Introductions to WeierstrassZetaHalfPeriodValues

Introduction to the Weierstrass utility functions

General

The creation and development of the elliptic functions' theory in the 18th century required the introduction of special supporting utility functions, which were frequently used for description of the properties of the elliptic functions. Among such utilities the basic role is played by so-called Weierstrass invariants and Weierstrass half-periods. These were given the unusual notations $\{g_2, g_3\}$ and $\{\omega_1, \omega_3\}$ instead of a consecutive numbering. The Weierstrass utility functions are a pair of bivariate functions that are inverses of each other:

$\{g_2, g_3\} = \{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\}$
$\{\omega_1, \omega_3\} = \{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\}$

$\{g_2, g_3\} = \{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\} /; \{\omega_1, \omega_3\} = \{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\}$.

Half-periods ω_1 and ω_3 (and $\omega_2 = -\omega_1 - \omega_3$) were mentioned in the works of C. G. J. Jacobi (1835), K. Weierstrass (1862), and A. Hurwitz (1905). The invariants g_2 and g_3 were mentioned in the works of A. Cayley and G. Boole (1845).

Numerous formulas of Weierstrass elliptic functions include values of the Weierstrass $\wp$ function and the Weierstrass zeta functions $\wp(z; g_2, g_3)$ and $\zeta(z; g_2, g_3)$ at the points $z = \omega_k(g_2, g_3) /; k = 1, 2, 3$. These values have the following widely used notations:

$\{e_1, e_2, e_3\} = \{e_1(g_2, g_3), e_2(g_2, g_3), e_3(g_2, g_3)\} = \{\wp(\omega_1; g_2, g_3), \wp(\omega_2; g_2, g_3), \wp(\omega_3; g_2, g_3)\} /;$
$\{\omega_1, \omega_3\} = \{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\} \bigwedge \omega_2 = -\omega_1 - \omega_3$

$\{\eta_1, \eta_2, \eta_3\} = \{\eta_1(g_2, g_3), \eta_2(g_2, g_3), \eta_3(g_2, g_3)\} = \{\zeta(\omega_1; g_2, g_3), \zeta(\omega_2; g_2, g_3), \zeta(\omega_3; g_2, g_3)\} /;$
$\{\omega_1, \omega_3\} = \{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\} \bigwedge \omega_2 = -\omega_1 - \omega_3$.

Definitions of Weierstrass utilities

The Weierstrass half-periods $\{\omega_1, \omega_3\} = \{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\}$ and the Weierstrass invariants $\{g_2, g_3\} = \{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\}$, the Weierstrass $\wp$ function values at half-periods $\{e_1, e_2, e_3\} = \{e_1(g_2, g_3), e_2(g_2, g_3), e_3(g_2, g_3)\}$, and the Weierstrass zeta function values at half-periods $\{\eta_1, \eta_2, \eta_3\} = \{\eta_1(g_2, g_3), \eta_2(g_2, g_3), \eta_3(g_2, g_3)\}$ are defined by the following formulas:

$$\{\omega_1, \omega_3\} = \{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\} = \left\{ i \left(\frac{60}{g_2} \sum_{\substack{m, n=-\infty \\ \{m,n\} \neq \{0,0\}}}^{\infty} \frac{1}{(2m + 2nt)^4} \right)^{1/4}, i\, t \left(\frac{60}{g_2} \sum_{\substack{m, n=-\infty \\ \{m,n\} \neq \{0,0\}}}^{\infty} \frac{1}{(2m + 2nt)^4} \right)^{1/4} \right\} /; J(t) = \frac{g_2^3}{g_2^3 - 27 g_3^2}$$

$$\{g_2, g_3\} == \{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\} == \left\{60 \sum_{\substack{m, n=-\infty \\ \{m,n\} \neq \{0,0\}}}^{\infty} \frac{1}{(2\, m\, \omega_1 + 2\, n\, \omega_3)^4}, 140 \sum_{\substack{m, n=-\infty \\ \{m,n\} \neq \{0,0\}}}^{\infty} \frac{1}{(2\, m\, \omega_1 + 2\, n\, \omega_3)^6}\right\} /; \operatorname{Im}\left(\frac{\omega_3}{\omega_1}\right) > 0$$

$$\{e_1, e_2, e_3\} == \{e_1(g_2, g_3), e_2(g_2, g_3), e_3(g_2, g_3)\} == \{\wp(\omega_1; g_2, g_3), \wp(\omega_2; g_2, g_3), \wp(\omega_3; g_2, g_3)\} /; \{\omega_1, \omega_3\} == \{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\} \land \omega_2 == -\omega_1 - \omega_3$$

$$\{\eta_1, \eta_2, \eta_3\} == \{\eta_1(g_2, g_3), \eta_2(g_2, g_3), \eta_3(g_2, g_3)\} == \{\zeta(\omega_1; g_2, g_3), \zeta(\omega_2; g_2, g_3), \zeta(\omega_3; g_2, g_3)\} /; \{\omega_1, \omega_3\} == \{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\} \land \omega_2 == -\omega_1 - \omega_3,$$

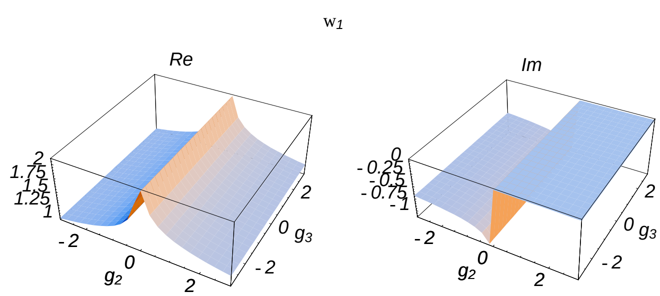
where $J(t)$ is the Klein invariant modular function, $\wp(z; g_2, g_3)$ is Weierstrass elliptic $\wp$ function, and $\zeta(z; g_2, g_3)$ is theWeierstrass zeta function.

The previous four vector functions are sometimes called the Weierstrass utilities because they are the basic elements of the Weierstrass theory of elliptic functions.

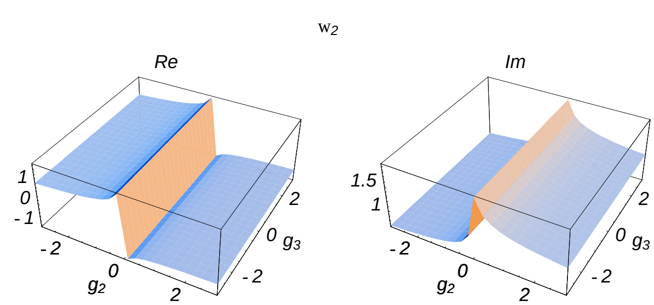
A quick look at the Weierstrass utilities

The following graphics show the values of the real and imaginary parts of the Weierstrass half-periods over the real g_2-g_3 plane.

```
WI[{g2_?NumberQ, g3_}] := With[{g2hp = Rationalize[g2, 0], g3hp =
Rationalize[g3, 0]},
                                    N[WeierstrassHalfPeriods[{g2hp, g2hp}], 30]]
```



