin(z)}{\partial z^n} = \sin\left(z + \frac{\pi n}{2}\right) \quad ; \quad n \in \mathbb{N}^+.$$

### Ordinary differential equation

The function  $\sin(z)$  satisfies the simplest possible linear differential equation with constant coefficients:

$$w''(z) + w(z) = 0 \quad ; \quad w(z) = \sin(z) \wedge w(0) = 0 \wedge w'(0) = 1.$$

The complete solution of this equation can be represented as a linear combination of  $\sin(z)$  and  $\cos(z)$  with arbitrary constant coefficients  $c_1$  and  $c_2$ :

$$w''(z) + w(z) = 0 \quad ; \quad w(z) = c_1 \cos(z) + c_2 \sin(z).$$

The function  $\sin(z)$  also satisfies first-order nonlinear differential equations:

$$w'(z) - \sqrt{1 - (w(z))^2} = 0 \quad ; \quad w(z) = \sin(z) \wedge w(0) = 0 \wedge |\operatorname{Re}(z)| < \frac{\pi}{2}.$$

### Series representation

The function  $\sin(z)$  has a simple series expansion at the origin that converges in the whole complex  $z$ -plane:

$$\sin(z) = z - \frac{z^3}{3!} + \dots = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)!}.$$

For real  $z = x$  this series can be interpreted as the imaginary part of the series expansion for the exponential function  $e^{ix}$ :

$$\sin(x) = x - \frac{x^3}{3!} + \dots = \operatorname{Im} \left( 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \dots \right) = \operatorname{Im} \left( \sum_{k=0}^{\infty} \frac{(ix)^k}{k!} \right) = \operatorname{Im}(e^{ix}).$$

### Product representation

The following famous infinite product representation for  $\sin(z)$  clearly illustrates that  $\sin(z) = 0$  at  $z = \pi k \wedge k \in \mathbb{Z}$ :

$$\sin(z) = z \prod_{k=1}^{\infty} \left( 1 - \frac{z^2}{\pi^2 k^2} \right).$$

### Indefinite integration

Indefinite integrals of expressions that contain the sine function can sometimes be expressed using elementary functions. However, special functions are frequently needed to express the results even when the integrands have a simple form (if they can be evaluated in closed form). Here are some examples:

$$\int \sin(z) dz = -\cos(z)$$

$$\int \sqrt{\sin(z)} dz = -2 E \left( \frac{1}{2} \left( \frac{\pi}{2} - z \right) \middle| 2 \right)$$

$$\int z^{\alpha-1} \sin^{\nu}(az) dz = \frac{2^{-\nu} (1 - \nu \bmod 2)}{\alpha} z^{\alpha} \left( \frac{\nu}{2} \right) - (2i)^{-\nu} z^{\alpha}$$

$$\sum_{j=0}^{\lfloor \frac{\nu-1}{2} \rfloor} (-1)^j \binom{\nu}{j} (\Gamma(\alpha, ia(2j-\nu)z) (ia(2j-\nu)z)^{-\alpha} + (-1)^{\nu} \Gamma(\alpha, ia(\nu-2j)z) (ia(\nu-2j)z)^{-\alpha}) /; \alpha \neq 0 \wedge \nu \in \mathbb{N}^+.$$

The last integral cannot be evaluated in closed form using the known classical special functions for arbitrary values of parameters  $\alpha$  and  $\nu$ .

### Definite integration

Definite integrals that contain the sine function are sometimes simple. For example, the famous Dirichlet and Fresnel integrals have the following values:

$$\int_0^{\infty} \frac{\sin(t)}{t} dt = \frac{\pi}{2}$$

$$\int_0^{\infty} \sin(t^2) dt = \frac{1}{2} \sqrt{\frac{\pi}{2}}.$$

Some special functions can be used to evaluate more complicated definite integrals. For example, elliptic integrals and gamma functions are needed to express the following integrals:

$$\int_0^\pi \sin^{\frac{1}{2}}(z) dz = 4 E\left(\frac{\pi}{4} \middle| 2\right)$$

$$\int_0^\pi \sin^{-\frac{1}{2}}(z) dz = \frac{\Gamma\left(\frac{1}{4}\right)^2}{\sqrt{2\pi}}$$

### Integral transforms

Integral transforms of expressions that contain the sine function may not be classically convergent, but they can be interpreted in a generalized functions (distributions) sense. For example, the exponential Fourier transform of the sine function  $\sin(z)$  does not exist in the classical sense, but it can be expressed using the Dirac delta function:

$$\mathcal{F}_t[\sin(t)](z) = i\sqrt{\frac{\pi}{2}} \delta(z-1) - i\sqrt{\frac{\pi}{2}} \delta(z+1).$$

Among other integral transforms of the sine function, the best known are the Fourier cosine and sine transforms, the Laplace, the Mellin, the Hilbert, and the Hankel transforms:

$$\mathcal{F}_{C_t}[\sin(at)](z) = \sqrt{\frac{2}{\pi}} \frac{a}{a^2 - z^2} \quad ; a \in \mathbb{R}$$

$$\mathcal{F}_{S_t}[\sin(at)](z) = \sqrt{\frac{\pi}{2}} (\delta(a-z) - \delta(a+z)) \quad ; a \in \mathbb{R}$$

$$\mathcal{L}_t[\sin(t)](z) = \frac{1}{z^2 + 1}$$

$$\mathcal{M}_t[\sin(t)](z) = \Gamma(z) \sin\left(\frac{\pi z}{2}\right) \quad ; -1 < \operatorname{Re}(z) < 1$$

$$\mathcal{H}_t[\sin(t)](x) = \cos(x)$$

$$\mathcal{H}_{t,\nu}[t^{\alpha-1} \sin(t)](z) =$$

$$-\frac{1}{\Gamma(\nu+1)} \left( 2^{-\nu} z^{\nu+\frac{1}{2}} \cos\left(\frac{1}{4}\pi(2\alpha+2\nu+3)\right) \Gamma\left(\alpha+\nu+\frac{1}{2}\right) {}_2F_1\left(\frac{1}{4}(2\alpha+2\nu+1), \frac{1}{4}(2\alpha+2\nu+3); \nu+1; z^2\right) \right) \quad ;$$

$$\operatorname{Re}(\alpha+\nu) > -\frac{3}{2} \wedge \operatorname{Re}(\alpha) < 1.$$

### Finite summation

The following finite sums from the sine can be expressed using trigonometric functions:

$$\sum_{k=0}^n \sin(ak) = \csc\left(\frac{a}{2}\right) \sin\left(\frac{a(n+1)}{2}\right) \sin\left(\frac{an}{2}\right)$$

$$\sum_{k=0}^n (-1)^k \sin(ak) = \cos\left(\frac{1}{2}(a+n(a+\pi))\right) \sec\left(\frac{a}{2}\right) \sin\left(\frac{1}{2}n(a+\pi)\right)$$

$$\sum_{k=0}^n \sin(ak+z) = \csc\left(\frac{a}{2}\right) \sin\left(\frac{1}{2}a(n+1)\right) \sin\left(\frac{an}{2}+z\right)$$

$$\sum_{k=0}^n (-1)^k \sin(a k + z) = \cos\left(\frac{1}{2}(a + n(a + \pi))\right) \sec\left(\frac{a}{2}\right) \sin\left(\frac{1}{2}n(a + \pi) + z\right)$$

$$\sum_{k=1}^n z^k \sin(k a) = \frac{z((z \sin(a n) - \sin(a(n + 1))) z^n + \sin(a))}{z^2 - 2 \cos(a) z + 1}$$

### Infinite summation

The following infinite sums can be expressed using elementary functions:

$$\sum_{k=1}^{\infty} \frac{\sin(a k)}{k} = \frac{\pi - a}{2} \quad ; \quad 0 < \operatorname{Re}(a) < 2\pi$$

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} \sin(a k)}{k} = \frac{a}{2} \quad ; \quad |\operatorname{Re}(a)| < \pi$$

$$\sum_{k=1}^{\infty} \frac{\sin(a k)}{k!} = e^{\cos(a)} \sin(\sin(a))$$

$$\sum_{k=1}^{\infty} \frac{z^k \sin(a k)}{k!} = e^{z \cos(a)} \sin(z \sin(a))$$

$$\sum_{k=1}^{\infty} z^k \sin(a k) = \frac{z \sin(a)}{z^2 - 2 \cos(a) z + 1} \quad ; \quad |z| < 1.$$

### Finite products

The following finite products from the sine can be expressed using trigonometric functions:

$$\prod_{k=1}^{n-1} \sin\left(\frac{\pi k}{n}\right) = 2^{1-n} n \quad ; \quad n \in \mathbb{N}^+$$

$$\prod_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} \sin\left(\frac{\pi k}{n}\right) = 2^{-\frac{1}{2}(n-1)} \sqrt{n} \quad ; \quad n \in \mathbb{N}^+$$

$$\prod_{k=1}^{n-1} \sin\left(\frac{\pi k}{n} + z\right) = \frac{2^{1-n} \sin(n z)}{\sin(z)} \quad ; \quad n \in \mathbb{N}^+$$

$$\prod_{k=1}^{n-1} \sin\left(\frac{2 \pi k}{n} + z\right) = (-2)^{1-n} \csc(z) \left( \cos\left(n z - \frac{\pi n}{2}\right) - \cos\left(\frac{n \pi}{2}\right) \right) \quad ; \quad n \in \mathbb{N}^+$$

$$\prod_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} \sin\left(\frac{(2 k + 1) \pi}{2 n}\right) = 2^{\frac{1-n}{2}} \csc\left(\frac{\pi}{2 n}\right) \quad ; \quad n \in \mathbb{N}^+.$$

### Infinite products

The following infinite product can be expressed using the sine function:

$$\prod_{k=1}^{\infty} \left( 1 - \frac{4}{3} \sin^2 \left( \frac{z}{3^k} \right) \right) = \frac{\sin(z)}{z}.$$

**Addition formulas**

The sine of a sum can be represented by the rule: "the sine of a sum is equal to the sum of the sine multiplied by the cosine and the cosine multiplied by the sine." A similar rule is valid for the sine of the difference:

$$\begin{aligned} \sin(a + b) &= \sin(a) \cos(b) + \cos(a) \sin(b) \\ \sin(a - b) &= \sin(a) \cos(b) - \cos(a) \sin(b). \end{aligned}$$

**Multiple arguments**

In the case of multiple arguments  $z, 2z, 3z, \dots$ , the function  $\sin(z)$  can be represented as the finite sum of terms that include powers of the sine and cosine:

$$\sin(2z) = 2 \sin(z) \cos(z)$$

$$\sin(3z) = 3 \sin(z) - 4 \sin^3(z)$$

$$\sin(nz) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^k \binom{n}{2k+1} \sin^{2k+1}(z) \cos^{n-2k-1}(z) \quad ; \quad n \in \mathbb{N}^+.$$

The function  $\sin(nz)$  can also be represented as the finite product including the sine of the linear argument of  $z$ :

$$\sin(nz) = 2^{n-1} \prod_{k=0}^{n-1} \sin \left( z + \frac{\pi k}{n} \right) \quad ; \quad n \in \mathbb{N}^+.$$

**Half-angle formulas**

The sine of the half-angle can be represented by the following simple formula that is valid in some vertical strips:

$$\sin \left( \frac{z}{2} \right) = \sqrt{\frac{1 - \cos(z)}{2}} \quad ; \quad 0 < \operatorname{Re}(z) < 2\pi \vee \operatorname{Re}(z) = 0 \wedge \operatorname{Im}(z) \geq 0 \vee \operatorname{Re}(z) = 2\pi \wedge \operatorname{Im}(z) \leq 0.$$

To make this formula correct for all complex  $z$ , a complicated prefactor is needed:

$$\sin \left( \frac{z}{2} \right) = c(z) \sqrt{\frac{1 - \cos(z)}{2}} \quad ; \quad c(z) = (-1)^{\lfloor \frac{\operatorname{Re}(z)}{2\pi} \rfloor} \left( 1 - \left( 1 + (-1)^{\lfloor \frac{\operatorname{Re}(z)}{2\pi} \rfloor + \lfloor -\frac{\operatorname{Re}(z)}{2\pi} \rfloor} \right) \theta(-\operatorname{Im}(z)) \right),$$

where  $c(z)$  contains the unit step, real part, imaginary part, and the floor functions.

**Sums of two direct functions**

The sum of two sine functions can be described by the rule: "the sum of the sines is equal to the doubled cosine of the half-difference multiplied by the sine of the half-sum." A similar rule is valid for the difference of two sines:

$$\sin(a) + \sin(b) = 2 \cos \left( \frac{a-b}{2} \right) \sin \left( \frac{a+b}{2} \right)$$

$$\sin(a) - \sin(b) = 2 \cos \left( \frac{a+b}{2} \right) \sin \left( \frac{a-b}{2} \right).$$

**Products involving the direct function**

The product of two sine functions and the product of the sine and cosine have the following representations:

$$\sin(a) \sin(b) = \frac{1}{2} (\cos(a - b) - \cos(a + b))$$

$$\sin(a) \cos(b) = \frac{1}{2} (\sin(a - b) + \sin(a + b)).$$

**Powers of the direct function**

The integer powers of the sine functions can be expanded as finite sums of cosine (or sine) functions with multiple arguments. These sums include binomial coefficients:

$$\sin^{2n}(z) = 2^{-2n} \binom{2n}{n} + 2^{1-2n} \sum_{k=0}^{n-1} (-1)^{n-k} \binom{2n}{k} \cos(2(n-k)z) ; n \in \mathbb{N}^+$$

$$\sin^{2n-1}(z) = 2^{2-2n} \sum_{k=0}^{n-1} (-1)^{n-k-1} \binom{2n-1}{k} \sin((2n-2k-1)z) ; n \in \mathbb{N}^+.$$

The previous formulas can be joined into the following one:

$$\sin^n(z) = 2^{-n} \binom{n}{\frac{n}{2}} (1 - n \bmod 2) - 2^{1-n} \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^{k+n-1} \binom{n}{k} \cos\left(\frac{\pi n}{2} + (n-2k)z\right) ; n \in \mathbb{N}^+.$$

**Inequalities**

The most famous inequalities for the sine function are:

$$|\sin(x)| \leq 1 ; x \in \mathbb{R}$$

$$\sin(x) \leq x ; x \geq 0.$$

**Relations with its inverse function**

There are simple relations between the function  $\sin(z)$  and its inverse function  $\sin^{-1}(z)$ :

$$\sin(\sin^{-1}(z)) = z \quad \sin^{-1}(\sin(z)) = z ; -\frac{\pi}{2} < \operatorname{Re}(z) < \frac{\pi}{2} \quad \bigvee \operatorname{Re}(z) = -\frac{\pi}{2} \wedge \operatorname{Im}(z) \geq 0 \quad \bigvee \operatorname{Re}(z) = \frac{\pi}{2} \wedge \operatorname{Im}(z) \leq 0 .$$

The second formula is valid at least in the vertical strip  $-\frac{\pi}{2} < \operatorname{Re}(z) < \frac{\pi}{2}$ . Outside of this strip a much more complicated relation (that contains the unit step, real part, and the floor functions) holds:

$$\sin^{-1}(\sin(z)) = (-1)^{\lfloor \frac{\operatorname{Re}(z)}{\pi} - \frac{1}{2} \rfloor} \left( \left( 1 + (-1)^{\lfloor \frac{\operatorname{Re}(z)}{\pi} + \frac{1}{2} \rfloor - \lfloor \frac{\operatorname{Re}(z)}{\pi} + \frac{1}{2} \rfloor} \right) \theta(-\operatorname{Im}(z)) - 1 \right) \left( z - \pi \left\lfloor \frac{\operatorname{Re}(z)}{\pi} - \frac{1}{2} \right\rfloor + \frac{\pi}{2} \left( 1 + (-1)^{\lfloor \frac{\operatorname{Re}(z)}{\pi} + \frac{1}{2} \rfloor - \lfloor \frac{\operatorname{Re}(z)}{\pi} + \frac{1}{2} \rfloor} \right) \theta(-\operatorname{Im}(z)) - \pi \right).$$

**Representations through other trigonometric functions**

Sine and cosine functions are connected by a very simple formula that contains the linear function in the argument:

$$\sin(z) = \cos\left(\frac{\pi}{2} - z\right).$$

Another famous formula, connecting  $\sin(z)$  and  $\cos(z)$ , is reflected in the well-known Pythagorean theorem:

$$\sin^2(z) = 1 - \cos^2(z)$$

$$\sin(z) = \sqrt{1 - \cos^2(z)} \quad /; 0 < \operatorname{Re}(z) < \pi \vee \operatorname{Re}(z) = 0 \wedge \operatorname{Im}(z) \geq 0 \vee \operatorname{Re}(z) = \pi \wedge \operatorname{Im}(z) \leq 0.$$

The last restriction on  $z$  can be removed, but the formula will get a complicated coefficient  $c(z)$  that contains the unit step, real part, imaginary part, and the floor function and  $|c(z)| = 1$ :

$$\sin(z) = c(z) \sqrt{1 - \cos^2(z)} \quad /; c(z) = (-1)^{\lfloor \frac{\operatorname{Re}(z)}{\pi} \rfloor} \left( 1 - \left( 1 + (-1)^{\lfloor \frac{\operatorname{Re}(z)}{\pi} \rfloor + \lfloor -\frac{\operatorname{Re}(z)}{\pi} \rfloor \right) \theta(-\operatorname{Im}(z)) \right).$$

The sine function can also be represented using other trigonometric functions by the following formulas:

$$\sin(z) = \frac{2 \tan\left(\frac{z}{2}\right)}{\tan^2\left(\frac{z}{2}\right) + 1} \quad \sin(z) = \frac{2 \cot\left(\frac{z}{2}\right)}{\cot^2\left(\frac{z}{2}\right) + 1}$$

$$\sin(z) = \frac{1}{\csc(z)} \quad \sin(z) = \frac{1}{\sec\left(\frac{\pi}{2} - z\right)}.$$

### Representations through hyperbolic functions

The sine function has representations using the hyperbolic functions:

$$\sin(z) = -i \sinh(iz) \quad \sin(iz) = i \sinh(z) \quad \sin(z) = \cosh\left(\frac{\pi i}{2} - iz\right)$$

$$\sin(z) = \frac{2i \tanh\left(\frac{iz}{2}\right)}{\tanh^2\left(\frac{iz}{2}\right) - 1} \quad \sin(z) = \frac{2i \coth\left(\frac{iz}{2}\right)}{1 - \coth^2\left(\frac{iz}{2}\right)} \quad \sin(z) = -\frac{i}{\operatorname{csch}(iz)} \quad \sin(z) = \frac{1}{\operatorname{sech}\left(\frac{\pi i}{2} - iz\right)}.$$

### Applications

The sine function is used throughout mathematics, the exact sciences, and engineering.

## Introduction to the Trigonometric Functions in *Mathematica*

### Overview

The following shows how the six trigonometric functions are realized in *Mathematica*. Examples of evaluating *Mathematica* functions applied to various numeric and exact expressions that involve the trigonometric functions or return them are shown. These involve numeric and symbolic calculations and plots.

### Notations

#### *Mathematica* forms of notations

All six trigonometric functions are represented as built-in functions in *Mathematica*. Following *Mathematica*'s general naming convention, the `StandardForm` function names are simply capitalized versions of the traditional mathematics names. Here is a list `trigFunctions` of the six trigonometric functions in `StandardForm`.



```

trigFunctions = {Sin[z], Cos[z], Tan[z], Cot[z], Sec[z], Cos[z]}
{Sin[z], Cos[z], Tan[z], Cot[z], Sec[z], Cos[z]}

```

Here is a list `trigFunctions` of the six trigonometric functions in `TraditionalForm`.

```

trigFunctions // TraditionalForm
{sin(z), cos(z), tan(z), cot(z), sec(z), cos(z)}

```

### Additional forms of notations

*Mathematica* also knows the most popular forms of notations for the trigonometric functions that are used in other programming languages. Here are three examples: `CForm`, `TeXForm`, and `FortranForm`.

```

trigFunctions /. {z -> 2 Pi z} // (CForm /@ #) &
{Sin (2 * Pi * z), Cos (2 * Pi * z), Tan (2 * Pi * z),
 Cot (2 * Pi * z), Sec (2 * Pi * z), Cos (2 * Pi * z)}

trigFunctions /. {z -> 2 Pi z} // (TeXForm /@ #) &
{\sin (2\, \pi \, z), \cos (2\, \pi \, z), \tan (2\, \pi \, z), \cot
 (2\, \pi \, z), \sec (2\, \pi \, z), \cos (2\, \pi \, z)}

trigFunctions /. {z -> 2 Pi z} // (FortranForm /@ #) &
{Sin (2 * Pi * z), Cos (2 * Pi * z), Tan (2 * Pi * z),
 Cot (2 * Pi * z), Sec (2 * Pi * z), Cos (2 * Pi * z)}

```

## Automatic evaluations and transformations

### Evaluation for exact, machine-number, and high-precision arguments

For a simple exact argument, *Mathematica* returns exact results. For instance, for the argument  $\pi/6$ , the `Sin` function evaluates to  $1/2$ .

```

Sin  $\left[\frac{\pi}{6}\right]$ 

```

$$\frac{1}{2}$$

```

{Sin[z], Cos[z], Tan[z], Cot[z], Csc[z], Sec[z]} /. z ->  $\frac{\pi}{6}$ 

```

$$\left\{\frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{1}{\sqrt{3}}, \sqrt{3}, 2, \frac{2}{\sqrt{3}}\right\}$$

For a generic machine-number argument (a numerical argument with a decimal point and not too many digits), a machine number is returned.

```

Cos[3.]
-0.989992

```

`{Sin[z], Cos[z], Tan[z], Cot[z], Csc[z], Sec[z]} /. z -> 2.`

`{0.909297, -0.416147, -2.18504, -0.457658, 1.09975, -2.403}`

The next inputs calculate 100-digit approximations of the six trigonometric functions at  $z = 1$ .