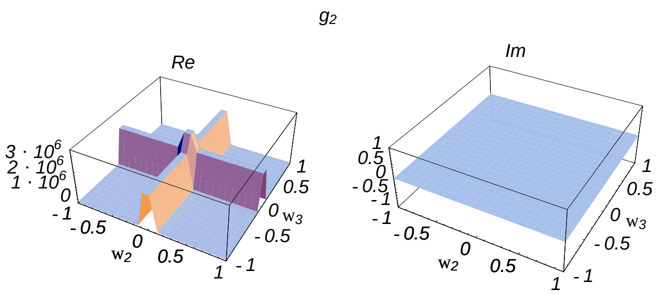
- GraphicsArray -



- GraphicsArray -

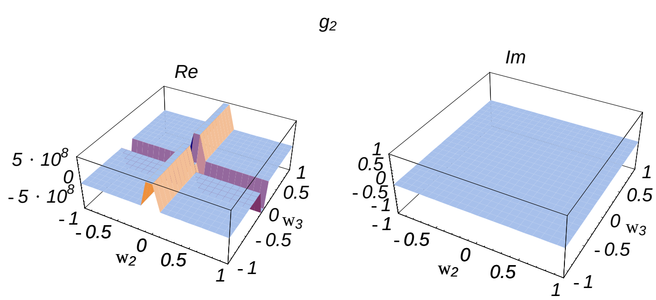
The following graphics show the values of the real and imaginary parts of the Weierstrass invarants over the ω_1-$i\,\omega_3$ plane.

```
WHP[{ω1_?NumberQ, ω3_}] := With[{ω1hp = Rationalize[
ω1, 0], ω3hp = Rationalize[ω3, 0]},

                               N[WeierstrassInvariants[{ω1hp,
ω3hp}], 30]]
```



- GraphicsArray -

```
Show[GraphicsArray[

Plot3D[# @ WHP[{ω1, I ω3}][[2]],

       {ω1, -1, 1}, {ω3, -1, 1}, PlotPoints -> 24, Mesh ->
False, PlotRange -> All,

       DisplayFunction -> Identity,

       AxesLabel -> {Subscript[ω, 2], Subscript[ω, 3], None},
PlotLabel -> #]& /@ {Re, Im}],

       PlotLabel -> Subscript[g, 2]]
```

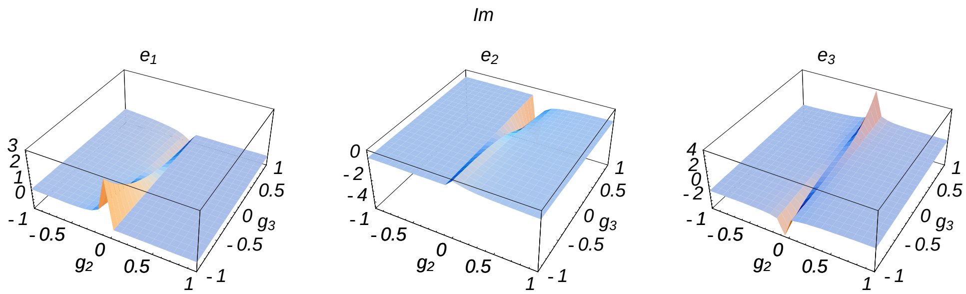


- GraphicsArray -

The following graphics show the values of the real and imaginary parts of the Weierstrass ℘ function at the half-periods over the real g_2-g_3 plane.

```
WE[{g2_?NumberQ, g3_}, k_] := Module[{g2hp = Rationalize[g2, 0], g3hp =
Rationalize[g3, 0],
                                         ω1, ω2, ω3},
                                     {ω1, ω3} =
N[WeierstrassHalfPeriods[{g2hp, g2hp}], 30];
                                     ω2 =-ω1 - ω3;
                                     WeierstrassP[{ω1, ω2,
ω3}[[k]], {g2, g3}]
                                     ]
```

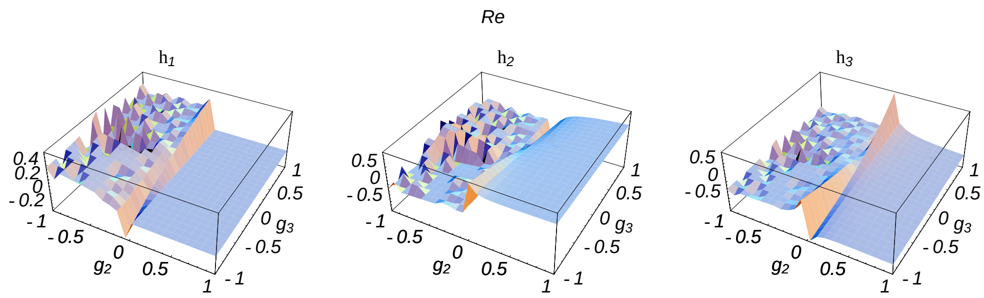
- GraphicsArray -



- GraphicsArray -

The following graphics show the values of the real and imaginary parts of the Weierstrass zeta function at the half-periods over the real g_2-g_3 plane.

```
WZ[{g2_?NumberQ, g3_}, k_] := Module[{g2hp = Rationalize[g2, 0], g3hp =
Rationalize[g3, 0], ω1, ω2, ω3},
                                     {ω1, ω3} =
N[WeierstrassHalfPeriods[{g2hp, g2hp}], 30];
                                     ω2 =-ω1 - ω3;
                                     WeierstrassZeta[{ω1, ω2,
ω3}[[k]], {g2, g3}]
                                     ]
```



- GraphicsArray -

Connections within the group of Weierstrass utilities and inverses and with other function groups

Representations through related equivalent functions

The Weierstrass half-periods $\{\omega_1, \omega_3\}$ can be represented through the complete elliptic integral $K(m)$ and the inverse elliptic nome $q^{-1}(z)$ by the formula:

$$\{\omega_1, \omega_3\} = \left\{\frac{K(m)}{\sqrt{e_1 - e_3}}, \frac{i\, K(1-m)}{\sqrt{e_1 - e_3}}\right\} /; \{e_1, e_2, e_3\} = \{\wp(\omega_1; g_2, g_3), \wp(\omega_1 + \omega_3; g_2, g_3), \wp(\omega_3; g_2, g_3)\} \bigwedge m = q^{-1}\left(\exp\left(\frac{i\, \pi\, \omega_3}{\omega_1}\right)\right) \bigwedge \{\omega_1, \omega_3\} = \{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\}.$$

The Weierstrass invariants $\{g_2, g_3\}$ can be represented through the complete elliptic integral $K(m)$, the inverse elliptic nome $q^{-1}(z)$, the modular lambda function $\lambda(z)$, and the theta functions $\vartheta_j(0, q)$ by the following formulas:

$$\{g_2, g_3\} = \left\{\frac{4\left(m^2 - m + 1\right) K(m)^4}{3\, \omega_1^4}, \frac{4\,(m-2)\,(2\,m-1)\,(m+1)\, K(m)^6}{27\, \omega_1^6}\right\} /; m = \lambda\left(\frac{\omega_3}{\omega_1}\right)$$

$$g_2^3\left(2\,m^3 - 3\,m^2 - 3\,m + 2\right)^2 - 108\, g_3^2\left(m^2 - m + 1\right)^3 = 0 /; m = q^{-1}\left(\exp\left(\frac{i\, \pi\, \omega_3}{\omega_1}\right)\right)$$

$$\{g_2, g_3\} = \left\{\frac{1}{12}\left(\frac{\pi}{\omega_1}\right)^4 \left(\vartheta_2(0, q)^8 - \vartheta_3(0, q)^4\, \vartheta_2(0, q)^4 + \vartheta_3(0, q)^8\right), \frac{1}{432}\left(\frac{\pi}{\omega_1}\right)^6 \left(2\left(\vartheta_2(0, q)^{12} + \vartheta_3(0, q)^{12}\right) - 3\left(\vartheta_2(0, q)^4 + \vartheta_3(0, q)^4\right) \vartheta_2(0, q)^4\, \vartheta_3(0, q)^4\right)\right\} /; q = \exp\left(\frac{i\, \pi\, \omega_3}{\omega_1}\right)$$

$$\{g_2, g_3\} = \left\{\frac{1}{24}\left(\frac{\pi}{\omega_1}\right)^4 \left(\vartheta_2(0, q)^8 + \vartheta_3(0, q)^8 + \vartheta_4(0, q)^8\right), \frac{1}{432}\left(\frac{\pi}{\omega_1}\right)^6 \left(\vartheta_2(0, q)^4 + \vartheta_3(0, q)^4\right)\left(\vartheta_3(0, q)^4 + \vartheta_4(0, q)^4\right)\left(\vartheta_4(0, q)^4 - \vartheta_2(0, q)^4\right)\right\} /; q = \exp\left(\frac{i\, \pi\, \omega_3}{\omega_1}\right).$$

The Weierstrass $\wp$ function values $\{e_1, e_2, e_3\}$ at half-periods can be represented through the complete elliptic integral
$K(m)$, the modular lambda function $\lambda(z)$, the Weierstrass sigma function $\sigma(z; g_2, g_3)$, and the theta functions $\vartheta_j(0, q)$ by the following formulas:

$$\{e_1, e_2, e_3\} = \left\{\frac{(2-m)K(m)^2}{3\omega_1^2}, \frac{(2m-1)K(m)^2}{3\omega_1^2}, -\frac{(m+1)K(m)^2}{3\omega_1^2}\right\} /; m = \lambda\left(\frac{\omega_3}{\omega_1}\right)$$

$$\frac{e_2-e_3}{e_1-e_3} = \lambda\left(\frac{\omega_3}{\omega_1}\right)$$

$$\frac{e_1-e_2}{e_1-e_3} = 1-\lambda\left(\frac{\omega_3}{\omega_1}\right)$$

$$e_i - e_j = \frac{\sigma_j(\omega_i; g_2, g_3)^2}{\sigma(\omega_i; g_2, g_3)^2} /; \{i, j\} \in \{1, 2, 3\} \bigwedge i \neq j$$

$$e_i - e_j = e^{-2\eta_j\omega_i} \frac{\sigma(\omega_k; g_2, g_3)^2}{\sigma(\omega_i; g_2, g_3)^2\, \sigma(\omega_j; g_2, g_3)^2} /; \{i, j, k\} \in \{1, 2, 3\} \bigwedge i \neq j \neq k$$

$$\{e_1, e_2, e_3\} = \frac{1}{3}\left(\frac{\pi}{2\omega_1}\right)^2 \left\{\vartheta_3(0, q)^4 + \vartheta_4(0, q)^4, \vartheta_2(0, q)^4 - \vartheta_4(0, q)^4, -\vartheta_2(0, q)^4 - \vartheta_3(0, q)^4\right\} /; q = \exp\left(\frac{i\pi\omega_3}{\omega_1}\right).$$

The Weierstrass zeta function values at half-periods $\{\eta_1, \eta_2, \eta_3\}$ can be represented through the complete elliptic integrals $K(m)$ and $E(m)$, the modular lambda function $\lambda(z)$, and the theta functions $\vartheta_j(0, q)$ by the following formulas:

$$\{\eta_1, \eta_2, \eta_3\} = \left\{\sqrt{e_1-e_3}\left(E(m) - \frac{e_1}{e_1-e_3}K(m)\right),\right.$$
$$\left.\frac{K(m)\,e_1 - (e_1-e_3)(E(m) - i\,E(1-m)) + i\,K(1-m)\,e_3}{\sqrt{e_1-e_3}}, -i\sqrt{e_1-e_3}\left(E(1-m) + \frac{e_3}{e_1-e_3}K(1-m)\right)\right\} /; m = \lambda\left(\frac{\omega_3}{\omega_1}\right)$$

$$\eta_1 = \frac{K(m)(3E(m) + (m-2)K(m))}{3\omega_1} /; m = \lambda\left(\frac{\omega_3}{\omega_1}\right)$$

$$\eta_1 = -\frac{\pi^2}{12\omega_1}\frac{\vartheta_1^{(3)}(0, q)}{\vartheta_1'(0, q)} /; q = \exp\left(\frac{i\pi\omega_3}{\omega_1}\right)$$

$$\eta_1 = -e_i\,\omega_1 - \frac{\pi^2}{4\omega_1}\frac{\vartheta_{i+1}''(0, q)}{\vartheta_{i+1}(0, q)} /; q = \exp\left(\frac{i\pi\omega_3}{\omega_1}\right) \bigwedge i \in \{1, 2, 3\}$$

$$\eta_1^2 = \left(\frac{g_2}{6} - e_i^2\right)\omega_1^2 - e_i - \frac{\pi^2\eta_1}{2\omega_1}\frac{\vartheta_{i+1}''(0, q)}{\vartheta_{i+1}(0, q)} - \frac{\pi^4}{48\omega_1^2}\frac{\vartheta_{i+1}^{(4)}(0, q)}{\vartheta_{i+1}(0, q)} /; q = \exp\left(\frac{i\pi\omega_3}{\omega_1}\right) \bigwedge i \in \{1, 2, 3\}.$$

Relations to inverse functions

The following formula shows that the Weierstrass half-periods $\{\omega_1, \omega_3\}$ play the role of inverse functions to the Weierstrass invariants $\{g_2, g_3\}$:

$$\{g_2, g_3\} = \{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\} /; \{\omega_1, \omega_3\} = \{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\}.$$

Representations through other Weierstrass utilities

The Weierstrass half-periods $\{\omega_1, \omega_3\}$, the invariants $\{g_2, g_3\}$, and the Weierstrass $\wp$ function values at half-periods $\{e_1, e_2, e_3\}$ are connected by the following formulas:

$$\{g_2, g_3\} = \{-4(e_1 e_2 + e_3 e_2 + e_1 e_3), 4 e_1 e_2 e_3\} /; \{\omega_1, \omega_2, \omega_3\} = \{\omega_1(g_2, g_3), -\omega_1(g_2, g_3) - \omega_3(g_2, g_3), \omega_3(g_2, g_3)\} \bigwedge n \in \{1, 2, 3\}$$

$$\{g_2, g_3\} = \{2(e_1^2 + e_2^2 + e_3^2), 4 e_1 e_2 e_3\} /; \{\omega_1, \omega_2, \omega_3\} = \{\omega_1(g_2, g_3), -\omega_1(g_2, g_3) - \omega_3(g_2, g_3), \omega_3(g_2, g_3)\} \bigwedge n \in \{1, 2, 3\}$$

$$g_2^2 = 8(e_1^4 + e_2^4 + e_3^4) /; \{\omega_1, \omega_2, \omega_3\} = \{\omega_1(g_2, g_3), -\omega_1(g_2, g_3) - \omega_3(g_2, g_3), \omega_3(g_2, g_3)\} \bigwedge n \in \{1, 2, 3\}$$

$$4 e_k^3 - g_2 e_k - g_3 = 0 /; k \in \{1, 2, 3\} \bigwedge \{\omega_1, \omega_2, \omega_3\} = \{\omega_1(g_2, g_3), -\omega_1(g_2, g_3) - \omega_3(g_2, g_3), \omega_3(g_2, g_3)\} \bigwedge n \in \{1, 2, 3\}$$

$$g_2^3 - 27 g_3^2 = 16 (e_1 - e_2)^2 (e_3 - e_1)^2 (e_3 - e_2)^2 /; \{\omega_1, \omega_2, \omega_3\} = \{\omega_1(g_2, g_3), -\omega_1(g_2, g_3) - \omega_3(g_2, g_3), \omega_3(g_2, g_3)\} \bigwedge n \in \{1, 2, 3\}.$$