`N[Tan[1], 40]`

`1.557407724654902230506974807458360173087`

`Cot[1] // N[#, 50] &`

`0.64209261593433070300641998659426562023027811391817`

`N[{Sin[z], Cos[z], Tan[z], Cot[z], Csc[z], Sec[z]} /. z -> 1, 100]`

`{0.8414709848078965066525023216302989996225630607983710656727517099919104043912396689  
9486397435430526959,  
0.5403023058681397174009366074429766037323104206179222276700972553811003947744717649  
5179518560871830893,  
1.5574077246549022305069748074583601730872507723815200383839466056988613971517272899  
555099965202242984,  
0.6420926159343307030064199865942656202302781139181713791011622804262768568391646729  
1984829197601968047,  
1.1883951057781212162615994523745510035278298340979626252652536663591843673571904879  
913663568030853023,  
1.8508157176809256179117532413986501934703966550940092988351582778588154112615967059  
921841413287306671}`

Within a second, it is possible to calculate thousands of digits for the trigonometric functions. The next input calculates 10000 digits for  $\sin(1)$ ,  $\cos(1)$ ,  $\tan(1)$ ,  $\cot(1)$ ,  $\sec(1)$ , and  $\csc(1)$  and analyzes the frequency of the occurrence of the digit  $k$  in the resulting decimal number.

`Map[Function[w, {First[#, Length[#]} & /@ Split[Sort[First[RealDigits[w]]]]],  
N[{Sin[z], Cos[z], Tan[z], Cot[z], Csc[z], Sec[z]} /. z -> 1, 10 000]]`

`{{{0, 983}, {1, 1069}, {2, 1019}, {3, 983}, {4, 972}, {5, 994},  
{6, 994}, {7, 988}, {8, 988}, {9, 1010}}, {{0, 998}, {1, 1034}, {2, 982},  
{3, 1015}, {4, 1013}, {5, 963}, {6, 1034}, {7, 966}, {8, 991}, {9, 1004}},  
{{0, 1024}, {1, 1025}, {2, 1000}, {3, 969}, {4, 1026}, {5, 944}, {6, 999},  
{7, 1001}, {8, 1008}, {9, 1004}}, {{0, 1006}, {1, 1030}, {2, 986},  
{3, 954}, {4, 1003}, {5, 1034}, {6, 999}, {7, 998}, {8, 1009}, {9, 981}},  
{{0, 1031}, {1, 976}, {2, 1045}, {3, 917}, {4, 1001}, {5, 996}, {6, 964},  
{7, 1012}, {8, 982}, {9, 1076}}, {{0, 978}, {1, 1034}, {2, 1016},  
{3, 974}, {4, 987}, {5, 1067}, {6, 943}, {7, 1006}, {8, 1027}, {9, 968}}}`

Here are 50-digit approximations to the six trigonometric functions at the complex argument  $z = 3 + 5i$ .

`N[Csc[3 + 5 i], 100]`

```
0.0019019704237010899966700172963208058404592525121712743108017196953928700340468202\
96847410109982878354 +
0.013341591397996678721837322466473194390132347157253190972075437462485814431570118\
67262664488519840339 i
```

```
N[{Sin[z], Cos[z], Tan[z], Cot[z], Csc[z], Sec[z]} /. z -> 3 + 5 i, 50]
```

```
{10.472508533940392276673322536853503271126419950388 -
 73.460621695673676366791192505081750407213922814475 i,
-73.467292212645262467746454594833950830814859165299 -
10.471557674805574377394464224329537808548330651734 i,
-0.000025368676207676032417806136707426288195560702602478 +
0.99991282015135380828209263013972954140566020462086 i,
-0.000025373100044545977383763346789469656754050037355986 -
1.0000871868058967743285316881045218577131612831891 i,
0.0019019704237010899966700172963208058404592525121713 +
0.013341591397996678721837322466473194390132347157253 i,
-0.013340476530549737487361100811100839468470481725038 +
0.0019014661516951513089519270013254277867588978133499 i}
```

*Mathematica* always evaluates mathematical functions with machine precision, if the arguments are machine numbers. In this case, only six digits after the decimal point are shown in the results. The remaining digits are suppressed, but can be displayed using the function `InputForm`.

```
{Sin[2.], N[Sin[2]], N[Sin[2], 16], N[Sin[2], 5], N[Sin[2], 20]}
```

```
{0.909297, 0.909297, 0.909297, 0.909297, 0.90929742682568169540}
```

```
% // InputForm
```

```
{0.9092974268256817, 0.9092974268256817, 0.9092974268256817, 0.9092974268256817,
0.909297426825681695396019865911745`20}
```

```
Precision[%%]
```

```
16
```

### Simplification of the argument

*Mathematica* uses symmetries and periodicities of all the trigonometric functions to simplify expressions. Here are some examples.

```
Sin[-z]
```

```
-Sin[z]
```

```
Sin[z +  $\pi$ ]
```

```
-Sin[z]
```

```
Sin[z + 2  $\pi$ ]
```

```
Sin[z]
```

```
Sin[z + 34  $\pi$ ]
```

Sin[z]

{Sin[-z], Cos[-z], Tan[-z], Cot[-z], Csc[-z], Sec[-z]}

{-Sin[z], Cos[z], -Tan[z], -Cot[z], -Csc[z], Sec[z]}

{Sin[z + π], Cos[z + π], Tan[z + π], Cot[z + π], Csc[z + π], Sec[z + π]}

{-Sin[z], -Cos[z], Tan[z], Cot[z], -Csc[z], -Sec[z]}

{Sin[z + 2 π], Cos[z + 2 π], Tan[z + 2 π], Cot[z + 2 π], Csc[z + 2 π], Sec[z + 2 π]}

{Sin[z], Cos[z], Tan[z], Cot[z], Csc[z], Sec[z]}

{Sin[z + 342 π], Cos[z + 342 π], Tan[z + 342 π], Cot[z + 342 π], Csc[z + 342 π], Sec[z + 342 π]}

{Sin[z], Cos[z], Tan[z], Cot[z], Csc[z], Sec[z]}

*Mathematica* automatically simplifies the composition of the direct and the inverse trigonometric functions into the argument.

{Sin[ArcSin[z]], Cos[ArcCos[z]], Tan[ArcTan[z]],  
Cot[ArcCot[z]], Csc[ArcCsc[z]], Sec[ArcSec[z]]}

{z, z, z, z, z, z}

*Mathematica* also automatically simplifies the composition of the direct and any of the inverse trigonometric functions into algebraic functions of the argument.

{Sin[ArcSin[z]], Sin[ArcCos[z]], Sin[ArcTan[z]],  
Sin[ArcCot[z]], Sin[ArcCsc[z]], Sin[ArcSec[z]]}

$$\left\{ z, \sqrt{1-z^2}, \frac{z}{\sqrt{1+z^2}}, \frac{1}{\sqrt{1+\frac{1}{z^2}}}, \frac{1}{z}, \sqrt{1-\frac{1}{z^2}} \right\}$$

{Cos[ArcSin[z]], Cos[ArcCos[z]], Cos[ArcTan[z]],  
Cos[ArcCot[z]], Cos[ArcCsc[z]], Cos[ArcSec[z]]}

$$\left\{ \sqrt{1-z^2}, z, \frac{1}{\sqrt{1+z^2}}, \frac{1}{\sqrt{1+\frac{1}{z^2}}}, \sqrt{1-\frac{1}{z^2}}, \frac{1}{z} \right\}$$

{Tan[ArcSin[z]], Tan[ArcCos[z]], Tan[ArcTan[z]],  
Tan[ArcCot[z]], Tan[ArcCsc[z]], Tan[ArcSec[z]]}

$$\left\{ \frac{z}{\sqrt{1-z^2}}, \frac{\sqrt{1-z^2}}{z}, z, \frac{1}{z}, \frac{1}{\sqrt{1-\frac{1}{z^2}}}, \sqrt{1-\frac{1}{z^2}} \right\}$$

{Cot[ArcSin[z]], Cot[ArcCos[z]], Cot[ArcTan[z]],  
Cot[ArcCot[z]], Cot[ArcCsc[z]], Cot[ArcSec[z]]}

$$\left\{ \frac{\sqrt{1-z^2}}{z}, \frac{z}{\sqrt{1-z^2}}, \frac{1}{z}, z, \sqrt{1-\frac{1}{z^2}}, z, \frac{1}{\sqrt{1-\frac{1}{z^2}}} \right\}$$

$$\{\text{Csc}[\text{ArcSin}[z]], \text{Csc}[\text{ArcCos}[z]], \text{Csc}[\text{ArcTan}[z]], \text{Csc}[\text{ArcCot}[z]], \text{Csc}[\text{ArcCsc}[z]], \text{Csc}[\text{ArcSec}[z]]\}$$

$$\left\{ \frac{1}{z}, \frac{1}{\sqrt{1-z^2}}, \frac{\sqrt{1+z^2}}{z}, \sqrt{1+\frac{1}{z^2}}, z, z, \frac{1}{\sqrt{1-\frac{1}{z^2}}} \right\}$$

$$\{\text{Sec}[\text{ArcSin}[z]], \text{Sec}[\text{ArcCos}[z]], \text{Sec}[\text{ArcTan}[z]], \text{Sec}[\text{ArcCot}[z]], \text{Sec}[\text{ArcCsc}[z]], \text{Sec}[\text{ArcSec}[z]]\}$$

$$\left\{ \frac{1}{\sqrt{1-z^2}}, \frac{1}{z}, \sqrt{1+z^2}, \sqrt{1+\frac{1}{z^2}}, \frac{1}{\sqrt{1-\frac{1}{z^2}}}, z \right\}$$

In cases where the argument has the structure  $\pi k/2 + z$  or  $\pi k/2 - z$ , and  $\pi k/2 + iz$  or  $\pi k/2 - iz$  with integer  $k$ , trigonometric functions can be automatically transformed into other trigonometric or hyperbolic functions. Here are some examples.

$$\text{Tan}\left[\frac{\pi}{2} - z\right]$$

$$\text{Cot}[z]$$

$$\text{Csc}[i z]$$

$$-i \text{Csch}[z]$$

$$\left\{ \text{Sin}\left[\frac{\pi}{2} - z\right], \text{Cos}\left[\frac{\pi}{2} - z\right], \text{Tan}\left[\frac{\pi}{2} - z\right], \text{Cot}\left[\frac{\pi}{2} - z\right], \text{Csc}\left[\frac{\pi}{2} - z\right], \text{Sec}\left[\frac{\pi}{2} - z\right] \right\}$$

$$\{\text{Cos}[z], \text{Sin}[z], \text{Cot}[z], \text{Tan}[z], \text{Sec}[z], \text{Csc}[z]\}$$

$$\{\text{Sin}[i z], \text{Cos}[i z], \text{Tan}[i z], \text{Cot}[i z], \text{Csc}[i z], \text{Sec}[i z]\}$$

$$\{i \text{Sinh}[z], \text{Cosh}[z], i \text{Tanh}[z], -i \text{Coth}[z], -i \text{Csch}[z], \text{Sech}[z]\}$$

### Simplification of simple expressions containing trigonometric functions

Sometimes simple arithmetic operations containing trigonometric functions can automatically produce other trigonometric functions.

$$1 / \text{Sec}[z]$$

$$\text{Cos}[z]$$

$$\{1 / \text{Sin}[z], 1 / \text{Cos}[z], 1 / \text{Tan}[z], 1 / \text{Cot}[z], 1 / \text{Csc}[z], 1 / \text{Sec}[z], \text{Sin}[z] / \text{Cos}[z], \text{Cos}[z] / \text{Sin}[z], \text{Sin}[z] / \text{Sin}[\pi/2 - z], \text{Cos}[z] / \text{Sin}[z]^2\}$$

{Csc[z], Sec[z], Cot[z], Tan[z], Sin[z], Cos[z], Tan[z], Cot[z], Tan[z], Cot[z] Csc[z]}

**Trigonometric functions arising as special cases from more general functions**

All trigonometric functions can be treated as particular cases of some more advanced special functions. For example, sin(z) and cos(z) are sometimes the results of auto-simplifications from Bessel, Mathieu, Jacobi, hypergeometric, and Meijer functions (for appropriate values of their parameters).

**BesselJ** $\left[\frac{1}{2}, z\right]$

$$\frac{\sqrt{\frac{2}{\pi}} \text{Sin}[z]}{\sqrt{z}}$$

**MathieuC** $[1, 0, z]$

Cos[z]

**JacobiSC** $[z, 0]$

Tan[z]

In[14]:= {BesselJ $\left[\frac{1}{2}, z\right]$ , MathieuS $[1, 0, z]$ , JacobiSN $[z, 0]$ ,  
HypergeometricPFQ $\left[\{\}, \left\{\frac{3}{2}\right\}, -\frac{z^2}{4}\right]$ , MeijerG $\left[\{\{\}, \{\}\}, \left\{\left\{\frac{1}{2}\right\}, \{0\}\right\}, \frac{z^2}{4}\right]$ }

Out[14]=  $\left\{\frac{\sqrt{\frac{2}{\pi}} \text{Sin}[z]}{\sqrt{z}}, \text{Sin}[z], \text{Sin}[z], \frac{\text{Sin}\left[\sqrt{z^2}\right]}{\sqrt{z^2}}, \frac{\sqrt{z^2} \text{Sin}[z]}{\sqrt{\pi} z}\right\}$

In[15]:= {BesselJ $\left[-\frac{1}{2}, z\right]$ , MathieuC $[1, 0, z]$ , JacobiCD $[z, 0]$ ,  
Hypergeometric0F1 $\left[\frac{1}{2}, -\frac{z^2}{4}\right]$ , MeijerG $\left[\{\{\}, \{\}\}, \left\{\{0\}, \left\{\frac{1}{2}\right\}\right\}, \frac{z^2}{4}\right]$ }

Out[15]=  $\left\{\frac{\sqrt{\frac{2}{\pi}} \text{Cos}[z]}{\sqrt{z}}, \text{Cos}[z], \text{Cos}[z], \text{Cos}\left[\sqrt{z^2}\right], \frac{\text{Cos}[z]}{\sqrt{\pi}}\right\}$

In[16]:= {JacobiSC $[z, 0]$ , JacobiCS $[z, 0]$ , JacobiDS $[z, 0]$ , JacobiDC $[z, 0]$ }

Out[16]= {Tan[z], Cot[z], Csc[z], Sec[z]}

**Equivalence transformations carried out by specialized *Mathematica* functions**

**General remarks**

Almost everybody prefers using  $\sin(z)/2$  instead of  $\cos(\pi/2 - z)\sin(\pi/6)$ . *Mathematica* automatically transforms the second expression into the first one. The automatic application of transformation rules to mathematical expressions can give overly complicated results. Compact expressions like  $\sin(2z)\sin(\pi/16)$  should not be automatically expanded into the more complicated expression  $\sin(z)\cos(z)\left(2 - (2 + 2^{1/2})^{1/2}\right)^{1/2}$ . *Mathematica* has special commands that produce these types of expansions. Some of them are demonstrated in the next section.

### TrigExpand

The function `TrigExpand` expands out trigonometric and hyperbolic functions. In more detail, it splits up sums and integer multiples that appear in the arguments of trigonometric and hyperbolic functions, and then expands out the products of the trigonometric and hyperbolic functions into sums of powers, using the trigonometric and hyperbolic identities where possible. Here are some examples.

```
TrigExpand[Sin[x - y]]
```

```
Cos[y] Sin[x] - Cos[x] Sin[y]
```

```
Cos[4 z] // TrigExpand
```

```
Cos[z]^4 - 6 Cos[z]^2 Sin[z]^2 + Sin[z]^4
```

```
TrigExpand[{{Sin[x + y], Sin[3 z]},
            {Cos[x + y], Cos[3 z]},
            {Tan[x + y], Tan[3 z]},
            {Cot[x + y], Cot[3 z]},
            {Csc[x + y], Csc[3 z]},
            {Sec[x + y], Sec[3 z]}}]
```

```
{ {Cos[y] Sin[x] + Cos[x] Sin[y], 3 Cos[z]^2 Sin[z] - Sin[z]^3},
  {Cos[x] Cos[y] - Sin[x] Sin[y], Cos[z]^3 - 3 Cos[z] Sin[z]^2},
  { Cos[y] Sin[x] / (Cos[x] Cos[y] - Sin[x] Sin[y]) + Cos[x] Sin[y] / (Cos[x] Cos[y] - Sin[x] Sin[y]),
    (3 Cos[z]^2 Sin[z] / (Cos[z]^3 - 3 Cos[z] Sin[z]^2) - Sin[z]^3 / (Cos[z]^3 - 3 Cos[z] Sin[z]^2)) },
  { Cos[x] Cos[y] / (Cos[y] Sin[x] + Cos[x] Sin[y]) - Sin[x] Sin[y] / (Cos[y] Sin[x] + Cos[x] Sin[y]),
    (Cos[z]^3 / (3 Cos[z]^2 Sin[z] - Sin[z]^3) - 3 Cos[z] Sin[z]^2 / (3 Cos[z]^2 Sin[z] - Sin[z]^3)) },
  { 1 / (Cos[y] Sin[x] + Cos[x] Sin[y]), 1 / (3 Cos[z]^2 Sin[z] - Sin[z]^3) },
  { 1 / (Cos[x] Cos[y] - Sin[x] Sin[y]), 1 / (Cos[z]^3 - 3 Cos[z] Sin[z]^2) } }
```

```
TableForm[ (# == TrigExpand[#]) & /@
```

```
Flatten[{{Sin[x + y], Sin[3 z]}, {Cos[x + y], Cos[3 z]}, {Tan[x + y], Tan[3 z]},
        {Cot[x + y], Cot[3 z]}, {Csc[x + y], Csc[3 z]}, {Sec[x + y], Sec[3 z]}}]
```

$$\begin{aligned} \sin[x + y] &== \cos[y] \sin[x] + \cos[x] \sin[y] \\ \sin[3z] &== 3 \cos[z]^2 \sin[z] - \sin[z]^3 \\ \cos[x + y] &== \cos[x] \cos[y] - \sin[x] \sin[y] \\ \cos[3z] &== \cos[z]^3 - 3 \cos[z] \sin[z]^2 \\ \tan[x + y] &== \frac{\cos[y] \sin[x]}{\cos[x] \cos[y] - \sin[x] \sin[y]} + \frac{\cos[x] \sin[y]}{\cos[x] \cos[y] - \sin[x] \sin[y]} \\ \tan[3z] &== \frac{3 \cos[z]^2 \sin[z]}{\cos[z]^3 - 3 \cos[z] \sin[z]^2} - \frac{\sin[z]^3}{\cos[z]^3 - 3 \cos[z] \sin[z]^2} \\ \cot[x + y] &== \frac{\cos[x] \cos[y]}{\cos[y] \sin[x] + \cos[x] \sin[y]} - \frac{\sin[x] \sin[y]}{\cos[y] \sin[x] + \cos[x] \sin[y]} \\ \cot[3z] &== \frac{\cos[z]^3}{3 \cos[z]^2 \sin[z] - \sin[z]^3} - \frac{3 \cos[z] \sin[z]^2}{3 \cos[z]^2 \sin[z] - \sin[z]^3} \\ \csc[x + y] &== \frac{1}{\cos[y] \sin[x] + \cos[x] \sin[y]} \\ \csc[3z] &== \frac{1}{3 \cos[z]^2 \sin[z] - \sin[z]^3} \\ \sec[x + y] &== \frac{1}{\cos[x] \cos[y] - \sin[x] \sin[y]} \\ \sec[3z] &== \frac{1}{\cos[z]^3 - 3 \cos[z] \sin[z]^2} \end{aligned}$$

### TrigFactor

The function `TrigFactor` factors trigonometric and hyperbolic functions. In more detail, it splits up sums and integer multiples that appear in the arguments of trigonometric and hyperbolic functions, and then factors the resulting polynomials in the trigonometric and hyperbolic functions, using the corresponding identities where possible. Here are some examples.

`TrigFactor[Sin[x] + Cos[y]]`

$$\left( \cos\left[\frac{x}{2} - \frac{y}{2}\right] + \sin\left[\frac{x}{2} - \frac{y}{2}\right] \right) \left( \cos\left[\frac{x}{2} + \frac{y}{2}\right] + \sin\left[\frac{x}{2} + \frac{y}{2}\right] \right)$$

`Tan[x] - Cot[y] // TrigFactor`

$$-\cos[x + y] \csc[y] \sec[x]$$

`TrigFactor[{Sin[x] + Sin[y],  
Cos[x] + Cos[y],  
Tan[x] + Tan[y],  
Cot[x] + Cot[y],  
Csc[x] + Csc[y],  
Sec[x] + Sec[y]}]`

$$\left\{ 2 \cos\left[\frac{x}{2} - \frac{y}{2}\right] \sin\left[\frac{x}{2} + \frac{y}{2}\right], 2 \cos\left[\frac{x}{2} - \frac{y}{2}\right] \cos\left[\frac{x}{2} + \frac{y}{2}\right], \sec[x] \sec[y] \sin[x + y], \right. \\ \left. \csc[x] \csc[y] \sin[x + y], \frac{1}{2} \cos\left[\frac{x}{2} - \frac{y}{2}\right] \csc\left[\frac{x}{2}\right] \csc\left[\frac{y}{2}\right] \sec\left[\frac{x}{2}\right] \sec\left[\frac{y}{2}\right] \sin\left[\frac{x}{2} + \frac{y}{2}\right], \right. \\ \left. \frac{2 \cos\left[\frac{x}{2} - \frac{y}{2}\right] \cos\left[\frac{x}{2} + \frac{y}{2}\right]}{\left(\cos\left[\frac{x}{2}\right] - \sin\left[\frac{x}{2}\right]\right) \left(\cos\left[\frac{x}{2}\right] + \sin\left[\frac{x}{2}\right]\right) \left(\cos\left[\frac{y}{2}\right] - \sin\left[\frac{y}{2}\right]\right) \left(\cos\left[\frac{y}{2}\right] + \sin\left[\frac{y}{2}\right]\right)} \right\}$$

### TrigReduce



The function `TrigReduce` rewrites products and powers of trigonometric and hyperbolic functions in terms of those functions with combined arguments. In more detail, it typically yields a linear expression involving trigonometric and hyperbolic functions with more complicated arguments. `TrigReduce` is approximately inverse to `TrigExpand` and `TrigFactor`. Here are some examples.

`TrigReduce[Sin[z]^3]`

$$\frac{1}{4} (3 \sin[z] - \sin[3z])$$

`Sin[x] Cos[y] // TrigReduce`

$$\frac{1}{2} (\sin[x-y] + \sin[x+y])$$

`TrigReduce[{Sin[z]^2, Cos[z]^2, Tan[z]^2, Cot[z]^2, Csc[z]^2, Sec[z]^2}]`

$$\left\{ \frac{1}{2} (1 - \cos[2z]), \frac{1}{2} (1 + \cos[2z]), \frac{1 - \cos[2z]}{1 + \cos[2z]}, \frac{-1 - \cos[2z]}{-1 + \cos[2z]}, -\frac{2}{-1 + \cos[2z]}, \frac{2}{1 + \cos[2z]} \right\}$$

`TrigReduce[TrigExpand[{{Sin[x+y], Sin[3z], Sin[x] Sin[y]},  
{Cos[x+y], Cos[3z], Cos[x] Cos[y]},  
{Tan[x+y], Tan[3z], Tan[x] Tan[y]},  
{Cot[x+y], Cot[3z], Cot[x] Cot[y]},  
{Csc[x+y], Csc[3z], Csc[x] Csc[y]},  
{Sec[x+y], Sec[3z], Sec[x] Sec[y]}]]]`

$$\left\{ \left\{ \sin[x+y], \sin[3z], \frac{1}{2} (\cos[x-y] - \cos[x+y]) \right\}, \right. \\
\left. \left\{ \cos[x+y], \cos[3z], \frac{1}{2} (\cos[x-y] + \cos[x+y]) \right\}, \right. \\
\left. \left\{ \tan[x+y], \tan[3z], \frac{\cos[x-y] - \cos[x+y]}{\cos[x-y] + \cos[x+y]} \right\}, \right. \\
\left. \left\{ \cot[x+y], \cot[3z], \frac{\cos[x-y] + \cos[x+y]}{\cos[x-y] - \cos[x+y]} \right\}, \right. \\
\left. \left\{ \csc[x+y], \csc[3z], \frac{2}{\cos[x-y] - \cos[x+y]} \right\}, \right. \\
\left. \left\{ \sec[x+y], \sec[3z], \frac{2}{\cos[x-y] + \cos[x+y]} \right\} \right\}$$

`TrigReduce[TrigFactor[{{Sin[x] + Sin[y], Cos[x] + Cos[y],  
Tan[x] + Tan[y], Cot[x] + Cot[y], Csc[x] + Csc[y], Sec[x] + Sec[y]}]]]`

$$\left\{ \sin[x] + \sin[y], \cos[x] + \cos[y], \frac{2 \sin[x+y]}{\cos[x-y] + \cos[x+y]}, \right. \\
\left. \frac{2 \sin[x+y]}{\cos[x-y] - \cos[x+y]}, \frac{2 (\sin[x] + \sin[y])}{\cos[x-y] - \cos[x+y]}, \frac{2 (\cos[x] + \cos[y])}{\cos[x-y] + \cos[x+y]} \right\}$$

**TrigToExp**

The function `TrigToExp` converts direct and inverse trigonometric and hyperbolic functions to exponential or logarithmic functions. It tries, where possible, to give results that do not involve explicit complex numbers. Here are some examples.

`TrigToExp[Sin[2 z]]`

$$\frac{1}{2} i e^{-2 i z} - \frac{1}{2} i e^{2 i z}$$

`Sin[z] Tan[2 z] // TrigToExp`

$$-\frac{(e^{-i z} - e^{i z})(e^{-2 i z} - e^{2 i z})}{2(e^{-2 i z} + e^{2 i z})}$$

`TrigToExp[{Sin[z], Cos[z], Tan[z], Cot[z], Csc[z], Sec[z]}]`

$$\left\{ \frac{1}{2} i e^{-i z} - \frac{1}{2} i e^{i z}, \frac{e^{-i z}}{2} + \frac{e^{i z}}{2}, \frac{i(e^{-i z} - e^{i z})}{e^{-i z} + e^{i z}}, -\frac{i(e^{-i z} + e^{i z})}{e^{-i z} - e^{i z}}, -\frac{2 i}{e^{-i z} - e^{i z}}, \frac{2}{e^{-i z} + e^{i z}} \right\}$$

### ExpToTrig

The function `ExpToTrig` converts exponentials to trigonometric or hyperbolic functions. It tries, where possible, to give results that do not involve explicit complex numbers. It is approximately inverse to `TrigToExp`. Here are some examples.

`ExpToTrig[e^{i x \beta}]`

$$\cos[x \beta] + i \sin[x \beta]$$

`\frac{e^{i x \alpha} - e^{i x \beta}}{e^{i x \gamma} + e^{i x \delta}} // ExpToTrig`

$$\frac{\cos[x \alpha] - \cos[x \beta] + i \sin[x \alpha] - i \sin[x \beta]}{\cos[x \gamma] + \cos[x \delta] + i \sin[x \gamma] + i \sin[x \delta]}$$

`ExpToTrig[TrigToExp[{Sin[z], Cos[z], Tan[z], Cot[z], Csc[z], Sec[z]}]]`

$$\{\sin[z], \cos[z], \tan[z], \cot[z], \csc[z], \sec[z]\}$$

`ExpToTrig[{ \alpha e^{-i x \beta} + \alpha e^{i x \beta}, \alpha e^{-i x \beta} + \gamma e^{i x \beta} }]`

$$\{2 \alpha \cos[x \beta], \alpha \cos[x \beta] + \gamma \cos[x \beta] - i \alpha \sin[x \beta] + i \gamma \sin[x \beta]\}$$

### ComplexExpand

The function `ComplexExpand` expands expressions assuming that all the occurring variables are real. The value option `TargetFunctions` is a list of functions from the set `{Re, Im, Abs, Arg, Conjugate, Sign}`. `ComplexExpand` tries to give results in terms of the specified functions. Here are some examples

`ComplexExpand[Sin[x + i y] Cos[x - i y]]`

$$\begin{aligned} & \text{Cos}[x] \text{Cosh}[y]^2 \text{Sin}[x] - \text{Cos}[x] \text{Sin}[x] \text{Sinh}[y]^2 + \\ & i \left( \text{Cos}[x]^2 \text{Cosh}[y] \text{Sinh}[y] + \text{Cosh}[y] \text{Sin}[x]^2 \text{Sinh}[y] \right) \end{aligned}$$

**Csc[x + i y] Sec[x - i y] // ComplexExpand**

$$\begin{aligned} & -\frac{4 \text{Cos}[x] \text{Cosh}[y]^2 \text{Sin}[x]}{(\text{Cos}[2x] - \text{Cosh}[2y]) (\text{Cos}[2x] + \text{Cosh}[2y])} + \frac{4 \text{Cos}[x] \text{Sin}[x] \text{Sinh}[y]^2}{(\text{Cos}[2x] - \text{Cosh}[2y]) (\text{Cos}[2x] + \text{Cosh}[2y])} + \\ & i \left( \frac{4 \text{Cos}[x]^2 \text{Cosh}[y] \text{Sinh}[y]}{(\text{Cos}[2x] - \text{Cosh}[2y]) (\text{Cos}[2x] + \text{Cosh}[2y])} + \right. \\ & \left. \frac{4 \text{Cosh}[y] \text{Sin}[x]^2 \text{Sinh}[y]}{(\text{Cos}[2x] - \text{Cosh}[2y]) (\text{Cos}[2x] + \text{Cosh}[2y])} \right) \end{aligned}$$

In[17]:= **li1 = {Sin[x + i y], Cos[x + i y], Tan[x + i y], Cot[x + i y], Csc[x + i y], Sec[x + i y]}**

Out[17]= {Sin[x + i y], Cos[x + i y], Tan[x + i y], Cot[x + i y], Csc[x + i y], Sec[x + i y]}

In[18]:= **ComplexExpand[li1]**

$$\begin{aligned} \text{Out[18]= } & \left\{ \text{Cosh}[y] \text{Sin}[x] + i \text{Cos}[x] \text{Sinh}[y], \text{Cos}[x] \text{Cosh}[y] - i \text{Sin}[x] \text{Sinh}[y], \right. \\ & \frac{\text{Sin}[2x]}{\text{Cos}[2x] + \text{Cosh}[2y]} + \frac{i \text{Sinh}[2y]}{\text{Cos}[2x] + \text{Cosh}[2y]}, -\frac{\text{Sin}[2x]}{\text{Cos}[2x] - \text{Cosh}[2y]} + \frac{i \text{Sinh}[2y]}{\text{Cos}[2x] - \text{Cosh}[2y]}, \\ & \left. -\frac{2 \text{Cosh}[y] \text{Sin}[x]}{\text{Cos}[2x] - \text{Cosh}[2y]} + \frac{2 i \text{Cos}[x] \text{Sinh}[y]}{\text{Cos}[2x] - \text{Cosh}[2y]}, \frac{2 \text{Cos}[x] \text{Cosh}[y]}{\text{Cos}[2x] + \text{Cosh}[2y]} + \frac{2 i \text{Sin}[x] \text{Sinh}[y]}{\text{Cos}[2x] + \text{Cosh}[2y]} \right\} \end{aligned}$$

In[19]:= **ComplexExpand[Re[#] & /@ li1, TargetFunctions -> {Re, Im}]**

$$\begin{aligned} \text{Out[19]= } & \left\{ \text{Cosh}[y] \text{Sin}[x], \text{Cos}[x] \text{Cosh}[y], \frac{\text{Sin}[2x]}{\text{Cos}[2x] + \text{Cosh}[2y]}, \right. \\ & \left. -\frac{\text{Sin}[2x]}{\text{Cos}[2x] - \text{Cosh}[2y]}, -\frac{2 \text{Cosh}[y] \text{Sin}[x]}{\text{Cos}[2x] - \text{Cosh}[2y]}, \frac{2 \text{Cos}[x] \text{Cosh}[y]}{\text{Cos}[2x] + \text{Cosh}[2y]} \right\} \end{aligned}$$

In[20]:= **ComplexExpand[Im[#] & /@ li1, TargetFunctions -> {Re, Im}]**

$$\begin{aligned} \text{Out[20]= } & \left\{ \text{Cos}[x] \text{Sinh}[y], -\text{Sin}[x] \text{Sinh}[y], \frac{\text{Sinh}[2y]}{\text{Cos}[2x] + \text{Cosh}[2y]}, \right. \\ & \left. \frac{\text{Sinh}[2y]}{\text{Cos}[2x] - \text{Cosh}[2y]}, \frac{2 \text{Cos}[x] \text{Sinh}[y]}{\text{Cos}[2x] - \text{Cosh}[2y]}, \frac{2 \text{Sin}[x] \text{Sinh}[y]}{\text{Cos}[2x] + \text{Cosh}[2y]} \right\} \end{aligned}$$

In[21]:= **ComplexExpand[Abs[#] & /@ li1, TargetFunctions -> {Re, Im}]**

$$\text{Out[21]} = \left\{ \sqrt{\text{Cosh}[y]^2 \text{Sin}[x]^2 + \text{Cos}[x]^2 \text{Sinh}[y]^2}, \sqrt{\text{Cos}[x]^2 \text{Cosh}[y]^2 + \text{Sin}[x]^2 \text{Sinh}[y]^2}, \right. \\ \sqrt{\frac{\text{Sin}[2x]^2}{(\text{Cos}[2x] + \text{Cosh}[2y])^2} + \frac{\text{Sinh}[2y]^2}{(\text{Cos}[2x] + \text{Cosh}[2y])^2}}, \\ \sqrt{\frac{\text{Sin}[2x]^2}{(\text{Cos}[2x] - \text{Cosh}[2y])^2} + \frac{\text{Sinh}[2y]^2}{(\text{Cos}[2x] - \text{Cosh}[2y])^2}}, \\ \sqrt{\frac{4 \text{Cosh}[y]^2 \text{Sin}[x]^2}{(\text{Cos}[2x] - \text{Cosh}[2y])^2} + \frac{4 \text{Cos}[x]^2 \text{Sinh}[y]^2}{(\text{Cos}[2x] - \text{Cosh}[2y])^2}}, \\ \left. \sqrt{\frac{4 \text{Cos}[x]^2 \text{Cosh}[y]^2}{(\text{Cos}[2x] + \text{Cosh}[2y])^2} + \frac{4 \text{Sin}[x]^2 \text{Sinh}[y]^2}{(\text{Cos}[2x] + \text{Cosh}[2y])^2}} \right\}$$

In[22]:= % // Simplify[#, {x, y} ∈ Reals] &

$$\text{Out[22]} = \left\{ \frac{\sqrt{-\text{Cos}[2x] + \text{Cosh}[2y]}}{\sqrt{2}}, \frac{\sqrt{\text{Cos}[2x] + \text{Cosh}[2y]}}{\sqrt{2}}, \frac{\sqrt{\text{Sin}[2x]^2 + \text{Sinh}[2y]^2}}{\text{Cos}[2x] + \text{Cosh}[2y]}, \right. \\ \left. \sqrt{-\frac{\text{Cos}[2x] + \text{Cosh}[2y]}{\text{Cos}[2x] - \text{Cosh}[2y]}}, \frac{\sqrt{2}}{\sqrt{-\text{Cos}[2x] + \text{Cosh}[2y]}}, \frac{\sqrt{2}}{\sqrt{\text{Cos}[2x] + \text{Cosh}[2y]}} \right\}$$

In[23]:= ComplexExpand[Arg[#] & /@ l11, TargetFunctions → {Re, Im}]

$$\text{Out[23]} = \left\{ \text{ArcTan}[\text{Cosh}[y] \text{Sin}[x], \text{Cos}[x] \text{Sinh}[y]], \text{ArcTan}[\text{Cos}[x] \text{Cosh}[y], -\text{Sin}[x] \text{Sinh}[y]], \right. \\ \text{ArcTan}\left[\frac{\text{Sin}[2x]}{\text{Cos}[2x] + \text{Cosh}[2y]}, \frac{\text{Sinh}[2y]}{\text{Cos}[2x] + \text{Cosh}[2y]}\right], \\ \text{ArcTan}\left[-\frac{\text{Sin}[2x]}{\text{Cos}[2x] - \text{Cosh}[2y]}, \frac{\text{Sinh}[2y]}{\text{Cos}[2x] - \text{Cosh}[2y]}\right], \\ \text{ArcTan}\left[-\frac{2 \text{Cosh}[y] \text{Sin}[x]}{\text{Cos}[2x] - \text{Cosh}[2y]}, \frac{2 \text{Cos}[x] \text{Sinh}[y]}{\text{Cos}[2x] - \text{Cosh}[2y]}\right], \\ \left. \text{ArcTan}\left[\frac{2 \text{Cos}[x] \text{Cosh}[y]}{\text{Cos}[2x] + \text{Cosh}[2y]}, \frac{2 \text{Sin}[x] \text{Sinh}[y]}{\text{Cos}[2x] + \text{Cosh}[2y]}\right]\right\}$$

In[24]:= ComplexExpand[Conjugate[#] & /@ l11, TargetFunctions → {Re, Im}] // Simplify

$$\text{Out[24]} = \left\{ \text{Cosh}[y] \text{Sin}[x] - i \text{Cos}[x] \text{Sinh}[y], \text{Cos}[x] \text{Cosh}[y] + i \text{Sin}[x] \text{Sinh}[y], \right. \\ \frac{\text{Sin}[2x] - i \text{Sinh}[2y]}{\text{Cos}[2x] + \text{Cosh}[2y]}, -\frac{\text{Sin}[2x] + i \text{Sinh}[2y]}{\text{Cos}[2x] - \text{Cosh}[2y]}, \\ \left. \frac{1}{\text{Cosh}[y] \text{Sin}[x] - i \text{Cos}[x] \text{Sinh}[y]}, \frac{1}{\text{Cos}[x] \text{Cosh}[y] + i \text{Sin}[x] \text{Sinh}[y]} \right\}$$

**Simplify**

The function `Simplify` performs a sequence of algebraic transformations on its argument, and returns the simplest form it finds. Here are two examples.

`Simplify[Sin[2 z] / Sin[z]]`

`2 Cos[z]`

`Sin[2 z] / Cos[z] // Simplify`

`2 Sin[z]`

Here is a large collection of trigonometric identities. All are written as one large logical conjunction.

$$\begin{aligned} & \text{Simplify[\#] \& /@ \left( \text{Cos}[z]^2 + \text{Sin}[z]^2 == 1 \wedge \right. \\ & \text{Sin}[z]^2 == \frac{1 - \text{Cos}[2 z]}{2} \wedge \text{Cos}[z]^2 == \frac{1 + \text{Cos}[2 z]}{2} \wedge \\ & \text{Tan}[z]^2 == \frac{1 - \text{Cos}[2 z]}{1 + \text{Cos}[2 z]} \wedge \text{Cot}[z]^2 == \frac{1 + \text{Cos}[2 z]}{1 - \text{Cos}[2 z]} \wedge \\ & \text{Sin}[2 z] == 2 \text{Sin}[z] \text{Cos}[z] \wedge \text{Cos}[2 z] == \text{Cos}[z]^2 - \text{Sin}[z]^2 == 2 \text{Cos}[z]^2 - 1 \wedge \\ & \text{Sin}[a + b] == \text{Sin}[a] \text{Cos}[b] + \text{Cos}[a] \text{Sin}[b] \wedge \text{Sin}[a - b] == \text{Sin}[a] \text{Cos}[b] - \text{Cos}[a] \text{Sin}[b] \wedge \\ & \text{Cos}[a + b] == \text{Cos}[a] \text{Cos}[b] - \text{Sin}[a] \text{Sin}[b] \wedge \text{Cos}[a - b] == \text{Cos}[a] \text{Cos}[b] + \text{Sin}[a] \text{Sin}[b] \wedge \\ & \text{Sin}[a] + \text{Sin}[b] == 2 \text{Sin}\left[\frac{a + b}{2}\right] \text{Cos}\left[\frac{a - b}{2}\right] \wedge \text{Sin}[a] - \text{Sin}[b] == 2 \text{Cos}\left[\frac{a + b}{2}\right] \text{Sin}\left[\frac{a - b}{2}\right] \wedge \\ & \text{Cos}[a] + \text{Cos}[b] == 2 \text{Cos}\left[\frac{a + b}{2}\right] \text{Cos}\left[\frac{a - b}{2}\right] \wedge \text{Cos}[a] - \text{Cos}[b] == 2 \text{Sin}\left[\frac{a + b}{2}\right] \text{Sin}\left[\frac{b - a}{2}\right] \wedge \\ & \text{Tan}[a] + \text{Tan}[b] == \frac{\text{Sin}[a + b]}{\text{Cos}[a] \text{Cos}[b]} \wedge \text{Tan}[a] - \text{Tan}[b] == \frac{\text{Sin}[a - b]}{\text{Cos}[a] \text{Cos}[b]} \wedge \\ & A \text{Sin}[z] + B \text{Cos}[z] == A \sqrt{1 + \frac{B^2}{A^2}} \text{Sin}\left[z + \text{ArcTan}\left[\frac{B}{A}\right]\right] \wedge \\ & \text{Sin}[a] \text{Sin}[b] == \frac{\text{Cos}[a - b] - \text{Cos}[a + b]}{2} \wedge \\ & \text{Cos}[a] \text{Cos}[b] == \frac{\text{Cos}[a - b] + \text{Cos}[a + b]}{2} \wedge \text{Sin}[a] \text{Cos}[b] == \frac{\text{Sin}[a + b] + \text{Sin}[a - b]}{2} \wedge \\ & \text{Sin}\left[\frac{z}{2}\right]^2 == \frac{1 - \text{Cos}[z]}{2} \wedge \text{Cos}\left[\frac{z}{2}\right]^2 == \frac{1 + \text{Cos}[z]}{2} \wedge \\ & \left. \text{Tan}\left[\frac{z}{2}\right] == \frac{1 - \text{Cos}[z]}{\text{Sin}[z]} == \frac{\text{Sin}[z]}{1 + \text{Cos}[z]} \wedge \text{Cot}\left[\frac{z}{2}\right] == \frac{\text{Sin}[z]}{1 - \text{Cos}[z]} == \frac{1 + \text{Cos}[z]}{\text{Sin}[z]} \right) \end{aligned}$$

`True`

The function `Simplify` has the `Assumption` option. For example, *Mathematica* knows that  $-1 \leq \sin(x) \leq 1$  for all real  $x$ , and uses the periodicity of trigonometric functions for the symbolic integer coefficient  $k$  of  $k\pi$ .

```
Simplify[Abs[Sin[x]] ≤ 1, x ∈ Reals]
```

```
True
```

```
Abs[Sin[x]] ≤ 1 // Simplify[#, x ∈ Reals] &
```

```
True
```

```
Simplify[{Sin[z + 2 k π], Cos[z + 2 k π], Tan[z + k π],  
Cot[z + k π], Csc[z + 2 k π], Sec[z + 2 k π]}, k ∈ Integers]
```

```
{Sin[z], Cos[z], Tan[z], Cot[z], Csc[z], Sec[z]}
```

```
Simplify[{Sin[z + k π] / Sin[z], Cos[z + k π] / Cos[z], Tan[z + k π] / Tan[z],  
Cot[z + k π] / Cot[z], Csc[z + k π] / Csc[z], Sec[z + k π] / Sec[z]}, k ∈ Integers]
```

```
{(-1)k, (-1)k, 1, 1, (-1)k, (-1)k}
```