The best-known properties and formulas for Weierstrass utilities

Specific values

The Weierstrass invariants $\{g_2, g_3\}$ have the following values at infinities:

$$g_2(\omega_1, \tilde{\infty})^3 - 27 g_3(\omega_1, \tilde{\infty})^2 = 0$$

$$\frac{9 g_3(\omega_1, \tilde{\infty})}{2 g_2(\omega_1, \tilde{\infty})} = \left(\frac{\pi}{2\omega_1}\right)^2$$

$$\{g_2(\tilde{\infty}, \tilde{\infty}), g_3(\tilde{\infty}, \tilde{\infty})\} = \{0, 0\}.$$

The Weierstrass $\wp$ function values at half-periods $\{e_1, e_2, e_3\}$ can be evaluated at closed forms for some values of arguments g_2, g_3:

$$\{e_1, e_2, e_3\} = \left\{\frac{1}{\sqrt[3]{4}}, \frac{1}{\sqrt[3]{4}} e^{\frac{4\pi i}{3}}, \frac{1}{\sqrt[3]{4}} e^{\frac{2\pi i}{3}}\right\} /; \{g_2, g_3\} = \{0, 1\}$$

$$\{e_1, e_2, e_3\} = \left\{\frac{1}{2}, 0, -\frac{1}{2}\right\} /; \{g_2, g_3\} = \{1, 0\}$$

$$e_1(g_2(\omega_1, \tilde{\infty}), g_3(\omega_1, \tilde{\infty})) = \frac{\pi^2}{6\omega_1^2}$$

$$\{e_1(g_2(\omega_1, \tilde{\infty}), g_3(\omega_1, \tilde{\infty})), e_2(g_2(\omega_1, \tilde{\infty}), g_3(\omega_1, \tilde{\infty})), e_3(g_2(\omega_1, \tilde{\infty}), g_3(\omega_1, \tilde{\infty}))\} = \left\{\frac{3 g_3}{g_2}, -\frac{3 g_3}{g_2}, -\frac{3 g_3}{g_2}\right\}$$

$$e_3(g_2(\tilde{\infty}, \omega_3), g_3(\tilde{\infty}, \omega_3)) == \frac{\pi^2}{6\,\omega_3^2}$$

$$\{e_1(g_2(\tilde{\infty}, \tilde{\infty}), g_3(\tilde{\infty}, \tilde{\infty})), e_2(g_2(\tilde{\infty}, \tilde{\infty}), g_3(\tilde{\infty}, \tilde{\infty})), e_3(g_2(\tilde{\infty}, \tilde{\infty}), g_3(\tilde{\infty}, \tilde{\infty}))\} == \left\{\frac{1}{\omega_1^2}, \frac{1}{\omega_2^2}, \frac{1}{\omega_3^2}\right\}.$$

The Weierstrass zeta function values at half-periods $\{\eta_1, \eta_2, \eta_3\}$ can also be evaluated at closed forms for some values of arguments g_2, g_3:

$$\{\eta_1, \eta_2, \eta_3\} == \left\{\frac{\pi}{2\,\omega_1\sqrt{3}}, \frac{\pi}{2\,\omega_1\sqrt{3}}\, e^{2\pi i/3}, \frac{\pi}{2\,\omega_1\sqrt{3}}\, e^{4\pi i/3}\right\} /; \{g_2, g_3\} == \{0, 1\}$$

$$\{\eta_1, \eta_2, \eta_3\} == \left\{\frac{\pi}{4\,\omega_1}, \frac{\pi}{4\,\omega_1}(1-i), -\frac{\pi\, i}{4\,\omega_1}\right\} /; \{g_2, g_3\} == \{1, 0\}$$

$$\eta_1(g_2(\omega_1, \tilde{\infty}), g_3(\omega_1, \tilde{\infty})) == \frac{\pi^2}{12\,\omega_1}$$

$$\eta_3(g_2(\tilde{\infty}, \omega_3), g_3(\tilde{\infty}, \omega_3)) == \frac{\pi^2}{12\,\omega_3}$$

$$\{\eta_1(g_2(\tilde{\infty}, \tilde{\infty}), g_3(\tilde{\infty}, \tilde{\infty})), \eta_2(g_2(\tilde{\infty}, \tilde{\infty}), g_3(\tilde{\infty}, \tilde{\infty})), \eta_3(g_2(\tilde{\infty}, \tilde{\infty}), g_3(\tilde{\infty}, \tilde{\infty}))\} == \left\{\frac{1}{\omega_1}, \frac{1}{\omega_2}, \frac{1}{\omega_3}\right\}.$$

Analyticity

The Weierstrass half-periods $\{\omega_1, \omega_3\} == \{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\}$, the Weierstrass $\wp$ function values at half-periods $\{e_1, e_2, e_3\} == \{e_1(g_2, g_3), e_2(g_2, g_3), e_3(g_2, g_3)\}$, and the Weierstrass zeta function values at half-periods $\{\eta_1, \eta_2, \eta_3\} == \{\eta_1(g_2, g_3), \eta_2(g_2, g_3), \eta_3(g_2, g_3)\}$ are vector-valued functions of g_2 and g_3 that are analytic in each vector component, and they are defined over $\mathbb{C}^2$.

The Weierstrass invariants $\{g_2, g_3\} == \{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\}$ is a vector-valued function of ω_1 and ω_3 that is analytic in each vector component, and it is defined over $\mathbb{C}^2$ (for $\omega_1 \neq a\,\omega_3$, $a \in \mathbb{R}$).

Periodicity

The Weierstrass invariants $\{g_2, g_3\} == \{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\}$ with $\omega_1 == 1$ is a periodic function with period 1:

$$\{g_2(1, \omega_3 + n), g_3(1, \omega_3 + n)\} == \{g_2(1, \omega_3), g_3(1, \omega_3)\} /; n \in \mathbb{Z}.$$

The other Weierstrass utility functions $\{\omega_1, \omega_3\}$, $\{e_1, e_2, e_3\}$, and $\{\eta_1, \eta_2, \eta_3\}$ are not periodic functions.

Parity and symmetry

The Weierstrass half-periods $\{\omega_1, \omega_3\} == \{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\}$ and Weierstrass zeta function values at half-periods $\{\eta_1, \eta_2, \eta_3\} == \{\eta_1(g_2, g_3), \eta_2(g_2, g_3), \eta_3(g_2, g_3)\}$ have mirror symmetry:

$$\{\omega_1(\overline{g_2}, \overline{g_3}), \omega_3(\overline{g_2}, \overline{g_3})\} == \overline{\{\omega_1(g_2, g_3), -\omega_3(g_2, g_3)\}}$$

$$\{\eta_1(\overline{g_2}, \overline{g_3}), \eta_3(\overline{g_2}, \overline{g_3})\} == \overline{\{\eta_1(g_2, g_3), -\eta_3(g_2, g_3)\}}$$

The Weierstrass invariants $\{g_2, g_3\} == \{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\}$ and the Weierstrass $\wp$ function values at half-periods $\{e_1, e_2, e_3\} == \{e_1(g_2, g_3), e_2(g_2, g_3), e_3(g_2, g_3)\}$ have standard mirror symmetry:

$$\{g_2(\overline{\omega_1}, \overline{\omega_3}), g_3(\overline{\omega_1}, \overline{\omega_3})\} == \overline{\{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\}}$$

$$\{e_1(\overline{g_2}, \overline{g_3}), e_2(\overline{g_2}, \overline{g_3}), e_3(\overline{g_2}, \overline{g_3})\} == \overline{\{e_1(g_2, g_3), e_2(g_2, g_3), e_3(g_2, g_3)\}}.$$

The Weierstrass invariants $\{g_2, g_3\} == \{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\}$ have permutation symmetry and are homogeneous:

$$\{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\} == \{g_2(\omega_3, \omega_1), g_3(\omega_3, \omega_1)\}$$

$$\{g_2(\lambda\,\omega_1, \lambda\,\omega_3), g_3(\lambda\,\omega_1, \lambda\,\omega_3)\} == \left\{\frac{1}{\lambda^4}\, g_2(\omega_1, \omega_3), \frac{1}{\lambda^6}\, g_3(\omega_1, \omega_3)\right\} /;\ \lambda \in \mathbb{C}.$$

The Weierstrass invariants $\{g_2, g_3\} == \{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\}$ are the invariants under the change of variables $\omega_1 \to a\,\omega_1 + b\,\omega_3$ and $\omega_3 \to c\,\omega_1 + d\,\omega_3$ with integers a, b, c, and d, satisfying the restriction $a\,d - b\,c == \pm 1$ (modular transformations):

$$\{g_2(a\,\omega_1 + b\,\omega_3, c\,\omega_1 + d\,\omega_3), g_3(a\,\omega_1 + b\,\omega_3, c\,\omega_1 + d\,\omega_3)\} == \{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\} /;\ \{a, b, c, d\} \in \mathbb{Z} \bigwedge a\,d - b\,c == \pm 1.$$

This property leads to similar properties of the Weierstrass $\wp$ function values at half-periods $\{e_1, e_2, e_3\} == \{e_1(g_2, g_3), e_2(g_2, g_3), e_3(g_2, g_3)\}$ and the Weierstrass zeta function values at half-periods $\{\eta_1, \eta_2, \eta_3\} == \{\eta_1(g_2, g_3), \eta_2(g_2, g_3), \eta_3(g_2, g_3)\}$:

$$\{\{e_1(g_2(a\,\omega_1 + b\,\omega_3, c\,\omega_1 + d\,\omega_3), g_3(a\,\omega_1 + b\,\omega_3, c\,\omega_1 + d\,\omega_3)), e_2(g_2(a\,\omega_1 + b\,\omega_3, c\,\omega_1 + d\,\omega_3), g_3(a\,\omega_1 + b\,\omega_3, c\,\omega_1 + d\,\omega_3)), e_3(g_2(a\,\omega_1 + b\,\omega_3, c\,\omega_1 + d\,\omega_3), g_3(a\,\omega_1 + b\,\omega_3, c\,\omega_1 + d\,\omega_3))\}\} == \{\{e_1(g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)), e_2(g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)), e_3(g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3))\}\} /;\ \{a, b, c, d\} \in \mathbb{Z} \bigwedge a\,d - b\,c == \pm 1$$

$$\{\{\eta_1(g_2(a\,\omega_1 + b\,\omega_3, c\,\omega_1 + d\,\omega_3), g_3(a\,\omega_1 + b\,\omega_3, c\,\omega_1 + d\,\omega_3)), \eta_2(g_2(a\,\omega_1 + b\,\omega_3, c\,\omega_1 + d\,\omega_3), g_3(a\,\omega_1 + b\,\omega_3, c\,\omega_1 + d\,\omega_3)), \eta_3(g_2(a\,\omega_1 + b\,\omega_3, c\,\omega_1 + d\,\omega_3), g_3(a\,\omega_1 + b\,\omega_3, c\,\omega_1 + d\,\omega_3))\}\} == \{\{\eta_1(g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)), \eta_2(g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)), \eta_3(g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3))\}\} /;\ \{a, b, c, d\} \in \mathbb{Z} \bigwedge a\,d - b\,c == \pm 1.$$

Series representations

The Weierstrass half-periods $\{\omega_1, \omega_3\} == \{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\}$ and invariants $\{g_2, g_3\} == \{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\}$ have the following double series expansions:

$$\{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\} == \left\{i\left(\frac{60}{g_2} \sum_{\substack{m, n=-\infty \\ \{m,n\}\neq\{0,0\}}}^{\infty} \frac{1}{(2\,m + 2\,n\,t)^4}\right)^{1/4}, i\,t\left(\frac{60}{g_2} \sum_{\substack{m, n=-\infty \\ \{m,n\}\neq\{0,0\}}}^{\infty} \frac{1}{(2\,m + 2\,n\,t)^4}\right)^{1/4}\right\} /;\ J(t) == \frac{g_2^3}{g_2^3 - 27\,g_3^2}$$

$$\{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\} == \left\{60 \sum_{\substack{m, n=-\infty \\ \{m,n\}\neq\{0,0\}}}^{\infty} \frac{1}{(2\,m\,\omega_1 + 2\,n\,\omega_3)^4}, 140 \sum_{\substack{m, n=-\infty \\ \{m,n\}\neq\{0,0\}}}^{\infty} \frac{1}{(2\,m\,\omega_1 + 2\,n\,\omega_3)^6}\right\} /;\ \mathrm{Im}\left(\frac{\omega_3}{\omega_1}\right) > 0,$$

where $J(t)$ is a Klein invariant modular function.

The last double series can be rewritten in the following forms:

$$\{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\} = \left\{\frac{\pi^4}{12}\left(\frac{1}{\omega_3^4} + \frac{1}{\omega_1^4}\right) + \frac{15}{2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{1}{(m\,\omega_1 - n\,\omega_3)^4} + \frac{1}{(m\,\omega_1 + n\,\omega_3)^4}\right),\right.$$
$$\left.\frac{\pi^6}{216}\left(\frac{1}{\omega_3^6} + \frac{1}{\omega_1^6}\right) + \frac{35}{8}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{1}{(m\,\omega_1 - n\,\omega_3)^6} + \frac{1}{(m\,\omega_1 + n\,\omega_3)^6}\right)\right\}$$

$$\{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\} = \left\{\frac{\pi^4}{12}\frac{1}{\omega_1^4} + \frac{15}{2}\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}\frac{1}{(m\,\omega_1 + n\,\omega_3)^4}, \frac{\pi^6}{216}\frac{1}{\omega_1^6} + \frac{35}{8}\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}\frac{1}{(m\,\omega_1 + n\,\omega_3)^6}\right\}$$

$$\{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\} = \left\{\frac{\pi^4}{12}\frac{1}{\omega_3^4} + \frac{15}{2}\sum_{n=-\infty}^{\infty}\sum_{m=1}^{\infty}\frac{1}{(m\,\omega_1 + n\,\omega_3)^4}, \frac{\pi^6}{216}\frac{1}{\omega_3^6} + \frac{35}{8}\sum_{n=-\infty}^{\infty}\sum_{m=1}^{\infty}\frac{1}{(m\,\omega_1 + n\,\omega_3)^6}\right\}.$$