*Mathematica* also knows that the composition of inverse and direct trigonometric functions produces the value of the inner argument under the appropriate restriction. Here are some examples.

```
Simplify[{ArcSin[Sin[z]], ArcTan[Tan[z]], ArcCot[Cot[z]], ArcCsc[Csc[z]]},  
-π / 2 < Re[z] < π / 2]
```

```
{z, z, z, z}
```

```
Simplify[{ArcCos[Cos[z]], ArcSec[Sec[z]]}, 0 < Re[z] < π]
```

```
{z, z}
```

### FunctionExpand (and Together)

While the trigonometric functions auto-evaluate for simple fractions of  $\pi$ , for more complicated cases they stay as trigonometric functions to avoid the build up of large expressions. Using the function `FunctionExpand`, such expressions can be transformed into explicit radicals.

$$\text{Cos}\left[\frac{\pi}{32}\right]$$

$$\text{Cos}\left[\frac{\pi}{32}\right]$$

$$\text{FunctionExpand}\left[\text{Cos}\left[\frac{\pi}{32}\right]\right]$$

$$\frac{1}{2} \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}}$$

$$\text{Cot}\left[\frac{\pi}{24}\right] // \text{FunctionExpand}$$

$$\frac{\frac{\sqrt{2-\sqrt{2}}}{4} + \frac{1}{4}\sqrt{3(2+\sqrt{2})}}{-\frac{1}{4}\sqrt{3(2-\sqrt{2})} + \frac{\sqrt{2+\sqrt{2}}}{4}}$$

$$\left\{ \sin\left[\frac{\pi}{16}\right], \cos\left[\frac{\pi}{16}\right], \tan\left[\frac{\pi}{16}\right], \cot\left[\frac{\pi}{16}\right], \csc\left[\frac{\pi}{16}\right], \sec\left[\frac{\pi}{16}\right] \right\}$$

$$\left\{ \sin\left[\frac{\pi}{16}\right], \cos\left[\frac{\pi}{16}\right], \tan\left[\frac{\pi}{16}\right], \cot\left[\frac{\pi}{16}\right], \csc\left[\frac{\pi}{16}\right], \sec\left[\frac{\pi}{16}\right] \right\}$$

**FunctionExpand[%]**

$$\left\{ \frac{1}{2}\sqrt{2-\sqrt{2+\sqrt{2}}}, \frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2}}}, \sqrt{\frac{2-\sqrt{2+\sqrt{2}}}{2+\sqrt{2+\sqrt{2}}}}, \right.$$

$$\left. \sqrt{\frac{2+\sqrt{2+\sqrt{2}}}{2-\sqrt{2+\sqrt{2}}}}, \frac{2}{\sqrt{2-\sqrt{2+\sqrt{2}}}}, \frac{2}{\sqrt{2+\sqrt{2+\sqrt{2}}}} \right\}$$

$$\left\{ \sin\left[\frac{\pi}{60}\right], \cos\left[\frac{\pi}{60}\right], \tan\left[\frac{\pi}{60}\right], \cot\left[\frac{\pi}{60}\right], \csc\left[\frac{\pi}{60}\right], \sec\left[\frac{\pi}{60}\right] \right\}$$

$$\left\{ \sin\left[\frac{\pi}{60}\right], \cos\left[\frac{\pi}{60}\right], \tan\left[\frac{\pi}{60}\right], \cot\left[\frac{\pi}{60}\right], \csc\left[\frac{\pi}{60}\right], \sec\left[\frac{\pi}{60}\right] \right\}$$

**Together[FunctionExpand[%]]**

$$\left\{ \frac{1}{16} \left( -\sqrt{2} - \sqrt{6} + \sqrt{10} + \sqrt{30} + 2\sqrt{5+\sqrt{5}} - 2\sqrt{3(5+\sqrt{5})} \right), \right.$$

$$\frac{1}{16} \left( \sqrt{2} - \sqrt{6} - \sqrt{10} + \sqrt{30} + 2\sqrt{5+\sqrt{5}} + 2\sqrt{3(5+\sqrt{5})} \right),$$

$$\frac{-1 - \sqrt{3} + \sqrt{5} + \sqrt{15} + \sqrt{2(5+\sqrt{5})} - \sqrt{6(5+\sqrt{5})}}{1 - \sqrt{3} - \sqrt{5} + \sqrt{15} + \sqrt{2(5+\sqrt{5})} + \sqrt{6(5+\sqrt{5})}},$$

$$\frac{-1 + \sqrt{3} + \sqrt{5} - \sqrt{15} - \sqrt{2(5+\sqrt{5})} - \sqrt{6(5+\sqrt{5})}}{1 + \sqrt{3} - \sqrt{5} - \sqrt{15} - \sqrt{2(5+\sqrt{5})} + \sqrt{6(5+\sqrt{5})}},$$

$$\frac{16}{-\sqrt{2} - \sqrt{6} + \sqrt{10} + \sqrt{30} + 2\sqrt{5+\sqrt{5}} - 2\sqrt{3(5+\sqrt{5})}},$$

$$\left. \frac{16}{\sqrt{2} - \sqrt{6} - \sqrt{10} + \sqrt{30} + 2\sqrt{5+\sqrt{5}} + 2\sqrt{3(5+\sqrt{5})}} \right\}$$

If the denominator contains squares of integers other than 2, the results always contain complex numbers (meaning that the imaginary number  $i = \sqrt{-1}$  appears unavoidably).

$$\left\{ \text{Sin}\left[\frac{\pi}{9}\right], \text{Cos}\left[\frac{\pi}{9}\right], \text{Tan}\left[\frac{\pi}{9}\right], \text{Cot}\left[\frac{\pi}{9}\right], \text{Csc}\left[\frac{\pi}{9}\right], \text{Sec}\left[\frac{\pi}{9}\right] \right\}$$

$$\left\{ \text{Sin}\left[\frac{\pi}{9}\right], \text{Cos}\left[\frac{\pi}{9}\right], \text{Tan}\left[\frac{\pi}{9}\right], \text{Cot}\left[\frac{\pi}{9}\right], \text{Csc}\left[\frac{\pi}{9}\right], \text{Sec}\left[\frac{\pi}{9}\right] \right\}$$

**FunctionExpand[%] // Together**



$$\left\{ \frac{1}{8} \left( -i 2^{2/3} (-1 - i \sqrt{3})^{1/3} + 2^{2/3} \sqrt{3} (-1 - i \sqrt{3})^{1/3} + i 2^{2/3} (-1 + i \sqrt{3})^{1/3} + 2^{2/3} \sqrt{3} (-1 + i \sqrt{3})^{1/3} \right), \right. \\ \frac{1}{8} \left( 2^{2/3} (-1 - i \sqrt{3})^{1/3} + i 2^{2/3} \sqrt{3} (-1 - i \sqrt{3})^{1/3} + 2^{2/3} (-1 + i \sqrt{3})^{1/3} - i 2^{2/3} \sqrt{3} (-1 + i \sqrt{3})^{1/3} \right), \\ \frac{-(-1 - i \sqrt{3})^{1/3} - i \sqrt{3} (-1 - i \sqrt{3})^{1/3} + (-1 + i \sqrt{3})^{1/3} - i \sqrt{3} (-1 + i \sqrt{3})^{1/3}}{-i (-1 - i \sqrt{3})^{1/3} + \sqrt{3} (-1 - i \sqrt{3})^{1/3} - i (-1 + i \sqrt{3})^{1/3} - \sqrt{3} (-1 + i \sqrt{3})^{1/3}}, \\ \frac{(-1 - i \sqrt{3})^{1/3} + i \sqrt{3} (-1 - i \sqrt{3})^{1/3} + (-1 + i \sqrt{3})^{1/3} - i \sqrt{3} (-1 + i \sqrt{3})^{1/3}}{-i (-1 - i \sqrt{3})^{1/3} + \sqrt{3} (-1 - i \sqrt{3})^{1/3} + i (-1 + i \sqrt{3})^{1/3} + \sqrt{3} (-1 + i \sqrt{3})^{1/3}}, \\ 8 \left/ \left( -i 2^{2/3} (-1 - i \sqrt{3})^{1/3} + 2^{2/3} \sqrt{3} (-1 - i \sqrt{3})^{1/3} + i 2^{2/3} (-1 + i \sqrt{3})^{1/3} + 2^{2/3} \sqrt{3} (-1 + i \sqrt{3})^{1/3} \right), \right. \\ \left. - (8 i) \left/ \left( -i 2^{2/3} (-1 - i \sqrt{3})^{1/3} + 2^{2/3} \sqrt{3} (-1 - i \sqrt{3})^{1/3} - i 2^{2/3} (-1 + i \sqrt{3})^{1/3} - 2^{2/3} \sqrt{3} (-1 + i \sqrt{3})^{1/3} \right) \right\}$$

Here the function `RootReduce` is used to express the previous algebraic numbers as numbered roots of polynomial equations.

`RootReduce[Simplify[%]]`

$$\{ \text{Root}[-3 + 36 \#1^2 - 96 \#1^4 + 64 \#1^6 \&, 4], \text{Root}[-1 - 6 \#1 + 8 \#1^3 \&, 3], \\ \text{Root}[-3 + 27 \#1^2 - 33 \#1^4 + \#1^6 \&, 4], \text{Root}[-1 + 33 \#1^2 - 27 \#1^4 + 3 \#1^6 \&, 6], \\ \text{Root}[-64 + 96 \#1^2 - 36 \#1^4 + 3 \#1^6 \&, 6], \text{Root}[-8 + 6 \#1^2 + \#1^3 \&, 3] \}$$

The function `FunctionExpand` also reduces trigonometric expressions with compound arguments or compositions, including hyperbolic functions, to simpler ones. Here are some examples.

`FunctionExpand[Cot[Sqrt[-z^2]]]`

$$\frac{\sqrt{-z} \text{Coth}[z]}{\sqrt{z}}$$

`Tan[Sqrt[i z^2]] // FunctionExpand`

$$\frac{(-1)^{3/4} \sqrt{-(-1)^{3/4} z} \sqrt{(-1)^{3/4} z} \text{Tan}\left[(-1)^{1/4} z\right]}{z}$$

$$\{\text{Sin}[\sqrt{z^2}], \text{Cos}[\sqrt{z^2}], \text{Tan}[\sqrt{z^2}], \text{Cot}[\sqrt{z^2}], \text{Csc}[\sqrt{z^2}], \text{Sec}[\sqrt{z^2}]\} // \text{FunctionExpand}$$

$$\left\{ \frac{\sqrt{-i z} \sqrt{i z} \text{Sin}[z]}{z}, \text{Cos}[z], \frac{\sqrt{-i z} \sqrt{i z} \text{Tan}[z]}{z}, \right.$$

$$\left. \frac{\sqrt{-i z} \sqrt{i z} \text{Cot}[z]}{z}, \frac{\sqrt{-i z} \sqrt{i z} \text{Csc}[z]}{z}, \text{Sec}[z] \right\}$$

Applying Simplify to the last expression gives a more compact result.

**Simplify[%]**

$$\left\{ \frac{\sqrt{z^2} \text{Sin}[z]}{z}, \text{Cos}[z], \frac{\sqrt{z^2} \text{Tan}[z]}{z}, \frac{\sqrt{z^2} \text{Cot}[z]}{z}, \frac{\sqrt{z^2} \text{Csc}[z]}{z}, \text{Sec}[z] \right\}$$

Here are some similar examples.

**Sin[2 ArcTan[z]] // FunctionExpand**

$$\frac{2 z}{1 + z^2}$$

**Cos[ $\frac{\text{ArcCot}[z]}{2}$ ] // FunctionExpand**

$$\frac{\sqrt{1 + \frac{\sqrt{-z} \sqrt{z}}{\sqrt{-1-z^2}}}}{\sqrt{2}}$$

**{Sin[2 ArcSin[z]], Cos[2 ArcCos[z]], Tan[2 ArcTan[z]], Cot[2 ArcCot[z]], Csc[2 ArcCsc[z]], Sec[2 ArcSec[z]]} // FunctionExpand**

$$\left\{ 2 \sqrt{1-z} z \sqrt{1+z}, -1+2 z^2, -\frac{2 z}{(-1+z)(1+z)}, \right.$$

$$\left. \frac{1}{2} \left( 1 + \frac{1}{z^2} \right) z \left( \frac{1}{-1-z^2} - \frac{z^2}{-1-z^2} \right), \frac{\sqrt{-i z} \sqrt{i z} z}{2 \sqrt{(-1+z)(1+z)}}, \frac{z^2}{2-z^2} \right\}$$

**{Sin[ $\frac{\text{ArcSin}[z]}{2}$ ], Cos[ $\frac{\text{ArcCos}[z]}{2}$ ], Tan[ $\frac{\text{ArcTan}[z]}{2}$ ], Cot[ $\frac{\text{ArcCot}[z]}{2}$ ], Csc[ $\frac{\text{ArcCsc}[z]}{2}$ ], Sec[ $\frac{\text{ArcSec}[z]}{2}$ ]} // FunctionExpand**

$$\left\{ \frac{z \sqrt{1 - \sqrt{1 - z}} \sqrt{1 + z}}{\sqrt{2} \sqrt{-i z} \sqrt{i z}}, \frac{\sqrt{1 + z}}{\sqrt{2}}, \frac{z}{1 + \sqrt{i}(-i + z) \sqrt{-i}(i + z)}, \right.$$

$$\left. z \left( 1 + \frac{\sqrt{-1 - z^2}}{\sqrt{-z} \sqrt{z}} \right), \frac{\sqrt{2} \sqrt{-\frac{i}{z}} \sqrt{\frac{i}{z}} z}{\sqrt{1 - \frac{\sqrt{(-1+z)(1+z)}}{\sqrt{-i z} \sqrt{i z}}}}, \frac{\sqrt{2} \sqrt{-z}}{\sqrt{-1 - z}} \right\}$$

**Simplify[%]**

$$\left\{ \frac{z \sqrt{1 - \sqrt{1 - z^2}}}{\sqrt{2} \sqrt{z^2}}, \frac{\sqrt{1 + z}}{\sqrt{2}}, \frac{z}{1 + \sqrt{1 + z^2}}, z + \frac{\sqrt{z} \sqrt{-1 - z^2}}{\sqrt{-z}}, \frac{\sqrt{2} \sqrt{\frac{1}{z^2}} z}{\sqrt{1 - \frac{\sqrt{z^2} \sqrt{-1 + z^2}}{z^2}}}, \frac{\sqrt{2}}{\sqrt{1 + \frac{1}{z}}} \right\}$$

**FullSimplify**

The function FullSimplify tries a wider range of transformations than Simplify and returns the simplest form it finds. Here are some examples that contrast the results of applying these functions to the same expressions.

$$\text{Cos}\left[\frac{1}{2} i \text{Log}[1 - i z] - \frac{1}{2} i \text{Log}[1 + i z]\right] // \text{Simplify}$$

$$\text{Cosh}\left[\frac{1}{2} (\text{Log}[1 - i z] - \text{Log}[1 + i z])\right]$$

$$\text{Cos}\left[\frac{1}{2} i \text{Log}[1 - i z] - \frac{1}{2} i \text{Log}[1 + i z]\right] // \text{FullSimplify}$$

$$\frac{1}{\sqrt{1 + z^2}}$$

$$\left\{ \text{Sin}\left[-i \text{Log}\left[i z + \sqrt{1 - z^2}\right]\right], \text{Cos}\left[-i \text{Log}\left[i z + \sqrt{1 - z^2}\right]\right], \right.$$

$$\text{Tan}\left[-i \text{Log}\left[i z + \sqrt{1 - z^2}\right]\right], \text{Cot}\left[-i \text{Log}\left[i z + \sqrt{1 - z^2}\right]\right],$$

$$\left. \text{Csc}\left[-i \text{Log}\left[i z + \sqrt{1 - z^2}\right]\right], \text{Sec}\left[-i \text{Log}\left[i z + \sqrt{1 - z^2}\right]\right] \right\} // \text{Simplify}$$

$$\left\{ z, \frac{1 - z^2 + i z \sqrt{1 - z^2}}{i z + \sqrt{1 - z^2}}, \frac{z (z - i \sqrt{1 - z^2})}{-i + i z^2 + z \sqrt{1 - z^2}}, \frac{1 - z^2 + i z \sqrt{1 - z^2}}{i z^2 + z \sqrt{1 - z^2}}, \frac{1}{z}, \frac{2 (i z + \sqrt{1 - z^2})}{1 + (i z + \sqrt{1 - z^2})^2} \right\}$$

$$\left\{ \text{Sin}\left[-i \text{Log}\left[i z + \sqrt{1 - z^2}\right]\right], \text{Cos}\left[-i \text{Log}\left[i z + \sqrt{1 - z^2}\right]\right], \right.$$

$$\text{Tan}\left[-i \text{Log}\left[i z + \sqrt{1 - z^2}\right]\right], \text{Cot}\left[-i \text{Log}\left[i z + \sqrt{1 - z^2}\right]\right],$$

$$\left. \text{Csc}\left[-i \text{Log}\left[i z + \sqrt{1 - z^2}\right]\right], \text{Sec}\left[-i \text{Log}\left[i z + \sqrt{1 - z^2}\right]\right] \right\} // \text{FullSimplify}$$

$$\left\{ z, \sqrt{1-z^2}, \frac{z}{\sqrt{1-z^2}}, \frac{\sqrt{1-z^2}}{z}, \frac{1}{z}, \frac{1}{\sqrt{1-z^2}} \right\}$$

## Operations carried out by specialized *Mathematica* functions

### Series expansions

Calculating the series expansion of trigonometric functions to hundreds of terms can be done in seconds. Here are some examples.

**Series[Sin[z], {z, 0, 5}]**

$$z - \frac{z^3}{6} + \frac{z^5}{120} + O[z]^6$$

**Normal[%]**

$$z - \frac{z^3}{6} + \frac{z^5}{120}$$

**Series[{Sin[z], Cos[z], Tan[z], Cot[z], Csc[z], Sec[z]}, {z, 0, 3}]**

$$\left\{ z - \frac{z^3}{6} + O[z]^4, 1 - \frac{z^2}{2} + O[z]^4, z + \frac{z^3}{3} + O[z]^4, \frac{1}{z} - \frac{z}{3} - \frac{z^3}{45} + O[z]^4, \frac{1}{z} + \frac{z}{6} + \frac{7z^3}{360} + O[z]^4, 1 + \frac{z^2}{2} + O[z]^4 \right\}$$