q-series representations

The Weierstrass invariants $\{g_2, g_3\} = \{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\}$, the Weierstrass $\wp$ function values at half-periods $\{e_1, e_2, e_3\} = \{e_1(g_2, g_3), e_2(g_2, g_3), e_3(g_2, g_3)\}$, and the Weierstrass zeta function values at half-periods $\{\eta_1, \eta_2, \eta_3\} = \{\eta_1(g_2, g_3), \eta_2(g_2, g_3), \eta_3(g_2, g_3)\}$ have numerous q-series representations, for example:

$$\{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\} = \left\{20\left(\frac{\pi}{2\,\omega_1}\right)^4\left(\frac{1}{15} + 16\sum_{k=1}^{\infty}\frac{k^3\,q^{2k}}{1 - q^{2k}}\right), 28\left(\frac{\pi}{2\,\omega_1}\right)^6\left(\frac{2}{189} - \frac{16}{3}\sum_{k=1}^{\infty}\frac{k^5\,q^{2k}}{1 - q^{2k}}\right)\right\}$$

$$\{e_1(g_2, g_3), e_2(g_2, g_3), e_3(g_2, g_3)\} =$$
$$\left\{\frac{\pi^2}{6\,\omega_1^2} + \frac{4\,\pi^2}{\omega_1^2}\sum_{k=1}^{\infty}(2\,k - 1)\,\frac{q^{4k-2}}{1 - q^{4k-2}}, -\frac{\pi^2}{12\,\omega_1^2} - \frac{2\,\pi^2}{\omega_1^2}\sum_{k=1}^{\infty}(-1)^k\,\frac{k\,q^k}{1 + (-1)^k\,q^k}, -\frac{\pi^2}{12\,\omega_1^2} - \frac{2\,\pi^2}{\omega_1^2}\sum_{k=1}^{\infty}\frac{k\,q^k}{1 + q^k}\right\}$$

$$\eta_1 = \frac{\pi^2}{12\,\omega_1} - \frac{2\,\pi^2}{\omega_1}\sum_{k=1}^{\infty}\frac{k\,q^{2k}}{1 - q^{2k}}$$

$$\eta_1 = \frac{\pi^2}{2\,\omega_1}\left(4\sum_{k=1}^{\infty}\frac{q^{2k}}{\left(q^{2k} + 1\right)^2} + \frac{1}{2}\right) - e_1\,\omega_1,$$

where $q = \exp\left(\frac{\pi\,i\,\omega_3}{\omega_1}\right) = \exp\left(\frac{\pi\,i\,\omega_3(g_2, g_3)}{\omega_1(g_2, g_3)}\right)$.

The following rational function of g_2 and g_3 is a modular function if considered as a function of $q = \exp\left(\frac{\pi\,i\,\omega_3}{\omega_1}\right) = \exp\left(\frac{\pi\,i\,\omega_3(g_2, g_3)}{\omega_1(g_2, g_3)}\right)$:

$$J\left(\frac{\omega_3}{\omega_1}\right) = \frac{g_2^3}{g_2^3 - 27\,g_3^2}\ /;\ \{g_2, g_3\} = \{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\} \bigwedge \operatorname{Im}\left(\frac{\omega_3}{\omega_1}\right) > 0.$$

Other series representations

The Weierstrass utilities have some other forms of series expansions, for example:

$$\{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\} == \left\{\frac{\pi^4}{\omega_1^4}\left(20\sum_{k=1}^{\infty}\sigma_3(k)\, q^{2k} + \frac{1}{12}\right), \frac{\pi^6}{\omega_1^6}\left(\frac{1}{216} - \frac{7}{3}\sum_{k=1}^{\infty}\sigma_5(k)\, q^{2k}\right)\right\} /; q == \exp\left(\frac{i\pi\omega_3}{\omega_1}\right)$$

$$\eta_i + e_i\,\omega_j == \frac{\pi^2}{2\,\omega_i}\sum_{n=1}^{\infty}\csc^2\left(\pi\,\frac{2n-1}{2}\,\frac{\omega_j}{\omega_i}\right) /; \{i, j\} \in \{1, 2, 3\} \wedge i \neq j$$

$$\eta_j == \frac{\pi^2}{2\,\omega_j}\left(\frac{1}{6} + \sum_{n=1}^{\infty}\frac{1}{\sin^2\left(\frac{\pi n \omega_k}{\omega_j}\right)}\right) /; \{j, k\} \in \{1, 2, 3\} \wedge j \neq k,$$

where $\sigma_n(k)$ is the divisor sigma function.

Integral representations

The Weierstrass half-periods $\{\omega_1, \omega_3\} == \{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\}$ and invariants $\{g_2, g_3\} == \{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\}$ have the following integral representations:

$$\{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\} == \left\{\int_{e_1}^{\infty}\frac{1}{\sqrt{4t^3 - g_2 t - g_3}}\, dt,\ i\int_{-\infty}^{e_3}\frac{1}{\sqrt{4t^3 - g_2 t - g_3}}\, dt\right\} /;$$

$$g_2 \in \mathbb{R} \bigwedge g_3 \in \mathbb{R} \bigwedge g_2^3 - 27 g_3^2 > 0 \bigwedge 4t^3 - g_2 t - g_3 == 4(t - e_1)(t - e_2)(t - e_3) \bigwedge e_1 > e_2 > e_3$$

$$\{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\} == \left\{\frac{5}{8}\int_0^{\infty} t^3 (u(t, \omega_1, \omega_3) + v(t, \omega_1, \omega_3))\, dt,\ \frac{7}{384}\int_0^{\infty} t^5 (u(t, \omega_1, \omega_3) - v(t, \omega_1, \omega_3))\, dt\right\} /;$$

$$u(t, \omega_1, \omega_3) == \frac{\cosh(t\,\omega_3) + e^{-\frac{t}{2}\omega_3}\sinh\left(\frac{t\,\omega_3}{2}\right)}{\sinh\left(\frac{1}{2}t(\omega_1 - \omega_3)\right)\sinh\left(\frac{1}{2}t(\omega_1 + \omega_3)\right)} \bigwedge v(t, \omega_1, \omega_3) == \frac{e^{\frac{i t}{2}\omega_3}\cos\left(\frac{t\,\omega_3}{2}\right)}{\sin\left(\frac{1}{2}t(\omega_1 - \omega_3)\right)\sin\left(\frac{1}{2}t(\omega_1 + \omega_3)\right)}.$$

Product representations

The Weierstrass utilities can have product representations. For example,
the Weierstrass $\wp$ function values at half-periods $\{e_1, e_2, e_3\} == \{e_1(g_2, g_3), e_2(g_2, g_3), e_3(g_2, g_3)\}$ can be expressed through the following products:

$$\{e_1(g_2, g_3), e_2(g_2, g_3), e_3(g_2, g_3)\} == \frac{\pi^2}{12\,\omega_1^2}\left(\prod_{n=1}^{\infty}(1 - q^{2n})\right)^4$$

$$\left\{\left(\prod_{n=1}^{\infty}(1 + q^{2n-1})\right)^8 + \left(\prod_{n=1}^{\infty}(1 - q^{2n-1})\right)^8, \left(\prod_{n=1}^{\infty}(1 + q^{2n-1})\right)^8 - 2\left(\prod_{n=1}^{\infty}(1 - q^{2n-1})\right)^8, 2\left(\prod_{n=1}^{\infty}(1 + q^{2n-1})\right)^8 - \left(\prod_{n=1}^{\infty}(1 - q^{2n-1})\right)^8\right\}$$

$$e_2 - e_3 == \frac{4\pi^2}{\omega_1^2}\, q\left(\prod_{n=1}^{\infty}(1 - q^{2n})\right)^4\left(\prod_{n=1}^{\infty}(1 + q^{2n})\right)^8$$

$$e_1 - e_3 == \frac{\pi^2}{4\,\omega_1^2}\left(\prod_{n=1}^{\infty}(1 - q^{2n})\right)^4\left(\prod_{n=1}^{\infty}(1 + q^{2n-1})\right)^8$$

$$e_1 - e_2 = \frac{\pi^2}{4\,\omega_1^2} \left(\prod_{n=1}^{\infty} \left(1 - q^{2n}\right)\right)^4 \left(\prod_{n=1}^{\infty} \left(1 - q^{2n-1}\right)\right)^8$$

$$(e_2 - e_3)\,(e_1 - e_3)\,(e_1 - e_2) = \frac{\pi^6}{4\,\omega_1^6}\, q \left(\prod_{n=1}^{\infty} \left(1 - q^{2n}\right)\right)^{12},$$

where $q = \exp\left(\frac{\pi\, i\, \omega_3}{\omega_1}\right) = \exp\left(\frac{\pi\, i\, \omega_3(g_2, g_3)}{\omega_1(g_2, g_3)}\right)$.

Identities

The Weierstrass utilities satisfy numerous identities, for example:

$$\Delta(\tau)^6 = \frac{1}{16\,777\,216} \left(\Delta\left(\frac{\tau}{2}\right)^2 \Delta\left(\frac{\tau+1}{2}\right)^4 + 2\,\Delta\left(\frac{\tau}{2}\right)^3 \Delta\left(\frac{\tau+1}{2}\right)^3 + \Delta\left(\frac{\tau}{2}\right)^4 \Delta\left(\frac{\tau+1}{2}\right)^2 - 2304\,\Delta(\tau)^2\,\Delta\left(\frac{\tau}{2}\right)^2 \Delta\left(\frac{\tau+1}{2}\right)^2 + 393\,216\,\Delta\left(\frac{\tau}{2}\right) \Delta(\tau)^4\, \Delta\left(\frac{\tau+1}{2}\right)\right) /;\ \Delta(\tau) = g_2^3 - 27\, g_3^2 \bigwedge \{g_2, g_3\} = \{g_2(1, \tau), g_3(1, \tau)\}$$

$$e_1 + e_2 + e_3 = 0$$

$$\eta_1 + \eta_2 + \eta_3 = 0$$

$$\eta_1\,\omega_3 - \eta_3\,\omega_1 = \frac{\pi\, i}{2}$$

$$\eta_i\,\omega_j - \eta_j\,\omega_i = \operatorname{sgn}\left(\operatorname{Re}\left(-i\,\frac{\omega_j}{\omega_i}\right)\right) \frac{\pi\, i}{2} /;\ \{i, j\} \in \{1, 2, 3\} \bigwedge i \neq j.$$

Representations of derivatives

The first derivatives of Weierstrass half-periods $\{\omega_1, \omega_3\}$ and the Weierstrass $\wp$ and zeta function values at half-periods $\{e_1, e_2, e_3\}$ and $\{\eta_1, \eta_2, \eta_3\}$ with respect to variable g_2 and g_3 have the following representations:

$$\frac{\partial\{\omega_1, \omega_3\}}{\partial g_2} = \left\{\frac{18\, g_3\, \eta_1 - g_2^2\, \omega_1}{4\left(g_2^3 - 27\, g_3^2\right)}, \frac{18\, g_3\, \eta_3 - g_2^2\, \omega_3}{4\left(g_2^3 - 27\, g_3^2\right)}\right\}$$

$$\frac{\partial\{\omega_1, \omega_3\}}{\partial g_3} = \left\{\frac{9\, g_3\, \omega_1 - 6\, g_2\, \eta_1}{2\left(g_2^3 - 27\, g_3^2\right)}, \frac{9\, g_3\, \omega_3 - 6\, g_2\, \eta_3}{2\left(g_2^3 - 27\, g_3^2\right)}\right\}$$

$$\frac{\partial\{e_1, e_2, e_3\}}{\partial g_2} = \frac{1}{4\left(g_2^3 - 27\, g_3^2\right)} \left(2\, g_2^2\, \{e_1, e_2, e_3\} + 6\, g_3\, g_2 - 36\, g_3\, \{e_1, e_2, e_3\}^2 + \{e_1', e_2', e_3'\} \left(g_2^2\, \{\omega_1, \omega_2, \omega_3\} - 18\, g_3\, \{\eta_1, \eta_2, \eta_3\}\right)\right)$$

$$\frac{\partial\{e_1, e_2, e_3\}}{\partial g_3} = \frac{1}{2\left(g_2^3 - 27\, g_3^2\right)} \left(12\, g_2\, \{e_1, e_2, e_3\}^2 - 18\, g_3\, \{e_1, e_2, e_3\} - 2\, g_2^2 + (6\, g_2\, \{\eta_1, \eta_2, \eta_3\} - g_3\, \{\omega_1, \omega_2, \omega_3\}) \{e_1', e_2', e_3'\}\right)$$

$$\frac{\partial\{\eta_1, \eta_2, \eta_3\}}{\partial g_2} = \frac{1}{8\left(g_2^3 - 27 g_3^2\right)} \left(18 g_3 \left\{e_1', e_2', e_3'\right\} - g_2 (3 g_3 + 2 g_2 \{e_1, e_2, e_3\}) \{\omega_1, \omega_2, \omega_3\} + 2\left(g_2^2 + 18 g_3 \{e_1, e_2, e_3\}\right) \{\eta_1, \eta_2, \eta_3\}\right)$$

$$\frac{\partial\{\eta_1, \eta_2, \eta_3\}}{\partial g_3} = \frac{1}{4\left(g_2^3 - 27 g_3^2\right)} \left(\left(g_2^2 + 18 g_3 \{e_1, e_2, e_3\}\right) \{\omega_1, \omega_2, \omega_3\} - 6 g_2 \left\{e_1', e_2', e_3'\right\} - 6 (3 g_3 + 2 g_2 \{e_1, e_2, e_3\}) \{\eta_1, \eta_2, \eta_3\}\right),$$

where $\left\{e_1', e_2', e_3'\right\} = \{\wp'(\omega_1; g_2, g_3), \wp'(\omega_2; g_2, g_3), \wp'(\omega_3; g_2, g_3)\}$ are the values of the derivative of the Weierstrass elliptic $\wp$ function $\frac{\partial\wp(z; g_2, g_3)}{\partial z} = \wp'(z; g_2, g_3)$ at half-period points $z = \omega_j /; j = 1, 2, 3$.