**Series[Cot[z], {z, 0, 100}] // Timing**

$$\left\{ 1.442 \text{ Second}, \frac{1}{z} - \frac{z}{3} - \frac{z^3}{45} - \frac{2z^5}{945} - \frac{z^7}{4725} - \frac{2z^9}{93555} - \frac{1382z^{11}}{638512875} - \frac{4z^{13}}{18243225} - \frac{3617z^{15}}{162820783125} - \frac{87734z^{17}}{38979295480125} - \frac{349222z^{19}}{1531329465290625} - \frac{310732z^{21}}{13447856940643125} - \frac{472728182z^{23}}{201919571963756521875} - \frac{2631724z^{25}}{11094481976030578125} - \frac{13571120588z^{27}}{56465366017076273671875} - \frac{13785346041608z^{29}}{5660878804669082674070015625} - \frac{7709321041217z^{31}}{3124511028551170603633203125} - \frac{303257395102z^{33}}{12130454581433748587292890625} - \frac{52630543106106954746z^{35}}{20777977561866588586487628662044921875} - \frac{61684082396644z^{37}}{2403467618492375776343276883984375} - \frac{522165436992898244102z^{39}}{20080431172289638826798401128390556640625} - \frac{6080390575672283210764z^{41}}{2307789189818960127712594427864667427734375} - \frac{10121188937927645176372z^{43}}{3791367954702577352670690845777679169921875} \right\}$$

$$\begin{aligned}
& \frac{207\ 461\ 256\ 206\ 578\ 143\ 748\ 856\ z^{45}}{7\ 670\ 102\ 214\ 448\ 301\ 053\ 033\ 358\ 480\ 610\ 212\ 529\ 462\ 890\ 625} - \\
& \frac{11\ 218\ 806\ 737\ 995\ 635\ 372\ 498\ 255\ 094\ z^{47}}{4\ 093\ 648\ 603\ 384\ 274\ 996\ 519\ 698\ 921\ 478\ 879\ 580\ 162\ 286\ 669\ 921\ 875} - \\
& \frac{79\ 209\ 152\ 838\ 572\ 743\ 713\ 996\ 404\ z^{49}}{285\ 258\ 771\ 457\ 546\ 764\ 463\ 363\ 635\ 252\ 374\ 414\ 183\ 254\ 365\ 234\ 375} - \\
& \frac{246\ 512\ 528\ 657\ 073\ 833\ 030\ 130\ 766\ 724\ z^{51}}{8\ 761\ 982\ 491\ 474\ 419\ 367\ 550\ 817\ 114\ 626\ 909\ 562\ 924\ 278\ 968\ 505\ 859\ 375} - \\
& \frac{233\ 199\ 709\ 079\ 078\ 899\ 371\ 344\ 990\ 501\ 528\ z^{53}}{81\ 807\ 125\ 729\ 900\ 063\ 867\ 074\ 959\ 072\ 425\ 603\ 825\ 198\ 823\ 017\ 351\ 806\ 640\ 625} - \\
& \frac{1\ 416\ 795\ 959\ 607\ 558\ 144\ 963\ 094\ 708\ 378\ 988\ z^{55}}{4\ 905\ 352\ 087\ 939\ 496\ 310\ 826\ 487\ 207\ 538\ 302\ 184\ 255\ 342\ 959\ 123\ 162\ 841\ 796\ 875} - \\
& \frac{23\ 305\ 824\ 372\ 104\ 839\ 134\ 357\ 731\ 308\ 699\ 592\ z^{57}}{796\ 392\ 368\ 980\ 577\ 121\ 745\ 974\ 726\ 570\ 063\ 253\ 238\ 310\ 542\ 073\ 919\ 837\ 646\ 484\ 375} - \\
& \frac{9\ 721\ 865\ 123\ 870\ 044\ 576\ 322\ 439\ 952\ 638\ 561\ 968\ 331\ 928\ z^{59}}{3\ 278\ 777\ 586\ 273\ 629\ 598\ 615\ 520\ 165\ 380\ 455\ 583\ 231\ 003\ 564\ 645\ 636\ 125\ 000\ 418\ 914\ 794\ 921\ 875} - \\
& \frac{6\ 348\ 689\ 256\ 302\ 894\ 731\ 330\ 601\ 216\ 724\ 328\ 336\ z^{61}}{21\ 132\ 271\ 510\ 899\ 613\ 925\ 529\ 439\ 369\ 536\ 628\ 424\ 678\ 570\ 233\ 931\ 462\ 891\ 949\ 462\ 890\ 625} - \\
& \frac{106\ 783\ 830\ 147\ 866\ 529\ 886\ 385\ 444\ 979\ 142\ 647\ 942\ 017\ z^{63}}{3\ 508\ 062\ 732\ 166\ 890\ 409\ 707\ 514\ 582\ 539\ 928\ 001\ 638\ 766\ 051\ 683\ 792\ 497\ 378\ 070\ 587\ 158\ 203\ 125} \\
& (267\ 745\ 458\ 568\ 424\ 664\ 373\ 021\ 714\ 282\ 169\ 516\ 771\ 254\ 382\ z^{65}) / \\
& 86\ 812\ 790\ 293\ 146\ 213\ 360\ 651\ 966\ 604\ 262\ 937\ 105\ 495\ 141\ 563\ 588\ 806\ 888\ 204\ 273\ 501\ 373\ 291\ 015 : \\
& 625 - (250\ 471\ 004\ 320\ 250\ 327\ 955\ 196\ 022\ 920\ 428\ 000\ 776\ 938\ z^{67}) / \\
& 801\ 528\ 196\ 428\ 242\ 695\ 121\ 010\ 267\ 455\ 843\ 804\ 062\ 822\ 357\ 897\ 831\ 858\ 125\ 102\ 407\ 684\ 326\ 171\ 875 \\
& - (172\ 043\ 582\ 552\ 384\ 800\ 434\ 637\ 321\ 986\ 040\ 823\ 829\ 878\ 646\ 884\ z^{69}) / \\
& 5\ 433\ 748\ 964\ 547\ 053\ 581\ 149\ 916\ 185\ 708\ 338\ 218\ 048\ 392\ 402\ 830\ 337\ 634\ 114\ 958\ 370\ 880\ 742\ 156 : \\
& 982\ 421\ 875 - (11\ 655\ 909\ 923\ 339\ 888\ 220\ 876\ 554\ 489\ 282\ 134\ 730\ 564\ 976\ 603\ 688\ 520\ 858\ z^{71}) / \\
& 3\ 633\ 348\ 205\ 269\ 879\ 230\ 856\ 840\ 004\ 304\ 821\ 536\ 968\ 049\ 780\ 112\ 803\ 650\ 817\ 771\ 432\ 558\ 560\ 793 : \\
& 458\ 452\ 606\ 201\ 171\ 875 - \\
& (3\ 692\ 153\ 220\ 456\ 342\ 488\ 035\ 683\ 646\ 645\ 690\ 290\ 452\ 790\ 030\ 604\ z^{73}) / \\
& 11\ 359\ 005\ 221\ 796\ 317\ 918\ 049\ 302\ 062\ 760\ 294\ 302\ 183\ 889\ 391\ 189\ 419\ 445\ 133\ 951\ 612\ 582\ 060\ 536 : \\
& 346\ 435\ 546\ 875 - (5\ 190\ 545\ 015\ 986\ 394\ 254\ 249\ 936\ 008\ 544\ 252\ 611\ 445\ 319\ 542\ 919\ 116\ z^{75}) / \\
& 157\ 606\ 197\ 452\ 423\ 911\ 112\ 934\ 066\ 120\ 799\ 083\ 442\ 801\ 465\ 302\ 753\ 194\ 801\ 233\ 578\ 624\ 576\ 089 : \\
& 941\ 806\ 793\ 212\ 890\ 625 - \\
& (255\ 290\ 071\ 123\ 323\ 586\ 643\ 187\ 098\ 799\ 718\ 199\ 072\ 122\ 692\ 536\ 861\ 835\ 992\ z^{77}) / \\
& 76\ 505\ 736\ 228\ 426\ 953\ 173\ 738\ 238\ 352\ 183\ 101\ 801\ 688\ 392\ 812\ 244\ 485\ 181\ 277\ 127\ 930\ 109\ 049\ 138 : \\
& 257\ 655\ 704\ 498\ 291\ 015\ 625 - \\
& (9\ 207\ 568\ 598\ 958\ 915\ 293\ 871\ 149\ 938\ 038\ 093\ 699\ 588\ 515\ 745\ 502\ 577\ 839\ 313\ 734\ z^{79}) / \\
& 27\ 233\ 582\ 984\ 369\ 795\ 892\ 070\ 228\ 410\ 001\ 578\ 355\ 986\ 013\ 571\ 390\ 071\ 723\ 225\ 259\ 349\ 721\ 067\ 988 : \\
& 068\ 852\ 863\ 296\ 604\ 156\ 494\ 140\ 625 - \\
& (163\ 611\ 136\ 505\ 867\ 886\ 519\ 332\ 147\ 296\ 221\ 453\ 678\ 803\ 514\ 884\ 902\ 772\ 183\ 572\ z^{81}) / \\
& 4\ 776\ 089\ 171\ 877\ 348\ 057\ 451\ 105\ 924\ 101\ 750\ 653\ 118\ 402\ 745\ 283\ 825\ 543\ 113\ 171\ 217\ 116\ 857\ 704 : \\
& 024\ 700\ 607\ 798\ 175\ 811\ 767\ 578\ 125 - \\
& (8\ 098\ 304\ 783\ 741\ 161\ 440\ 924\ 524\ 640\ 446\ 924\ 039\ 959\ 669\ 564\ 792\ 363\ 509\ 124\ 335\ 729\ 908\ z^{83}) /
\end{aligned}$$

$$\begin{aligned}
 & 2\ 333\ 207\ 846\ 470\ 426\ 678\ 843\ 707\ 227\ 616\ 712\ 214\ 909\ 162\ 634\ 745\ 895\ 349\ 325\ 948\ 586\ 531\ 533\ 393\ \backslash \\
 & 530\ 725\ 143\ 500\ 144\ 033\ 328\ 342\ 437\ 744\ 140\ 625\ - \\
 & (122\ 923\ 650\ 124\ 219\ 284\ 385\ 832\ 157\ 660\ 699\ 813\ 260\ 991\ 755\ 656\ 444\ 452\ 420\ 836\ 648\ z^{85}) / \\
 & 349\ 538\ 086\ 043\ 843\ 717\ 584\ 559\ 187\ 055\ 386\ 621\ 548\ 470\ 304\ 913\ 596\ 772\ 372\ 737\ 435\ 524\ 697\ 231\ \backslash \\
 & 069\ 047\ 713\ 981\ 709\ 496\ 784\ 210\ 205\ 078\ 125\ - \\
 & (476\ 882\ 359\ 517\ 824\ 548\ 362\ 004\ 154\ 188\ 840\ 670\ 307\ 545\ 554\ 753\ 464\ 961\ 562\ 516\ 323\ 845\ 108\ z^{87}) / \\
 & 13\ 383\ 510\ 964\ 174\ 348\ 021\ 497\ 060\ 628\ 653\ 950\ 829\ 663\ 288\ 548\ 327\ 870\ 152\ 944\ 013\ 988\ 358\ 928\ 114\ \backslash \\
 & 528\ 962\ 242\ 087\ 062\ 453\ 152\ 690\ 410\ 614\ 013\ 671\ 875\ - \\
 & (1\ 886\ 491\ 646\ 433\ 732\ 479\ 814\ 597\ 361\ 998\ 744\ 134\ 040\ 407\ 919\ 471\ 435\ 385\ 970\ 472\ 345\ 164\ 676\ 056 \\
 & z^{89}) / \\
 & 522\ 532\ 651\ 330\ 971\ 490\ 226\ 753\ 590\ 247\ 329\ 744\ 050\ 384\ 290\ 675\ 644\ 135\ 735\ 656\ 667\ 608\ 610\ 471\ \backslash \\
 & 400\ 391\ 047\ 234\ 539\ 824\ 350\ 830\ 981\ 313\ 610\ 076\ 904\ 296\ 875\ - \\
 & (450\ 638\ 590\ 680\ 882\ 618\ 431\ 105\ 331\ 665\ 591\ 912\ 924\ 988\ 342\ 163\ 281\ 788\ 877\ 675\ 244\ 114\ 763\ 912 \\
 & z^{91}) / \\
 & 1\ 231\ 931\ 818\ 039\ 911\ 948\ 327\ 467\ 370\ 123\ 161\ 265\ 684\ 460\ 571\ 086\ 659\ 079\ 080\ 437\ 659\ 781\ 065\ 743\ \backslash \\
 & 269\ 173\ 212\ 919\ 832\ 661\ 978\ 537\ 311\ 246\ 395\ 111\ 083\ 984\ 375\ - \\
 & (415\ 596\ 189\ 473\ 955\ 564\ 121\ 634\ 614\ 268\ 323\ 814\ 113\ 534\ 779\ 643\ 471\ 190\ 276\ 158\ 333\ 713\ 923\ 216 \\
 & z^{93}) / \\
 & 11\ 213\ 200\ 675\ 690\ 943\ 223\ 287\ 032\ 785\ 929\ 540\ 201\ 272\ 600\ 687\ 465\ 377\ 745\ 332\ 153\ 847\ 964\ 679\ 254\ \backslash \\
 & 692\ 602\ 138\ 023\ 498\ 144\ 562\ 090\ 675\ 557\ 613\ 372\ 802\ 734\ 375\ - \\
 & (423\ 200\ 899\ 194\ 533\ 026\ 195\ 195\ 456\ 219\ 648\ 467\ 346\ 087\ 908\ 778\ 120\ 468\ 301\ 277\ 466\ 840\ 101\ 336\ \backslash \\
 & 699\ 974\ 518\ z^{95}) / \\
 & 112\ 694\ 926\ 530\ 960\ 148\ 011\ 367\ 752\ 417\ 874\ 063\ 473\ 378\ 698\ 369\ 880\ 587\ 800\ 838\ 274\ 234\ 349\ 237\ \backslash \\
 & 591\ 647\ 453\ 413\ 782\ 021\ 538\ 312\ 594\ 164\ 677\ 406\ 144\ 702\ 434\ 539\ 794\ 921\ 875\ - \\
 & (5\ 543\ 531\ 483\ 502\ 489\ 438\ 698\ 050\ 411\ 951\ 314\ 743\ 456\ 505\ 773\ 755\ 468\ 368\ 087\ 670\ 306\ 121\ 873\ 229\ \backslash \\
 & 244\ z^{97}) / \\
 & 14\ 569\ 479\ 835\ 935\ 377\ 894\ 165\ 191\ 004\ 250\ 040\ 526\ 616\ 509\ 162\ 234\ 077\ 285\ 176\ 247\ 476\ 968\ 227\ 225\ \backslash \\
 & 810\ 918\ 346\ 966\ 001\ 491\ 701\ 692\ 846\ 112\ 140\ 419\ 483\ 184\ 814\ 453\ 125\ - \\
 & (378\ 392\ 151\ 276\ 488\ 501\ 180\ 909\ 732\ 277\ 974\ 887\ 490\ 811\ 366\ 132\ 267\ 744\ 533\ 542\ 784\ 817\ 245\ 581\ \backslash \\
 & 660\ 788\ 990\ 844\ z^{99}) / \\
 & 9\ 815\ 205\ 420\ 757\ 514\ 710\ 108\ 178\ 059\ 369\ 553\ 458\ 327\ 392\ 260\ 750\ 404\ 049\ 930\ 407\ 987\ 933\ 582\ 359\ \backslash \\
 & 080\ 767\ 225\ 644\ 716\ 670\ 683\ 512\ 153\ 512\ 547\ 802\ 166\ 033\ 089\ 160\ 919\ 189\ 453\ 125 + O[z]^{101} \}
 \end{aligned}$$

*Mathematica* comes with the add-on package `DiscreteMath`RSolve`` that allows finding the general terms of series for many functions. After loading this package, and using the package function `SeriesTerm`, the following  $n^{\text{th}}$  term for odd trigonometric functions can be evaluated.

```

<< DiscreteMath`RSolve`
SeriesTerm[{Sin[z], Tan[z], Cot[z], Csc[z], Cos[z], Sec[z]}, {z, 0, n}]

```

$$\left\{ \frac{i^{-1+n} \text{KroneckerDelta}[\text{Mod}[-1+n, 2]] \text{UnitStep}[-1+n]}{\text{Gamma}[1+n]}, \right. \\ \text{If}\left[\text{Odd}[n], \frac{i^{-1+n} 2^{1+n} (-1 + 2^{1+n}) \text{BernoulliB}[1+n]}{(1+n)!}, 0\right], \frac{i i^n 2^{1+n} \text{BernoulliB}[1+n]}{(1+n)!}, \\ \left. \frac{i i^n 2^{1+n} \text{BernoulliB}\left[1+n, \frac{1}{2}\right]}{(1+n)!}, \frac{i^n \text{KroneckerDelta}[\text{Mod}[n, 2]]}{\text{Gamma}[1+n]}, \frac{i^n \text{EulerE}[n]}{n!} \right\}$$

### Differentiation

*Mathematica* can evaluate derivatives of trigonometric functions of an arbitrary positive integer order.

**D[Sin[z], z]**

Cos[z]

**Sin[z] // D[#, z] &**

Cos[z]

**∂<sub>z</sub>{Sin[z], Cos[z], Tan[z], Cot[z], Csc[z], Sec[z]}**

{Cos[z], -Sin[z], Sec[z]<sup>2</sup>, -Csc[z]<sup>2</sup>, -Cot[z] Csc[z], Sec[z] Tan[z]}

**∂<sub>{z,2}</sub>{Sin[z], Cos[z], Tan[z], Cot[z], Csc[z], Sec[z]}**

{-Sin[z], -Cos[z], 2 Sec[z]<sup>2</sup> Tan[z], 2 Cot[z] Csc[z]<sup>2</sup>,  
Cot[z]<sup>2</sup> Csc[z] + Csc[z]<sup>3</sup>, Sec[z]<sup>3</sup> + Sec[z] Tan[z]<sup>2</sup>}

**Table[D[{Sin[z], Cos[z], Tan[z], Cot[z], Csc[z], Sec[z]}, {z, n}], {n, 4}]**

{ {Cos[z], -Sin[z], Sec[z]<sup>2</sup>, -Csc[z]<sup>2</sup>, -Cot[z] Csc[z], Sec[z] Tan[z]}, {-Sin[z], -Cos[z],  
2 Sec[z]<sup>2</sup> Tan[z], 2 Cot[z] Csc[z]<sup>2</sup>, Cot[z]<sup>2</sup> Csc[z] + Csc[z]<sup>3</sup>, Sec[z]<sup>3</sup> + Sec[z] Tan[z]<sup>2</sup>},  
{-Cos[z], Sin[z], 2 Sec[z]<sup>4</sup> + 4 Sec[z]<sup>2</sup> Tan[z]<sup>2</sup>, -4 Cot[z]<sup>2</sup> Csc[z]<sup>2</sup> - 2 Csc[z]<sup>4</sup>,  
-Cot[z]<sup>3</sup> Csc[z] - 5 Cot[z] Csc[z]<sup>3</sup>, 5 Sec[z]<sup>3</sup> Tan[z] + Sec[z] Tan[z]<sup>3</sup>},  
{Sin[z], Cos[z], 16 Sec[z]<sup>4</sup> Tan[z] + 8 Sec[z]<sup>2</sup> Tan[z]<sup>3</sup>,  
8 Cot[z]<sup>3</sup> Csc[z]<sup>2</sup> + 16 Cot[z] Csc[z]<sup>4</sup>, Cot[z]<sup>4</sup> Csc[z] + 18 Cot[z]<sup>2</sup> Csc[z]<sup>3</sup> + 5 Csc[z]<sup>5</sup>,  
5 Sec[z]<sup>5</sup> + 18 Sec[z]<sup>3</sup> Tan[z]<sup>2</sup> + Sec[z] Tan[z]<sup>4</sup>}}

### Finite summation

*Mathematica* can calculate finite sums that contain trigonometric functions. Here are two examples.

**Sum[Sin[a k], {k, 0, n}]**

$$\frac{1}{2} \left( \cos\left[\frac{a}{2}\right] - \cos\left[\frac{a}{2} + a n\right] \right) \text{Csc}\left[\frac{a}{2}\right]$$

$$\sum_{k=0}^n (-1)^k \text{Sin}[a k]$$

$$\frac{1}{2} \operatorname{Sec}\left[\frac{a}{2}\right] \left(-\operatorname{Sin}\left[\frac{a}{2}\right] + \operatorname{Sin}\left[\frac{a}{2} + a n + n \pi\right]\right)$$

### Infinite summation

*Mathematica* can calculate infinite sums that contain trigonometric functions. Here are some examples.

$$\sum_{k=1}^{\infty} z^k \operatorname{Sin}[k x]$$

$$\frac{i (-1 + e^{2 i x}) z}{2 (e^{i x} - z) (-1 + e^{i x} z)}$$

$$\sum_{k=1}^{\infty} \frac{\operatorname{Sin}[k x]}{k!}$$

$$\frac{1}{2} i (e^{-i x} - e^{i x})$$

$$\sum_{k=1}^{\infty} \frac{\operatorname{Cos}[k x]}{k}$$

$$\frac{1}{2} (-\operatorname{Log}[1 - e^{-i x}] - \operatorname{Log}[1 - e^{i x}])$$

### Finite products

*Mathematica* can calculate some finite symbolic products that contain the trigonometric functions. Here are two examples.

$$\operatorname{Product}\left[\operatorname{Sin}\left[\frac{\pi k}{n}\right], \{k, 1, n - 1\}\right]$$

$$2^{1-n} n$$

$$\prod_{k=1}^{n-1} \operatorname{Cos}\left[z + \frac{\pi k}{n}\right]$$

$$-(-1)^n 2^{1-n} \operatorname{Sec}[z] \operatorname{Sin}\left[\frac{1}{2} n (\pi - 2 z)\right]$$

### Infinite products

*Mathematica* can calculate infinite products that contain trigonometric functions. Here are some examples.

$$\text{In}[2]:= \prod_{k=1}^{\infty} \operatorname{Exp}\left[z^k \operatorname{Sin}[k x]\right]$$

$$\text{Out}[2]= e^{\frac{i (-1 + e^{2 i x}) z}{2 (z + e^{2 i x} z - e^{i x} (1 + z^2))}}$$



$$\text{In[3]:= } \prod_{k=1}^{\infty} \text{Exp}\left[\frac{\text{Cos}[k x]}{k!}\right]$$

$$\text{Out[3]= } e^{\frac{1}{2}(-2 + e^{-1} + e^1)}$$

### Indefinite integration

*Mathematica* can calculate a huge number of doable indefinite integrals that contain trigonometric functions. Here are some examples.

$$\int \text{Sin}[7 z] \, dz$$

$$-\frac{1}{7} \text{Cos}[7 z]$$

$$\int \left\{ \left\{ \text{Sin}[z], \text{Sin}[z]^a \right\}, \left\{ \text{Cos}[z], \text{Cos}[z]^a \right\}, \left\{ \text{Tan}[z], \text{Tan}[z]^a \right\}, \right. \\ \left. \left\{ \text{Cot}[z], \text{Cot}[z]^a \right\}, \left\{ \text{Csc}[z], \text{Csc}[z]^a \right\}, \left\{ \text{Sec}[z], \text{Sec}[z]^a \right\} \right\} dz$$

$$\left\{ \left\{ -\text{Cos}[z], -\text{Cos}[z] \text{Hypergeometric2F1}\left[\frac{1}{2}, \frac{1-a}{2}, \frac{3}{2}, \text{Cos}[z]^2\right] \text{Sin}[z]^{1+a} (\text{Sin}[z]^2)^{\frac{1}{2}(-1-a)} \right\}, \right.$$

$$\left. \left\{ \text{Sin}[z], -\frac{\text{Cos}[z]^{1+a} \text{Hypergeometric2F1}\left[\frac{1+a}{2}, \frac{1}{2}, \frac{3+a}{2}, \text{Cos}[z]^2\right] \text{Sin}[z]}{(1+a) \sqrt{\text{Sin}[z]^2}} \right\}, \right.$$

$$\left. \left\{ -\text{Log}[\text{Cos}[z]], \frac{\text{Hypergeometric2F1}\left[\frac{1+a}{2}, 1, 1 + \frac{1+a}{2}, -\text{Tan}[z]^2\right] \text{Tan}[z]^{1+a}}{1+a} \right\}, \right.$$

$$\left. \left\{ \text{Log}[\text{Sin}[z]], -\frac{\text{Cot}[z]^{1+a} \text{Hypergeometric2F1}\left[\frac{1+a}{2}, 1, 1 + \frac{1+a}{2}, -\text{Cot}[z]^2\right]}{1+a} \right\}, \right.$$

$$\left. \left\{ -\text{Log}\left[\text{Cos}\left[\frac{z}{2}\right]\right] + \text{Log}\left[\text{Sin}\left[\frac{z}{2}\right]\right], \right.$$

$$\left. -\text{Cos}[z] \text{Csc}[z]^{-1+a} \text{Hypergeometric2F1}\left[\frac{1}{2}, \frac{1+a}{2}, \frac{3}{2}, \text{Cos}[z]^2\right] (\text{Sin}[z]^2)^{\frac{1}{2}(-1+a)} \right\},$$

$$\left. \left\{ -\text{Log}\left[\text{Cos}\left[\frac{z}{2}\right] - \text{Sin}\left[\frac{z}{2}\right]\right] + \text{Log}\left[\text{Cos}\left[\frac{z}{2}\right] + \text{Sin}\left[\frac{z}{2}\right]\right], \right.$$

$$\left. \left. -\frac{\text{Hypergeometric2F1}\left[\frac{1-a}{2}, \frac{1}{2}, \frac{3-a}{2}, \text{Cos}[z]^2\right] \text{Sec}[z]^{-1+a} \text{Sin}[z]}{(1-a) \sqrt{\text{Sin}[z]^2}} \right\} \right\}$$

### Definite integration

*Mathematica* can calculate wide classes of definite integrals that contain trigonometric functions. Here are some examples.

$$\int_0^{\pi/2} \sqrt[3]{\text{Sin}[z]} \, dz$$

$$\frac{\sqrt{\pi} \text{Gamma}\left[\frac{2}{3}\right]}{2 \text{Gamma}\left[\frac{7}{6}\right]}$$

$$\int_0^{\pi/2} \left\{ \sqrt{\text{Sin}[z]}, \sqrt{\text{Cos}[z]}, \sqrt{\text{Tan}[z]}, \sqrt{\text{Cot}[z]}, \sqrt{\text{Csc}[z]}, \sqrt{\text{Sec}[z]} \right\} dz$$

$$\left\{ 2 \text{EllipticE}\left[\frac{\pi}{4}, 2\right], 2 \text{EllipticE}\left[\frac{\pi}{4}, 2\right], \frac{\pi}{\sqrt{2}}, \frac{\pi}{\sqrt{2}}, \frac{2 \sqrt{\pi} \text{Gamma}\left[\frac{5}{4}\right]}{\text{Gamma}\left[\frac{3}{4}\right]}, \frac{2 \sqrt{\pi} \text{Gamma}\left[\frac{5}{4}\right]}{\text{Gamma}\left[\frac{3}{4}\right]} \right\}$$

$$\int_0^{\frac{\pi}{2}} \left\{ \{\text{Sin}[z], \text{Sin}[z]^a\}, \{\text{Cos}[z], \text{Cos}[z]^a\}, \{\text{Tan}[z], \text{Tan}[z]^a\}, \right.$$

$$\left. \{\text{Cot}[z], \text{Cot}[z]^a\}, \{\text{Csc}[z], \text{Csc}[z]^a\}, \{\text{Sec}[z], \text{Sec}[z]^a\} \right\} dz$$

$$\left\{ \left\{ 1, \frac{\sqrt{\pi} \text{Gamma}\left[\frac{1+a}{2}\right]}{a \text{Gamma}\left[\frac{a}{2}\right]} \right\}, \left\{ 1, \frac{\sqrt{\pi} \text{Gamma}\left[\frac{1+a}{2}\right]}{a \text{Gamma}\left[\frac{a}{2}\right]} \right\}, \right.$$

$$\left. \left\{ \int_0^{\frac{\pi}{2}} \text{Tan}[z] dz, \text{If}\left[\text{Re}[a] < 1, \frac{1}{2} \pi \text{Sec}\left[\frac{a \pi}{2}\right], \int_0^{\frac{\pi}{2}} \text{Tan}[z]^a dz\right] \right\}, \right.$$

$$\left. \left\{ \int_0^{\frac{\pi}{2}} \text{Cot}[z] dz, \text{If}\left[\text{Re}[a] < 1, \frac{1}{2} \pi \text{Sec}\left[\frac{a \pi}{2}\right], \int_0^{\frac{\pi}{2}} \text{Cot}[z]^a dz\right] \right\}, \right.$$

$$\left. \left\{ \int_0^{\frac{\pi}{2}} \text{Csc}[z] dz, \frac{\sqrt{\pi} \text{Gamma}\left[\frac{1}{2} - \frac{a}{2}\right]}{2 \text{Gamma}\left[1 - \frac{a}{2}\right]} \right\}, \left\{ \int_0^{\frac{\pi}{2}} \text{Sec}[z] dz, \frac{\sqrt{\pi} \text{Gamma}\left[\frac{1}{2} - \frac{a}{2}\right]}{2 \text{Gamma}\left[1 - \frac{a}{2}\right]} \right\} \right\}$$

**Limit operation**

Mathematica can calculate limits that contain trigonometric functions.

$$\text{Limit}\left[\frac{\text{Sin}[z]}{z} + \text{Cos}[z]^3, z \rightarrow 0\right]$$

2

$$\text{Limit}\left[\left(\frac{\text{Tan}[x]}{x}\right)^{\frac{1}{x^2}}, x \rightarrow 0\right]$$

e<sup>1/3</sup>

**Solving equations**

The next input solves equations that contain trigonometric functions. The message indicates that the multivalued functions are used to express the result and that some solutions might be absent.

$$\text{Solve}\left[\text{Tan}[z]^2 + 3 \text{Sin}\left[z + \text{Pi}/6\right] == 4, z\right]$$

Solve::ifun: Inverse functions are being used by Solve, so some solutions may not be found.

$$\left\{ \left\{ z \rightarrow \text{ArcCos}\left[\text{Root}\left[4 - 40 \#1^2 + 12 \#1^3 + 73 \#1^4 - 60 \#1^5 + 36 \#1^6 \&, 1\right]\right]\right\}, \right. \\ \left. \left\{ z \rightarrow -\text{ArcCos}\left[\text{Root}\left[4 - 40 \#1^2 + 12 \#1^3 + 73 \#1^4 - 60 \#1^5 + 36 \#1^6 \&, 2\right]\right]\right\}, \right. \\ \left. \left\{ z \rightarrow -\text{ArcCos}\left[\text{Root}\left[4 - 40 \#1^2 + 12 \#1^3 + 73 \#1^4 - 60 \#1^5 + 36 \#1^6 \&, 3\right]\right]\right\}, \right. \\ \left. \left\{ z \rightarrow \text{ArcCos}\left[\text{Root}\left[4 - 40 \#1^2 + 12 \#1^3 + 73 \#1^4 - 60 \#1^5 + 36 \#1^6 \&, 4\right]\right]\right\}, \right. \\ \left. \left\{ z \rightarrow \text{ArcCos}\left[\text{Root}\left[4 - 40 \#1^2 + 12 \#1^3 + 73 \#1^4 - 60 \#1^5 + 36 \#1^6 \&, 5\right]\right]\right\}, \right. \\ \left. \left\{ z \rightarrow \text{ArcCos}\left[\text{Root}\left[4 - 40 \#1^2 + 12 \#1^3 + 73 \#1^4 - 60 \#1^5 + 36 \#1^6 \&, 6\right]\right]\right\} \right\}$$

Complete solutions can be obtained by using the function `Reduce`.

**Reduce[Sin[x] == a, x] // TraditionalForm**

```
// InputForm =
C[1] ∈ Integers && (x == Pi - ArcSin[a] + 2 * Pi * C[1] || x == ArcSin[a] + 2 * Pi * C[1])
```

**Reduce[Cos[x] == a, x] // TraditionalForm**

```
// InputForm = C[1] ∈ Integers && (x == -ArcCos[a] + 2 * Pi * C[1] || x == ArcCos[a] + 2 * Pi * C[1])
```

**Reduce[Tan[x] == a, x] // TraditionalForm**

```
// InputForm = C[1] ∈ Integers && 1 + a^2 ≠ 0 && x == ArcTan[a] + Pi * C[1]
```

**Reduce[Cot[x] == a, x] // TraditionalForm**

```
// InputForm = C[1] ∈ Integers && 1 + a^2 ≠ 0 && x == ArcCot[a] + Pi * C[1]
```

**Reduce[Csc[x] == a, x] // TraditionalForm**

$$c_1 \in \mathbb{Z} \wedge a \neq 0 \wedge \left( x = -\sin^{-1}\left(\frac{1}{a}\right) + 2\pi c_1 + \pi \vee x = \sin^{-1}\left(\frac{1}{a}\right) + 2\pi c_1 \right)$$

**Reduce[Sec[x] == a, x] // TraditionalForm**

```
// InputForm = C[1] ∈ Integers && a ≠ 0 &&
(x == -ArcCos[a^(-1)] + 2 * Pi * C[1] || x == ArcCos[a^(-1)] + 2 * Pi * C[1])
```

### Solving differential equations

Here are differential equations whose linear-independent solutions are trigonometric functions. The solutions of the simplest second-order linear ordinary differential equation with constant coefficients can be represented through  $\sin(z)$  and  $\cos(z)$ .

**DSolve[w''[z] + w[z] == 0, w[z], z]**

```
{{w[z] → C[1] Cos[z] + C[2] Sin[z]}}
```

**dsol1 = DSolve[2 w[z] + 3 w''[z] + w^(4)[z] == 0, w[z], z]**

```
{{{w[z] → C[3] Cos[z] + C[1] Cos[√2 z] + C[4] Sin[z] + C[2] Sin[√2 z]}}}
```

In the last input, the differential equation was solved for  $w(z)$ . If the argument is suppressed, the result is returned as a pure function (in the sense of the  $\lambda$ -calculus).

```
dsol2 = DSolve[2 w[z] + 3 w''[z] + w^(4)[z] == 0, w, z]
{{w -> Function[{z}, C[3] Cos[z] + C[1] Cos[Sqrt[2] z] + C[4] Sin[z] + C[2] Sin[Sqrt[2] z]]}}
```

The advantage of such a pure function is that it can be used for different arguments, derivatives, and more.

```
w'[ξ] /. dsol1
```

```
{w'[ξ]}
```

```
w'[ξ] /. dsol2
```

```
{C[4] Cos[ξ] + Sqrt[2] C[2] Cos[Sqrt[2] ξ] - C[3] Sin[ξ] - Sqrt[2] C[1] Sin[Sqrt[2] ξ]}
```

All trigonometric functions satisfy first-order nonlinear differential equations. In carrying out the algorithm to solve the nonlinear differential equation, *Mathematica* has to solve a transcendental equation. In doing so, the generically multivariate inverse of a function is encountered, and a message is issued that a solution branch is potentially missed.

```
DSolve[{w'[z] == Sqrt[1 - w[z]^2], w[0] == 0}, w[z], z]
```

Solve::ifun: Inverse functions are being used by Solve, so some solutions may not be found.

```
{{w[z] -> Sin[z]}}
```

```
DSolve[{w'[z] == Sqrt[1 - w[z]^2], w[0] == 1}, w[z], z]
```

Solve::ifun: Inverse functions are being used by Solve, so some solutions may not be found.

```
{{w[z] -> Cos[z]}}
```

```
DSolve[{w'[z] - w[z]^2 - 1 == 0, w[0] == 0}, w[z], z]
```

Solve::ifun: Inverse functions are being used by Solve, so some solutions may not be found.

```
{{w[z] -> Tan[z]}}
```

```
DSolve[{w'[z] + w[z]^2 + 1 == 0, w[Pi/2] == 0}, w[z], z]
```

Solve::ifun: Inverse functions are being used by Solve, so some solutions may not be found.

```
{{w[z] -> Cot[z]}}
```

```
DSolve[{w'[z] == Sqrt[w[z]^4 - w[z]^2], 1/w[0] == 0}, w[z], z] // Simplify[#, 0 < z < Pi/2] &
```

Solve::verif: Potential solution {C[1] -> Indeterminate} (possibly discarded by verifier) should be checked by hand. May require use of limits.

Solve::ifun: Inverse functions are being used by Solve, so some solutions may not be found.

Solve::verif: Potential solution {C[1] -> Indeterminate} (possibly discarded by verifier) should be checked by hand. May require use of limits.

Solve::ifun: Inverse functions are being used by Solve, so some solutions may not be found.

```
{{w[z] → -Csc[z]}, {w[z] → Csc[z]}}
```

```
DSolve[{w'[z] == Sqrt[w[z]^4 - w[z]^2], 1/w[z] == 0}, w[z], z] // Simplify[#, 0 < z < Pi/2] &
```

```
Solve::verif: Potential solution {C[1] → Indeterminate} (possibly discarded by verifier) should be checked by hand. May require use of limits.
```

```
Solve::ifun: Inverse functions are being used by Solve, so some solutions may not be found.
```

```
Solve::verif: Potential solution {C[1] → Indeterminate} (possibly discarded by verifier) should be checked by hand. May require use of limits.
```

```
Solve::ifun: Inverse functions are being used by Solve, so some solutions may not be found.
```

```
{{w[z] → -Sec[z]}, {w[z] → Sec[z]}}
```

### Integral transforms

*Mathematica* supports the main integral transforms like direct and inverse Fourier, Laplace, and Z transforms that can give results that contain classical or generalized functions. Here are some transforms of trigonometric functions.

```
LaplaceTransform[Sin[t], t, s]
```

$$\frac{1}{1 + s^2}$$

```
FourierTransform[Sin[t], t, s]
```

$$i \sqrt{\frac{\pi}{2}} \text{DiracDelta}[-1 + s] - i \sqrt{\frac{\pi}{2}} \text{DiracDelta}[1 + s]$$

```
FourierSinTransform[Sin[t], t, s]
```

$$\sqrt{\frac{\pi}{2}} \text{DiracDelta}[-1 + s] - \sqrt{\frac{\pi}{2}} \text{DiracDelta}[1 + s]$$

```
FourierCosTransform[Sin[t], t, s]
```

$$-\frac{1}{\sqrt{2\pi}(-1 + s)} + \frac{1}{\sqrt{2\pi}(1 + s)}$$

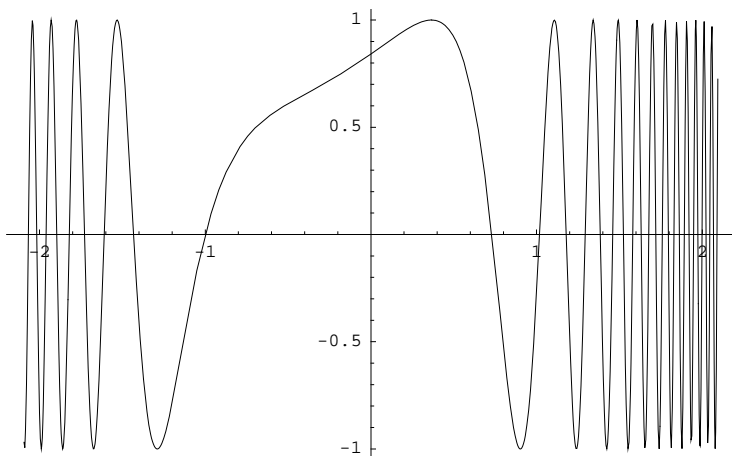
```
ZTransform[Sin[π t], t, s]
```

```
0
```

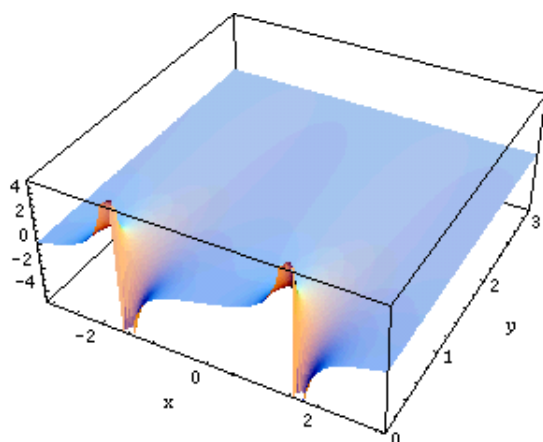
### Plotting

*Mathematica* has built-in functions for 2D and 3D graphics. Here are some examples.

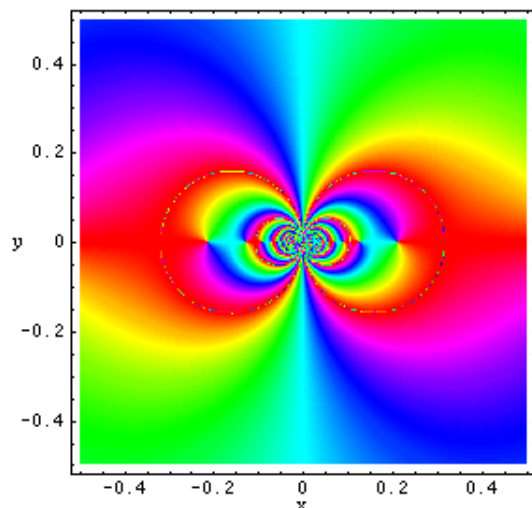
```
Plot[Sin[Sum[z^k, {k, 0, 5}], {z, -2π/3, 2π/3}];
```



```
Plot3D[Re[Tan[x + i y]], {x, -π, π}, {y, 0, π},
  PlotPoints → 240, PlotRange → {-5, 5},
  ClipFill → None, Mesh → False, AxesLabel → {"x", "y", None}];
```



```
ContourPlot[Arg[Sec[1/(x + i y)]], {x, -1/2, 1/2}, {y, -1/2, 1/2},
  PlotPoints → 400, PlotRange → {-π, π}, FrameLabel → {"x", "y", None, None},
  ColorFunction → Hue, ContourLines → False, Contours → 200];
```



## Introduction to the Sine Function in *Mathematica*

### Overview

The following shows how the sine function is realized in *Mathematica*. Examples of evaluating *Mathematica* functions applied to various numeric and exact expressions that involve the sine function or return it are shown. These involve numeric and symbolic calculations and plots.

### Notations

#### *Mathematica* forms of notations

Following *Mathematica*'s general naming convention, function names in `StandardForm` are just the capitalized versions of their traditional mathematics names. This shows the sine function in `StandardForm`.

```
Sin[z]
```

```
Sin[z]
```

This shows the sine function in `TraditionalForm`.

```
% // TraditionalForm
```

```
sin(z)
```

#### Additional forms of notations

*Mathematica* also knows the most popular forms of notations for the sine function that are used in other programming languages. Here are three examples: `CForm`, `TeXForm`, and `FortranForm`.

```
{CForm[Sin[2 Pi z]], FortranForm[Sin[2 Pi z]], TeXForm[Sin[2 Pi z]]}
```

```
{Sin(2 * Pi * z), Sin(2 * Pi * z), \sin(2 \, \pi \, z)}
```

### Automatic evaluations and transformations

### Evaluation for exact, machine-number, and high-precision arguments

For the exact argument  $z = \pi/4$ , *Mathematica* returns an exact result.

$$\text{Sin}\left[\frac{\pi}{4}\right]$$

$$\frac{1}{\sqrt{2}}$$

$$\text{Sin}[z] /. z \rightarrow \frac{\pi}{4}$$

$$\frac{1}{\sqrt{2}}$$

For a machine-number argument (a numerical argument with a decimal point and not too many digits), a machine number is also returned.

**Sin[5.]**

-0.958924

**Sin[z] /. z -> 3.**

0.14112

The next inputs calculate 100-digit approximations at  $z = 1$  and  $z = 2$ .

**N[Sin[z] /. z -> 1, 100]**

0.8414709848078965066525023216302989996225630607983710656727517099919104043912396689  
486397435430526959

**N[Sin[2], 100]**

0.9092974268256816953960198659117448427022549714478902683789730115309673015407835446  
201266889249593803

**Sin[2] // N[#, 100] &**

0.9092974268256816953960198659117448427022549714478902683789730115309673015407835446  
201266889249593803

Within a second, it is possible to calculate thousands of digits for the sine function. The next input calculates 10000 digits for  $\sin(1)$  and analyzes the frequency of the digit  $k$  in the resulting decimal number.