The first derivatives of Weierstrass invariants $\{g_2, g_3\}$ with respect to the variables ω_1 and ω_3 can be represented in different forms:

$$\frac{\partial\{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\}}{\partial\omega_1} = \left\{-30 \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} \frac{m}{(m\,\omega_1 + n\,\omega_3)^5}, -\frac{105}{4} \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} \frac{m}{(m\,\omega_1 + n\,\omega_3)^7}\right\}$$

$$\frac{\partial\{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\}}{\partial\omega_1} = \left\{-\frac{4 g_2}{\omega_1} - \frac{40\, i\, \pi^5\, \omega_3}{\omega_1^6} \sum_{k=1}^{\infty} \frac{k^4 q^{2k}}{\left(1 - q^{2k}\right)^2}, \frac{14\, i\, \pi^7\, \omega_3}{3\, \omega_1^8} \sum_{k=1}^{\infty} \frac{k^6 q^{2k}}{\left(1 - q^{2k}\right)^2} - \frac{6 g_3}{\omega_1}\right\} /; q = \exp\left(\frac{\pi\, i\, \omega_3}{\omega_1}\right)$$

$$\frac{\partial\{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\}}{\partial\omega_1} = \left\{-\frac{\omega_1}{\pi\,\omega_3} \sqrt{-\frac{\omega_3^2}{\omega_1^2}}\, (12 g_3\, \omega_3 - 8 g_2\, \eta_3), -\frac{\omega_1}{\pi\,\omega_3} \sqrt{-\frac{\omega_3^2}{\omega_1^2}} \left(\frac{2}{3} g_2^2\, \omega_3 - 12 g_3\, \eta_3\right)\right\}$$

$$\frac{\partial\{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\}}{\partial\omega_3} = \left\{-30 \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \frac{n}{(m\,\omega_1 + n\,\omega_3)^5}, -\frac{105}{4} \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \frac{n}{(m\,\omega_1 + n\,\omega_3)^7}\right\}$$

$$\frac{\partial\{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\}}{\partial\omega_3} = \left\{\frac{40\, i\, \pi^5}{\omega_1^5} \sum_{k=1}^{\infty} \frac{q^{2k} k^4}{\left(1 - q^{2k}\right)^2}, -\frac{14\, i\, \pi^7}{3\, \omega_1^7} \sum_{k=1}^{\infty} \frac{q^{2k} k^6}{\left(1 - q^{2k}\right)^2}\right\} /; q = \exp\left(\frac{\pi\, i\, \omega_3}{\omega_1}\right)$$

$$\frac{\partial\{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\}}{\partial\omega_3} = \left\{\frac{\omega_1}{\pi\,\omega_3} \sqrt{-\frac{\omega_3^2}{\omega_1^2}}\, (12 g_3\, \omega_1 - 8 g_2\, \eta_1), \frac{\omega_1}{\pi\,\omega_3} \sqrt{-\frac{\omega_3^2}{\omega_1^2}} \left(\frac{2}{3} g_2^2\, \omega_1 - 12 g_3\, \eta_1\right)\right\}.$$

The k^{th}-order derivatives of Weierstrass invariants $\{g_2, g_3\}$ with respect to the variables ω_1 and ω_3 have the following representations:

$$\frac{\partial^k\{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\}}{\partial\omega_1^k} = \left\{\frac{5}{4} \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} \frac{(-1)^k (k+3)!\, m^k}{(m\,\omega_1 + n\,\omega_3)^{4+k}}, \frac{7}{192} \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} \frac{(-1)^k (k+5)!\, m^k}{(m\,\omega_1 + n\,\omega_3)^{6+k}}\right\} /; k \in \mathbb{N}^+$$

$$\frac{\partial^k\{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\}}{\partial\omega_3^k} = \left\{\frac{5}{4} \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \frac{(-1)^k (k+3)!\, n^k}{(m\,\omega_1 + n\,\omega_3)^{k+4}}, \frac{7}{192} \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \frac{(-1)^k (k+5)!\, n^k}{(m\,\omega_1 + n\,\omega_3)^{k+6}}\right\} /; k \in \mathbb{N}^+.$$

Integration

The indefinite integrals of Weierstrass invariants $\{g_2, g_3\}$ with respect to the variable ω_1 have the following representations:

$$\left\{\int g_2(\omega_1, \omega_3)\, d\omega_1, \int g_3(\omega_1, \omega_3)\, d\omega_1\right\} = \left\{\frac{5\,\omega_1}{2\,\omega_3^3} \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \frac{m^2\,\omega_1^2 + 3\,m\,n\,\omega_3\,\omega_1 + 3\,n^2\,\omega_3^2}{n^3\,(m\,\omega_1 + n\,\omega_3)^3} - \frac{\pi^4}{36\,\omega_1^3},\right.$$
$$\left.\frac{7\,\omega_1}{8\,\omega_3^5} \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \left(m^4\,\omega_1^4 + 5\,m^3\,n\,\omega_3\,\omega_1^3 + 10\,m^2\,n^2\,\omega_3^2\,\omega_1^2 + 10\,m\,n^3\,\omega_3^3\,\omega_1 + 5\,n^4\,\omega_3^4\right) \big/ \left(n^5\,(m\,\omega_1 + n\,\omega_3)^5\right) - \frac{\pi^6}{1080\,\omega_1^5}\right\}.$$

Differential equations

The Weierstrass half-periods $\{\omega_1, \omega_3\} = \{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\}$ satisfy the following differential equations:

$$\left(g_2^3 - 27\,g_3^2\right) \frac{\partial\omega_1}{\partial g_2} + \frac{1}{4}\,\omega_1\,g_2^2 - \frac{9}{2}\,g_3\,\zeta(\omega_1; g_2, g_3) = 0 \;/;\; \{\omega_1, \omega_3\} = \{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\} \bigwedge$$
$$\left(g_2^3 - 27\,g_3^2\right) \frac{\partial\omega_3}{\partial g_2} + \frac{1}{4}\,\omega_3\,g_2^2 - \frac{9}{2}\,g_3\,\zeta(\omega_3; g_2, g_3) = 0 \;/;\; \{\omega_1, \omega_3\} = \{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\}$$

$$4\left(g_2^3 - 27\,g_3^2\right) \frac{\partial\{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\}}{\partial g_2} = 18\,g_3\,\eta_1 - g_2^2\,\{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\} \;/;$$
$$\{g_2, g_3\} = \left\{g_2\,(\eta_1, \eta_2), g_3\,(\eta_1, \eta_2)\right\} \bigwedge \{\eta_1, \eta_2\} = \{\zeta(\omega_1; g_2, g_3), \zeta(\omega_3; g_2, g_3)\} \bigwedge \{\omega_1, \omega_3\} = \{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\}.$$

The Weierstrass invariants $\{g_2, g_3\} = \{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\}$ satisfy the following differential equations:

$$\omega_1 \frac{\partial\{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\}}{\partial\omega_1} + \omega_3 \frac{\partial\{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\}}{\partial\omega_3} + \{4, 6\}\,\{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\} = 0$$

$$\omega_1 \frac{\partial g_2(\omega_1, \omega_3)}{\partial\omega_1} + \omega_3 \frac{\partial g_2(\omega_1, \omega_3)}{\partial\omega_3} + 4\,g_2(\omega_1, \omega_3) = 0$$

$$\omega_1 \frac{\partial g_3(\omega_1, \omega_3)}{\partial\omega_1} + \omega_3 \frac{\partial g_3(\omega_1, \omega_3)}{\partial\omega_3} + 6\,g_3(\omega_1, \omega_3) = 0.$$

The Weierstrass zeta function values at half-periods $\{\eta_1, \eta_2, \eta_3\} = \{\eta_1(g_2, g_3), \eta_2(g_2, g_3), \eta_3(g_2, g_3)\}$ satisfy the following differential equations:

$$72\,g_3\,\frac{\partial\eta_J}{\partial g_2} + 4\,g_2^2\,\frac{\partial\eta_J}{\partial g_3} - \omega_j\,g_2 = 0 \;/;\; j \in \{1, 2\}.$$

Applications of Weierstrass utilities

Applications of Weierstrass utilities include the application areas of the Weierstrass elliptic functions, such as integrable nonlinear differential equations, motion in cubic and quartic potentials, description of the movement of a spherical pendulum, and construction of minimal surfaces.

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