**Map[Function[w, {First[#], Length[#]} & /@ Split[Sort[First[RealDigits[w]]]],  
N[{Sin[z]} /. z -> 1, 10 000]]**

{{{0, 983}, {1, 1069}, {2, 1019}, {3, 983},  
{4, 972}, {5, 994}, {6, 994}, {7, 988}, {8, 988}, {9, 1010}}}

Here is a 50-digit approximation of the sine function at the complex argument  $z = 3 + 2i$ .



```

N[Sin[3 + 2 i], 50]
0.53092108624851980526704009066067655967277345095149 -
3.5905645899857799520125654477948167931949136757293 i

{N[Sin[z] /. z -> 3 + 2 i, 50], Sin[3 + 2 i] // N[#, 50] &}
{0.53092108624851980526704009066067655967277345095149 -
3.5905645899857799520125654477948167931949136757293 i,
0.53092108624851980526704009066067655967277345095149 -
3.5905645899857799520125654477948167931949136757293 i}

```

*Mathematica* automatically evaluates mathematical functions with machine precision, if the arguments of the function are machine-number elements. In this case, only six digits after the decimal point are shown in the results. The remaining digits are suppressed, but can be displayed using the function `InputForm`.

```

{Sin[2.], N[Sin[2]], N[Sin[2], 16], N[Sin[2], 5], N[Sin[2], 20]}
{0.909297, 0.909297, 0.9092974268256817, 0.90930, 0.90929742682568169540}

% // InputForm
{0.9092974268256817, 0.9092974268256817,
0.9092974268256816953960198659`16., 0.9092974268256816954`5.,
0.90929742682568169539601986591174484267`20.}

```

### Simplification of the argument

*Mathematica* knows the symmetry and periodicity of the sine function. Here are some examples.

```

Sin[-3]
-Sin[3]

{Sin[-z], Sin[z + π], Sin[z + 2 π], Sin[z + 342 π], Sin[-z + 21 π]}
{-Sin[z], -Sin[z], Sin[z], Sin[z], Sin[z]}

```

*Mathematica* automatically simplifies the composition of the direct and the inverse sine functions into its argument.

```

Sin[ArcSin[z]]
z

```

*Mathematica* also automatically simplifies the composition of the direct and any of the inverse trigonometric functions into algebraic functions of the argument.

```

{Sin[ArcSin[z]], Sin[ArcCos[z]], Sin[ArcTan[z]],
Sin[ArcCot[z]], Sin[ArcCsc[z]], Sin[ArcSec[z]]}

```

$$\left\{ z, \sqrt{1-z^2}, \frac{z}{\sqrt{1+z^2}}, \frac{1}{\sqrt{1+\frac{1}{z^2}}}, \frac{1}{z}, \sqrt{1-\frac{1}{z^2}} \right\}$$

If the argument has the structure  $\pi k/2 + z$  or  $\pi k/2 - z$ , and  $\pi k/2 + iz$  or  $\pi k/2 - iz$  with integer  $k$ , the sine function can be automatically transformed into trigonometric or hyperbolic sine or cosine functions.

$$\begin{aligned} & \text{Sin}\left[\frac{\pi}{2} - 4\right] \\ & \text{Cos}[4] \\ & \left\{ \text{Sin}\left[\frac{\pi}{2} - z\right], \text{Sin}\left[\frac{\pi}{2} + z\right], \text{Sin}\left[-\frac{\pi}{2} - z\right], \text{Sin}\left[-\frac{\pi}{2} + z\right], \text{Sin}[\pi - z], \text{Sin}[\pi + z] \right\} \\ & \{ \text{Cos}[z], \text{Cos}[z], -\text{Cos}[z], -\text{Cos}[z], \text{Sin}[z], -\text{Sin}[z] \} \end{aligned}$$

In[50]:= Sin[5 i]

Out[50]= i Sinh[5]

$$\begin{aligned} & \left\{ \text{Sin}[i z], \text{Sin}\left[\frac{\pi}{2} - i z\right], \text{Sin}\left[\frac{\pi}{2} + i z\right], \text{Sin}[\pi - i z], \text{Sin}[\pi + i z] \right\} \\ & \{ i \text{Sinh}[z], \text{Cosh}[z], \text{Cosh}[z], i \text{Sinh}[z], -i \text{Sinh}[z] \} \end{aligned}$$

### Simplification of simple expressions containing the sine function

Sometimes simple arithmetic operations containing the sine function can automatically produce other trigonometric functions.

$$\begin{aligned} & 1/\text{Sin}[4] \\ & \text{Csc}[4] \\ & \left\{ 1/\text{Sin}[z], 1/\text{Sin}[\pi/2 - z], \text{Sin}[\pi/2 - z]/\text{Sin}[z], \right. \\ & \quad \left. \text{Sin}[z]/\text{Sin}[\pi/2 - z], 1/\text{Sin}[\pi/2 - z], \text{Sin}[\pi/2 - z]/\text{Sin}[z]^2 \right\} \\ & \{ \text{Csc}[z], \text{Sec}[z], \text{Cot}[z], \text{Tan}[z], \text{Sec}[z], \text{Cot}[z] \text{Csc}[z] \} \end{aligned}$$

### The sine function arising as special cases from more general functions

The sine function can be treated as a particular case of some more general special functions. For example,  $\sin(z)$  can appear automatically from Bessel, Mathieu, Jacobi, hypergeometric, and Meijer functions for appropriate values of their parameters.

$$\begin{aligned} & \left\{ \text{BesselJ}\left[\frac{1}{2}, z\right], \text{MathieuS}[1, 0, z], \text{JacobiSN}[z, 0], \right. \\ & \quad \left. \text{HypergeometricPFQ}\left[\{\}, \left\{\frac{3}{2}\right\}, -\frac{z^2}{4}\right], \text{MeijerG}\left[\{\{\}, \{\}\}, \left\{\left\{\frac{1}{2}\right\}, \{0\}\right\}, \frac{z^2}{4}\right] \right\} \\ & \left\{ \frac{\sqrt{\frac{2}{\pi}} \text{Sin}[z]}{\sqrt{z}}, \text{Sin}[z], \text{Sin}[z], \frac{\text{Sin}[\sqrt{z^2}]}{\sqrt{z^2}}, \frac{\sqrt{z^2} \text{Sin}[z]}{\sqrt{\pi} z} \right\} \end{aligned}$$

### Equivalence transformations carried out by specialized *Mathematica* functions

### General remarks

Almost everybody prefers using  $\sin(z)/2$  instead of  $\cos(\pi/2 - z) \sin(\pi/6)$ . *Mathematica* automatically transforms the second expression into the first one. The automatic application of transformation rules to mathematical expressions can give overly complicated results. Compact expressions like  $\sin(2z) \sin(\pi/6)$  should not be automatically expanded into the more complicated expression  $\sin(z) \cos(z) \left(2 - (2 + 2^{1/2})^{1/2}\right)^{1/2}$ . *Mathematica* has special functions that produce such expansions. Some are demonstrated in the next section.

### TrigExpand

The function `TrigExpand` expands out trigonometric and hyperbolic functions. In more detail, it splits up sums and integer multiples that appear in the arguments of trigonometric and hyperbolic functions, and then expands out the products of trigonometric and hyperbolic functions into sums of powers, using trigonometric and hyperbolic identities where possible. Here are some examples.

`TrigExpand[Sin[x - y]]`

$$\cos[y] \sin[x] - \cos[x] \sin[y]$$

`Sin[4 z] // TrigExpand`

$$4 \cos[z]^3 \sin[z] - 4 \cos[z] \sin[z]^3$$

`Sin[2 z]^2 // TrigExpand`

$$\frac{1}{2} - \frac{\cos[z]^4}{2} + 3 \cos[z]^2 \sin[z]^2 - \frac{\sin[z]^4}{2}$$

`TrigExpand[{Sin[x + y + z], Sin[3 z]}]`

$$\left\{ \cos[y] \cos[z] \sin[x] + \cos[x] \cos[z] \sin[y] + \cos[x] \cos[y] \sin[z] - \sin[x] \sin[y] \sin[z], \right. \\ \left. 3 \cos[z]^2 \sin[z] - \sin[z]^3 \right\}$$

### TrigFactor

The function `TrigFactor` factors trigonometric and hyperbolic functions. In more detail, it splits up sums and integer multiples that appear in the arguments of trigonometric and hyperbolic functions, and then factors the resulting polynomials into trigonometric and hyperbolic functions, using trigonometric and hyperbolic identities where possible. Here are some examples.

`TrigFactor[Sin[x] + Sin[y]]`

$$2 \cos\left[\frac{x}{2} - \frac{y}{2}\right] \sin\left[\frac{x}{2} + \frac{y}{2}\right]$$

`Sin[x] - Cos[y] // TrigFactor`

$$-\left(\cos\left[\frac{x}{2} - \frac{y}{2}\right] - \sin\left[\frac{x}{2} - \frac{y}{2}\right]\right) \left(\cos\left[\frac{x}{2} + \frac{y}{2}\right] - \sin\left[\frac{x}{2} + \frac{y}{2}\right]\right)$$

### TrigReduce

The function `TrigReduce` rewrites the products and powers of trigonometric and hyperbolic functions in terms of trigonometric and hyperbolic functions with combined arguments. In more detail, it typically yields a linear expression involving trigonometric and hyperbolic functions with more complicated arguments. `TrigReduce` is approximately inverse to `TrigExpand` and `TrigFactor`. Here are some examples.

**TrigReduce[Sin[x] Sin[y]]**

$$\frac{1}{2} (\cos[x - y] - \cos[x + y])$$

**Sin[x] Cos[y] // TrigReduce**

$$\frac{1}{2} (\sin[x - y] + \sin[x + y])$$

**Table[TrigReduce[Sin[z]^n], {n, 2, 5}]**

$$\left\{ \frac{1}{2} (1 - \cos[2z]), \frac{1}{4} (3 \sin[z] - \sin[3z]), \right. \\ \left. \frac{1}{8} (3 - 4 \cos[2z] + \cos[4z]), \frac{1}{16} (10 \sin[z] - 5 \sin[3z] + \sin[5z]) \right\}$$

**TrigReduce[TrigExpand[{Sin[x + y + z], Sin[3z], Sin[x] Sin[y]}]]**

$$\left\{ \sin[x + y + z], \sin[3z], \frac{1}{2} (\cos[x - y] - \cos[x + y]) \right\}$$

**TrigFactor[Sin[x] + Sin[y]] // TrigReduce**

$$\sin[x] + \sin[y]$$

### TrigToExp

The function `TrigToExp` converts trigonometric and hyperbolic functions to exponentials. It tries, where possible, to give results that do not involve explicit complex numbers. Here are some examples.

**TrigToExp[Sin[z]]**

$$\frac{1}{2} i e^{-iz} - \frac{1}{2} i e^{iz}$$

**Sin[a z] + Sin[b z] // TrigToExp**

$$\frac{1}{2} i e^{-iaz} - \frac{1}{2} i e^{iaz} + \frac{1}{2} i e^{-ibz} - \frac{1}{2} i e^{ibz}$$

### ExpToTrig

The function `ExpToTrig` converts exponentials to trigonometric and hyperbolic functions. It is approximately inverse to `TrigToExp`. Here are some examples.

**ExpToTrig[TrigToExp[Sin[z]]]**

$$\sin[z]$$

```
{ $\alpha e^{-i x \beta} + \alpha e^{i x \beta}$ ,  $\alpha e^{-i x \beta} + \gamma e^{i x \beta}$ } // ExpToTrig
{2  $\alpha \cos[x \beta]$ ,  $\alpha \cos[x \beta] + \gamma \cos[x \beta] - i \alpha \sin[x \beta] + i \gamma \sin[x \beta]$ }
```

### ComplexExpand

The function `ComplexExpand` expands expressions assuming that all the variables are real. The value option `TargetFunctions` is a list of functions from the set {`Re`, `Im`, `Abs`, `Arg`, `Conjugate`, `Sign`}. `ComplexExpand` tries to give results in terms of the functions specified. Here are some examples.

```
ComplexExpand[Sin[x + i y]]
```

```
Cosh[y] Sin[x] + i Cos[x] Sinh[y]
```

```
Sin[x + i y] + Sin[x - i y] // ComplexExpand
```

```
2 Cosh[y] Sin[x]
```

```
ComplexExpand[Re[Sin[x + i y]], TargetFunctions -> {Re, Im}]
```

```
Cosh[y] Sin[x]
```

```
ComplexExpand[Im[Sin[x + i y]], TargetFunctions -> {Re, Im}]
```

```
Cos[x] Sinh[y]
```

```
ComplexExpand[Abs[Sin[x + i y]], TargetFunctions -> {Re, Im}]
```

```
 $\sqrt{\cosh^2[y] \sin^2[x] + \cos^2[x] \sinh^2[y]}$ 
```

```
ComplexExpand[Abs[Sin[x + i y]], TargetFunctions -> {Re, Im}] //
Simplify[#, {x, y} ∈ Reals] &
```

```
 $\frac{\sqrt{-\cos[2 x] + \cosh[2 y]}}{\sqrt{2}}$ 
```

```
ComplexExpand[Re[Sin[x + i y]] + Im[Sin[x + i y]], TargetFunctions -> {Re, Im}]
```

```
Cosh[y] Sin[x] + Cos[x] Sinh[y]
```

```
ComplexExpand[Arg[Sin[x + i y]], TargetFunctions -> {Re, Im}]
```

```
ArcTan[Cosh[y] Sin[x], Cos[x] Sinh[y]]
```

```
ComplexExpand[Arg[Sin[x + i y]], TargetFunctions -> {Re, Im}] //
Simplify[#, {x, y} ∈ Reals] &
```

```
ArcTan[Cosh[y] Sin[x], Cos[x] Sinh[y]]
```

```
ComplexExpand[Conjugate[Sin[x + i y]], TargetFunctions -> {Re, Im}] // Simplify
```

```
Cosh[y] Sin[x] - i Cos[x] Sinh[y]
```

### Simplify

The function `Simplify` performs a sequence of algebraic transformations on its argument, and returns the simplest form it finds. Here are some examples.

```
Simplify[Sin[2 ArcSin[z]] / Sqrt[1 - z^2]]
```

2 z

```
{Simplify[Sin[2 z] / Sin[z]], Sin[2 z] / Cos[z] // Simplify}
```

```
{2 Cos[z], 2 Sin[z]}
```

Here is a large collection of trigonometric identities. All are written as one large logical conjunction.

```
Simplify[#] & /@ {Cos[z]^2 + Sin[z]^2 == 1 &
Sin[z]^2 == (1 - Cos[2 z]) / 2 &
Sin[2 z] == 2 Sin[z] Cos[z] &
Sin[a + b] == Sin[a] Cos[b] + Cos[a] Sin[b] &
Sin[a - b] == Sin[a] Cos[b] - Cos[a] Sin[b] &
Sin[a] + Sin[b] == 2 Sin[(a + b) / 2] Cos[(a - b) / 2] &
Sin[a] - Sin[b] == 2 Cos[(a + b) / 2] Sin[(a - b) / 2] &
A Sin[z] + B Cos[z] == A Sqrt[1 + B^2 / A^2] Sin[z + ArcTan[B / A]] &
Sin[a] Sin[b] == (Cos[a - b] - Cos[a + b]) / 2 &
Sin[a] Cos[b] == (Sin[a + b] + Sin[a - b]) / 2 &
Sin[z / 2]^2 == (1 - Cos[z]) / 2}
```

True

The function `Simplify` has the `Assumption` option. For example, *Mathematica* knows that  $-1 \leq \sin(x) \leq 1$  for all real  $x$ , and knows about the periodicity of trigonometric functions for the symbolic integer coefficient  $k$  of  $k\pi$ .

```
Simplify[Abs[Sin[x]] <= 1, x ∈ Reals]
```

True

```
Abs[Sin[x]] <= 1 // Simplify[#, x ∈ Reals] &
```

True

```
Simplify[{Sin[z + 2 k π], Sin[z + k π] / Sin[z]}, k ∈ Integers]
```

```
{Sin[z], (-1)^k}
```

*Mathematica* also knows that the composition of the inverse and direct sine produces the value of the internal argument under the corresponding restriction.

**ArcSin[Sin[z]]**

ArcSin[Sin[z]]

**Simplify[ArcSin[Sin[z]], -π/2 < Re[z] < π/2]**

z

**FunctionExpand (and Together)**

While the sine function auto-evaluates for simple fractions of π, for more complicated cases it stays as a sine function to avoid the build up of large expressions. Using the function `FunctionExpand`, the sine function can sometimes be transformed into explicit radicals. Here are some examples.

In[1]:= {Sin[π/16], Sin[π/17], Sin[π/60]}

Out[1]:= {Sin[π/16], Sin[π/17], Sin[π/60]}

**FunctionExpand[%]**

$$\left\{ \frac{1}{2} \sqrt{2 - \sqrt{2 + \sqrt{2}}}, -\frac{\frac{1}{8} \sqrt{3} (-1 + \sqrt{5}) - \frac{1}{4} \sqrt{\frac{1}{2} (5 + \sqrt{5})}}{\sqrt{2}} + \frac{\frac{1}{8} (-1 + \sqrt{5}) - \frac{1}{4} \sqrt{\frac{3}{2} (5 + \sqrt{5})}}{\sqrt{2}} \right\}$$

**Together[%]**

$$\left\{ \frac{1}{2} \sqrt{2 - \sqrt{2 + \sqrt{2}}}, \frac{1}{16} \left( -\sqrt{2} - \sqrt{6} + \sqrt{10} + \sqrt{30} + 2\sqrt{5 + \sqrt{5}} - 2\sqrt{3(5 + \sqrt{5})} \right) \right\}$$

If the denominator contains squares of integers other than 2, the results always contain complex numbers (meaning that the imaginary number  $i = \sqrt{-1}$  appears unavoidably).

{Sin[π/9]}

{Sin[π/9]}

**FunctionExpand[%] // Together**

$$\left\{ \frac{1}{8} \left( -i 2^{2/3} (-1 - i \sqrt{3})^{1/3} + 2^{2/3} \sqrt{3} (-1 - i \sqrt{3})^{1/3} + i 2^{2/3} (-1 + i \sqrt{3})^{1/3} + 2^{2/3} \sqrt{3} (-1 + i \sqrt{3})^{1/3} \right) \right\}$$

Here the function `RootReduce` is used to express the previous algebraic numbers as roots of polynomial equations.

**RootReduce[Simplify[%]]**

$$\{\text{Root}[-3 + 36 \#1^2 - 96 \#1^4 + 64 \#1^6 \&, 4]\}$$

The function `FunctionExpand` also reduces trigonometric expressions with compound arguments or compositions, including inverse trigonometric functions, to simpler ones. Here are some examples.

$$\{\text{Sin}[\sqrt{z^2}], \text{Sin}\left[\frac{\text{ArcSin}[z]}{2}\right], \text{Sin}[3 \text{ArcSin}[z]]\} // \text{FunctionExpand}$$

$$\left\{\frac{\sqrt{-i z} \sqrt{i z} \text{Sin}[z]}{z}, \frac{z \sqrt{1 - \sqrt{1 - z} \sqrt{1 + z}}}{\sqrt{2} \sqrt{-i z} \sqrt{i z}}, -z^3 + 3 z (1 - z^2)\right\}$$

Applying `Simplify` to the last expression gives a more compact result.

`Simplify[%]`

$$\left\{\frac{\sqrt{z^2} \text{Sin}[z]}{z}, \frac{z \sqrt{1 - \sqrt{1 - z^2}}}{\sqrt{2} \sqrt{z^2}}, 3 z - 4 z^3\right\}$$

### FullSimplify

The function `FullSimplify` tries a wider range of transformations than `Simplify` and returns the simplest form it finds. Here are some examples that contrast the results of applying the functions `Simplify` and `FullSimplify` to the same expressions.

$$\begin{aligned} \text{set1} = & \left\{ \text{Sin}\left[-i \text{Log}\left[i z + \sqrt{1 - z^2}\right]\right], \text{Sin}\left[\frac{\pi}{2} + i \text{Log}\left[i z + \sqrt{1 - z^2}\right]\right], \right. \\ & \text{Sin}\left[\frac{1}{2} i \text{Log}[1 - i z] - \frac{1}{2} i \text{Log}[1 + i z]\right], \text{Sin}\left[\frac{1}{2} i \text{Log}\left[1 - \frac{i}{z}\right] - \frac{1}{2} i \text{Log}\left[1 + \frac{i}{z}\right]\right], \\ & \left. \text{Sin}\left[-i \text{Log}\left[\sqrt{1 - \frac{1}{z^2}} + \frac{i}{z}\right]\right], \text{Sin}\left[\frac{\pi}{2} + i \text{Log}\left[\sqrt{1 - \frac{1}{z^2}} + \frac{i}{z}\right]\right] \right\} \\ & \left\{ -\frac{i \left(-1 + \left(i z + \sqrt{1 - z^2}\right)^2\right)}{2 \left(i z + \sqrt{1 - z^2}\right)}, \frac{1 + \left(i z + \sqrt{1 - z^2}\right)^2}{2 \left(i z + \sqrt{1 - z^2}\right)}, i \text{Sinh}\left[\frac{1}{2} \text{Log}[1 - i z] - \frac{1}{2} \text{Log}[1 + i z]\right], \right. \\ & \left. i \text{Sinh}\left[\frac{1}{2} \text{Log}\left[1 - \frac{i}{z}\right] - \frac{1}{2} \text{Log}\left[1 + \frac{i}{z}\right]\right], -\frac{i \left(-1 + \left(\sqrt{1 - \frac{1}{z^2}} + \frac{i}{z}\right)^2\right)}{2 \left(\sqrt{1 - \frac{1}{z^2}} + \frac{i}{z}\right)}, \frac{1 + \left(\sqrt{1 - \frac{1}{z^2}} + \frac{i}{z}\right)^2}{2 \left(\sqrt{1 - \frac{1}{z^2}} + \frac{i}{z}\right)} \right\} \end{aligned}$$

`set1 // Simplify`



$$\left\{ z, \frac{1 - z^2 + i z \sqrt{1 - z^2}}{i z + \sqrt{1 - z^2}}, i \operatorname{Sinh}\left[\frac{1}{2} (\operatorname{Log}[1 - i z] - \operatorname{Log}[1 + i z])\right], \right. \\ \left. i \operatorname{Sinh}\left[\frac{1}{2} \left(\operatorname{Log}\left[\frac{-i + z}{z}\right] - \operatorname{Log}\left[\frac{i + z}{z}\right]\right)\right], \frac{1}{z}, \frac{-1 + i \sqrt{1 - \frac{1}{z^2}} z + z^2}{z \left(i + \sqrt{1 - \frac{1}{z^2}} z\right)} \right\}$$

set1 // FullSimplify

$$\left\{ z, \sqrt{1 - z^2}, \frac{z}{\sqrt{1 + z^2}}, \frac{1}{\sqrt{1 + \frac{1}{z^2} z}}, \frac{1}{z}, \sqrt{1 - \frac{1}{z^2}} \right\}$$

### Operations carried out by specialized *Mathematica* functions

#### Series expansions

Calculating the series expansion of a sine function to hundreds of terms can be done in seconds.

`Series[Sin[z], {z, 0, 3}]`

$$z - \frac{z^3}{6} + O[z]^4$$

`Normal[%]`

$$z - \frac{z^3}{6}$$

`Series[Sin[z], {z, 0, 100}] // Timing`

$$\left\{ 0. \text{ Second, } z - \frac{z^3}{6} + \frac{z^5}{120} - \frac{z^7}{5040} + \frac{z^9}{362880} - \frac{z^{11}}{39916800} + \frac{z^{13}}{6227020800} - \frac{z^{15}}{1307674368000} + \right. \\ \frac{z^{17}}{355687428096000} - \frac{z^{19}}{121645100408832000} + \frac{z^{21}}{5109094217170944000} - \\ \frac{z^{23}}{2585201673888497664000} + \frac{z^{25}}{1551121004333098598400000} - \\ \frac{z^{27}}{1088886945041835216076800000} + \frac{z^{29}}{884176199373970195454361600000} - \\ \frac{z^{31}}{82283865417792281772556288000000} + \frac{z^{33}}{86833176181188649551819440128000000} - \\ \frac{z^{35}}{103331479663861449296666513375232000000} + \\ \left. \frac{z^{37}}{137637530912263450463159795815809024000000} \right\}$$

$$\begin{aligned}
& \frac{z^{39}}{20\ 397\ 882\ 081\ 197\ 443\ 358\ 640\ 281\ 739\ 902\ 897\ 356\ 800\ 000\ 000} + \\
& \frac{z^{41}}{33\ 452\ 526\ 613\ 163\ 807\ 108\ 170\ 062\ 053\ 440\ 751\ 665\ 152\ 000\ 000\ 000} - \\
& \frac{z^{43}}{60\ 415\ 263\ 063\ 373\ 835\ 637\ 355\ 132\ 068\ 513\ 997\ 507\ 264\ 512\ 000\ 000\ 000} + \\
& \frac{z^{45}}{119\ 622\ 220\ 865\ 480\ 194\ 561\ 963\ 161\ 495\ 657\ 715\ 064\ 383\ 733\ 760\ 000\ 000\ 000} - \\
& \frac{z^{47}}{258\ 623\ 241\ 511\ 168\ 180\ 642\ 964\ 355\ 153\ 611\ 979\ 969\ 197\ 632\ 389\ 120\ 000\ 000\ 000} + \\
& \frac{z^{49}}{608\ 281\ 864\ 034\ 267\ 560\ 872\ 252\ 163\ 321\ 295\ 376\ 887\ 552\ 831\ 379\ 210\ 240\ 000\ 000\ 000} - \\
& \frac{z^{51}}{1\ 551\ 118\ 753\ 287\ 382\ 280\ 224\ 243\ 016\ 469\ 303\ 211\ 063\ 259\ 720\ 016\ 986\ 112\ 000\ 000\ 000\ 000} + \\
& \frac{z^{53}}{4\ 274\ 883\ 284\ 060\ 025\ 564\ 298\ 013\ 753\ 389\ 399\ 649\ 690\ 343\ 788\ 366\ 813\ 724\ 672\ 000\ 000\ 000\ 000} - \frac{z^{55}}{12\ 696\ 403\ 353\ 658\ 275\ 925\ 965\ 100\ 847\ 566\ 516\ 959\ 580\ 321\ 051\ 449\ 436\ 762\ 275\ 840\ 000\ 000\ 000\ 000} + \\
& \frac{z^{57}}{40\ 526\ 919\ 504\ 877\ 216\ 755\ 680\ 601\ 905\ 432\ 322\ 134\ 980\ 384\ 796\ 226\ 602\ 145\ 184\ 481\ 280\ 000\ 000\ 000} \div \\
& \frac{z^{59}}{138\ 683\ 118\ 545\ 689\ 835\ 737\ 939\ 019\ 720\ 389\ 406\ 345\ 902\ 876\ 772\ 687\ 432\ 540\ 821\ 294\ 940\ 160\ 000} \div \\
& \frac{z^{61}}{507\ 580\ 213\ 877\ 224\ 798\ 800\ 856\ 812\ 176\ 625\ 227\ 226\ 004\ 528\ 988\ 036\ 003\ 099\ 405\ 939\ 480\ 985\ 600} \div \\
& \frac{z^{63}}{1\ 982\ 608\ 315\ 404\ 440\ 064\ 116\ 146\ 708\ 361\ 898\ 137\ 544\ 773\ 690\ 227\ 268\ 628\ 106\ 279\ 599\ 612\ 729\ 753} \div \\
& \frac{z^{65}}{8\ 247\ 650\ 592\ 082\ 470\ 666\ 723\ 170\ 306\ 785\ 496\ 252\ 186\ 258\ 551\ 345\ 437\ 492\ 922\ 123\ 134\ 388\ 955\ 774} \div \\
& \frac{z^{67}}{36\ 471\ 110\ 918\ 188\ 685\ 288\ 249\ 859\ 096\ 605\ 464\ 427\ 167\ 635\ 314\ 049\ 524\ 593\ 701\ 628\ 500\ 267\ 962\ 436} \div \\
& \frac{z^{69}}{171\ 122\ 452\ 428\ 141\ 311\ 372\ 468\ 338\ 881\ 272\ 839\ 092\ 270\ 544\ 893\ 520\ 369\ 393\ 648\ 040\ 923\ 257\ 279} \div \\
& \frac{z^{71}}{850\ 478\ 588\ 567\ 862\ 317\ 521\ 167\ 644\ 239\ 926\ 010\ 288\ 584\ 608\ 120\ 796\ 235\ 886\ 430\ 763\ 388\ 588\ 680} \div \\
& \frac{z^{73}}{4\ 470\ 115\ 461\ 512\ 684\ 340\ 891\ 257\ 138\ 125\ 051\ 110\ 076\ 800\ 700\ 282\ 905\ 015\ 819\ 080\ 092\ 370\ 422\ 104} \div \\
& \frac{z^{75}}{24\ 809\ 140\ 811\ 395\ 398\ 091\ 946\ 477\ 116\ 594\ 033\ 660\ 926\ 243\ 886\ 570\ 122\ 837\ 795\ 894\ 512\ 655\ 842\ 677} \div \\
& \frac{z^{77}}{145\ 183\ 092\ 028\ 285\ 869\ 634\ 070\ 784\ 086\ 308\ 284\ 983\ 740\ 379\ 224\ 208\ 358\ 846\ 781\ 574\ 688\ 061\ 991} \div \\
& \frac{z^{79}}{349\ 156\ 420\ 080\ 065\ 207\ 861\ 248\ 000\ 000\ 000\ 000\ 000\ 000} - \frac{z^{81}}{894\ 618\ 213\ 078\ 297\ 528\ 685\ 144\ 171\ 539\ 831\ 652\ 069\ 808\ 216\ 779\ 571\ 907\ 213\ 868\ 063\ 227\ 837\ 990} \div \\
& \frac{z^{83}}{5\ 797\ 126\ 020\ 747\ 367\ 985\ 879\ 734\ 231\ 578\ 109\ 105\ 412\ 357\ 244\ 731\ 625\ 958\ 745\ 865\ 049\ 716\ 390\ 179} \div \\
& \frac{z^{85}}{693\ 892\ 056\ 256\ 184\ 534\ 249\ 745\ 940\ 480\ 000\ 000\ 000\ 000\ 000\ 000} - \frac{z^{87}}{\dots} \div
\end{aligned}$$

```

39 455 239 697 206 586 511 897 471 180 120 610 571 436 503 407 643 446 275 224 357 528 369 751 562 \
996 629 334 879 591 940 103 770 870 906 880 000 000 000 000 000 000 000 000 000 000 + z85 /
281 710 411 438 055 027 694 947 944 226 061 159 480 056 634 330 574 206 405 101 912 752 560 026 \
159 795 933 451 040 286 452 340 924 018 275 123 200 000 000 000 000 000 000 000 000 - z87 /
2 107 757 298 379 527 717 213 600 518 699 389 595 229 783 738 061 356 212 322 972 511 214 654 115 \
727 593 174 080 683 423 236 414 793 504 734 471 782 400 000 000 000 000 000 000 000 000 + z89 /
16 507 955 160 908 461 081 216 919 262 453 619 309 839 666 236 496 541 854 913 520 707 833 171 034 \
378 509 739 399 912 570 787 600 662 729 080 382 999 756 800 000 000 000 000 000 000 000 - z91 /
135 200 152 767 840 296 255 166 568 759 495 142 147 586 866 476 906 677 791 741 734 597 153 670 \
771 559 994 765 685 283 954 750 449 427 751 168 336 768 008 192 000 000 000 000 000 000 000 000 +
z93 /
1 156 772 507 081 641 574 759 205 162 306 240 436 214 753 229 576 413 535 186 142 281 213 246 807 \
121 467 315 215 203 289 516 844 845 303 838 996 289 387 078 090 752 000 000 000 000 000 000 000 -
z95 /
10 329 978 488 239 059 262 599 702 099 394 727 095 397 746 340 117 372 869 212 250 571 234 293 987 \
594 703 124 871 765 375 385 424 468 563 282 236 864 226 607 350 415 360 000 000 000 000 000 000 \
000 + z97 /
96 192 759 682 482 119 853 328 425 949 563 698 712 343 813 919 172 976 158 104 477 319 333 745 612 \
481 875 498 805 879 175 589 072 651 261 284 189 679 678 167 647 067 832 320 000 000 000 000 000 \
000 000 - z99 /
933 262 154 439 441 526 816 992 388 562 667 004 907 159 682 643 816 214 685 929 638 952 175 999 \
932 299 156 089 414 639 761 565 182 862 536 979 208 272 237 582 511 852 109 168 640 000 000 000 \
000 000 000 000 + O[z]101 }

```

Mathematica comes with the add-on package `DiscreteMath`RSolve`` that allows finding the general terms of series for many functions. After loading this package, and using the package function `SeriesTerm`, the following  $n^{\text{th}}$  term of  $\sin(z)$  can be evaluated.

```
In[54]:= << DiscreteMath`RSolve`
```

```
In[55]:= SeriesTerm[Sin[z], {z, 0, n}] z^n
```

```
Out[55]= 
$$\frac{i^{-1+n} z^n \text{KroneckerDelta}[\text{Mod}[-1+n, 2]] \text{UnitStep}[-1+n]}{\text{Gamma}[1+n]}$$

```

This result can be checked by the following process.

```
In[56]:= FunctionExpand[Sum[Evaluate[%], {n, 0, Infinity}]]
```

```
Out[56]= Sin[z]
```

### Differentiation

Mathematica can evaluate derivatives of the sine function of an arbitrary positive integer order.

```
 $\partial_z \text{Sin}[z]$ 
```

```
Cos[z]
```

$\partial_{\{z,2\}} \text{Sin}[z]$

$-\text{Sin}[z]$

$\text{Table}[\text{D}[\text{Sin}[z], \{z, n\}], \{n, 10\}]$

$\{\text{Cos}[z], -\text{Sin}[z], -\text{Cos}[z], \text{Sin}[z], \text{Cos}[z], -\text{Sin}[z], -\text{Cos}[z], \text{Sin}[z], \text{Cos}[z], -\text{Sin}[z]\}$

### Finite summation

*Mathematica* can calculate finite symbolic sums that contain the sine function. Here are some examples.

$$\sum_{k=1}^n \text{Sin}[a k]$$

$$\frac{1}{2} \left( \text{Cos}\left[\frac{a}{2}\right] - \text{Cos}\left[\frac{a}{2} + a n\right] \right) \text{Csc}\left[\frac{a}{2}\right]$$

$$\sum_{k=1}^n (-1)^k \text{Sin}[a k]$$

$$\frac{1}{2} \text{Sec}\left[\frac{a}{2}\right] \left( -\text{Sin}\left[\frac{a}{2}\right] + \text{Sin}\left[\frac{a}{2} + a n + n \pi\right] \right)$$

### Infinite summation

*Mathematica* can calculate infinite sums that contain sine function. Here are some examples.

$$\sum_{k=1}^{\infty} z^k \text{Sin}[k x]$$

$$\frac{i \left( -1 + e^{2 i x} \right) z}{2 \left( e^{i x} - z \right) \left( -1 + e^{i x} z \right)}$$

$$\sum_{k=1}^{\infty} \frac{\text{Sin}[k x]}{k!}$$

$$\frac{1}{2} i \left( e^{-i x} - e^{i x} \right)$$

$$\sum_{k=1}^{\infty} \frac{\text{Sin}[k x]}{k}$$

$$-\frac{1}{2} i \left( \text{Log}\left[1 - e^{-i x}\right] - \text{Log}\left[1 - e^{i x}\right] \right)$$

### Finite products

*Mathematica* can calculate some finite symbolic products that contain the sine function. Here are two examples.

$$\prod_{k=1}^{n-1} \text{Sin}\left[\frac{k \pi}{n}\right]$$

$$2^{1-n} n$$

$$\prod_{k=1}^{n-1} \text{Sin}\left[z + \frac{\pi k}{n}\right]$$

$$2^{1-n} \text{Csc}[z] \text{Sin}[n z]$$

### Indefinite integration

*Mathematica* can calculate a huge number of doable indefinite integrals that contain the sine function. Here are some examples.

$$\int \text{Sin}[z] \, dz$$

$$-\text{Cos}[z]$$

$$\int \text{Sin}[z]^a \, dz$$

$$-\text{Cos}[z] \text{Hypergeometric2F1}\left[\frac{1}{2}, \frac{1-a}{2}, \frac{3}{2}, \text{Cos}[z]^2\right] \text{Sin}[z]^{1+a} (\text{Sin}[z]^2)^{\frac{1}{2}(-1-a)}$$

### Definite integration

*Mathematica* can calculate wide classes of definite integrals that contain the sine function. Here are some examples.

$$\int_0^{\pi/2} \sqrt{\text{Sin}[z]} \, dz$$

$$2 \text{EllipticE}\left[\frac{\pi}{4}, 2\right]$$

$$\int_0^{\frac{\pi}{2}} \text{Sin}[z]^a \, dz$$

$$\frac{\sqrt{\pi} \text{Gamma}\left[\frac{1+a}{2}\right]}{a \text{Gamma}\left[\frac{a}{2}\right]}$$

### Limit operation

*Mathematica* can calculate limits that contain the sine function. Here are some examples.

$$\text{Limit}\left[\frac{\text{Sin}[2 z]}{z}, z \rightarrow 0\right]$$

$$2$$

$$\text{Limit}\left[\frac{\text{Sin}\left[\sqrt{z^2}\right]}{z}, z \rightarrow 0, \text{Direction} \rightarrow 1\right]$$

$$-1$$

$$\text{Limit}\left[\frac{\sin\left[\sqrt{z^2}\right]}{z}, z \rightarrow 0, \text{Direction} \rightarrow -1\right]$$

1

### Solving equations

The next inputs solve two equations that contain the sine function. Because of the multivalued nature of the inverse sine function, a message indicates that only some of the possible solutions are returned.

$$\text{Solve}\left[\sin[z]^2 + 3 \sin[z + \text{Pi}/6] = 4, z\right]$$

Solve::ifun: Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. More...

$$\left\{ \left\{ z \rightarrow -\text{ArcCos}\left[\frac{3}{4} + \frac{1}{2} \sqrt{-\frac{31}{4} + \frac{38}{(505 + 9 i \sqrt{2271})^{1/3}} + \frac{1}{2} (505 + 9 i \sqrt{2271})^{1/3}}\right] - \right.$$

$$\left. \frac{1}{4} \sqrt{\left(-62 - \frac{152}{(505 + 9 i \sqrt{2271})^{1/3}} - 2 (505 + 9 i \sqrt{2271})^{1/3} - \right.}\right.$$

$$\left. \left. \frac{81}{\sqrt{-\frac{31}{4} + \frac{38}{(505 + 9 i \sqrt{2271})^{1/3}} + \frac{1}{2} (505 + 9 i \sqrt{2271})^{1/3}}}\right] \right\},$$

$$\left\{ \left\{ z \rightarrow -\text{ArcCos}\left[\frac{3}{4} + \frac{1}{2} \sqrt{-\frac{31}{4} + \frac{38}{(505 + 9 i \sqrt{2271})^{1/3}} + \frac{1}{2} (505 + 9 i \sqrt{2271})^{1/3}}\right] + \right.$$

$$\left. \frac{1}{4} \sqrt{\left(-62 - \frac{152}{(505 + 9 i \sqrt{2271})^{1/3}} - 2 (505 + 9 i \sqrt{2271})^{1/3} - \right.}\right.$$

$$\left. \left. \frac{81}{\sqrt{-\frac{31}{4} + \frac{38}{(505 + 9 i \sqrt{2271})^{1/3}} + \frac{1}{2} (505 + 9 i \sqrt{2271})^{1/3}}}\right] \right\},$$

$$\left\{ z \rightarrow \text{ArcCos} \left[ \frac{3}{4} - \frac{1}{2} \sqrt{-\frac{31}{4} + \frac{38}{(505 + 9i\sqrt{2271})^{1/3}} + \frac{1}{2} (505 + 9i\sqrt{2271})^{1/3}} \right. \right. \\ \left. \left. \frac{1}{4} \sqrt{\left( -62 - \frac{152}{(505 + 9i\sqrt{2271})^{1/3}} - 2 (505 + 9i\sqrt{2271})^{1/3} + \frac{81}{\sqrt{-\frac{31}{4} + \frac{38}{(505 + 9i\sqrt{2271})^{1/3}} + \frac{1}{2} (505 + 9i\sqrt{2271})^{1/3}}} \right)} \right] \right\}, \\ \left\{ z \rightarrow \text{ArcCos} \left[ \frac{3}{4} - \frac{1}{2} \sqrt{-\frac{31}{4} + \frac{38}{(505 + 9i\sqrt{2271})^{1/3}} + \frac{1}{2} (505 + 9i\sqrt{2271})^{1/3}} \right. \right. \\ \left. \left. \frac{1}{4} \sqrt{\left( -62 - \frac{152}{(505 + 9i\sqrt{2271})^{1/3}} - 2 (505 + 9i\sqrt{2271})^{1/3} + \frac{81}{\sqrt{-\frac{31}{4} + \frac{38}{(505 + 9i\sqrt{2271})^{1/3}} + \frac{1}{2} (505 + 9i\sqrt{2271})^{1/3}}} \right)} \right] \right\}$$

**Solve[Sin[x] == a, x]**

Solve::ifun: Inverse functions are being used by Solve, so some solutions may not be found.

{x -> ArcSin[a]}

A complete solution of the previous equation can be obtained using the function Reduce.

**Reduce[Sin[x] == a, x] // InputForm**

C[1] ∈ Integers && (x == π - ArcSin[a] + 2 π C[1] || x == ArcSin[a] + 2 π C[1])

**Solving differential equations**

Here are differential equations whose linear independent solutions include the sine function. The solutions of the simplest second-order linear ordinary differential equation with constant coefficients can be represented using  $\sin(z)$  and  $\cos(z)$ .

```
DSolve[w''[z] + w[z] == 0, w[z], z]
```

```
{{w[z] -> C[1] Cos[z] + C[2] Sin[z]}}
```

```
dsol1 = DSolve[4 w[z] + 5 w''[z] + w^(4)[z] == 0, w[z], z]
```

```
{{w[z] -> C[3] Cos[z] + C[1] Cos[2 z] + C[4] Sin[z] + C[2] Sin[2 z]}}
```

In the last input, the differential equation was solved for  $w(z)$ . If the argument is suppressed, the result is returned as a pure function (in the sense of the  $\lambda$ -calculus).

```
dsol2 = DSolve[4 w[z] + 5 w''[z] + w^(4)[z] == 0, w, z]
```

```
{{w -> Function[{z}, C[3] Cos[z] + C[1] Cos[2 z] + C[4] Sin[z] + C[2] Sin[2 z]]}}
```

The advantage of such a pure function is that it can be used for different arguments, derivatives, and more.

```
w'[ξ] /. dsol1
```

```
{w'[ξ]}
```

```
w'[ξ] /. dsol2
```

```
{C[4] Cos[ξ] + 2 C[2] Cos[2 ξ] - C[3] Sin[ξ] - 2 C[1] Sin[2 ξ]}
```

In carrying out the algorithm to solve the following nonlinear differential equation, *Mathematica* has to solve a transcendental equation. In doing so, the generically multivariate inverse of a function is encountered, and a message is issued that a solution branch is potentially missed.

```
DSolve[{w'[z] == Sqrt[1 - w[z]^2], w[0] == 0}, w[z], z]
```

```
Solve::ifun: Inverse functions are being used by Solve, so some solutions may not be found.
```

```
{{w[z] -> Sin[z]}}
```

## Integral transforms

*Mathematica* supports the main integral transforms like direct and inverse Fourier, Laplace, and Z transforms that can give results that contain classical or generalized functions.

```
LaplaceTransform[Sin[t], t, s]
```

$$\frac{1}{1 + s^2}$$

```
FourierTransform[Sin[t], t, s]
```

$$i \sqrt{\frac{\pi}{2}} \text{DiracDelta}[-1 + s] - i \sqrt{\frac{\pi}{2}} \text{DiracDelta}[1 + s]$$



**FourierSinTransform**[Sin[t], t, s]

$$\sqrt{\frac{\pi}{2}} \text{DiracDelta}[-1 + s] - \sqrt{\frac{\pi}{2}} \text{DiracDelta}[1 + s]$$

**FourierCosTransform**[Sin[t], t, s]

$$-\frac{1}{\sqrt{2\pi}(-1+s)} + \frac{1}{\sqrt{2\pi}(1+s)}$$

**ZTransform**[Sin[π t], t, s]

0

### Plotting

*Mathematica* has built-in functions for 2D and 3D graphics. Here are some examples.

```
Plot[Sin[Sum[z^k, {k, 0, 5}], {z, -2 π/3, 2 π/3}];
```

```
Plot3D[Re[Sin[x + i y]], {x, -π, π}, {y, 0, π},
  PlotPoints → 240, PlotRange → {-5, 5},
  ClipFill → None, Mesh → False, AxesLabel → {"x", "y", None}];
```

```
ContourPlot[Arg[Sin[1/(x + i y)]], {x, -1/2, 1/2}, {y, -1/2, 1/2},
  PlotPoints → 400, PlotRange → {-π, π}, FrameLabel → {"x", "y", None, None},
  ColorFunction → Hue, ContourLines → False, Contours → 200];
```

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